SIGN-GRADED POSETS, UNIMODALITY OF W-POLYNOMIALS AND THE CHARNEY-DAVIS CONJECTURE

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Abstract. We generalize the notion of graded posets to what we call sign-graded (labeled) posets. We prove that the W-polynomial of a sign-graded poset is symmetric and unimodal. This extends a recent result of Reiner and Welker who proved it for graded posets by associating a simplicial polytopal sphere to each graded poset P. By proving that the W-polynomials of sign-graded posets has the right sign at -1, we are able to prove the Charney-Davis Conjecture for these spheres (whenever they are flag).

1. Introduction and preliminaries

Recently Reiner and Welker [8] proved that the W-polynomial of a graded naturally labeled poset P has unimodal coefficients. They proved this by associating to P a simplicial polytopal sphere, $\Delta_{eq}(P)$, whose h-polynomial is the W-polynomial of P, and invoking McMullen's g-theorem [11]. Whenever this sphere is flag, i.e., its minimal non-faces all have cardinality two, they noted that the Neggers-Stanley Conjecture implies the Charney-Davis Conjecture for $\Delta_{eq}(P)$. In this paper we give a completely different proof of the unimodality of W-polynomials of graded posets, and we also prove the Charney-Davis Conjecture for $\Delta_{eq}(P)$ (whenever they are flag). Our proof is by studying a family of labeled posets, which we call sign-graded posets, of which the class of graded naturally labeled posets is a sub-class.

In this paper all posets will be finite. For undefined terminology on posets we refer the reader to [13]. We denote the cardinality of a poset P with a small letter p. Let P be a poset and let $\omega : P \to \{1, 2, \dots, p\}$ be a bijection. The pair (P, ω) is called a *labeled poset*. If ω is order-preserving then (P, ω) is said to be *naturally labeled*. A (P, ω)-partition is a map $\sigma : P \to \{1, 2, 3, \dots\}$ such that

- σ is order reversing, that is, if $x \leq y$ then $\sigma(x) \geq \sigma(y)$,
- if $x < y$ and $\omega(x) > \omega(y)$ then $\sigma(x) > \sigma(y)$.

The theory of (P, ω)-partitions was developed by Stanley in [10]. The number of (P, ω)-partitions $\sigma : P \to \{1, 2, \dots, n\}$ is a polynomial of degree p in n called the *order polynomial* of (P, ω) and is denoted $\Omega(P, \omega; n)$. The W-polynomial of (P, ω) is defined by

$$\sum_{n \geq 0} \Omega(P, \omega; n) t^n = \frac{tW(P, \omega; t)}{(1-t)^{p+1}}.$$

The *Jordan-Hölder set*, $\mathcal{L}(P, \omega)$, of (P, ω) is the set of permutations $\omega(x_1), \omega(x_2), \dots, \omega(x_p)$ where $x_1, x_2, \dots, x_p$ is a linear extension of P. A *descent* in a permutation $\pi =$

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$\pi_1\pi_2\cdots\pi_p$ is an index $1 \leq i \leq p-1$ such that $\pi_i > \pi_{i+1}$. The number of descents of π is denoted $\text{des}(\pi)$. A result of Stanley's [10] implies that the W-polynomial can be written as

$$W(P,\omega;t) = \sum_{\pi \in \mathcal{L}(P,\omega)} t^{\text{des}(\pi)},$$

The Neggers-Stanley Conjecture is the following:

Conjecture 1.1 (Neggers-Stanley). *For any labeled poset (P,ω) the polynomial $W(P,\omega;t)$ has only real zeros.*

It was first conjectured by Neggers [6] in 1978 for natural labelings and by Stanley in 1986 for arbitrary labelings. The conjecture has been proved for special cases, see [1, 2, 8, 14] for the state of the art. If a polynomial has only real non-positive zeros then its coefficients form a unimodal sequence. For the W-polynomials of graded posets unimodality was first proved by Gasharov [5] whenever the rank is at most 2, and as mentioned by Reiner and Welker for all graded posets.

For the relevant definitions concerning the topology behind the Charney-Davis Conjecture we refer the reader to [3, 8, 12].

