321-polygon-avoiding permutations and Chebyshev polynomials

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Submitted: July 22, 2002; Accepted: January 6, 2003.
MR Subject Classifications: 05A05, 05A15, 30B70, 42C05

Abstract

A 321-*k-gon-avoiding permutation* π avoids 321 and the following four patterns:

$$\begin{array}{l} k(k+2)(k+3)\cdots(2k-1)1(2k)23\cdots(k-1)(k+1), \\ k(k+2)(k+3)\cdots(2k-1)(2k)12\cdots(k-1)(k+1), \\ (k+1)(k+2)(k+3)\cdots(2k-1)1(2k)23\cdots k, \\ (k+1)(k+2)(k+3)\cdots(2k-1)(2k)123\cdots k. \end{array}$$

The 321-4-gon-avoiding permutations were introduced and studied by Billey and Warrington [BW] as a class of elements of the symmetric group whose Kazhdan-Lusztig, Poincaré polynomials, and the singular loci of whose Schubert varieties have fairly simple formulas and descriptions. Stankova and West [SW1] gave an exact enumeration in terms of linear recurrences with constant coefficients for the cases $k = 2, 3, 4$. In this paper, we extend these results by finding an explicit expression for the generating function for the number of 321-k-gon-avoiding permutations on n letters. The generating function is expressed via Chebyshev polynomials of the second kind.

1 Introduction

Definition 1 *Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. Then α contains τ if there exists a subsequence $1 \le i_1 < i_2 < \ldots < i_k \le n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a pattern; α avoids τ, or is τ-avoiding, if α does not contain such a subsequence. The set of all τ-avoiding permutations in S_n is* *denoted by $S_n(\tau)$. For a collection of patterns T, α avoids T if α avoids all $\tau \in T$; the corresponding subset of S_n is denoted by $S_n(T)$.*

While the case of permutations avoiding a single pattern has attracted much attention (for example, see [BaWe, BWX, S, SW2]), the case of multiple pattern avoidance remains less investigated. In particular, it is natural to consider permutations avoiding pairs of patterns τ_1, τ_2. The enumeration problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SS]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]). For $\tau_1, \tau_2 \in S_4$ the classification into Wilf classes has been completed and enumeration formulae are known for many Wilf classes exact (see [Bo1, Km] and references therein). Several recent papers [CW, MV1, Kr, MV2, MV3, MV4] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2. The tools involved in these papers include Fibonacci numbers, Catalan numbers, Chebyshev polynomials, continued fractions, and Dyck words, e.g. in [MV2]:

Theorem 2 (Mansour, Vainshtein) *Let $U_m(\cos\theta) = \sin(m+1)\theta/\sin\theta$ be the Chebyshev polynomial of the second kind. When $2 \le d+1 \le k$, the generating function for the number of permutations in $S_n(321, (d+1)\cdots k12\cdots d)$ is given by*

$$\frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}.$$

Recently, a special class of restricted permutations has arisen in representation theory.

Definition 3 *A permutation π is k-*gon-avoiding *if it avoids each pattern in the set $\mathcal{P}_k$:*

$$\begin{array}{l}\{k(k+2)(k+3)\cdots(2k-1)1(2k)23\cdots(k-1)(k+1),\\ \;k(k+2)(k+3)\cdots(2k-1)(2k)12\cdots(k-1)(k+1),\\ \;(k+1)(k+2)(k+3)\cdots(2k-1)1(2k)23\cdots k,\\ \;(k+1)(k+2)(k+3)\cdots(2k-1)(2k)123\cdots k\}.\end{array}$$

We say that π is a 321-k-gon-*avoiding permutation if it is both k-gon-avoiding and* 321-*avoiding. The number of* 321-*k-gon-avoiding permutations in S_n is denoted by $f_k(n)$. The corresponding generating function is $f_k(x) = \sum_{n\ge 0} f_k(n)x^n$.*

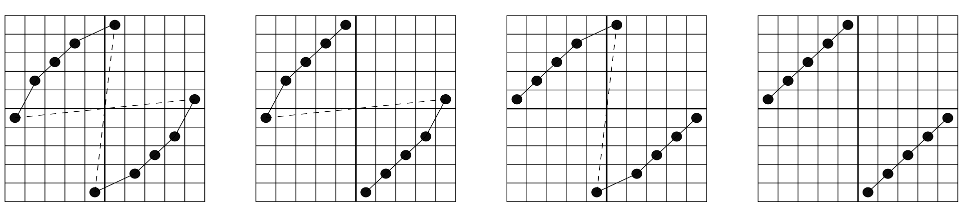

Figure 1: $\mathcal{B}_5$: all four 5-gons

Note that $f_k(n) = \frac{1}{n+1}\binom{2n}{n}$ for $n \in [0, 2k-1]$, as these count the permutations in $S_n(321)$ (see [Kn]).

