WORDS RESTRICTED BY 3-LETTER GENERALIZED MULTIPERMUTATION PATTERNS

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Abstract. We find exact formulas and/or generating functions for the number of words avoiding 3-letter generalized multipermutation patterns and find which of them are equally avoided.

1. Introduction

A *generalized pattern* τ is a (possibly hyphenated) string in $[\ell]^m$ which contains all letters in $[\ell] = \{1, \ldots, \ell\}$. We say that the string $\sigma \in [k]^n$ *contains* a generalized pattern τ if σ contains a subsequence isomorphic to τ in which the entries corresponding to consecutive entries of τ not separated by a hyphen must be adjacent. Otherwise, we say that σ *avoids* τ and write $\sigma \in [k]^n(\tau)$. Thus, $[k]^n(\tau)$ denotes the set of strings in $[k]^n$ (i.e. n-long k-ary strings) which avoid τ.

Example 1.1. An string $\pi = a_1a_2\ldots a_n$ avoids 13-2 if π has no subsequence $a_ia_{i+1}a_j$ with $j > i+1$ and $a_i < a_j < a_{i+1}$.

Classical patterns are generalized patterns with all possible hyphens (say, 2-1-3), in other words, those that place no adjacency requirements on σ. The first case of classical patterns studied was that of permutations avoiding a permutation pattern of length 3. Knuth [6] found that, for any $\tau \in S_3$, $|S_n(\tau)| = C_n$, the nth Catalan number. Later, Simion and Schmidt [8] determined the number $|S_n(P)|$ of permutations in S_n simultaneously avoiding any given set of patterns $P \subseteq S_3$. Burstein [2] extended this to $|[k]^n(P)|$ with $P \subseteq S_3$. Burstein and Mansour [3] considered forbidden patterns with repeated letters.

Generalized permutation patterns were introduced by Babson and Steingrímsson [1] with the purpose of the study of Mahonian statistics. Later, Claesson [4] and Claesson and Mansour [5] considered the number of permutations avoiding one or two generalized patterns with one hyphen.

In this paper, we consider the case of words avoiding a single generalized pattern of length 3.

We say that two patterns τ_1 and τ_2 are *Wilf-equivalent* or belong to the same *Wilf class* if $|[k]^n(\tau_1)| = |[k]^n(\tau_2)|$ for all integers $k, n \geq 0$. Given a generalized pattern τ we define its *reversal* $r(\tau)$ to be τ read right-to-left (including hyphens). For example, $r(13\text{-}2) = 2\text{-}31$. We also define *complement* of τ, denoted $c(\tau)$, to be the pattern obtained by substituting $\ell+1-\tau(i)$ for $\tau(i)$ and leaving hyphens in the same positions (for example, $c(13\text{-}2) = 31\text{-}2$). Clearly, $c \circ r = r \circ c$, so $\langle c, r\rangle = D_2$, the group of symmetries of a rectangle. We call the set $\{\tau, r(\tau), c(\tau), c(r(\tau))\}$ the *symmetry class* of τ. Obviously, all patterns in the same symmetry class (e.g. 13-2,

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2-31, 31-2, 2-13) are Wilf-equivalent, so we only need to consider one representative of each symmetry class to determine Wilf classes.

Let τ be any generalized pattern; we define $\#\tau(\sigma)$ as the number of occurrences of τ in σ. Let $F_\tau(n,k;q;a_1,\dots,a_d)$ be the *occurrence polynomial*, whose coefficient at q^r is the number of words in $[k]^n$ having exactly r occurrences of τ and ending on $a_1\dots a_d$; that is,

$$F_\tau(n,k;q;a_1,\dots,a_d) = \sum_{\sigma\in[k]^{n-d}} q^{\#\tau(\sigma a_1\dots a_d)}.$$

For $d=0$, we denote $F_\tau(n,k;q)=F_\tau(n,k;q;\emptyset)$. We also denote the generating function for the sequence $\{F_\tau(n,k;q)\}_{n\ge0}$ by $F_\tau(x;k;q)$; that is,

$$F_\tau(x,k;q)=\sum_{n\ge0}x^nF_\tau(n,k;q)=\sum_{n\ge0}\sum_{\sigma\in[k]^n}q^{\#\tau(\sigma)}x^n.$$

2. Two-letter generalized patterns

Here, the only symmetry classes with repeated letters are those of generalized patterns 11 and 1-1. However, avoiding 1-1 simply means having no repeated letters, so

$$|[k]^n(\text{1-1})| = \binom{k}{n}n! = (k)_n,$$

the n-th lower factorial of k, which is 0 when $n>k$. Avoiding 11 is the same as having no repeated adjacent letters, so (see Theorem 2.1)

$$|[k]^n(11)| = k(k-1)^{n-1},$$

for all $n\ge1$. The remaining symmetry classes is those of patterns 12 and 1-2. A word avoiding 1-2 is just a non-increasing string, so

$$|[k]^n(\text{1-2})| = \binom{n+k-1}{n}.$$

Avoiding 12 means there is no adjacent pair of letters in increasing order, in other words, a string avoiding 12 is a non-increasing string so (see Theorem 2.3)

$$|[k]^n(12)| = \binom{n+k-1}{n} = |[k]^n(\text{1-2})|.$$

Indeed, the strings avoiding 12 are exactly those avoiding 1-2.