Conjecture 1.2 (Charney-Davis, [3]). *Let Δ be a flag simplicial homology $(d-1)$-sphere, where d is even. Then the h-vector, $h(\Delta,t)$, of Δ satisfies*

$$(-1)^{d/2}h(\Delta,-1) \geq 0.$$

Recall that the nth *Eulerian polynomial*, $A_n(x)$, is the W-polynomial of an anti-chain of n elements. The Eulerian polynomials can be written as

$$A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{n,i}x^i(1+x)^{n-1-2i},$$

where $a_{n,i}$ is a non-negative integer for all i. This was proved by Foata and Schützenberger in [4] and combinatorially by Shapiro, Getu and Woan in [9]. From this expansion we see immediately that $A_n(x)$ is symmetric and that the coefficients in the standard basis are unimodal. It also follows that $(-1)^{(n-1)/2}A_n(-1) \geq 0$.

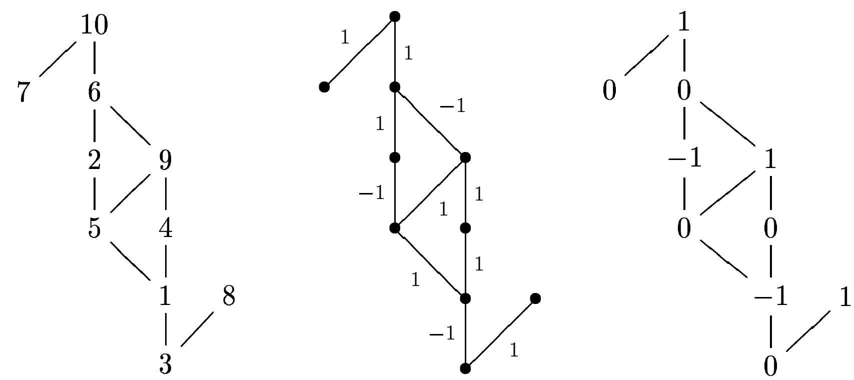
We will in Section 2 define a class of labeled poset whose members we call sign-graded posets. This class includes the class of naturally labeled graded posets. In Section 4 we show that the W-polynomial of a sign-graded poset (P,ω) of rank r can be expanded, just as the Eulerian polynomial, as

$$W(P,\omega;t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P,\omega)t^i(1+t)^{p-r-1-2i}, \tag{1.1}$$

where $a_i(P,\omega)$ are non-negative integers. Hence, symmetry and unimodality follow, and $W(P,\omega;t)$ has the right sign at -1. Consequently, whenever the associated sphere $\Delta_{eq}(P)$ of a graded poset P is flag the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$. We also note that all symmetric polynomials with non-positive zeros only, admits an expansion such as (1.1). Hence, that $W(P,\omega;t)$ has such an expansion can be seen as further evidence for the Neggers-Stanley Conjecture.

In [7] the Charney-Davis quantity of a graded naturally labeled poset (P,ω) of rank r was defined to be $(-1)^{(p-1-r)/2}W(P,\omega;-1)$. In Section 5 we give a combinatorial interpretation of the Charney-Davis quantity as counting certain reverse alternating permutations. Finally in Section 6 we give a characterization of sign-graded posets in terms of properties of order polynomials.

FIGURE 1. A sign-graded poset, its two labelings and the corresponding rank function.



2. SIGN-GRADED POSETS

Let (P, ω) be a labeled poset and let $E = E(P) = \{(x, y) \in P \times P : x \prec y\}$ be the covering relations of P. An element y covers x, written $x \prec y$, if $x < y$ and $x < z < y$ for no $z \in P$. We associate a labeling $\epsilon : E \to \{-1, 1\}$ of the Hasse-diagram of P by

$$\epsilon(x, y) = \begin{cases} 1 \text{ if } & \omega(x) < \omega(y), \\ -1 \text{ if } & \omega(x) > \omega(y). \end{cases}$$

Note that the definition of a (P, ω)-partition only depends on the function ϵ. In what follows we will often refer to ϵ as the labeling and write $\Omega(P, \epsilon; t)$.

Definition 2.1. Let $\epsilon : E \to \{-1, 1\}$ be a labeling of E. We say that P is *sign-graded with respect to* ϵ (or ϵ-graded for short) if for every maximal chain $x_0 \prec x_1 \prec \cdots \prec x_n$ the sum

$$\sum_{i=1}^{n} \epsilon(x_{i-1}, x_i)$$

is the same. The common value, $r(\epsilon)$, of the above sum is called the *rank* of ϵ. The *rank function*, $\rho : P \to \mathbb{Z}$ is defined by

$$\rho(x) = \sum_{i=1}^{m} \epsilon(x_{i-1}, x_i),$$

where $x_0 \prec x_1 \prec \cdots \prec x_m = x$ is any saturated chain from a minimal element to x.