Billey and Warrington [BW] introduced the 321-4-gon-avoiding (or *321-hexagon-avoiding*) permutations as a class in S_n whose Kazhdan-Lusztig and Poincaré polynomials, and the singular loci of whose Schubert varieties have fairly simple formulas and descriptions. Upon their request, Stankova and West [SW1] presented an exact enumeration for the cases $k = 2, 3, 4$ by using generating trees, the symmetries in the set of the $\mathcal{P}_k$, and the structure of the 321-avoiding permutations via Schensted's 321–subsequences decomposition.

Theorem 4 (Stankova,West) *For $k = 2, 3, 4$, the sequences $f_k(n)$ satisfy the recursive relations*

$$\begin{array}{l} f_4(n) = 6f_4(n-1) - 11f_4(n-2) + 9f_4(n-3) - 4f_4(n-4) - 4f_4(n-5) + f_4(n-6),\ n \geq 6; \\ f_3(n) = 4f_3(n-1) - 4f_3(n-2) + 3f_3(n-3) + f_3(n-4) - f_3(n-5),\ n \geq 5; \\ f_2(n) = 3f_2(n-1) - 3f_2(n-2) + f_2(n-3) = (n-1)^2 + 1,\ n \geq 3. \end{array}$$

In this paper we present an approach to the study of 321- k-gon- avoiding permutations in S_n which generalizes the methods in [SW1] and [MV3]. As a consequence, we extend the results in [SW1] to all 321-k-gon-avoiding permutations, and derive a number of other related results. The main theorem of the paper is formulated as follows.

Theorem 5 *For $k \geq 3$ and $s \geq 1$, define $L_n^k(s) = \sum_{j=0}^{s} (-1)^j \binom{s-j}{j} f_k(n-j)$. When $n \geq 2k$, this sequence satisfies the linear recursive relation with constant coefficients:*

$$L_n^k(2k-2) = L_{n-1}^k(2k-2) + L_{n-3}^k(2k-5) + L_{n-3}^k(2k-4) + L_{n-4}^k(2k-5) + L_{n-4}^k(2k-4).$$

Corollary 6 *For $k \geq 3$,*

$$f_k(x) = \frac{(1+2x^2+x^3)U_{2k-3}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1+x)U_{2k-4}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}\left[(1+2x^2+x^3)U_{2k-2}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1+x)U_{2k-3}\left(\frac{1}{2\sqrt{x}}\right)\right]}.$$

The proofs of Theorem 5 and Corollary 6 are presented in Section 2. Note that Corollary 6 implies the previously known results for the cases $k = 3, 4$ (see [SW1]):

$$\begin{aligned} f_3(x) &= \frac{1-3x+2x^2-2x^3-2x^4}{1-4x+4x^2-3x^3-x^4+x^5}, \\ f_4(x) &= \frac{1-5x+7x^2-5x^3+x^4+3x^5}{1-6x+11x^2-9x^3+4x^4+4x^5-x^6}. \end{aligned}$$

In Section 3, we describe several generalizations of Theorem 5 and Corollary 6, following similar arguments from their proofs.

2 Proof of Theorem 5

2.1 Refinement of the numbers $f_k(n)$

Definition 7 *For m, n with $1 \leq m \leq n$ and distinct $i_1, i_2, ..., i_m \in \mathbb{N}$, we denote by $f_k(n; i_1, \ldots, i_m)$ the number of 321-k-gon-avoiding permutations $\pi \in S_n$ which start with $i_1 i_2 \cdots i_m$: $\pi_1\pi_2\cdots\pi_m = i_1 i_2 \cdots i_m$. The corresponding subset of S_n is denoted by*

$$\mathcal{F}_k(n; i_1, \ldots, i_m).$$

Here follow basic properties of the numbers $f_k(n; i_1, \ldots, i_m)$, easily deduced from the definitions.

