ON LINEAR TRANSFORMATIONS PRESERVING THE PÓLYA FREQUENCY PROPERTY

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Abstract. We prove that certain linear operators preserve the Pólya frequency property and real-rootedness, and apply our results to settle some conjectures and open problems in combinatorics proposed by Bóna, Brenti and Reiner-Welker.

1. Introduction

Many sequences encountered in various areas of mathematics, statistics and computer science are known or conjectured to be unimodal or log-concave, see [8, 32, 34]. A sufficient condition for a sequence to enjoy these properties is that it is a Pólya frequency (PF for short) sequence, or equivalently for finite sequences, that its generating function has only real and non-positive zeros. It is often the case that the generating function of a finite PF-sequence has more transparent properties when expanded in a basis other than the standard basis $\{x^i\}_{i\geq 0}$ of $\mathbb{R}[x]$. Therefore it is natural to investigate how PF-sequences translate when expressed in various basis. This amounts to studying properties of the linear operator that maps one basis to another. A systematic study of this was first pursued by Brenti in [7]. This is also the theme of this paper.

In Section 3 we will study linear operators of the type

$$\phi_F = \sum_{k=0}^{n} Q_k(x)\frac{d^k}{dx^k},$$

where $F(x,z) = \sum_{k=0}^{n} Q_k(x)z^k \in \mathbb{R}[x,z]$. Here we will give sufficient conditions on F for ϕ_F to preserve the PF-property. The results attained generalizes and unifies theorems of Hermite, Poulain, Pólya and Schur. We will also in this section give a sufficient condition for a family of natural $\mathbb{R}$-bilinear forms to preserve the PF-property in both arguments. This generalizes results of Wagner [11, 37, 38].

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An important linear operator in combinatorics is the operator defined by $\mathcal{E}(\binom{x}{i}) = x^i$, for all $i \in \mathbb{N}$. In Section 4 we will prove that whenever a polynomial f of degree d has nonnegative coefficients when expanded in the basis $\{x^i(x+1)^{d-i}\}_{i=0}^d$ the polynomial $\mathcal{E}(f)$ will have only real, non-positive and simple zeros.

In the remainder of the paper we use the theory developed to settle some conjectures and open problems raised in combinatorics. Highlights are; we prove that the q-Eulerian polynomials, $A_n(x;q)$, defined by Foata and Schützenberger [17] and further studied by Brenti in [10] have only real zeros for all integers q. This settles a conjecture raised by Brenti. We will also continue the study of the W-Eulerian polynomials, defined for any finite Coxeter group W and the q-analog $B_n(x;q)$, initiated by Brenti in [9].

In Section 7 we prove that the h-vectors of a family simplicial complexes associated to finite Weyl groups defined by Fomin and Zelevinski [18] are PF, thus settling an open problem raised by Reiner and Welker [30]. In Section 5 we prove that the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ of t-stack sortable permutations in $\mathcal{S}_n$ with k descents form PF-sequences when $t = 2, n-2$, and thereby settling two new cases of an open problem proposed by Bóna [2, 3].

2. Notation and preliminaries

In this section we collect definitions, notation and results that will be used frequently in the rest of the paper. Let $\{a_i\}_{i=0}^\infty$ be a sequence of real numbers. It is *unimodal* if there is a number p such that $a_0 \leq a_1 \leq \cdots \leq a_p \geq a_{p+1} \geq \cdots$, and *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i > 0$.

An infinite matrix $A = (a_{ij})_{i,j\geq 0}$ of real numbers is *totally positive*, TP, if all minors of A are nonnegative. An infinite sequence $\{a_i\}_{i=0}^\infty$ of real numbers is a *Pólya frequency sequence*, PF-sequence, if the matrix $(a_{i-j})_{i,j\geq 0}$ is TP. Thus a PF-sequence is by definition log-concave and therefore also unimodal. A finite sequence $a_0, a_1, a_2, \dots, a_n$ is said to be PF if the infinite sequence $a_0, a_1, a_2, \dots, a_n, 0, 0, \dots$ is PF. A sequence $\{a_i\}_{i=0}^\infty$ is said to be PF_r if all minors of size r of $(a_{i-j})_{i,j\geq 0}$ are nonnegative. If the polynomials $\{b_i(x)\}_{i=0}^d$ are linearly independent over $\mathbb{R}$ and $r \in \mathbb{N}$ we define the set $PF_r[\{b_i(x)\}_{i=0}^d]$ to be

$$PF_r[\{b_i(x)\}_{i=1}^d] = \{\sum_{i=0}^d \lambda_i b_i(x) : \{\lambda_i\}_{i=0}^\infty \text{ is } PF_r\},$$

and $PF[\{b_i(x)\}_{i=1}^d] = \bigcap_{r=0}^\infty PF_r[\{b_i(x)\}_{i=1}^d]$.

The following theorem characterizes PF-sequences. It was conjectured by Schoenberg and proved by Edrei [16], see also [24].

Theorem 2.1. *Let $\{a_i\}_{i=0}^\infty$ be a sequence of real numbers with $a_0 = 1$. Then it is a PF-sequence if and only if the generating function can be* *expanded, in a neighborhood of the origin, as*

$$\sum_{i\geq 0} a_i z^i = e^{\gamma z}\frac{\prod_{i\geq 0}(1+\alpha_i z)}{\prod_{i\geq 0}(1-\beta_i z)},$$

where $\gamma \geq 0$*,* $\alpha_i, \beta_i > 0$ *and* $\sum_{i\geq 0}(\alpha_i + \beta_i) < \infty$*.*

A consequence of this theorem is that a finite sequence is PF if and only if its generating function is a polynomial with only real non positive zeros.

Let $f, g \in \mathbb{R}[x]$ be real-rooted with zeros: $\alpha_1 \leq \cdots \leq \alpha_i$ and $\beta_1 \leq \cdots \leq \beta_j$, respectively. We say that f *interlaces* g, denoted $f \preceq g$, if $j = i + 1$ and

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \beta_{j-1} \leq \alpha_{j-1} \leq \beta_j.$$

We say that f *alternates left of* g, denoted $f \ll g$, if $i = j$ and

$$\alpha_1 \leq \beta_1 \leq \cdots \leq \beta_{i-1} \leq \alpha_i \leq \beta_i.$$

If in addition f and g have no common zero then we say that f *strictly interlaces* g and f *strictly alternates left of* g, respectively. We also say that two polynomials f and g *alternate* if one of the polynomials alternates left of or interlaces the other. We will need two simple lemmata concerning these concepts. A polynomial is said to be *standard* if its leading coefficient is positive.

Lemma 2.2. *Let* g *and* $\{f_i\}_{i=1}^n$ *be real-rooted standard polynomials.*

(i) *If for each* $1 \leq i \leq n$ *we have either* $g \ll f_i$ *or* $g \preceq f_i$*. Then the sum* $F = f_1 + f_2 + \cdots + f_n$ *is real-rooted with* $g \preceq F$ *or* $g \ll F$*, depending on the degree of* F*.*

(ii) *If for each* $1 \leq i \leq n$ *we have either* $f_i \ll g$ *or* $f_i \preceq g$*. Then the sum* $F = f_1 + f_2 + \cdots + f_n$ *is real-rooted with* $F \preceq g$ *or* $F \ll g$*, depending on the degree of* F*.*

Proof. The lemma follows easily by counting the sign-changes of F at the zeros of g, see e.g., [39, Prop. 3.5]. □

The next lemma is obvious:

Lemma 2.3. *If* $f_0, f_1, \ldots, f_n$ *are real-rooted polynomials with* $f_0 \ll f_n$ *and* $f_{i-1} \ll f_i$ *for all* $1 \leq i \leq n$*, then* $f_i \ll f_j$ *for all* $0 \leq i \leq j \leq n$*.*

The following theorem is a characterization of alternating polynomials due to Obreschkoff [26] and Dedieu [14]:

Theorem 2.4. *Let* $f, g \in \mathbb{R}[x]$*. Then* f *and* g *alternate (strictly alternate) if and only if all polynomials in the space*

$$\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}\},$$

have only real (real and simple) zeros.