See Fig. 1 for an example of a sign-graded poset. Note that if ϵ is identically equal to 1, then a sign-graded poset with respect to ϵ is just a graded poset. Note also that if P is ϵ-graded then P is also $-\epsilon$-graded, where $-\epsilon$ is defined by $(-\epsilon)(x, y) = -\epsilon(x, y)$. It may come as a surprise to the reader that when it comes to order-polynomials of sign-graded posets, the specific labeling does not matter:

Theorem 2.2. *Let P be ϵ-graded and μ-graded. Then*

$$\Omega(P, \epsilon; t - \frac{r(\epsilon)}{2}) = \Omega(P, \mu; t - \frac{r(\mu)}{2}).$$

Proof. Let ρ_ϵ and ρ_μ denote the rank functions of (P,ϵ) and (P,μ) respectively, and let $\mathcal{A}(\epsilon)$ denote the set of (P,ϵ)-partitions. Define a function $\xi : \mathcal{A}(\epsilon) \to \mathbb{Q}^P$ by $\xi\sigma(x) = \sigma(x) + \Delta(x)$, where

$$\Delta(x) = \frac{r(\epsilon) - \rho_\epsilon(x)}{2} - \frac{r(\mu) - \rho_\mu(x)}{2}.$$

The four possible combinations of labelings of a covering-relation $(x,y) \in E$ are

TABLE 1

$\epsilon(x,y)$	$\mu(x,y)$	σ	Δ	$\xi\sigma$
1	1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) \geq \xi\sigma(y)$
1	-1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y) + 1$	$\xi\sigma(x) > \xi\sigma(y)$
-1	1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y) - 1$	$\xi\sigma(x) \geq \xi\sigma(y)$
-1	-1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) > \xi\sigma(y)$

given in Table 1.

According to the table $\xi\sigma$ is a (P,μ)-partition provided that $\xi\sigma(x) > 0$ for all $x \in P$. But $\xi\sigma$ is order-reversing so it attains its minima on maximal elements. If z is a maximal element we have $\xi\sigma(z) = \sigma(z)$ so $\xi : \mathcal{A}(\epsilon) \to \mathcal{A}(\mu)$. By symmetry we also have a map $\eta : \mathcal{A}(\mu) \to \mathcal{A}(\epsilon)$ defined by

$$\eta\sigma(x) = \sigma(x) + \frac{r(\mu) - \rho_\mu(x)}{2} - \frac{r(\epsilon) - \rho_\epsilon(x)}{2}.$$

Hence, $\eta = \xi^{-1}$ and ξ is a bijection.

Since σ and $\xi\sigma$ are order-reversing they attain their maxima on minimal elements. But if z is a minimal element then $\xi\sigma(z) = \sigma(z) + \frac{r(\epsilon)-r(\mu)}{2}$, which gives

$$\Omega(P,\mu;n) = \Omega(P,\epsilon;n + \frac{r(\mu) - r(\epsilon)}{2}),$$

and proves the theorem. □

Theorem 2.3. *Let P be ϵ-graded. Then*

$$\Omega(P,\epsilon;t) = (-1)^p\Omega(P,\epsilon;-t-r(\epsilon)).$$

Proof. We have the following reciprocity for order polynomials, see [10]:

$$\Omega(P,-\epsilon;t) = (-1)^p\Omega(P,\epsilon;-t). \tag{2.1}$$

Note that $r(-\epsilon) = -r(\epsilon)$, so by Theorem 2.2 we have:

$$\Omega(P,-\epsilon;t) = \Omega(P,\epsilon,t - r(\epsilon)),$$

which, combined with (2.1), gives the desired result. □

Corollary 2.4. *Let P be an ϵ-graded poset. Then $W(P,\epsilon,t)$ is symmetric with center of symmetry $(p - r(\epsilon) - 1)/2$. If P is also μ-graded then*

$$W(P,\mu;t) = t^{(r(\epsilon)-r(\mu))/2}W(P,\epsilon;t).$$

Proof. It is known, see [10], that if $W(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)t^i$ then $\Omega(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+p-1-i}{p}$. Let $r = r(\epsilon)$. Theorem 2.3 gives:

$$\begin{aligned}
\Omega(P,\epsilon;t) &= \sum_{i\geq 0} w_i(P,\epsilon)(-1)^p\binom{-t-r+p-1-i}{p} \\
&= \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+r+i}{p} \\
&= \sum_{i\geq 0} w_{p-r-1-i}(P,\epsilon)\binom{t+p-1-i}{p},
\end{aligned}$$

so $w_i(P,\epsilon) = w_{p-r-1-i}(P,\epsilon)$ for all i, and the symmetry follows. The relationship between the W-polynomials of ϵ and μ follows from Theorem 2.2 and the expansion of order-polynomials in the basis $\binom{t+p-1-i}{p}$. □