Lemma 8 *Let $n \geq 3$, $1 \leq m \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.*

(a) If $m \leq n-2$ and $3 \leq i_1$, then $f_k(n; i_1, \ldots, i_m, j) = 0$ for $2 \leq j \leq i_m - 1$. Consequently,

$$f_k(n; i_1, \ldots, i_m) = f_k(n; i_1, \ldots, i_m, 1) + \sum_{j=i_m+1}^{n} f_k(n; i_1, \ldots, i_m, j).$$

(b) If $m \leq k-1$ and $2 \leq i_1$, then $f_k(n; i_1, \ldots, i_m, 1) = f_k(n-1; i_1 - 1, \ldots, i_m - 1)$.

(c) If $i_1 \leq k-1$, then $f_k(n; i_1, \ldots, i_m) = f_k(n-1; i_2 - 1, \ldots, i_m - 1)$.

Proof For (a), observe that if $\pi \in \mathcal{F}_k(n; i_1, \ldots, i_m, j)$ then the entry $\langle i_m, j, 1\rangle$ gives an occurrence of 321 in π. For the second part of (a), consider the entry π_{m+1} of π. Again, avoiding 321 forces $\pi_{m+1} = 1$ or $\pi_{m+1} > i_m$. For (b), denote by π' the permutation obtained from π by deleting its smallest entry and decreasing all other entries by 1 and the permutation π obtained from π' by $\pi = (\pi'_1 + 1, \ldots, \pi'_m + 1, 1, \pi'_{m+1} + 1, \ldots, \pi'_{n-1} + 1)$. Then $\pi \in \mathcal{F}_k(n; i_1, \ldots, i_m, 1)$ if and only if $\pi' \in \mathcal{F}_k(n-1; i_1 - 1, \ldots, i_m - 1)$, since entry 1 placed as in (b) cannot be used in an occurrence of 321 or $\tau \in \mathcal{P}_k$ in π. For (c), observe that if $\pi_1\pi_2\cdots\pi_m = i_1 i_2 \cdots i_m$ then the entry i_1 cannot appear in any occurrences of $\tau \in \mathcal{P}_k$; further, if there is an occurrence xyz of 321 such that $x = i_1$ then there is an occurrence $i_2 y x$ of 321 in π. □

Lemma 8 implies an explicit formula for $f_k(n; s)$ for the first values of s.

Proposition 9 *For $1 \leq s \leq \min\{k-1, n\}$,*

$$f_k(n; s) = \sum_{j=0}^{s-1} (-1)^j \binom{s-1-j}{j} f_k(n-1-j).$$

Proof By Lemma 8(a)-(c) the proposition holds for $s = 1, 2$. For $s \geq 3$, Lemma 8(a) says

$$
\begin{aligned}
f_k(n;s) &= f_k(n;s,1) + \sum_{j=s+1}^{n} f_k(n;s,j) \\
\Rightarrow\ f_k(n;s) &= f_k(n-1;s-1) + \sum_{j=s+1}^{n} f_k(n-1;j-1) = \sum_{j=s-1}^{n-1} f_k(n-1;j).
\end{aligned}
$$

Equivalently,

$$
f_k(n;s) = f_k(n-1) - \sum_{j=1}^{s-2} f_k(n-1;j). \tag{1}
$$

Using induction on s, we assume that the proposition holds for all $1 \leq j \leq s-1$. Then (1) yields

$$
f_k(n;s) = f_k(n-1) - \sum_{j=1}^{s-2} \sum_{i=0}^{j-1} (-1)^i \binom{j-1-i}{i} f_k(n-2-i).
$$

Switching the summation for indices i and j, applying the familiar equality

$$
\binom{1}{a} + \binom{2}{a} + \cdots + \binom{b}{a} = \binom{b+1}{a+1},
$$

and finally relabelling the remaining index i to j, we obtain for all $2 \leq s \leq k-1$

$$
\begin{aligned}
f_k(n;s) &= f_k(n-1) + \sum_{j=1}^{s-1} (-1)^j \binom{s-1-j}{j} f_k(n-1-j) \\
&= \sum_{j=0}^{s-1} (-1)^j \binom{s-1-j}{j} f_k(n-1-j).
\end{aligned}
$$

Next we introduce objects $A_d(n,m)$ which organize suitably the information about the numbers $f_k(n; i_1, i_2, ..., i_m)$ and play an important role in the proof of the main result. □

Definition 10 *For* $1 \leq d \leq n+1-m$ *and* $1 \leq m \leq n$ *set*

$$
A_d(n,m) = \sum_{d \leq i_1 < i_2 < \cdots < i_m \leq n} f_k(n; i_1, \ldots, i_m).
$$

In the following subsections 2.2–2.3 we derive two expressions for $A_k(n, k-1)$, compare them in subsection 2.4, and thus complete the proof of Theorem 5.