An immediate but non-trivial consequence of this theorem is:

Corollary 2.5. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Then ϕ preserves the real-rootedness property (real- and simple-rootedness property) if and only if ϕ preserves the alternating property (strictly alternating property).*

We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$. The symmetric group of bijections $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is denoted by $\mathcal{S}_n$. A *descent* in a permutation $\pi \in \mathcal{S}_n$ is an index $1 \leq i \leq n$, such that $\pi(i) > \pi(i+1)$. Let $\operatorname{des}(\pi)$ denote the number of descents in π. The *Eulerian polynomials*, $A_n(x)$, are defined by $A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{des}(\pi)+1}$ and satisfies, see e.g. [12]

$$\sum_{k \geq 0} k^n x^n = \frac{A_n(x)}{(1-x)^{n+1}}.$$

The binomial polynomials are defined by $\binom{x}{0} = 1$ and $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ for $k \geq 1$.

In several proofs we will implicitly use the fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. In particular the limit of a sequence of real-rooted polynomials is again real-rooted. For a treatment of these matters we refer the reader to [25].

3. A class of linear operators preserving the PF-property

For any polynomial $F(x, z) = \sum_{k=0}^{n} Q_k(x) z^k \in \mathbb{R}[x, z]$ we define a linear operator $\phi_F : \mathbb{R}[x] \to \mathbb{R}[x]$ by,

$$\phi_F(f) := \sum_{k=0}^{n} Q_k(x) \frac{d^k}{dx^k} f(x).$$

In this section we will investigate for which $F \in \mathbb{R}[x, z]$ the linear operator ϕ_F preserves real-rootedness- and the PF-property .

We will need some terminology and a theorem from [5]. For $\xi \in \mathbb{R}$ let $T_\xi : \mathbb{R}[x] \to \mathbb{R}[x]$ be the translation operator defined by $T_\xi(f(x)) = f(x + \xi)$. For any linear operator $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ we define a linear transform $\mathcal{L}_\phi : \mathbb{R}[x] \to \mathbb{R}[x, z]$ by

$$\begin{aligned} \mathcal{L}_\phi(f) &:= \phi(T_z(f)) \\ &= \sum_n \phi(f^{(n)})(x) \frac{z^n}{n!} \qquad (3.1) \\ &= \sum_n \frac{\phi(x^n)}{n!} f^{(n)}(z). \end{aligned}$$

Definition 3.1. Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Define a function $d_\phi : \mathbb{R}[x] \to \mathbb{N} \cup \{-\infty\}$ by: If $\phi(f^{(n)}) = 0$ for all $n \in \mathbb{N}$, we let

$d_\phi(f) = -\infty$. Otherwise let $d_\phi(f)$ be the smallest integer d such that $\phi(f^{(n)}) = 0$ for all $n > d$. Hence $d_\phi(f) \le \deg f$ for all $f \in \mathbb{R}[x]$.

The set $\mathscr{A}(\phi)$ is defined as follows: If $d_\phi(f) \in \{-\infty, 0\}$ and $\phi(f)$ is standard real- and simple-rooted, then $f \in \mathscr{A}(\phi)$. Moreover, $f \in \mathscr{A}(\phi)$ if $d = d_\phi(f) \ge 1$ and all of the following conditions are satisfied:

(i) $\phi(f^{(i)})$ all have leading coefficients of the same sign and $\deg(\phi(f^{(i-1)})) = \deg(\phi(f^{(i)})) + 1$ for $1 \le i \le d$,
(ii) $\phi(f)$ and $\phi(f')$ have no common real zero,
(iii) $\phi(f^{(d)})$ strictly interlaces $\phi(f^{(d-1)})$,
(iv) for all $\xi \in \mathbb{R}$ the polynomial $\mathcal{L}_\phi(f)(\xi, z)$ is real-rooted.

The following theorem is proved in [5]:

Theorem 3.2. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. If $f \in \mathscr{A}(\phi)$ then $\phi(f)$ is real- and simple-rooted.*

We will also need the following classical theorem of Hermite and Poulain. For a proof see [26].

Theorem 3.3. *Let $f = a_0 + a_1x + \cdots + a_nx^n$ and g be real-rooted polynomials. Then the polynomial*

$$f(\frac{d}{dx})g := a_0g(x) + a_1g'(x) + \cdots + a_ng^{(n)}(x)$$

is real-rooted. Moreover, if $f(\frac{d}{dx})g \neq 0$ then any multiple zero of $f(\frac{d}{dx})g$ is a multiple zero of g.

The following theorem gives a sufficient condition for a polynomial to be mapped onto a real-rooted polynomial.

Theorem 3.4. *Let $F = \sum_{k=0}^n Q_k(x)z^k$ be such that $Q_0 \neq 0$ and*

(I) *For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted,*
(II) *Q_0 strictly interlaces or strictly alternates left of Q_1, and $\deg Q_0 = 0$ or Q_0 and Q_1 have leading coefficients of the same sign.*

Suppose that

(III) *f is real- and simple-rooted and that for $0 \le k \le \deg f$ the polynomials $\phi_F(f^{(k)})$ have their leading term of the same sign with*

$$\deg \phi_F(f^{(k)}) = \deg Q_0 + \deg f - k.$$

Then $\phi_F(f)$ is real- and simple-rooted.

Proof. We will show that the set of real- and simple-rooted polynomials satisfying (III) is a subset of $\mathscr{A}(\phi_F)$ by verifying conditions (i)-(iv) of Definition 3.1. Condition (i) follows immediately from (III). For condition (iv) note that

$$\mathcal{L}_\phi(f)(\xi, z) = \sum_{k=0}^n Q_k(\xi)f^{(k)}(\xi + z),$$

so by the Theorem 3.3 condition (iv) is satisfied. Suppose that η is a common zero of $\phi_F(f)$ and $\phi_F(f')$. By (3.1) we have that 0 is a multiple zero of $\mathcal{L}_\phi(f)(\eta, z)$. Moreover, since $\mathcal{L}_\phi(f)(\eta, z)$ is not identically equal to zero, by (II), Theorem 3.3 tells us that 0 is a multiple zero of $f(\eta+z)$. This means that η is multiple zero of f contrary to the assumption that f is simple-rooted, and verifies condition (ii).

For condition (iii) we have to show that for all $\alpha \in \mathbb{R}$ such that $x+\alpha$ satisfies (III) the polynomial $\phi_F(1) = Q_0$ strictly interlaces $f(x) := \phi_F(x+\alpha) = (x+\alpha)Q_0 + Q_1$. This follows from (II) when analyzing the sign of $f(x) := \phi_F(x+\alpha)$ at the zeros of Q_0: Let $\alpha_k < \alpha_{k-1} < \cdots < \alpha_1$ be the zeros of Q_0 ordered by size. Suppose that Q_0 and Q_1 are standard and that Q_0 strictly interlaces or strictly alternates left of Q_1. Then $\operatorname{sgn} f(\alpha_i) = \operatorname{sgn} Q_1(\alpha_i) = (-1)^i$ for $1 \leq i \leq k$. By Rolle's theorem we know that f has a zero in each interval (α_i, α_{i+1}). This accounts for $k-1$ real zeros of f. Since Q_0 has positive sign, so does f by condition (III). Now, because $f(\alpha_1) < 0$ and f is standard, f must have a zero to the right of α_1. We now know that f has k zeros real. The signs at α_i forces the remaining zero to be in the interval $(-\infty, \alpha_k)$. Thus Q_0 strictly interlaces Q_1 as was to be shown.