The following theorem tells us that the class of sign-graded posets is considerably greater than the class of graded posets.

Theorem 2.5. *Let P be a finite poset. Then there exists a labeling $\epsilon : E \to \{-1,1\}$ such that (P,ϵ) is sign-graded if and only if all maximal chains in P have the same parity (cardinality modulo 2).*

Moreover, the labeling ϵ can be chosen so that the corresponding rank function has values in $\{0,1\}$.

Proof. It is clear that if P is ϵ-graded then all maximal chains have the same parity. Let P be a poset whose maximal chains have the same parity. Then, for any $x \in P$, all saturated chains starting at a minimal element and ending at x has the same length modulo 2. Hence, we may define a labeling $\epsilon : P \to \{-1,1\}$ by $\epsilon(x,y) = (-1)^{\ell(x)}$, where $\ell(x)$ is the length of any saturated chain starting at a minimal element and ending at x. It follows that P is ϵ-graded and that its rank function has values in $\{0,1\}$. □

We say that $\omega : P \to \{1,2,\ldots,p\}$ is *canonical* if (P,ω) has a rank-function ρ with values in $\{0,1\}$, and $\rho(x) < \rho(y)$ implies $\omega(x) < \omega(y)$. By Theorem 2.5 we know that P admits a canonical labeling if P is sign-graded with respect to some ϵ.

3. The Jordan-Hölder set of a sign-graded poset

Let (P,ω) be sign-graded. We may assume that $\omega(x) < \omega(y)$ whenever $\rho(x) < \rho(y)$. Assume that $x,y \in P$ are incomparable and that $\rho(y) = \rho(x)+1$. Then the Jordan-Hölder set of (P,ω) can be partitioned into two sets: One where in all permutations $\omega(x)$ comes before $\omega(y)$ and one where $\omega(y)$ comes before $\omega(x)$. This means that

$$\mathcal{L}(P,\omega) = \mathcal{L}(P',\omega) \sqcup \mathcal{L}(P'',\omega), \tag{3.1}$$

where P' is the transitive closure of $E \cup \{x \prec y\}$, and P'' is the transitive closure of $E \cup \{y \prec x\}$.

Lemma 3.1. *With definitions as above (P',ω) and (P'',ω) are sign-graded with the same rank-function as that for (P,ω).*

Proof. Let $C : z_0 \prec z_1 \prec \cdots \prec z_k = z$ be a saturated chain in P'', where z_0 is a minimal element in P''. Of course z_0 is also a minimal element in P. We have to prove that

$$\rho(z) = \sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}),$$

where ϵ'' is the "edge"-labeling of P'' and ρ is the rank-function of (P, ω).

All covering relations in P'', except $y \prec x$, are also covering relations in P. Note that $\epsilon''(y, x) = -1$. If y and x do not appear in C, then C is a saturated chain in P and we have nothing to prove. Otherwise

$$C : y_0 \prec \cdots \prec y_i = y \prec x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z.$$

Note that if $s_0 \prec s_1 \prec \cdots \prec s_\ell$ is any saturated chain in P then $\sum_{i=0}^{\ell-1} \epsilon(s_i, s_{i+1}) = \rho(s_\ell) - \rho(s_0)$. Since $y_0 \prec \cdots \prec y_i = y$ and $x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z$ are saturated chains in P we have

$$\begin{aligned} \sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}) &= \rho(y) + \epsilon''(y, x) + \rho(z) - \rho(x) \\ &= \rho(y) - 1 - \rho(x) + \rho(z) \\ &= \rho(z), \end{aligned}$$

as was to be proved. The statement for (P', ω) follows similarly. □

We say that a sign-graded poset (P, ω) is *saturated* if for all $x, y \in P$ we have that x and y are comparable whenever $|\rho(y) - \rho(x)| = 1$. Let P and Q be posets on the same set. Then Q *extends* P if $x <_Q y$ whenever $x <_P y$.