2.2 First expression of $A_k(n, k-1)$

The numbers $A_d(n, m)$ satisfy the following recurrence.

Lemma 11 *For* $2 \leq d \leq k$ *and* $1 \leq m \leq \min\{k-1, n\}$,

$$A_d(n, m) = A_{d-1}(n, m) - A_{d-1}(n-1, m-1).$$

Proof By Definition 4, for all $2 \leq d \leq k$ we have:

$$A_d(n, m) = A_{d-1}(n, m) - \sum_{d \leq i_2 < \cdots < i_m \leq n} f_k(n; d-1, i_2, ..., i_m).$$

Lemma 8(c) and Definition 4 imply

$$\begin{aligned} A_d(n, m) &= A_{d-1}(n, m) - \sum_{d-1 = i_2 < i_3 < \cdots < i_m \leq n-1} f_k(n-1; i_2, i_3 \ldots, i_m) \\ &= A_{d-1}(n, m) - A_{d-1}(n-1, m-1). \end{aligned}$$

□

We next find an explicit expression for $A_1(n, m)$ in terms of $f_k(n)$.

Lemma 12 *Let* $1 \leq m \leq \min\{k-1, n\}$. *Then*

$$A_1(n, m) = \sum_{j=0}^{m} (-1)^j \binom{m-j}{j} f_k(n-j).$$

Proof For $m = 1$, $A_1(n, 1) = \sum_{1 \leq i_1 \leq n} f_k(n; i_1) = f_k(n)$, which equals the required expression. Assume the lemma for m and all appropriate n. Comparing the $(m+1)^{\text{st}}$ entry j of π with i_m,

$$A_1(n, m) = A_1(n, m+1) + \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} \sum_{j=1}^{i_m - 1} f_k(n; i_1, i_2, ..., i_m, j).$$

For the summation part on the right–hand side, avoidance of (321) implies that all numbers $1, 2, ..., j-1$ appear before the entry j, and hence $j \leq m$. From Lemma 8,

$$\begin{aligned} A_1(n, m) &= A_1(n, m+1) + \sum_{j=1}^{m} \sum_{j+1 \leq i_j < i_{j+1} < \ldots < i_m \leq n} f_k(n; 1, 2, ..., j-1, i_j, i_{j+1}, ..., i_m, j) \\ &= A_1(n, m+1) + \sum_{j=1}^{m} \sum_{2 \leq i_j < i_{j+1} < \ldots < i_m \leq n-j+1} f_k(n-j+1; i_j, i_{j+1}, ..., i_m, 1) \\ &= A_1(n, m+1) + \sum_{j=1}^{m} \sum_{1 \leq i_j < i_{j+1} < \ldots < i_m \leq n-j} f_k(n-j; i_j, i_{j+1}, ..., i_m) \\ &= A_1(n, m+1) + \sum_{j=1}^{m} A_1(n-j, m-j+1) = \sum_{j=0}^{m} A_1(n-j, m-j+1) \end{aligned}$$

Applying the above to $A_1(n-1,m-1)$, subtracting the results, using the induction hypothesis and the Pascal triangle identity $\binom{m-j}{j}+\binom{m-j}{j-1}=\binom{m-j+1}{j}$, we arrive at

$$\begin{aligned}
A_1(n,m) &= A_1(n,m+1)+A_1(n-1,m-1)\\
\Rightarrow\ A_1(n,m+1) &= A_1(n,m)-A_1(n-1,m-1)\\
&= \sum_{j=0}^{m}(-1)^j\binom{m-j}{j}f_k(n-j)-\sum_{j=0}^{m-1}(-1)^j\binom{m-j-1}{j}f_k(n-1-j)\\
&= \sum_{j=0}^{m+1}(-1)^j\binom{m-j+1}{j}f_k(n-j).
\end{aligned}$$

□

Lemmas 11–12 can be combined to derive the following explicit expression for $A_d(n,m)$, which is easily proven by induction on d.

Corollary 13 *Let $1\le d\le k$, $1\le m\le \min\{k-1,n\}$. Then*

$$A_d(n,m)=\sum_{j=0}^{d+m-1}(-1)^j\binom{d+m-1-j}{j}f_k(n-j)=L_n^k(d+m-1).$$

In particular,

$$A_k(n,k-1)=\sum_{j=0}^{2k-2}(-1)^j\binom{2k-2-j}{j}f_k(n-j)=L_n^k(2k-2).$$

2.3 Second expression of $A_k(n,k-1)$

We start by introducing three objects related to $A_d(n,m)$.