Now, if $Q_0 = A \in \mathbb{R}$ then $\deg Q_1 \leq 1$. Suppose that $Q_1 = B \in \mathbb{R}$. Then clearly A strictly interlaces $(x+\alpha)A + B$. If $Q_0 = A$ and $Q_1 = Cx + D$ where $A, B, C \in \mathbb{R}$, then $f = (A+C)x + A\alpha + D$, so by (III) we have that Q_0 strictly interlaces f. This concludes the proof. $\square$

In some cases it may be convenient to have sharper hypothesis. Therefore we state the following form of the theorem.

Corollary 3.5. *Let $d \in \mathbb{N}$ be given and let $F = \sum_{k=0}^{n} Q_k(x)z^k$ be such that $Q_0 \neq 0$ and*

(i) *For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted,*
(ii) *Q_0 strictly interlaces or strictly alternates left of Q_1, and $\deg Q_0 = 0$ or Q_0 and Q_1 have leading coefficients of the same sign.*
(iii) *The polynomials $\phi_F(x^k)$, $0 \leq k \leq d$ have the same sign and*

$$\deg \phi_F(x^k) = \deg Q_0 + k.$$

Then $\phi_F(f)$ is real-rooted (real- and simple-rooted) if f is real-rooted (real and simple-rooted) and $\deg(f) \leq d$.

Proof. The case of real- and simple-rooted f follows immediately from Theorem 3.4 since (iii) implies (III). If f is a real-rooted polynomial of degree at most d, then f is the limit of a sequence $\{f_k\}_{k=0}^{\infty}$ of real- and simple-rooted polynomials of degree at most d. It follows that $\phi_F(f)$ is the limit of $\phi_F(f_k)$, and the thesis follows by continuity. $\square$

In the language of PF-sequences we have:

Theorem 3.6. *Let $d \in \mathbb{N}$ be given and let $F = \sum_{k=0}^{n} Q_k(x)z^k \in \mathbb{R}[x, z]$ be such that $Q_0 \neq 0$ and*

(i) *For all* $\xi \in \mathbb{R}$, $F(\xi, z)$ *is real-rooted,*
(ii) $\phi_F(1)$ *strictly interlaces* $\phi_F(x)$.
(iii) *For all* $0 \leq k \leq d$
$$\deg \phi_F(x^k) = \deg Q_0 + k,$$
and $\phi_F(x^k) \in PF_1$.

Then $PF[\{\phi_F(x^i)\}_{i=0}^d] \subseteq PF[x^i]$.

Several old results can be derived from these last few theorems. In [27, p. 163] Pólya gave a theorem which he states probably was the most general theorem on real-rootedness known at the time. "Dieser Satz gehört wohl zu den allgemeinsten bekannten Sätzen über Wurzelrealität.":

Theorem 3.7. *Let* $f(x)$ *be a real-rooted polynomial of degree* n*, and let*
$$b_0 + b_1 x + \cdots + b_{n+m} x^{n+m}, \quad (m \geq 0)$$
be a real-rooted polynomial such that $b_i > 0$ *for* $0 \leq i \leq n$. *Then the equation*
$$G(x, y) := b_0 f(y) + b_1 x f'(y) + b_2 x^2 f''(y) + \cdots + b_n f^{(n)}(y) = 0,$$
has n *real intersection points, (counted with multiplicity), with the line*
$$sx - ty + u = 0,$$
provided that $s, t \geq 0$, $s + t > 0$ *and* $u \in \mathbb{R}$.

Proof. We may assume that $s, t > 0$ since the other cases follows by continuity when s and/or t tends to zero. Thus we may write the equation as
$$a_0 g(x) + a_1 x g'(x) + a_2 x^2 g''(x) + \cdots + a_n g^{(n)}(x) = 0,$$
where $g(x) = f(st^{-1}x + ut^{-1})$ and $a_i = s^i t^{-i} b_i$. Now, we see that all hypothesis of Corollary 3.5 are satisfied for
$$F(x, z) = a_0 + a_1 xz + a_2 x^2 z^2 + \cdots + a_{n+m} x^{n+m} z^{n+m},$$
when $d = n$. □

We will later need one famous consequence of this theorem, $t = 1, s = u = 0$, due to Schur [31].

Theorem 3.8. *Let* $f = \sum_{k=0}^n a_k x^k$ *and* $g = \sum_{k=0}^m b_k x^k$ *be two real-rooted polynomials such that* g *has all zeros of the same sign. Then the polynomial*
$$(fSg)(x) = \sum_{k \geq 0}^{M} k! a_k b_k x^k,$$
where $M = \min(m, n)$ *has only real zeros.*

3.1. **Multiplier-sequences.** A multiplier-sequence is a sequence $T = \{\gamma_i\}_{i=0}^{\infty}$ of real numbers such that if a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ has only real zeros, then the polynomial

$$T[p(x)] := a_0\gamma_0 + a_1\gamma_1 x + \cdots + a_n\gamma_n x^n,$$

also has only real zeros. There is a characterization of multiplier-sequences due to Pólya and Schur [27, p. 100-124]:

Theorem 3.9. *Let $T = \{\gamma_i\}_{i=0}^{\infty}$ be a sequence of real numbers and let $\phi(x) = T[e^x] = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ be its exponential generating function. Then T is a multiplier-sequence if and only if ϕ is a real entire function which can be written as*

$$\phi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + \delta_k x) e^{-\delta_k x},$$

where $c, \beta, \delta_k \in \mathbb{R}$, $c \neq 0$, $\alpha \geq 0$, $n \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \delta_k^2 < \infty$.

The following lemma is well-known but elementary, so we give a proof here.

Lemma 3.10. *A multiplier-sequence is strictly log-concave. In particular, a nonnegative multiplier-sequence has no internal zeros.*

Proof. If $f(x) = a_m x^m + a_{m+1} x^{m+1} + \cdots + a_n x^n$ is real-rooted with $a_m a_n \neq 0$, then the coefficients satisfy (see [21, p. 52]):

$$\frac{a_i^2}{\binom{n}{i}^2} > \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} \quad (m < i < n).$$

Now, if $\Gamma = \{\gamma_i\}_{i=0}^{\infty}$ is a multiplier-sequence, then $\Gamma[(x+1)^n]$ is real-rooted for all $n \in \mathbb{N}$, which implies

$$\gamma_i > \gamma_{i-1}\gamma_{i+1},$$

for all i such that there are integers $m < i < n$ with $\gamma_m \gamma_n \neq 0$.

□

Theorem 3.11. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a non-negative multiplier-sequence, and let $\alpha < \beta \in \mathbb{R}$ be given. Define two $\mathbb{R}$-bilinear forms $\mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x]$ by*

$$\begin{aligned} f \cdot g &:= \sum_{k \geq 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x-\alpha)^k (x-\beta)^k, \\ f \circ g &:= \sum_{k \geq 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x-\alpha)^k, \end{aligned}$$

If f is real-rooted and g is $[\alpha, \beta]$-rooted, then $f \cdot g$ is real-rooted. If f is real-rooted and g is $[-\infty, \alpha]$-rooted, then $f \circ g$ is real-rooted.