Corollary 3.2. *Let (P, ω) be a sign-graded poset. Then the Jordan-Hölder set of (P, ω) is uniquely decomposed as the disjoint union*

$$\mathcal{L}(P, \omega) = \bigsqcup_Q \mathcal{L}(Q, \omega),$$

where the union is over all saturated sign-graded posets (Q, ω), which extend (P, ω) and has the same rank-function as (P, ω).

Proof. That the union exhausts $\mathcal{L}(P, \omega)$ follows from (3.1) and Lemma 3.1. Let (Q_1, ω) and (Q_2, ω) be two different saturated sign-graded posets that extends (P, ω) and have the same rank-function as (P, ω). Then we may assume that there is a covering relation $x \prec y$ in Q_1 which is not a covering relation in Q_2. Since $|\rho(x) - \rho(y)| = 1$ we must have $y \prec x$ in Q_2. Thus $\omega(x)$ precedes $\omega(y)$ in any permutation in $\mathcal{L}(Q_1, \omega)$, and $\omega(y)$ precedes $\omega(x)$ in any permutation in $\mathcal{L}(Q_2, \omega)$. Hence, the union is disjoint. □

We need two operations on labeled posets: Let (P, ϵ) and (Q, μ) be two labeled posets. The *ordinal sum*, $P \oplus Q$, of two non-empty posets P and Q is the poset with the disjoint union of P and Q as underlying set and with partial order defined by $x \le y$ if, either $x \le_P y$ or $x \le_Q y$, or $x \in P, y \in Q$. Define two labelings of

$E(P \oplus Q)$ by

$$\begin{aligned}
(\epsilon \oplus_1 \mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y) \in E(P),\\
(\epsilon \oplus_1 \mu)(x,y) &= \mu(x,y) \text{ if } (x,y) \in E(Q) \text{ and}\\
(\epsilon \oplus_1 \mu)(x,y) &= 1 \text{ otherwise.}\\
(\epsilon \oplus_{-1} \mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y) \in E(P),\\
(\epsilon \oplus_{-1} \mu)(x,y) &= \mu(x,y) \text{ if } (x,y) \in E(Q) \text{ and}\\
(\epsilon \oplus_{-1} \mu)(x,y) &= -1 \text{ otherwise.}
\end{aligned}$$

With a slight abuse of notation we write $P \oplus_{\pm 1} Q$ when the labelings of P and Q are understood from the context. Note that ordinal sums are associative, i.e., $(P \oplus_{\pm 1} Q) \oplus_{\pm 1} R = P \oplus_{\pm 1} (Q \oplus_{\pm 1} R)$, and preserve the property of being sign-graded. The following result is obtained easily by combinatorial reasoning, see [2, 14]:

Proposition 3.3. *Let (P, ω) and (Q, ν) be two labeled posets. Then*

$$W(P \oplus Q, \omega \oplus_1 \nu; t) = W(P, \omega; t)W(Q, \nu; t)$$

and

$$W(P \oplus Q, \omega \oplus_{-1} \nu; t) = tW(P, \omega; t)W(Q, \nu; t).$$

Proposition 3.4. *Suppose that (P, ω) is a saturated canonically labeled sign-graded poset. Then (P, ω) is the direct sum*

$$(P, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the A_is are anti-chains.

Proof. Let $\pi \in \mathcal{L}(P, \omega)$. Then we may write π as $\pi = w_0 w_1 \cdots w_k$ where the w_is are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. Then $\pi \in J(Q, \omega)$ where

$$(Q, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

and A_i is the anti-chain consisting of the elements $\omega^{-1}(a)$, where a is a letter of w_i (A_i is an anti-chain, since if $x < y$ where $x, y \in A_i$ there would be a letter in π between $\omega(x)$ and $\omega(y)$ whose rank was different than that of x, y). Now, (Q, ω) is saturated so $P = Q$. $\square$

Note that the argument in the above proof also can be used to give a simple proof of Corollary 3.2 when ω is canonical. However, we wanted to prove Corollary 3.2 in its generality even though we only need it for canonical labelings.