Theorem 2.1. *Let $\tau = 11\dots1\in[1]^l$ be a generalized pattern. Then*

$$F_\tau(x;k;q) = \frac{1+(1-q)x\sum_{j=0}^{l-2}(kx)^j-(1-q)(k-1)\sum_{d=2}^{l-1}x^d\sum_{j=0}^{l-1-d}(kx)^j}{1-(k-1+q)x-(k-1)(1-q)(1-x^{l-2})\frac{x^2}{1-x}}.$$

Proof. Let $\langle j\rangle_d = jj\dots j$, a string of d letters j. Then, by definition,

$$\begin{aligned}F_\tau(n,k;q;\langle j\rangle_d) &= \sum_{\sigma\in[k]^{n-d}} q^{\#\tau(\sigma,\langle j\rangle_d)} = \sum_{i=1}^{k}\sum_{\sigma\in[k]^{n-d-1}} q^{\#\tau(\sigma,i,\langle j\rangle_d)} = \\ &= \sum_{i\ne j}\sum_{\sigma\in[k]^{n-d-1}} q^{\#\tau(\sigma,i,\langle j\rangle_d)} + \sum_{\sigma\in[k]^{n-d-1}} q^{\#\tau(\sigma,\langle j\rangle_{d+1})}.\end{aligned}$$

If we sum over all $j = 1, 2, \dots, k$, then for all $d \le l-2$ we have

$$\sum_{j=1}^{k} F_\tau(n,k;q;\langle j\rangle_d) = (k-1)F_\tau(n-d;k;q) + \sum_{j=1}^{k} F_\tau(n,k;q;\langle j\rangle_{d+1}),$$

hence

$$F_\tau(n,k;q) = (k-1)\sum_{d=1}^{l-2} F_\tau(n-d,k;q) + G(n,k;q),$$

where $G(n,k;q) = \sum_{j=1}^{k}\sum_{\sigma\in[k]^{n-(l-1)}} q^{\#\tau(\sigma,\langle j\rangle_{l-1})}$. Again, by the same above argument it is easy to see

$$G(n,k;q) = (k-1)F_\tau(n-(l-1),k;q) + qG(n-1,k;q).$$

Now consider $G_\tau(n,k;q) - qG_\tau(n-1,k;q)$ together with the two equations above to obtain

$$F_\tau(n,k;q) = (k-1+q)F_\tau(n-1,k;q) + (k-1)(1-q)\sum_{d=2}^{l-1} F_\tau(n-d,k;q),$$

for all $n \ge l$. Besides, $F_\tau(n,k;q) = k^n$ for all $n \le l-1$, hence, taking the generating functions of both sides, we see that the theorem holds. □

Example 2.2. Theorem 2.1 yields for all $n \ge 1$,

$$F_{11}(n,k;q) = k(q+k-1)^{n-1}.$$

and

$$F_{111}(x;k;q) = \frac{1 + x(1+x)(1-q)}{1-(k-1+q)x-(k-1)(1-q)x^2}.$$

Letting $q = 0$, we get $|[k]^n(11)| = k(k-1)^{n-1}$, and

$$\sum_{n\ge 0} |[k]^n(\underbrace{11\dots1}_{l})|x^n = \frac{1+x+\cdots+x^{l-1}}{1-(k-1)x-\cdots-(k-1)x^{l-1}}.$$

For a different approach to the last formula, see [7], Example 6.4 ff., pp. 1102–1103, for an easily generalizable case of $k = 2$.

We also obtain from the above that the number of strings in $[k]^n$ with exactly j occurrences of the generalized pattern 11 is $\binom{n-1}{j}k(k-1)^{n-j-1}$.

Theorem 2.3. *For all $n \ge 2$ and $k \ge 1$,*

$$F_{12}(x;k;q) = \frac{1}{1+\frac{1-(1+(1-q)x)^k}{1-q}}.$$

Proof. By definitions,

$$\begin{aligned}F_{12}(n,k;q;j) &= \sum_{i=1}^{k}\sum_{w\in[k]^{n-2}} q^{\#12(wij)} = \\ &= q\sum_{i=1}^{j-1}\sum_{w\in[k]^{n-2}} q^{\#12(wi)} + \sum_{i=j}^{k}\sum_{w\in[k]^{n-2}} q^{\#12(wi)} = \\ &= (q-1)\sum_{i=1}^{j-1}\sum_{w\in[k]^{n-2}} q^{\#12(wi)} + \sum_{i=1}^{k}\sum_{w\in[k]^{n-2}} q^{\#12(wi)} = \\ &= F_{12}(n-1,k;q) + (q-1)\sum_{i=1}^{j-1}\sum_{w\in[k]^{n-2}} q^{\#12(wi)}.\end{aligned}$$