Proof. We prove the statement for $\cdot$ since the case of $\circ$ is similar. We may assume that $\lambda_0 > 0$. Clearly the theorem is true if $\lambda_i = 0$ for all $i > 0$, so by Lemma 3.10 we may assume that $\lambda_1 > 0$. Let g have all zeros simple and in the interval (α, β), and let ϕ be the linear operator defined by $\phi(f) = f \cdot g$. Then $\phi = \phi_F$, where

$$F(x,z) = \sum_{k\geq 0} \lambda_k \frac{g^{(k)}(x)}{k!}(x-\alpha)^k(x-\beta)^k z^k.$$

Since $\{\lambda_k\}_{k\geq 0}$ is a multiplier sequence $F(\xi, z)$ is real-rooted for all real choices of ξ. Now, $Q_0 = \lambda_0 g(x)$ and $Q_1 = \lambda_1(x-\alpha)(x-\beta)g'(x)$, so Q_0 strictly interlaces Q_1. Moreover, $\deg \phi(x^k) = \deg Q_0 + k$ for all k, so all the hypothesis of Corollary 3.5 are fulfilled. Since any $[\alpha, \beta]$-rooted polynomial is the limit of polynomial which are (α, β)- and simple-rooted the thesis follows by continuity. $\square$

A sequence of real numbers $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ is called an *multiplier n-sequence* if for any real-rooted polynomial $f = a_0 + a_1x + \cdots + a_nx^n$ of degree at most n the polynomial $\Gamma[f] := a_0\gamma_0 + a_1\gamma_1 x + \cdots + a_n\gamma_n x^n$ is real-rooted. There is a simple algebraic characterization of multiplier n-sequences [13]:

Theorem 3.12. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then Γ is a multiplier n-sequence if and only if $\Gamma[(x+1)^n]$ is real-rooted with all its zeros of the same sign.*

Recall the definition of the *hypergeometric function* ${}_2F_1$:

$${}_2F_1\left(a, b; c; z\right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m m!},$$

where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ when $m \geq 1$. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be expressed as follows [28, p. 254]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right), \quad (3.2)$$

We need the following lemma:

Lemma 3.13. *Let n be a positive integer and r a non-negative real number. Then $\Gamma = \{\binom{-n-r}{k}\}_{k=0}^{\infty}$ is a multiplier n-sequence.*

Proof. Let $r > 0$. Then

$$\begin{aligned} \Gamma[(x+1)^n] &= \sum_{k=0}^{n} \binom{-n-r}{k}\binom{n}{k} x^k \\ &= {}_2F_1\left(-n, n+r; 1; x\right) \\ &= P_n^{(0,r-1)}(1-2x), \end{aligned}$$

where the last equality follows from (3.2). Since the Jacobi polynomials are known, see [28], to have all their zeros in $[-1, 1]$ when $\alpha, \beta > -1$, we have that $\Gamma[(x+1)^n]$ has all its zeros in $[0,1]$. The case $r=0$ follows by continuity when we let r tend to zero from above. $\square$

For any real number q let $\Gamma_q := \{q+k\}_{k=0}^{\infty}$.

Corollary 3.14. *Let $n>1$ be a positive integer. Then Γ_q is an n-sequence if and only if $q \notin (-n,0)$.*

Proof. Let $q \in \mathbb{R}$ be given. We have to determine for which $n>1$ the zeros of $\Gamma_q[(x+1)^n]$ are all real and of the same sign. Now,

$$\Gamma_q[(x+1)^n] = (x+1)^{n-1}\{(n+q)x+q\}.$$

If $q \geq 0$ or $n=-q$ then all zeros are negative so we may assume that $q<0$ and $n \neq -q$. If $n>-q$ then $q/(n+q)$ is negative so $\Gamma_q[(x+1)^n]$ has zeros of different signs. If on the other hand $n<-q$ then $q/(n+q)$ is positive which gives that all zeros of $\Gamma_q[(x+1)^n]$ are negative, and the lemma follows. $\square$

4. The E-transformation

The *E-transformation* is the invertible linear operator, $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$, defined by

$$\mathcal{E}(\binom{x}{i}) = x^i,$$

for all $i \in \mathbb{N}$. The PF-preserving properties of this linear operator was first studied in [7] and later in [38, 39] and [5]. It is important in the theory of (P,ω)-partitions since it maps the order-polynomial of a labeled poset to the E-polynomial of the same labeled poset, see [7, 38]. In, [7] Brenti proved the following theorem. Let $\lambda(f)$ and $\Lambda(f)$ denote the smallest and the largest real zero of the polynomial f, respectively.

Theorem 4.1. *Suppose that $f \in \mathbb{R}[x]$ has only real zeros and that $f(n)=0$ for all $n \in ([\lambda(f),-1] \cup [0,\Lambda(f)]) \cap \mathbb{Z}$. Then $\mathcal{E}(f)$ has all zeros real and non-positive.*

In this section we will prove the following theorem:

Theorem 4.2. *For all $n \in \mathbb{N}$ we have*

$$PF_1[\{x^i(x+1)^{n-i}\}_{i=0}^n] \subseteq PF[\binom{x}{i}]$$

Moreover if $f \in PF_1[\{x^i(x+1)^{n-i}\}_{i=0}^n]$ then $\mathcal{E}(f)$ has simple zeros and

$$\mathcal{E}((x+1)^d) \ll \mathcal{E}(f) \ll \mathcal{E}(x^d).$$

The *diamond product* of two polynomials in $\mathbb{R}[x]$ is the $\mathbb{R}$-bilinear form defined by

$$(f \diamond g)(x) := \mathcal{E}(\mathcal{E}^{-1}(f)\mathcal{E}^{-1}(g)). \tag{4.1}$$

This product was first studied by Wagner in [38, 39] and further studied in [5]. See also Section 8 of this paper. Using the Vandermonde identity

$$\binom{x}{i}\binom{x}{j} = \sum_{k\geq 0}\binom{k}{k-i,\, i+j-k,\, k-j}\binom{x}{k},$$

it follows, see [39], that

$$(f \diamond g)(x) = \sum_{k\geq 0} \frac{f^{(k)}(x)}{k!}\frac{g^{(k)}(x)}{k!}x^k(x+1)^k. \tag{4.2}$$

We will later need a symmetry property of $\mathcal{E}$. Let $\mathcal{R} : \mathbb{R}[x] \to \mathbb{R}[x]$ be the algebra automorphism defined by $\mathcal{R}(x) = -1 - x$.

Lemma 4.3.

$$\mathcal{R}\mathcal{E} = \mathcal{E}\mathcal{R}$$

Proof. From (4.2) it follows that

$$\mathcal{R}(f \diamond g) = \mathcal{R}(f) \diamond \mathcal{R}(g) \tag{4.3}$$

Note that $\mathcal{R}\mathcal{E}(f) = \mathcal{E}\mathcal{R}(f)$ whenever f is linear. Now, suppose that f, g are polynomials such that $\mathcal{R}\mathcal{E}(f) = \mathcal{E}\mathcal{R}(f)$ and $\mathcal{R}\mathcal{E}(g) = \mathcal{E}\mathcal{R}(g)$. Then

$$\begin{aligned}
\mathcal{R}\mathcal{E}(fg) &= \mathcal{R}\big(\mathcal{E}(f) \diamond \mathcal{E}(g)\big) && \text{by (4.1)}\\
&= \big(\mathcal{R}\mathcal{E}(f)\big) \diamond \big(\mathcal{R}\mathcal{E}(g)\big) && \text{by (4.3)}\\
&= \big(\mathcal{E}\mathcal{R}(f)\big) \diamond \big(\mathcal{E}\mathcal{R}(g)\big)\\
&= \mathcal{E}\big(\mathcal{R}(f)\mathcal{R}(g)\big)\\
&= \mathcal{E}\mathcal{R}(fg).
\end{aligned}$$