4. The W-polynomial of a sign-graded poset

The space, S^d, of symmetric polynomials in $\mathbb{R}[t]$ with center of symmetry $d/2$ has a basis

$$B_d = \{t^i(1+t)^{d-2i}\}_{i=0}^{\lfloor d/2 \rfloor}.$$

If $h \in S^d$ has non-negative coefficients in this basis it follows immediately that the coefficients of h in the standard basis are unimodal. Let S^d_+ be the non-negative span of B_d. Thus S^d_+ is a cone. Another property of S^d_+ is that if $h \in S^d_+$ then it has the correct sign at -1 i.e.,

$$(-1)^{d/2}h(-1) \geq 0.$$

Lemma 4.1. *Let $c, d \in \mathbb{N}$. Then*
$$\begin{aligned} S^c S^d &\subset S^{c+d} \\ S_+^c S_+^d &\subset S_+^{c+d}. \end{aligned}$$
Suppose further that $h \in S^d$ has positive leading coefficient and that all zeros of h are real and non-positive. Then $h \in S_+^d$.

Proof. The inclusions are obvious. Since $t \in S_+^2$ and $(1+t) \in S_+^1$ we may assume that none of them divides h. But then we may collect the zeros of h in pairs θ and θ^{-1}. Let $A_\theta = -\theta - \theta^{-1}$. Then
$$h = C \prod_{\theta < -1} (t^2 + A_\theta t + 1),$$
where $C > 0$. Since $A_\theta > 2$ we have
$$t^2 + A_\theta t + 1 = (t+1)^2 + (A_\theta - 2)t \in S_+^2,$$
and the lemma follows. □

We can now prove our main theorem.

Theorem 4.2. *Suppose that (P, ω) is a sign-graded poset of rank r. Then $W(P, \omega; t) \in S_+^{p-r-1}$.*

Proof. By Corollary 2.4 and Lemma 2.5 we may assume that (P, ω) is canonically labeled. By Corollary 3.2 we know that
$$W(P, \omega; t) = \sum_Q W(Q, \omega; t),$$
where (Q, ω) are saturated and sign-graded with the same rank function as that of (P, ω). The W-polynomials of anti-chains are the Eulerian polynomials, which only have real non-negative zeros. By Proposition 3.4 and Proposition 3.3 the polynomial $W(Q, \omega; t)$ has only real non-positive zeros so by Lemma 4.1 and Corollary 2.4 we have $W(Q, \omega; t) \in S_+^{p-r-1}$. The Theorem now follows since S_+^{p-r-1} is a cone. □

Corollary 4.3. *Let (P, ω) be sign-graded of rank r then $W(P, \omega; t)$ is symmetric and its coefficients are unimodal. Moreover, $W(P, \omega; t)$ has the correct sign at -1, i.e.,*
$$(-1)^{(p-1-r)/2} W(P, \omega; -1) \geq 0.$$

Corollary 4.4. *Let P be a (naturally labeled) graded poset. Suppose that $\Delta_{eq}(P)$ is flag. Then the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$.*

If $h(t)$ is any polynomial with integer coefficients and $h(t) \in S^d$, it follows that $h(t)$ has integer coefficients in the basis $t^i(1+t)^{d-2i}$. Thus we know that if (P, ω) is sign-graded of rank r, then
$$W(P, \omega; t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P, \omega) t^i (1+t)^{p-r-1-2i},$$
where $a_i(P, \omega)$ are non-negative integers. It would be interesting to have a combinatorial interpretation of these coefficients, and thus a combinatorial proof of Theorem 4.2.

Let (P,ϵ) be a labeled poset. We say that (P,ϵ) *admits a rank function* if for every $x \in P$ and saturated chain $x_0 \prec x_1 \prec \cdots \prec x_k = x$, where x_0 is a minimal element, the quantity

$$\rho(x) = \sum_{i=1}^{k} \epsilon(x_{i-1}, x_i)$$

is the same. Hence, a labeled poset (P,ϵ) with a rank function is sign-graded if and only if ρ is constant on maximal elements.