Definition 14 *For $1\le d\le n-m+1$ and $1\le m\le n$ set*

$$\begin{aligned}
B_d(n,m) &= \sum_{d+1\le i_1<i_2<\cdots<i_m\le n} f_k(n;d,i_1,\ldots,i_m);\\
C_d(n,m) &= \sum_{d\le i_1<i_2<\cdots<i_m\le n} f_k(n;i_1,\ldots,i_m,1);\\
D_d(n,m) &= \sum_{d+1\le i_1<i_2<\cdots<i_m\le n} f_k(n;d,i_1,\ldots,i_m,1).
\end{aligned}$$

Note that by Lemma 8(a), for $k\ge 2$:

$$A_k(n,k-1)=A_k(n,k)+C_k(n,k-1). \tag{2}$$

The following Propositions 15–16 describe $A_k(n,k)$ and $C_k(n,k-1)$ in terms of $f_k(n)$.

Proposition 15 *Let $n\ge k-1$. Then*

(a) $C_k(n,k-1)=C_{k-1}(n-1,k-1)+A_{k-1}(n-3,k-2)$;

(b) $C_k(n,k-1)=C_k(n-1,k-1)+A_{k-2}(n-3,k-2)+A_{k-1}(n-3,k-2)$.

Proof For (a), similarly to Lemma 8(b) (with $k \geq 3$), we have

$$C_k(n,k-1) = \sum_{k \leq i_1 < \cdots < i_{k-1} \leq n} f_k(n; i_1, \ldots, i_{k-1}, 1, 2) + \sum_{k \leq i_1 < \cdots < i_{k-1} < i_k \leq n} f_k(n; i_1, \ldots, i_{k-1}, 1, i_k). \quad (3)$$

If π starts with $i_1, \ldots, i_{k-1}, 1, 2$ as in the first sum in (3), then the entry 2 cannot participate in an occurrence of 321 or of $\tau \in \mathcal{P}_k$. Hence

$$f_k(n; i_1, \ldots, i_{k-1}, 1, 2) = f_k(n-1; i_1 - 1, \ldots, i_{k-1} - 1, 1). \quad (4)$$

For the second sum in (3), if π starts with $i_1, \ldots, i_{k-1}, 1, i_k$, avoidance of 321 and both

$$(k+1)(k+2)(k+3)\cdots(2k-1)1(2k)23\cdots k,$$
$$k(k+2)(k+3)\cdots(2k-1)1(2k)23\cdots(k-1)(k+1),$$

implies $i_1 = k$ and $i_2 = k+1$. Now, if π starts with $k, k+1, i_3, \ldots, i_{k-1}, 1, i_k$ where $k+2 \leq i_3 < \cdots < i_k \leq n$, then note that no occurrence of $\tau \in \mathcal{P}_k$ can contain the entries k or $k+1$; further, an occurrence of 321 containing k will exist in π if and only if there is such an occurrence containing $k+1$. Using this and Lemma 8,

$$\begin{aligned} f_k(n; k, k+1, i_3, \ldots, i_{k-1}, 1, i_{k+1}) &= f_k(n-1; k, i_3 - 1, \ldots, i_{k-1} - 1, 1, i_{k+1} - 1) \\ &= f_k(n-2; k-1, i_3 - 2, \ldots, i_{k-1} - 2, i_{k+1} - 2) \\ &= f_k(n-3; i_3 - 3, \ldots, i_{k-1} - 3, i_{k+1} - 3). \end{aligned}$$

Combining the last equality with (3), (4) and the definitions of $C_d(n,m)$ and $A_d(n,m)$, we obtain the desired equality

$$C_k(n,k-1) = C_{k-1}(n-1,k-1) + A_{k-1}(n-3,k-2).$$

For (b), by definitions of $C_d(n,m)$ and $D_d(n,m)$, we have

$$C_{k-1}(n-1,k-1) = C_k(n-1,k-1) + D_{k-1}(n-1,k-2).$$

Combining with (a), it is enough to show $D_{k-1}(n-1,k-2) = A_{k-2}(n-3,k-2)$. To this end, note that if $\pi \in \mathcal{F}_k(n-1; k-1, i_1, \ldots, i_{k-2}, 1)$ where $k+1 \leq i_1 < \cdots < i_{k-2} \leq n-1$, then by Lemma 8

$$f_k(n-1; k-1, i_1, \ldots, i_{k-2}, 1) = f_k(n-3; i_1 - 2, \ldots, i_{k-2} - 2).$$