Since we may view $\mathcal{E}$ and $\mathcal{R}$ as $\mathbb{C}$-linear operators on $\mathbb{C}[x]$, and $\diamond$ as a $\mathbb{C}$-bilinear form on $\mathbb{C}[x]$, the lemma follows from the fundamental theorem of algebra. □

Lemma 4.4. *Let $\alpha \in [-1, 0]$ and let f be a polynomial such that $\mathcal{E}(f)$ is $[-1, 0]$-rooted. Then $\mathcal{E}((x-\alpha)f)$ is $[-1, 0]$-rooted and $\mathcal{E}(f)$ interlaces $\mathcal{E}((x-\alpha)f)$. If $\mathcal{E}(f)$ in addition only has simple zeros, then so does $\mathcal{E}((x-\alpha)f)$.*

Proof. Let $g = \mathcal{E}(f)$ and let $\alpha \in [-1, 0]$. By (4.2) we have that

$$\mathcal{E}((x-\alpha)f) = (x-\alpha)g + x(x+1)g'. \tag{4.4}$$

Since g interlaces $(x-\alpha)g$ and $x(x+1)g'$ it also interlaces the sum, by Lemma 2.2. Also, if $x \notin [-1, 0]$ then the summands have the same sign so $\mathcal{E}((x-\alpha)f)$ cannot have any zeros outside $[-1, 0]$. Suppose that g has only simple zeros. Then by (4.4) the only possible common zeros of g and $\mathcal{E}((x-\alpha)f)$ are 0 and -1. If $\deg(f) \geq 1$ it also follows from (4.4) that the multiplicities of 0 and -1 of $\mathcal{E}((x-\alpha)f)$ are the same as those of g. Hence the (simple) zeros of g separates the zeros of

$\mathcal{E}((x-\alpha)f)$ except possibly at $0, -1$, and we conclude that $\mathcal{E}((x-\alpha)f)$ has only simple zeros. $\square$

Lemma 4.5. *For all integers $n \geq 1$ we have*

$$(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n).$$

Proof. We may write

$$x^n = \sum_{k=1}^{n} a_k \binom{x}{k},$$

where $a_k \in \mathbb{R}$. Thus

$$\begin{aligned} \mathcal{E}((x+1)^n) &= \sum_{k=1}^{n} a_k \mathcal{E}[\binom{x}{k} + \binom{x}{k-1}] \\ &= \sum_{k=1}^{n} a_k (x^k + x^{k-1}) \\ &= (x+1)x^{-1}\mathcal{E}(x^n). \end{aligned}$$

$\square$

For $i \in \mathbb{N}$ and let RR_n denote the set of real-rooted monic polynomials of degree n. We define a partial order $\leq$ on RR_n as follows: If $f, g \in RR_n$ have zeros $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ respectively then $f \leq g$, if $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$.

Theorem 4.6. *Suppose that f and g are $[-1, 0]$-rooted with $f \leq g$. Then $\mathcal{E}(f)$ and $\mathcal{E}(g)$ are $[-1, 0]$- and simple-rooted, with $\mathcal{E}(f) \ll \mathcal{E}(g)$.*

Proof. By Lemma 4.4 and induction we only have to show that $\mathcal{E}(f) \ll \mathcal{E}(g)$. If f and g have the same zeros except for one, i.e., $f = (x-\alpha)h$ and $g = (x-\beta)h$, where $\alpha < \beta$, then

$$\mathcal{E}(g) = \mathcal{E}(f) - (\beta - \alpha)\mathcal{E}(h),$$

and since $\mathcal{E}(h)$ interlaces $\mathcal{E}(f)$ we have $\mathcal{E}(f) \ll \mathcal{E}(g)$ by Lemma 2.2.

Now, suppose that f and g are $[-1, 0]$-rooted polynomials of degree n such that $f \leq g$. Then there are $[-1, 0]$-rooted polynomials $\{h_i\}_{i=0}^{M}$ with

$$(x+1)^n = h_0 \leq h_1 \leq \cdots \leq h_M = x^n,$$

such that $f, g \in \{h_i\}_{i=0}^{M}$ and h_{i-1} and h_i only differ in one zero for $1 \leq i \leq n$. We therefore have

$$\mathcal{E}(h_0) \ll \mathcal{E}(h_1) \ll \cdots \ll \mathcal{E}(h_M),$$

and since $\mathcal{E}(h_0) \ll \mathcal{E}(h_M)$, by Lemma 4.5, the theorem follows from Lemma 2.3. $\square$

A consequence of Theorem 4.6 is that if $\{f_i\}_{i=1}^{m}$ is a sequence of standard $[-1, 0]$-rooted polynomials of the same degree d, then by Lemma

2.2 and Theorem 4.6, the image under $\mathcal{E}$ of any non-negative sum $F = \sum_{i=1}^m \mu_i f_i$ will be $[-1,0]$-rooted with

$$\mathcal{E}((x+1)^d) \ll \mathcal{E}(F) \ll \mathcal{E}(x^d).$$

It is easy to see that a standard polynomial f of degree d is $[-1,0]$-rooted if and only if f can be written as

$$f(x) = (x+1)^d g(\frac{x}{x+1}),$$

where g is a standard and $(-\infty, 0)$-rooted. On the other hand, since $x^i(x+1)^{d-i}$ is $[-1,0]$-rooted we have that F can be written as a non-negative sum of standard $[-1,0]$-rooted polynomial of degree d if and only if

$$F(x) = \sum_{i=0}^d a_i x^i (x+1)^{d-i},$$

where $a_i \geq 0$. This proves Theorem 4.2.

5. t-stack sortable permutations

For relevant definitions regarding t-stack sortable permutations we refer the reader to [2]. Let $W_t(n,k)$ be the number of t-stack sortable permutations in the symmetric group, $\mathcal{S}_n$, with k descents. Recently, Bóna [1, 3] showed that for fixed n and t the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ form a unimodal sequence. When $t = n-1$ and $t = 1$ we get the Eulerian and the Narayana numbers (see [36] and [33, Exercise 6.36]), respectively. These are known to be PF-sequences and Bóna [2, 3] has raised the question if this is true for general t. Here we will settle the problem to the affirmative for $t = 2$ and $t = n-2$.

The numbers $W_2(n,k)$ are surprisingly hard to determine despite their compact and simple form. It was recently shown that

$$W_2(n,k) = \frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!}.$$

See [4, 15, 20, 23] for proofs and more information on 2-stack sortable permutations.

From the case $r = 0$ in Lemma 3.13 and the identity

$$\sum_{k=0}^n \binom{2n-k-1}{n-1}\binom{n}{k} x^k = (-1)^n \sum_{k=0}^n \binom{-n}{k}\binom{n}{k}(-x)^{n-k},$$

it follows that $\binom{2n-k-1}{n-1}$ is an n-sequence.