Besides, $F_{12}(n,k;q;1) = F_{12}(n-1,k;q)$, hence it is easy to see by induction on j that

$$F_{12}(n,k;q;j) = \sum_{i=0}^{j-1}\binom{j-1}{i}(q-1)^i F_{12}(n-1-i,k;q).$$

Therefore, for all $n \geq 1$,

$$\begin{aligned}F_{12}(n,k;q) &= \sum_{j=1}^{k}\sum_{i=0}^{j-1}\binom{j-1}{i}(q-1)^i F_{12}(n-1-i,k;q) = \\ &= \sum_{j=1}^{k}\binom{k}{j}(q-1)^{j-1}F_{12}(n-j,k;q).\end{aligned}$$

Besides, $F_{12}(0,k;q) = 1$, hence the theorem holds for the generating function $F_{12}(x;k;q)$. □

3. Three-letter generalized patterns

This section is divided into three subsections corresponding to the three cases of 3-letter generalized patterns: classical patterns, patterns with exactly one adjacent pair of letters, and patterns without internal dashes (i.e. with three consecutive letters).

3.1. **Classical patterns.**

The symmetry class representatives are 1-2-3, 1-3-2, 1-1-2, 1-2-1, 1-1-1. It is known [2] that

$$|[k]^n(\text{1-2-3})| = |[k]^n(\text{1-3-2})| = 2^{n-2(k-2)}\sum_{j=0}^{k-2} a_{k-2,j}\binom{n+2j}{n},$$

where

$$a_{k,j} = \sum_{m=j}^{k} C_m D_{k-m},\quad D_t = \binom{2t}{t},\quad C_t = \frac{1}{t+1}\binom{2t}{t}.$$

Recently, it was shown in [3] that

$$|[k]^n(\text{1-1-1})| = \sum_{i=0}^{k}\binom{k}{i}\binom{i}{n-i}\frac{n!}{2^{n-i}} = \sum_{i=0}^{k} B(i,n-i)(k)_i,$$

where $(k)_i$ is the falling factorial, and $B(r,s) = \frac{(r+s)!}{2^s(r-s)!s!}$ is the Bessel number of the first kind. In particular, $f_{1\text{-}1\text{-}1}(n,k) = 0$ when $n > 2k$. It was also proved in [3] that

$$|[k]^n(1\text{-}2\text{-}1)| = |[k]^n(1\text{-}1\text{-}2)| = \sum_{j=0}^{k} \binom{n+k-j-1}{n} c(n, n-j),$$

where $c(n,j)$ is the signless Stirling number of the first kind.

3.2. **Generalized patterns with exactly one adjacent pair of letters.** Let $F_\tau(x;k)$ be the generating function for the sequence $\{f_\tau(n,k)\}_{n\geq 0}$ where $f_\tau(n,k) = |[k]^n(\tau)|$; that is,

$$F_\tau(x;k) = \sum_{n\geq 0} f_\tau(n,k)x^n.$$

In the current subsection, we find explicit formulas or recurrence formulas for $F_\tau(x;k)$ where τ is a three-letter generalized pattern with exactly one adjacent pair of letters. The symmetry class representatives are 11-1, 11-2, 21-1, 21-2, 12-3, 21-3 and 13-2. We will now find $F_\tau(x;k)$ for each of the above patterns.

Theorem 3.1. *For all $k \geq 1$,*

$$F_{11\text{-}1}(x;k) = \begin{vmatrix} B_k(x) & -A_k(x) & 0 & \cdots & 0 \\ B_{k-1}(x) & 1 & -A_{k-1}(x) & \cdots & 0 \\ B_{k-2}(x) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_2(x) & 0 & 0 & \ddots & -A_2(x) \\ B_1(x)+A_1(x) & 0 & 0 & \cdots & 1 \end{vmatrix}$$
$$= \prod_{i=1}^{k} A_i(x) + \sum_{j=1}^{k} \left(B_j(x) \prod_{i=j+1}^{k} A_i(x) \right)$$

where $A_j(x) = \frac{jx^2}{1-(j-1)x}$ and $B_j(x) = \frac{1+x}{1-(j-1)x}$.

Proof. Let $a_{n,k} = |[k]^n(11\text{-}1)|$, and let $a_{n,k}(i_1,\ldots,i_d)$ be the number of all words $\sigma \in [k]^n(11\text{-}1)$ such that $\sigma_j = i_j$ for all $j = 1,2,\ldots d$. Then it follows that $a_{n,k} = \sum_{i=1}^{k} a_{n,k}(i)$. On the other hand, $a_{n,k}(i) = \sum_{j=1}^{k} a_{n,k}(i,j)$, $a_{n,k}(i,j) = a_{n-1,k}(j)$ for $i \neq j$, and $a_{n,k}(i,i) = a_{n-2,k-1}$. Therefore,

$$a_{n,k} = k(a_{n-1,k} + a_{n-2,k-1}) - a_{n-1,k}$$

for all $n \geq 2$ and $k \geq 1$. Besides, $a_{1,k} = k$ and $a_{0,k} = 1$, hence, for all $k \geq 1$,

$$F_{11\text{-}1}(x;k) = \frac{1+x}{1-(k-1)x} + \frac{kx^2}{1-(k-1)x} F_{11\text{-}1}(x;k-1),$$