Theorem 4.5. *Suppose that (P,ϵ) admits a rank-function with values in $\{0,1\}$. Then $W(P,\epsilon;t)$ has unimodal coefficients.*

Proof. One may check that the proofs of Lemma 3.1, Corollary 3.2 and Proposition 3.4 holds for this case too. But then

$$W(P,\epsilon;t) = \sum_{Q} W(Q,\epsilon;t),$$

where $W(Q,\epsilon;t)$ is unimodal and symmetric with center of symmetry $(p-1)/2$ or $(p-2)/2$. The sum of such polynomials is again unimodal. □

5. The Charney-Davis quantity

In [7] Reiner, Stanton and Welker defined the *Charney-Davis quantity* of a graded naturally labeled poset (P,ω) of rank r to be

$$CD(P,\omega) = (-1)^{(p-1-r)/2} W(P,\omega;-1).$$

We may define it in the exact same way for sign-graded posets. Since the particular labeling does not matter we write $CD(P)$. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be any permutation. We say that π is *alternating* if $\pi_1 > \pi_2 < \pi_3 > \cdots$ and *reverse alternating* if $\pi_1 < \pi_2 > \pi_3 < \cdots$. Let (P,ω) be a canonically labeled sign-graded poset. If $\pi \in \mathcal{L}(P,\omega)$ then we may write π as $\pi = w_0w_1\cdots w_k$ where w_i are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. The words w_i are called the *components* of π. The following theorem is well known, see for example [9], and gives the Charney-Davis quantity of an anti-chain.

Proposition 5.1. *Let $n \geq 0$ be an integer. Then $(-1)^{(n-1)/2}A_n(-1)$ is equal to 0 if n is even and equal to the number of (reverse) alternating permutations of the set $\{1,2,\ldots,n\}$ if n is odd.*

Theorem 5.2. *Let (P,ω) be a canonically labeled sign-graded poset. Then the Charney-Davis quantity, $CD(P)$, is equal to the number of reverse alternating permutations in $\mathcal{L}(P,\omega)$ such that all components have an odd numbers of letters.*

Proof. It suffices to prove the theorem when (P,ω) is saturated. By Proposition 3.4 we know that

$$(P,\omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the A_is are anti-chains. This means that $CD(P) = CD(A_0)CD(A_1)\cdots CD(A_k)$. Let $\pi = w_0w_1\cdots w_k \in \mathcal{L}(P,\omega)$ where w_i is a permutation of $\omega(A_i)$. Then π is a reverse alternating such that all components have an odd numbers of letters if and only if, for all i, w_i is reverse alternating if i is even and alternating if i is odd. Hence, by Proposition 5.1, the number of such permutations is indeed $CD(A_0)CD(A_1)\cdots CD(A_k)$. □

6. A characterization of sign-graded posets

Here we give a characterization of sign-graded posets along the lines of the characterization of graded posets given by Stanley in [10]. Let (P, ϵ) be any labeled poset. Define a function $\delta = \delta_\epsilon : P \to \mathbb{Z}$ by

$$\delta(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)\},$$

where $x = x_0 \prec x_1 \prec \cdots \prec x_\ell$ is any saturated chain starting at x and ending at a maximal element x_ℓ. Define a map $\Phi = \Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathbb{Z}^P$ by

$$\Phi\sigma = \sigma + \delta.$$

We have

$$\delta(x) \geq \delta(y) + \epsilon(x, y). \tag{6.1}$$

This means that $\Phi\sigma(x) > \Phi\sigma(y)$ if $\epsilon(x, y) = 1$ and $\Phi\sigma(x) \geq \Phi\sigma(y)$ if $\epsilon(x, y) = -1$. Thus $\Phi\sigma$ is a $(P, -\epsilon)$-partition provided that $\Phi\sigma(x) > 0$ for all $x \in P$. But $\Phi\sigma$ is order reversing so it attains its minimum at maximal elements and for maximal elements, z, we have $\Phi\sigma(z) = \sigma(z)$. This shows that $\Phi : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is an injection.

We say that a labeling ϵ of a poset P satisfies the *δ-chain condition* if for every $x \in P$ and saturated chain $x = x_0 \prec x_1 \prec \cdots \prec x_\ell$, where x_ℓ is a maximal element, the quantity

$$\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)$$

is the same.

Proposition 6.1. *Let (P, ϵ) be labeled poset. Then $\Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is a bijection if and only if ϵ satisfies the δ-chain condition.*

Proof. If ϵ satisfies the δ-chain condition, then so does $-\epsilon$ and $\delta_{-\epsilon}(x) = -\delta_\epsilon(x)$ for all $x \in P$. Thus the if part follows since the inverse of Φ_ϵ is $\Phi_{-\epsilon}$.

For the only if direction note that ϵ satisfies the δ-chain condition if and only if for all $(x, y) \in E$ we have

$$\delta(x) = \delta(y) + \epsilon(x, y)$$

If ϵ fails to satisfy the δ-chain property we have, by (6.1), that there is a covering relation $(x, y) \in E$ such that either $\epsilon(x, y) = 1$ and $\delta(x) \geq \delta(y) + 2$ or $\epsilon(x, y) = -1$ and $\delta(x) \geq \delta(y)$.