Theorem 5.1. *For all $n \geq 0$ the sequence $\{W_2(n,k)\}_{k=0}^{n-1}$, which records 2-stack sortable permutations by descents, is PF.*

Proof. We may write $W_2(n,k)$ as

$$W_2(n,k) = \frac{\binom{2n-k-1}{n-1}\binom{n+k}{n-1}\binom{2n}{2k+1}}{n^2\binom{2n}{n}}.$$

An simple consequence of the notion of PF-sequences reads as follows: If $\{a_i\}_{i\geq 0}$ is PF then so is $\{a_{ki}\}_{i\geq 0}$, where k is any positive integer. Applying this to the polynomial $x(1+x)^{2n}$ we see that $\sum_k \binom{2n}{2k+1}x^k$ is real-rooted. Therefore the polynomial,

$$\sum_{k=0}^{n-1}\binom{n+k}{n-1}\binom{2n}{2k+1}x^k = \sum_{k=0}^{n-1}\binom{2n-k-1}{n-1}\binom{2n}{2k+1}x^{n-1-k},$$

is real-rooted. Another application of Lemma 3.13 gives that $W_{n,2}(x)$ is real-rooted. □

It is easy to see that a permutation $\pi \in \mathcal{S}_n$ is $(n-2)$-stack sortable if and only if it is not of the form $\sigma n 1$. Thus the generating function satisfies

$$xW_{n,n-2}(x) = A_n(x) - xA_{n-2}(x),$$

where $A_n(x)$ is the nth Eulerian polynomial.

Theorem 5.2. *For all real numbers $t > -2$ and integers $n > 2$, the polynomial*

$$A_n(t,x) = A_n(x) + txA_{n-2}(x),$$

is real- and simple-rooted. Moreover, $A_n(t,x)/x$ strictly interlaces $A_{n+1}(t,x)/x$ for $-2 < t \leq 3$.

Corollary 5.3. *For all $n \geq 2$ we have that $\{W_{n-2}(n,k)\}_{k=0}^{n-1}$ is PF. Moreover, $W_{n,n-2}(x)$ strictly interlaces $W_{n+1,n-1}(x)$.*

Proof of Theorem 5.2. It is well known that $A_{n-1}(x) \ll xA_{n-2}(x)$ and $A_{n-1}(x) \preceq A_n(x)$. So by Lemma 2.2 we have that $A_n(t,x)$ is real- and simple-rooted for $t \geq 0$. However, when $t < 0$ a similar argument does not apply.

Let $E_n(t,x) = A_n(t, \frac{x}{1+x})$. Then

$$E_n(t,x) = E_n(x) + tx(1+x)E_{n-2}(x),$$

where the coefficient to x^k in $E_n(x)$ counts the number of surjections $\sigma : [n] \to [k]$, see [7, 38]. These polynomials satisfy the recursion:

$$E_n(x) = x\frac{d}{dx}((1+x)E_{n-1}(x)),$$

with initial condition $E_1(x) = x$. Thus, if we let $G_n(x) = E_{n+1}(x)/x$ we have the following recursion:

$$G_n(x) = \frac{d}{dx}(x(1+x)G_{n-1}(x)), \tag{5.1}$$

with $G_0(x) = 1$. Obviously $G_n(x)$ is real- and simple-rooted. If we apply (5.1) two times we get the equation:

$$G_n(x) = (1 + 6x + 6x^2)G_{n-2}(x) + 3x(1 + 2x)(1 + x)G'_{n-2}(x)+ \\ x^2(1 + x)^2G''_{n-2}(x),$$

and

$$G_n(t, x) = (1+(6+t)x+(6+t)x^2)G_{n-2}(x)+3x(1+2x)(1+x)G'_{n-2}(x)+ \\ x^2(1 + x)^2G''_{n-2}(x).$$

To apply Theorem 3.4 we need show that for all $\xi \in \mathbb{R}$ and $-2 < t < 0$ the polynomial

$$F(\xi, z) := (1 + (6 + t)\xi + (6 + t)\xi^2) + 3\xi(1 + 2\xi)(1 + \xi)z + \xi^2(1 + \xi)^2z^2$$

is real-rooted. The discriminant of $F(\xi, z)$,

$$\Delta(F(\xi, z)) = \xi^2(1 + \xi)^2(2 + t + (3 - t)(1 + 2\xi)^2),$$

is non-negative when $-2 \leq t \leq 3$, so $F(\xi, z)$ real-rooted for these t. Since all the Q_ks are standard it is easy to see that condition (III) in the statement of Theorem 3.4 is satisfied. Moreover, $1+(6+t)x+(6+t)x^2$ strictly interlaces $3x(1 + 2x)(1 + x)$ when $t > -2$ so Theorem 3.4 applies. Since G_n strictly interlaces G_{n+1} we have by Theorem 3.4 and Corollary 2.5 that $\phi_F(G_n)$ strictly interlaces $\phi_F(G_{n+1})$. Thus $A_n(t, x)$ strictly interlaces $A_{n+1}(t, x)$. □

6. q-Eulerian and W-Eulerian polynomials

A q-analog of the Eulerian polynomials was introduced and studied in [17] and further studied in [10]. It is defined by

$$A_n(x; q) := \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)}q^{\mathrm{c}(\pi)},$$

where $\mathrm{c}(\pi)$ and $\mathrm{exc}(\pi)$ denotes the number of *cycles* and *excedances* in π respectively. These polynomials satisfy the recursion

$$A_{n+1}(x; q) = (nx + q)A_n(x; q) - x(x - 1)\frac{\partial}{\partial x}A_n(x; q),$$

with initial condition $A_0(x; q) := 1$. See [10] for a proof. The following theorem appears in [10].

Theorem 6.1. *Let $q \in \mathbb{R}$, $q > 0$. Then the polynomials $A_n(x, q)$ have only real non-positive simple zeros.*

Brenti also makes the following conjecture:

Conjecture 6.2. *Let $n, m \in \mathbb{N}$. Then $A_n(x; -m)$ has only real zeros.*

In what follows we will prove this conjecture using multiplier n-sequences. For $n \in \mathbb{N}$ define the polynomials $E_n(x;q)$ by:

$$E_n(x;q) := (1+x)^n A_n(\frac{x}{1+x};q).$$

It is clear that $E_n(x;q)$ is real-rooted if and only if $A_n(x;q)$ is real-rooted. These polynomials satisfy a somewhat easier recursion. Namely,

$$E_{n+1}(x;q) = (1+x)\{qE_n(x;q) + x\frac{\partial}{\partial x}E_n(x;q)\}, \tag{6.1}$$

with initial condition $E_0(x;q) = 1$. Now, for $q \in \mathbb{R}$ let $\Gamma_q : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear operator defined by $\Gamma_q(f(x)) = qf(x) + xf'(x)$. Since $\Gamma_q(x^n) = (q+n)x^n$ we may apply Lemma 3.14.

Theorem 6.3. *Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$. If $q \geq 0$, $n \leq -q$ or $q \in \mathbb{Z}$ then $E_n(x;q)$ has only real zeros.*

Proof. We may write (6.1) as

$$E_{n+1}(x;q) = (x+1)\Gamma_q[E_n(x;q)].$$

The cases $q \geq 0$ and $n \leq -q$ follow from Lemma 3.14 by induction. We may therefore assume that $q = -m$ is a negative integer. We claim that $\deg E_n(x;q) = n$ if $n \leq m$ and $\deg E_n(x;q) = m$ if $n \geq m$. From this the real-rootedness follows by Lemma 3.14 and induction. The case $n \leq m$ is clear since $\Gamma_q[x^{n-1}] = -(m-n+1) < 0$. The case $n > m$ also follows by induction. Suppose that $n \geq m$ and that $\deg E_n(x;q) = m$. Then by the recursion we have that $\deg E_{n+1}(x;q) \leq m+1$. Moreover, since $\Gamma_q[x^m] = 0$ we have that $\deg E_{n+1}(x;q) \leq m$. Let $a \neq 0$ be the coefficient to x^m of $E_n(x;q)$. Then the coefficient to x^m of $E_{n+1}(x;q)$ is $a\Gamma_q[x^{m-1}] = -a$, so $\deg E_{n+1}(x;q) = m$, and the thesis follows. □