and $F_{11\text{-}1}(x;0) = 1$. The rest follows by induction on k. □

Theorem 3.2. *For all $k \geq 1$,*

$$F_{11\text{-}2}(x;k) = \prod_{j=0}^{k-1} \frac{1-(j-1)x}{1-(j+x)x}.$$

Proof. Let $\sigma \in [k]^n(11\text{-}2)$; then there are $f_{11\text{-}2}(n, k-1)$ such words where $\sigma_j \neq k$ for all j. Let $f_{11\text{-}2}(n, k; j)$ be the number of words $\sigma \in [k]^n(11\text{-}2)$ where $\sigma_j = k$ and j is minimal, then $\sigma = \sigma_1 k \sigma_2$ for some $\sigma_1 \in [k-1]^{j-1}(11)$ and $\sigma_2 \in [k]^{n-j}(11\text{-}2)$. Moreover, for all σ_1 and σ_2 as above, we have $\sigma = \sigma_1 k \sigma_2 \in [k]^n(11\text{-}2)$, so

$$f_{11\text{-}2}(n,k) = f_{11\text{-}2}(n,k-1) + \sum_{j=1}^{n} f_{11\text{-}2}(n,k;j)$$
$$f_{11\text{-}2}(n,k;j) = f_{11}(j-1,k-1) f_{11\text{-}2}(n-j,k)$$

so

$$\sum_{j=1}^{n} f_{11\text{-}2}(n,k;j) = f_{11\text{-}2}(n-1,k) + \sum_{j=2}^{n} f_{11}(j-1,k-1) f_{11\text{-}2}(n-j,k).$$

Hence, from the Example 2.2, we get that, for all $n \geq 1$,

$$f_{11\text{-}2}(n,k) = f_{11\text{-}2}(n,k-1) + f_{11\text{-}2}(n-1,k) + \sum_{j=0}^{n-2} (k-1)(k-2)^j f_{11\text{-}2}(n-2-j,k).$$

In addition, $f_{11\text{-}2}(0,k) = \delta_{0,k}$, hence

$$F_{11\text{-}2}(x;k) = \frac{1-(k-2)x}{1-(k-1)x-x^2} F_{11\text{-}2}(x;k-1).$$

The rest follows easily. □

Example 3.3. For $k = 1, 2$, Theorem 3.2 yields $F_{11\text{-}2}(x;1) = \frac{1}{1-x}$ and $F_{11\text{-}2}(x;2) = \frac{1}{(1-x)(1-x-x^2)}$. In other words, $f_{11\text{-}2}(n,1) = 1$ and $f_{11\text{-}2}(n,2) = F_{n+3} - 1$, where F_{n+3} is the $(n+3)$-rd Fibonacci number.

Theorem 3.4. *Patterns* 21-1 *and and* 21-2 *are Wilf-equivalent, and we have, for* $k \geq 0$,

$$F_{21\text{-}2}(x;k) = F_{21\text{-}1}(x;k) = 1 + \sum_{d=0}^{k-1} \left(x^{d+1} F_{21\text{-}1}(x;k-d) \sum_{i=d}^{k-1} (1-x)^{i-d} \binom{i}{d} \right).$$

Proof. Let us find $F_{21\text{-}1}(x;k)$ and $F_{21\text{-}2}(x;k)$ to prove that the two are equal.

Let us derive the formula for $F_{21\text{-}1}(x;k)$.

Let $g_{n;k}(i_1, \ldots, i_d)$ be the number of words w in $[k]^n(21\text{-}1)$ such that $w_j = i_j$ for all $j = 1, 2, \ldots, d$ (in other words, w begins with the string $i_1 \ldots i_d$).

Consider $g_{n,k}(i,j)$. There are two cases. If $j \geq i$, then the first i cannot be part of any 21-1 in a word w beginning with (i,j) and places no further restrictions on the rest of the w, so $g_{n,k}(i,j) = g_{n-1,k}(j)$ if $j \geq i$. If $j < i$, then deleting the first i from w starting with (i,j), we get a word $w' \in [k]^{n-1}(21\text{-}1)$ which contains exactly one j, namely, as the first letter. Now let $w'' \in [k-1]^{n-1}$ be the word obtained from w' by subtracting 1 from each letter $> j$. Obviously, this is mapping is a bijection onto the set of words in $[k-1]^{n-1}(21\text{-}1)$ starting with j (since neither it nor its inverse, i.e. adding one to each letter $\geq j$ except the first letter, creates any new occurrences of 21-1). Thus, $g_{n,k}(i,j) = g_{n-1,k-1}(j)$ if $j < i$.

$$\begin{aligned} g_{n,k}(i) &= \sum_{j=1}^{k} g_{n,k}(i,j) = \sum_{j=1}^{i-1} g_{n-1,k-1}(j) + \sum_{j=i}^{k} g_{n-1,k}(j) \\ &= f_{21\text{-}1}(n-1,k) + \sum_{j=1}^{i-1} (g_{n-1,k-1}(j) - g_{n-1,k}(j)). \end{aligned} \tag{3.1}$$

We can show by induction on d that (3.1) implies

$$g_{n,k}(d) = \sum_{j=0}^{d-1} \binom{d-1}{j} \sum_{i=0}^{d-1-j} \binom{d-1-j}{i} (-1)^i f_{21\text{-}1}(n-1-i-j, k-j).$$

Hence for all $n \geq 1$,

$$f_{21\text{-}1}(n,k) = \sum_{d=1}^{k} \sum_{j=0}^{d-1} \binom{d-1}{j} \sum_{i=0}^{d-1-j} \binom{d-1-j}{i} (-1)^i f_{21\text{-}1}(n-1-i-j, k-j)$$

with $f_{21\text{-}1}(0,k) = 1$.