Suppose that $\epsilon(x, y) = 1$. It is clear that there is a $\sigma \in \mathcal{A}(-\epsilon)$ such that $\sigma(x) = \sigma(y) + 1$. But then

$$\sigma(x) - \delta(x) \leq \sigma(y) - \delta(y) - 1,$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$.

Similarly, if $\epsilon(x, y) = -1$ then we can find a partition $\sigma \in \mathcal{A}(-\epsilon)$ with $\sigma(x) = \sigma(y)$, and then

$$\sigma(x) - \delta(x) \leq \sigma(y) - \delta(y),$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$. □

Define $r(\epsilon)$ by

$$r(\epsilon) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) : x_0 \prec x_1 \prec \cdots \prec x_\ell \text{ is maximal}\}.$$

We then have:

$$\begin{aligned} \max\{\Phi\sigma(x) : x \in P\} &= \max\{\sigma(x) + \delta_\epsilon(x) : x \text{ is minimal}\} \\ &\leq \max\{\sigma(x) : x \in P\} + r(\epsilon). \end{aligned}$$

So if we let $\mathcal{A}_n(\epsilon)$ be the (P,ϵ)-partitions with largest part at most n we have that $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection. A labeling ϵ of P is said to satisfy the λ-*chain condition* if for every $x \in P$ there is a maximal chain $c : x_0 \prec x_1 \prec \cdots \prec x_\ell$ containing x such that $\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) = r(\epsilon)$.

Lemma 6.2. *Suppose that n is a non-negative integer such that $\Omega(P,\epsilon;n) \neq 0$. If*

$$\Omega(P, -\epsilon; n + r(\epsilon)) = \Omega(P, \epsilon; n)$$

then ϵ satisfies the λ-chain condition.

Proof. Define $\delta^* : P \to \mathbb{Z}$ by

$$\delta^*(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)\},$$

where the maximum is taken over all maximal chains starting at a minimal element and ending at x. Then

$$\delta(x) + \delta^*(x) \leq r(\epsilon) \tag{6.2}$$

for all x, and ϵ satisfies the λ-chain condition if and only if we have equality in (6.2) for all $x \in P$. It is easy to see that the map $\Phi^* : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ defined by

$$\Phi^*\sigma(x) = \sigma(x) + r(\epsilon) - \delta^*(x),$$

is well-defined and is an injection. By (6.2) we have $\Phi\sigma(x) \leq \Phi^*\sigma(x)$ for all σ and all $x \in P$, with equality if and only if x is in a maximal chain of maximal weight. This means that in order for $\Phi : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ to be a bijection it is necessary for ϵ to satisfy the λ-chain condition. □

Theorem 6.3. *Let ϵ be a labeling of P. Then*

$$\Omega(P, \epsilon; t) = (-1)^p \Omega(P, \epsilon; -t - r(\epsilon))$$

if and only if P is ϵ-graded of rank $r(\epsilon)$.

Proof. The "if" part is Theorem 2.3, so suppose that the equality of the theorem holds. By reciprocity we have

$$(-1)^p \Omega(P, \epsilon; -t - r(\epsilon)) = \Omega(P, -\epsilon; t + r(\epsilon)),$$

and since $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection it is also a bijection. By Proposition 6.1, ϵ satisfies the δ-chain condition, and, by Lemma 6.2, we have that all minimal elements are members of maximal chains of maximal weight. In other words P is ϵ-graded. □

It should be noted that it is not necessary for P to be ϵ-graded in order for $W(P,\epsilon;t)$ to be symmetric. For example, if (P,ϵ) is any labeled poset then the W-polynomial of the disjoint union of (P,ϵ) and $(P,-\epsilon)$ is easily seen to be symmetric. However, we have the following:

Corollary 6.4. *Suppose that*

$$\Omega(P,\epsilon;t) = \Omega(P,-\epsilon;t+s),$$

for some $s \in \mathbb{Z}$. Then $-r(-\epsilon) \leq s \leq r(\epsilon)$, with equality if and only if P is ϵ-graded.

Proof. We have an injection $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$. This means that $s \leq r(\epsilon)$. The lower bound follows from the injection $\Phi_{-\epsilon}$, and the statement of equality follows from Theorem 6.3. □

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