The Eulerian polynomial, $P(W,x)$, of a finite Coxeter group W is the polynomial,

$$P(W,x) = \sum_{\sigma \in W} x^{d_W(\sigma)},$$

where $d_W(\sigma)$ is the number of W-descents of σ, see [9]. This polynomial is also the generating function for the h-vector of the Coxeter complex associated to (W,S). For Coxeter groups of type A_n we have that $P(A_n,x) = A_n(x)/x$, the shifted Eulerian polynomial. Also, for Coxeter groups of type B_n it is known, see [9], that $P(B_n,x)$, has only real zeros. It is easy to see that $P(W_1 \times W_2, x) = P(W_1,x)P(W_2,x)$ for finite Coxeter groups W_1 and W_2. Also, the real-rootedness can be checked ad hoc for the exceptional groups. Thus, by the classification of finite irreducible Coxeter groups, to prove that $P(W,x)$ has only real zeros for all finite Coxeter groups it suffices to prove that $P(D_n,x)$ is real-rooted for Coxeter groups of type D_n. The real-rootedness of

$P(D_n, x)$ is conjectured by Brenti in [9]. It is known that the Eulerian polynomials of type A_n, B_n and D_n are related by, see [9, 29, 35]:

$$P(D_n, x) = P(B_n, x) - n2^{n-1}xP(A_{n-1}, x).$$

This relationship was first noticed by Stembridge [35]. One step towards proving the real-rootedness of $P(D_n, x)$ is to learn more about the relationships between the zeros of $P(B_n, x)$ and $P(A_n, x)$.

Brenti [9] introduced a q-analog of $P(B_n, x)$

$$B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{d_B(\sigma)}, \tag{6.2}$$

where $d_B(\sigma)$ is the number of B_n-descents of σ and $N(\sigma)$ is the number of negative entries of σ, see [9]. He proved that

$$\sum_{i \geq 0} ((1+q)i + 1)^n x^i = \frac{B_n(x; q)}{(1-x)^{n+1}}, \tag{6.3}$$

and that $B_n(x; q)$ is real- and simple-rooted for all $q \geq 0$. Suppose that $f(i)$ is a polynomial in i of degree d, then the polynomial $W(f)$ is defined by

$$\sum_{i \geq 0} f(i) x^i = \frac{W(f)(x)}{(1-x)^{d+1}},$$

One can show, see [7], that $\mathcal{E}(f)$ and $W(f)$ are related by:

$$\mathcal{E}(f)(x) = (1+x)^{\deg(f)} W(f)(\frac{x}{1+x}). \tag{6.4}$$

It follows that $W(f)$ has only real non-positive roots if and only if $\mathcal{E}(f)$ is $[-1, 0]$-rooted. Since $((1+q)i + 1)^n$ is a $[-1, 0]$-rooted polynomial in i for any $q \geq 0$ it follows from e.g. Theorem 4.2 that $B_n(x; q)$ is real-rooted in x for any fixed $q \geq 0$. It is natural to generalize $B_n(x; q)$ to have $n+1$ parameters as $B_n(x; \mathbf{q}) := W\big(\prod_{i=0}^n \big((1+q_i)x+1\big)\big)$. This polynomial has a nice combinatorial interpretation:

Theorem 6.4. *For all $n \in \mathbb{N}$ we have:*

$$B_n(x, \mathbf{q}) = \sum_{\sigma \in B_n} q_1^{\chi_1(\sigma)} q_2^{\chi_2(\sigma)} \cdots q_n^{\chi_n(\sigma)} t^{d_B(\sigma)},$$

where

$$\chi_i(\sigma) = \begin{cases} 1 & \textit{if } \sigma_i < 0, \\ 0 & \textit{if } \sigma_i > 0. \end{cases}$$

Proof. The proof is an obvious generalization of the proof of Theorem 3.4 of [9]. □

Note that this theorem gives a semi-combinatorial interpretation of the W-transform of any $[-1, 0)$-rooted polynomial.

Corollary 6.5. *Let $n \in \mathbb{N}$ and let $q_1, q_2, \ldots, q_n$ be non-negative real numbers. Then $B_n(x; \mathbf{q})$ has only real and simple zeros.*

We need the following lemma on the degree of $W(f)$.

Lemma 6.6. *Let* $f \in \mathbb{R}[x]$. *Then*

$$\deg W(f) = \deg f - \mathrm{mult}(-1, \mathcal{E}(f)).$$

Moreover, $\mathrm{mult}(-1, \mathcal{E}(f))$ *is equal to the maximal integer* k *such that* $(x+1)(x+2)\cdots(x+k)$ *divides* f.

Proof. Since $\deg \mathcal{E}(f) = \deg f$ for all f we have by (6.4) that $\deg W(f) = \deg f - \mathrm{mult}(-1, \mathcal{E}(f))$. If we expand f in the basis $\{\binom{-x-1}{i}\}$ as:

$$\begin{aligned} f(x) &= \sum_{i\geq 0}(-1)^i a_i \binom{-x-1}{i}, \\ &= \sum_{i\geq 0} \frac{a_i}{i!}(x+1)\cdots(x+i), \end{aligned}$$

we have by Lemma 4.3 that

$$\mathcal{E}(f)(x) = \sum_{i\geq 0} a_i (x+1)^i,$$

and the lemma follows. □

We now have more precise knowledge of the location of the zeros of $B_n(x;q)$ for any given $q \geq 0$.

Theorem 6.7. *Let* $0 < q < t \in \mathbb{R}$ *and* $n > 0$ *be an integer. Then*

$$B_n(x;0) \preceq B_n(x;t) \ll B_n(x;q) \ll xB_n(x;0),$$

where the three first polynomials have no common zeros.

Proof. Let $0 < r < s < 1$. Then by the proof of Lemma 4.4 we have

$$\begin{aligned} \mathcal{E}(x^n) \ll \mathcal{E}(x(x{+}r)^{n-1}) \ll_{strict} \mathcal{E}((x{+}r)^n) \ll_{strict} \mathcal{E}((x{+}r)^{n-1}(x{+}s)) \ll \\ \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+s)^{n-1}(x+1)) \ll \mathcal{E}((x+1)^n), \end{aligned}$$

where $\ll_{strict}$ means strictly alternating left of. Since $(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n)$ this implies

$$\mathcal{E}(x^n) \ll_{strict} \mathcal{E}((x+r)^n) \ll_{strict} \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+1)^n).$$

Now since

$$B_n(x;q) = (q{+}1)^n W((x{+}\frac{1}{1+q})^n) = (q{+}1)^n(1{-}x)^n\mathcal{E}((x{+}\frac{1}{1+q})^n)(\frac{x}{1-x}),$$

we see by Lemma 6.6 that $\deg B_n(x;0) = n-1$ and $\deg B_n(x;q) = n$ if $q \neq 0$. Moreover, the alternating property is preserved under the operation (6.4) and the theorem follows. □

It follows from (6.2) that $P(B_n, x) = B_n(x;1)$ and $P(A_n, x) = B_n(x;0)$.

Corollary 6.8. *For all integers* $n \geq 1$ *we have that* $P(A_n, x)$ *strictly interlaces* $P(B_n, x)$.

Since $P(A_n, x) \ll xP(A_{n-1}, x)$ and $P(A_n, x) \preceq P(B_n, x)$, we have by Lemma 2.2 that for all $t \geq 0$ the polynomial $P(B_n, x) + txP(A_{n-1}, x)$ is real-rooted. Unfortunately a similar argument does not apply when $t < 0$.