If we multiply through by x^n and sum over all $n \geq 1$, we get

$$F_{21\text{-}1}(x;k) - 1 = \sum_{d=1}^{k} \sum_{j=0}^{d-1} \binom{d-1}{j} x^{j+1} (1-x)^{d-1-j} F_{21\text{-}1}(x;k-j),$$

which means that

$$F_{21\text{-}1}(x;k) - 1 = \sum_{d=0}^{k-1} \left(x^{d+1} F_{21\text{-}1}(x;k-d) \sum_{i=d}^{k-1} (1-x)^{i-d} \binom{i}{d} \right).$$

Now let us find the formula for $F_{21\text{-}2}(x;k)$.

Similarly, we define $h_{n,k}(i_1,\dots,i_d)$ to be the number of words w in $[k]^n(21\text{-}2)$ such that $w_j = i_j$ for all $j = 1,2,\dots,d$. As before, $h_{n,k}(i,j) = h_{n-1,k}(j)$ if $j \geq i$. If $j < i$ and $w \in [k]^n(21\text{-}2)$ starts with (i,j), then w must contain only one i. Thus, deleting i and subtracting 1 from each letter $> i$ is a bijection from the set of words in $[k]^n(21\text{-}2)$ that start with (i,j) onto the set of words in $[k-1]^{n-1}(21\text{-}2)$ that start with j. Therefore, $h_{n,k}(i,j) = h_{n-1,k-1}(j)$ if $j < i$, so it is easy to see that $h_{n,k}(i)$ satisfies the same recurrence 3.1 as $g_{n,k}(i)$, i.e.

$$h_{n,k}(i) = f_{21\text{-}2}(n-1,k) + \sum_{j=1}^{i-1} (h_{n-1,k-1}(j) - h_{n-1,k}(j)).$$

Noting that $g_{n,k}(i)$ and $h_{n,k}(i)$ are the same when $n = 0,1$ or $k = 0,1$, we see that $F_{21\text{-}1}(x;k) = F_{21\text{-}2}(x;k)$ for $k \geq 0$. □

Example 3.5.

$$F_{21\text{-}2}(x;0) = F_{21\text{-}1}(x;0) = 1$$

$$F_{21\text{-}2}(x;1) = F_{21\text{-}1}(x;1) = \frac{1}{1-x}$$

$$F_{21\text{-}2}(x;2) = F_{21\text{-}1}(x;2) = \frac{1}{(1-x)^2} + \frac{x^2}{(1-x)^3}$$

$$F_{21\text{-}2}(x;3) = F_{21\text{-}1}(x;3) = \frac{1-3x+6x^2-5x^3+3x^4-x^5}{(1-x)^6}$$

Theorem 3.6. *Patterns* 12-3 *and* 21-3 *are Wilf-equivalent, and for all* $k \geq 1$,

$$F_{12\text{-}3}(x;k) = F_{21\text{-}3}(x;k) = \prod_{j=0}^{k-1} \frac{1}{1 - \frac{x}{(1-x)^j}}.$$

Proof. Let $\sigma = (\sigma', k, \sigma'') \in [k]^n$ such that $\sigma_j = k$ and j is minimal. (Note that there are $f_{12\text{-}3}(n;k-1)$ (respectively, $f_{21\text{-}3}(n;k-1)$) words avoiding 12-3 (respectively, 21-3) where such j does not exist.) Hence, σ avoids 12-3 (respectively, 21-3) if and only if $\sigma' \in [k-1]^{j-1}$ avoids 12 (respectively, 21) and $\sigma'' \in [k]^{n-j}$ avoids 12-3 (respectively, 21-3). Theorem 2.3 yields for all $n \geq 0$ and $k \geq 2$,

$$f_{12\text{-}3}(n,k) = f_{12\text{-}3}(n,k-1) + \sum_{j=1}^{n} \binom{j-1+k-2}{k-2} f_{12\text{-}3}(n-j,k),$$
$$f_{21\text{-}3}(n,k) = f_{21\text{-}3}(n,k-1) + \sum_{j=1}^{n} \binom{j-1+k-2}{k-2} f_{21\text{-}3}(n-j,k).$$

Since $f_{12\text{-}3}(n,1) = f_{21\text{-}3}(n,1) = 1$ as well, we get that $F_{12\text{-}3}(x;k) = F_{21\text{-}3}(x;k)$ and

$$F_{12\text{-}3}(x;k) = F_{12\text{-}3}(x;k-1) + \frac{x}{(1-x)^{k-1}} F_{12\text{-}3}(x;k).$$

□

Lemma 3.7. *Let* $a_{n,k}(i)$ *be the number of words* $\sigma \in [k]^n(13\text{-}2)$ *such that* $\sigma_1 = i$. *Then, for* $n \geq 1$, $k \geq 0$

$$f_{13\text{-}2}(n,k) = \sum_{j=1}^{k} a_{n,k}(j), \quad a_{n,k}(j) = \sum_{i=1}^{j+1} a_{n-1,k}(i) + \sum_{i=j+1}^{k-1} a_{n-1,i}(j+1).$$

In addition, $f_{13\text{-}2}(n,1) = 1$ *for all* n.

Proof. The first identity holds by definition, and

$$a_{n,k}(j) = \sum_{i=1}^{k} a_{n,k}(j,i) = \sum_{i=1}^{j+1} a_{n,k}(j,i) + \sum_{i=j+2}^{k} a_{n,k}(j,i),$$

where $a_{n,k}(j,i)$ is the number of $\sigma \in [k]^n(13\text{-}2)$ such that $\sigma_1 = j$ and $\sigma_2 = i$. If $i \leq j+1$, then the first letter j places no restriction on the rest of σ, so $a_{n,k}(j,i) = a_{n-1,k}(i)$. If $j+2 \leq i \leq k$, then the rest of σ (positions 3 to n) may not contain any letters from $j+1$ to $i-1$ (for a total of $i-j-1$ letters), so $a_{n,k}(j,i) = a_{n-1,k+j+1-i}(j+1)$. Hence

$$a_{n,k}(j) = \sum_{i=1}^{j+1} a_{n-1,k}(i) + \sum_{i=j+2}^{k} a_{n-1,k+j+1-i}(j+1).$$

The rest is easy to obtain. □

We also note that $a_{n,k}(k-1) = a_{n,k}(k) = f(n-1,k)$ and that the second formula in Lemma 3.7 implies

$$a_{n,k}(j) - a_{n,k}(j-1) - a_{n,k-1}(j) + a_{n,k-1}(j-1) = \\ = \begin{cases} a_{n-1,k}(j+1) - a_{n-1,k-1}(j), & \text{for } 2 \le j \le k-1, \\ a_{n-1,k}(j+1) - a_{n-1,k-1}(j) + a_{n-1,k}(j), & \text{for } j = 1, \\ 0, & \text{for } j = k. \end{cases}$$

From Lemma 3.7, it is easy to obtain explicit formulas for small values of k.

Theorem 3.8.

$$F_{13\text{-}2}(x;1) = \frac{1}{1-x},$$
$$F_{13\text{-}2}(x;2) = \frac{1}{1-2x},$$
$$F_{13\text{-}2}(x;3) = \frac{(1-x)^2}{(1-2x)(1-3x+x^2)},$$
$$F_{13\text{-}2}(x;4) = \frac{1-4x+6x^2-3x^3}{(1-3x)(1-2x)(1-3x+x^2)}.$$

3.3. **Generalized Patterns without internal dashes.** The symmetry class representatives are 111, 122, 212, 123, 213. In the current subsection, we find explicit formulas for $F_\tau(x;k)$ for each of these representatives τ. Example 2.2 yields the following result for 111.

Theorem 3.9. *For all* $k \ge 1$

$$F_{111}(x;k) = \frac{1+x+x^2}{1-(k-1)x-(k-1)x^2}.$$

Theorem 3.10.

$$F_{122}(x;k) = \frac{x}{(1-x^2)^k-(1-x)}, \quad k \ge 0.$$

Proof. Let $\sigma \in [k]^n(122)$ where σ contains j letters k. If $j = 0$, then there are $f_{122}(n,k-1)$ such words. Let $j \ge 1$, $\sigma = (\sigma', k, \sigma'')$, and let $\sigma_r = k$ where r minimal (so σ' does not contain k). If $\sigma_{r+1} \ne k$ then σ avoids 122 if and only if $\sigma' \in [k-1]^{r-1}(122)$, and $\sigma'' \in [k]^{n-r}(122)$ such that $\sigma''_1 \ne k$, so there are

$$\sum_{r=1}^{n} f_{122}(r-1,k-1)(f_{122}(n-r,k) - f_{122}(n-1-r,k))$$

such words. If $\sigma_{r+1} = k$, then $\sigma' = \emptyset$, and σ avoids 122 if and only if σ'' avoids 122, so there are $f_{122}(n-2,k)$ such words. Hence

$$f_{122}(n,k) = f_{122}(n,k-1) + f_{122}(n-2,k) \\ + \sum_{i=1}^{n} f_{122}(i-1,k-1)(f_{122}(n-i,k) - f_{122}(n-1-i,k))$$

for all $n \geq 2$ and $k \geq 1$, therefore,

$$f_{122}(n,k) = f_{122}(n,k-1) + f_{122}(n-2,k) + \sum_{i=0}^{n-1} f_{122}(i,k-1)f_{122}(n-1-i,k) - \sum_{i=0}^{n-1} f_{122}(i,k-1)f_{122}(n-2-i,k)).$$