One can extract more from (6.3). Brenti [9] proved that the polynomial

$$\sum_{\sigma \in B_n, N(\sigma) \in \{k, n-k\}} x^{d_B(\sigma)},$$

is real-rooted for all choices of $0 \leq k \leq n$. Using Theorem 4.6 we can extend this result to:

Corollary 6.9. *Let S be any subset of $[0, n]$. Then the polynomial*

$$P(B_n, S; x) := \sum_{\sigma \in B_n, N(\sigma) \in S} x^{d_B(\sigma)},$$

has only real and simple zeros.

Proof. Comparing the coefficient of q^i in both sides of (6.3) we see that $P(B_n, S; x) = W(f_n(S; x))$ where

$$f_n(S; x) = \sum_{s \in S} \binom{n}{s} x^s (x+1)^{n-s}.$$

So the theorem follows from Theorem 4.2. $\square$

One instance of Theorem 6.9 is particularly interesting. Recall that a Coxeter group of type D_n is isomorphic to the subgroup

$$D_n = \{\sigma \in B_n : \quad 2 \mid N(\sigma)\}$$

Hence, we have the following corollary

Corollary 6.10. *For all $n \in \mathbb{N}$ the polynomial*

$$\sum_{\sigma \in D_n} x^{d_B(\sigma)}$$

has only real and simple zeros.

Note that the above polynomial is not $P(D_n, x)$, since B_n-descents and D_n-descents are not the same.

7. The h-vector of a family of simplicial complexes defined by Fomin and Zelevinsky

Fomin and Zelevinsky [18] recently associated to any finite Weyl group W a simplicial complex $\Delta_{FZ}(W)$. For the classical Weyl groups these polynomials are given by

$$\begin{aligned} h(\Delta_{FZ}(A_{n-1}),x) &= \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}x^k, \\ h(\Delta_{FZ}(B_n),x) &= \sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}x^k, \\ h(\Delta_{FZ}(D_n),x) &= h(\Delta_{FZ}(B_n),x) - nxh(\Delta_{FZ}(A_{n-2}),x). \end{aligned}$$

It is known that the h-polynomials corresponding to A_n and B_n have only real zeros. We will here show that so has $h(\Delta_{FZ}(D_n),x)$.

Theorem 7.1. *Let $\alpha,\beta \in \mathbb{R}$ be such that $\alpha \geq 0, 2\alpha+\beta > 0$ and let $n \geq 2$ be an integer. Then the polynomial*

$$F_n(\alpha,\beta) := \alpha h(\Delta_{FZ}(B_n),x) + \beta nxh(\Delta_{FZ}(A_{n-2}),x),$$

is real- and simple-rooted. Moreover, $h(\Delta_{FZ}(B_{n-1}),x)$ strictly interlaces $F_n(\alpha,\beta)$ if $\alpha > 0$ and strictly alternates left of $F_n(\alpha,\beta)$ if $\alpha = 0$.

Corollary 7.2. *Let W be a finite Weyl group. Then $h(\Delta_{FZ}(W),x)$ has only real and simple zeros. For the classical Weyl groups we have the following relationships:*

$$\begin{aligned} h(\Delta_{FZ}(A_{n-1}),x) &\preceq h(\Delta_{FZ}(A_n),x), \\ h(\Delta_{FZ}(B_{n-1}),x) &\preceq h(\Delta_{FZ}(B_n),x), \\ h(\Delta_{FZ}(B_{n-1}),x) &\preceq h(\Delta_{FZ}(D_n),x), \\ h(\Delta_{FZ}(A_{n-1}),x) &\preceq h(\Delta_{FZ}(B_n),x), \end{aligned}$$

where the interlacing is strict.

Proof. For the exceptional Weyl group one can check the real-rootedness ad hoc, see [30]. That $\{h(\Delta_{FZ}(A_n),x)\}_{n\geq 0}$ form a Sturm sequence is proved in [6]. The other cases follows from Theorem 7.1. □

The *Hadamard product* of two polynomials

$$\begin{aligned} p(x) &= a_0 + a_1x + \cdots + a_mx^m \\ q(x) &= b_0 + b_1x + \cdots + b_nx^n \end{aligned}$$

is the polynomial

$$(p \star q)(x) = a_0b_0 + a_1b_1x + \cdots + a_Nb_Nx^N,$$

where $N = \min(m,n)$. Málo proved that if the zeros of p are real and the zeros of q are real and of the same sign then the zeros of $p \star q$ are real as well. This also follows from Theorem 3.8 since $p \star q = \Gamma[pSq]$ where Γ is the multiplier sequence $\{\frac{1}{k!}\}_{k=0}^{\infty}$. It is known, see e.g. [19], that if f has only real zeros then all zeros of $\Gamma[f]$ are real and simple except for possibly at the origin.

Proof of Theorem 7.1. Let $F_n(\alpha,\beta) = \alpha h(\Delta_{FZ}(B_n), x) + \beta n x h(\Delta_{FZ}(A_{n-2}), x)$. We may write $F_n(\alpha,\beta)$ as

$$F_n(\alpha,\beta) = \alpha(x+1)^n \star (x+1)^n + \beta\big(x(x+1)^{n-1}\big) \star (x+1)^{n-1},$$

where the first summand can be written as

$$(x+1)\big((x+1)^{n-1} \star (x+1)^{n-1}\big) + 2\big(x(x+1)^{n-1}\big) \star (x+1)^{n-1}.$$

Thus $F_n(\alpha,\beta) = \alpha(x+1)f + (2\alpha+\beta)g$ where $f = (x+1)^{n-1} \star (x+1)^{n-1}$ and $g = \big(x(x+1)^{n-1}\big) \star (x+1)^{n-1}$. By the discussion before this proof we have that for all real choices of $\gamma, \delta \in \mathbb{R}$ the polynomial

$$\gamma f + \delta g = \big((\gamma + \delta x)(x+1)^{n-1}\big) \star (x+1)^{n-1},$$

is real- and simple-rooted. By the Obreschkoff theorem we infer that f strictly alternates left of g. Now, since $f \preceq (x+1)f$ and $f \ll g$ we know by Lemma 2.2 that f either interlaces or alternates left of $F_n(\alpha,\beta)$ for all $\alpha,\beta \in \mathbb{R}$ such that $\operatorname{sgn}(\alpha) = \operatorname{sgn}(2\alpha+\beta)$. Moreover, since g and f have no common zeros nor does $F_n(\alpha,\beta)$ and f (provided that $2\alpha + \beta \neq 0$). □

8. Two bilinear forms

There are a few bilinear forms on polynomials that occur frequently in combinatorics. Let $\# : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by

$$(f\#g)(x) := \sum_{k\geq 0} f^{(k)}(x) g^{(k)}(x) \frac{x^k}{k!}.$$

This product is important when analyzing how the the zeros of σ-polynomials behave under disjoint union of graphs, see [11].

Theorem 8.1. *Let f be real-rooted and let g have only real zeros of the same sign. Then $f\#g$ is real-rooted.*

Proof. The theorem follows from Theorem 3.11, since $\{1\}_{k=0}^{\infty}$ is trivially a multiplier-sequence. □

This generalizes a result of Wagner, who proved that $f\#g$ is real-rooted whenever f and g have only non-negative zeros, see [11, 37].

Recall the definition, (4.2), of the diamond product. This product is important in the theory of (P,ω)-partitions and the Neggers-Stanley conjecture, see [38]. Applying Theorem 3.11 with the multiplier-sequence $\{\frac{1}{k!}\}_{k\geq 0}$ we get:

Theorem 8.2. *Let f be real-rooted and let g have all zeros in the interval $[-1,0]$. Then $f \diamond g$ is real-rooted.*

This was first proved by Wagner [39] under the additional hypothesis that f has all zeros in $[-1,0]$, and generalized by the present author in [5].

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