In addition, $f_{122}(1,1) = f_{122}(0,1) = f_{122}(0,0) = 1$ and $f_{122}(n,k) = 0$ for $n < 0$, hence, taking the generating functions, we see that

$$F_{122}(x;k) = F_{122}(x;k-1) + x^2F_{122}(x;k) + (x-x^2)F_{122}(x;k)F_{122}(x;k-1).$$

Now, after dividing through by $F_{112}(x;k)F_{112}(x;k-1)$ and some routine manipulations, we see that the theorem holds. □

Example 3.11. Theorem 3.10 yields $F_{122}(x;2) = \frac{1}{(1-x)(1-x-x^2)}$, which means $f_{122}(n,2) = F_{n+3} - 1$, where F_{n+3} is the $(n+3)$-rd Fibonacci number.

Theorem 3.12.

$$F_{212}(x;k) = \frac{1}{1 - x\sum_{j=0}^{k-1}\frac{1}{1+jx^2}}, \quad k \geq 0.$$

Proof. Let $d_{212}(n,k)$ be the number of words $\sigma \in [k]^{n+1}(212)$ such that $\sigma_1 = k$, and let $D_{212}(x;k)$ be the generating function for $d_{212}(n,k)$, that is $D_{212}(x;k) = \sum_{n\geq 0} d_{212}(n,k)x^n$.

Let $\sigma = (\sigma', k, \sigma'') \in [k]^n(212)$ so that σ' does not contain k. If $\sigma = \sigma'$, then there are $f_{212}(n,k-1)$ such words. Otherwise, $\sigma' \in [k-1]^{j-1}$ for some $j = 1,\dots,n$, so there are $f_{212}(j-1,k-1)d_{212}(n-j,k)$ such words. Therefore, for all $n \geq 1$,

$$f_{212}(n,k) = f_{212}(n,k-1) + \sum_{j=1}^{n} f_{212}(j-1,k-1)d_{212}(n-j,k).$$

In addition, $f_{212}(0,k) = 0$ for all $k \geq 1$, hence

$$F_{212}(x;k) = (1 + xD_{212}(x;k))F_{212}(x;k-1).$$

Now let $\sigma = (k,\sigma'') \in [k]^{n+1}(212)$. The first k is not part of any occurrence of 212, so we can delete it to get any word $\sigma'' \in [k]^n(212)$. We can obtain an occurrence of 212 by adding the first k back only if $\sigma = (k,i,k,\sigma''')$ for some $i < k$ and $(k,\sigma''') \in [k]^{n-1}(212)$. Then neither the second nor the third letter of σ can start 212, so the number of such "bad" words is $(k-1)d_{212}(n-2,k)$. Hence,

$$d_{212}(n,k) = f_{212}(n,k) - (k-1)d_{212}(n-2,k), \quad n \geq 2.$$

Taking generating functions of both sides, we obtain

$$D_{212}(x;k) = F_{212}(x;k) - (k-1)x^2D_{212}(x;k).$$

Now, solving the two generating function recurrences above, we get

$$\frac{1}{F_{212}(x;k)} = \frac{1}{F_{212}(x;k-1)} - \frac{x}{1+(k-1)x^2},$$

which implies the theorem. □

Theorem 3.13.
$$F_{123}(x;k) = \frac{1}{\sum_{j=0}^{k} a_j \binom{k}{j} x^j}, \quad k \geq 0,$$
where $a_{3m} = 1$, $a_{3m+1} = -1$, $a_{3m+2} = 0$ *for all* $m \geq 0$.

Proof. By definition, $f_{123}(n;k) = k^n$ for $k = 0,1,2$, so the formulas for $F_{123}(x;k)$ hold for $k = 0,1,2$. Let $d_{123}(n,k)$ be the number of words $\sigma \in [k]^n(123)$ such that $(\sigma, k+1)$ also avoids 123, and let $D_{123}(x;k)$ be the generating function for $d_{123}(n,k)$ with k fixed.

Lemma 3.14.
$$D_{123}(x;3) = \frac{1-3x^2+x^3}{1-3x+x^3}, \quad F_{123}(x;3) = \frac{1}{1-3x+x^3}.$$

Proof. Let $\sigma \in [3]^n(123)$, and let $f_{123}(n,k;a_1,\dots,a_d)$ be the number of words $\sigma \in [3]^n(123)$ such that $\sigma_i = a_i$ for all $i = 1,2\dots,d$. Deleting the first letter $i \in \{1,2,3\}$ of σ, we get a word $\sigma' \in [3]^{n-1}(123)$. Adjoining i in front of any word $\sigma' \in [3]^{n-1}(123)$, we get a word $\sigma \in [k]^n(123)$ or $\sigma = (1,2,3,\sigma'')$ for some $\sigma'' \in [3]^{n-3}(123)$. Hence,
$$f_{123}(n,3) = 3f_{123}(n-1,3) - f_{123}(n-3,3), \quad n \geq 3.$$
Besides, $f_{123}(n,3) = 3^n$ for $n = 0,1,2$, hence the formula for $F_{123}(x;3)$ holds.

Similarly, we have
$$d_{123}(n,3) = 3d_{123}(n-1,3) - d_{123}(n-3,3), \quad n \geq 4.$$
Besides, $d_{123}(0,3) = 1$, $d_{123}(1,3) = 3$, $d_{123}(2,3) = 6$, and $d_{123}(3,3) = 18$, hence the formula for $D_{123}(x;3)$ holds as well. □

Now, we are ready to prove the two main recurrences.

Lemma 3.15. *For all* $k \geq 4$,
$$F_{123}(x;k) = \frac{F_{123}(x;k-1)}{1-xD_{123}(x;k-1)}, \quad D_{123}(x;k) = \frac{x+(1-x)D_{123}(x;k-1)}{1-xD_{123}(x;k-1)}.$$

Proof. Let $\sigma = (\sigma', k, \sigma'') \in [k]^n(123)$ be such that $\sigma_j = k$ and $\sigma' \in [k-1]^{j-1}$ (the leftmost k is at position j). If $\sigma = \sigma'$ (i.e. σ has no k), then there are $f_{123}(n,k-1)$ such words, otherwise, there are $d_{123}(j-1,k-1)f_{123}(n-j,k)$ of them. Therefore,
$$f_{123}(n,k) = f_{123}(n,k-1) + \sum_{j=1}^{n} d_{123}(j-1,k-1)f_{123}(n-j,k),$$
for all $n \geq 1$. Besides, $f_{123}(0,k) = 0$ for $k \geq 1$, hence, taking generating functions, we get
$$F_{123}(x;k) = \frac{F_{123}(x;k-1)}{1-xD_{123}(x;k-1)}.$$
Similarly, we have for all $n \geq 1$,
$$d_{123}(n,k) = d_{123}(n,k-1) + \sum_{i=1}^{n-1} d_{123}(i-1,k-1)d_{123}(n-i,k),$$
or, equivalently,
$$d_{123}(n,k) = d_{123}(n,k-1) - d_{123}(n-1,k-1) + \sum_{i=1}^{n} d_{123}(i-1,k-1)d_{123}(n-i,k)$$

for all $n \geq 1$. Besides, $d_{123}(0,k) = 1$, and $d_{123}(n,k) = 0$ for $n < 0$, hence, taking generating functions, we get

$$D_{123}(x;k) = \frac{x + (1-x)D_{123}(x;k-1)}{1 - xD_{123}(x;k-1)}.$$

□

Finally, Lemmas 3.14 and 3.15 together yield us Theorem 3.13. We also note that the same Lemmas yield that

$$D_{123}(x;k) = \frac{\sum_{j=0}^{k} b_j \binom{k}{j} x^j}{\sum_{j=0}^{k} a_j \binom{k}{j} x^j}, \quad k \geq 0,$$

where $b_{3m} = a_{3m} = 1$, $b_{3m+1} = a_{3m+2} = 0$, $b_{3m+2} = a_{3m+1} = -1$ for all $m \geq 0$. □

Theorem 3.16.

$$F_{213}(x;k) = \frac{1}{1 - x - x\sum_{i=0}^{k-2}\prod_{j=0}^{i}(1 - jx^2)}, \quad k \geq 1,$$

and $F_{213}(x;0) = 1$.

Proof. For $k = 0$ the theorem is trivial, so we may assume $k \geq 1$. Let $d_{213}(n,k)$ be the number of all $\sigma \in [k]^n(213)$ such that $(\sigma, k+1)$ also avoids 213, and let $D_{213}(x;k)$ be the generating function for $d_{213}(n,k)$ with k fixed.

Similarly to Lemma 3.15 we have

$$f_{213}(n,k) = f_{213}(n,k-1) + \sum_{j=1}^{n} d_{213}(j-1,k-1) f_{213}(n-j,k), \quad n \geq 1.$$

Also, $f_{213}(0,k) = 0$ for $k \geq 1$, hence, taking generating functions, we get

$$F_{213}(x;k) = \frac{F_{213}(x;k-1)}{1 - xD_{213}(x;k-1)}.$$

Similarly to Lemma 3.15, we have for all $n \geq 1$,

$$d_{213}(n,k) = d_{213}(n,k-1) + d_{213}(n-1,k-1) + d_{213}(n-2,k-1) + \sum_{i=1}^{n-2} d_{213}(i-1,k-1) d_{213}(n-i,k),$$

or, equivalently,

$$d_{213}(n,k) = d_{213}(n,k-1) - (k-1)d_{213}(n-2,k-1) + \sum_{i=1}^{n} d_{213}(i-1,k-1) d_{213}(n-i,k),$$

for all $n \geq 1$. Besides, $d_{213}(0,k) = 1$, and $d_{213}(n,k) = 0$ for $n < 0$, hence, taking generating functions, we obtain

$$D_{213}(x;k) = \frac{(1 - (k-1)x^2) D_{213}(x;k-1)}{1 - xD_{213}(x;k-1)}.$$

□

We remark that it would be interesting to see to what extent the proofs of the theorems in this section can be automated by using enumeration schemes introduced by Zeilberger [9].

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