By definitions of $D_d(n,m)$ and $A_d(n,m)$, we obtain the required equality. □

Proposition 16 *Let $n \geq k-1$. The sequences A_k, B_k, C_k and D_k satisfy the relations:*

(a) $A_k(n,k) = B_k(n-1,k-2)$;

(b) $B_k(n-1,k-2) = D_k(n-1,k-2) + B_k(n-1,k-1)$;

(c) $B_k(n-1,k-1) = B_k(n-2,k-2)$;

(d) $D_k(n-1,k-2) = A_{k-2}(n-4,k-2) + A_{k-1}(n-4,k-2)$;

(e) $A_k(n,k) - A_k(n-1,k) = A_{k-2}(n-4,k-2) + A_{k-1}(n-4,k-2)$.

Proof For (a), if $\pi \in F_k(n; i_1, i_2, \ldots, i_k)$ so that $k \leq i_1 < i_2 < \cdots < i_k \leq n$, then avoidance of 321,

$$(k+1)(k+2)(k+3)\cdots(2k-1)(2k)123\cdots k,$$

and

$$k(k+2)(k+3)\cdots(2k-1)(2k)12\cdots(k-1)(k+1),$$

implies $i_1 = k$ and $i_2 = k+1$. Since no occurrence of $\tau \in \mathcal{P}_k$ in π can contain both entries k and $k+1$, it follows that $f_k(n; k, k+1, i_3, \ldots, i_k) = f_k(n-1; k, i_3-1, \ldots, i_k-1)$, and hence (a). For (b), if $\pi \in F_k(n-1; k, i_1, \ldots, i_{k-2})$ so that $k+1 \leq i_1 < i_2 < \cdots < i_{k-2} \leq n-1$, then by Lemma 8(a)-(b) we get

$$f_k(n-1; k, i_1, \ldots, i_{k-2}) = f_k(n-1; k, i_1, \ldots, i_{k-2}, 1) + \sum_{i_{k-1}=i_{k-2}+1}^{n-1} f_k(n-1; k, i_1, \ldots, i_{k-1}),$$

so, if summing over $k+1 \leq i_1 < i_2 < \cdots < i_{k-2} \leq n-1$ then by Definition 5 we have (b). For (c), if $\pi \in F_k(n-1; k, i_1, \ldots, i_{k-1})$ where $k+1 \leq i_1 < \cdots < i_{k-1} \leq n-1$, then avoidance of

$$321 \text{ and } k(k+2)(k+3)\cdots(2k-1)(2k)12\cdots(k-1)(k+1)$$

implies again $i_1 = k+1$. As in (a), $f_k(n-1; k, k+1, i_2, \ldots, i_{k-1}) = f_k(n-2; k, i_2-1, \ldots, i_{k-1}-1)$, and (c) follows. For (d), consider $D_k(n-1, k-2)$ along with a similar argument to the one in Lemma 8(b):

$$\begin{aligned} D_k(n-1, k-2) &= \sum_{k+1 \leq i_1 < \cdots < i_{k-2} \leq n} f_k(n; k, i_1, \ldots, i_{k-2}, 1, 2) \\ &+ \sum_{k+1 \leq i_1 < \cdots < i_{k-1} \leq n} f_k(n; k, i_1, \ldots, i_{k-2}, 1, i_{k-1}), \end{aligned}$$

Part (d) follows from here as in the proof of (4) and (5). Finally, (a)–(d) yield (e). □

2.4 Proofs of Theorem 5 and Corollary 6

Theorem 17 *For $k \geq 3$ and $s \geq 1$, define $L_n^k(s) = \sum_{j=0}^{s} (-1)^j \binom{s-j}{j} f_k(n-j)$. When $n \geq 2k$, this sequence satisfies the linear recursive relation with constant coefficients:*

$$L_n^k(2k-2) = L_{n-1}^k(2k-2) + L_{n-3}^k(2k-5) + L_{n-3}^k(2k-4) + L_{n-4}^k(2k-5) + L_{n-4}^k(2k-4).$$

Proof By (2), $A_k(n,k-1) - A_k(n,k) = C_k(n,k-1)$. Replacing n by $n-1$ and subtracting yields

$$(A_k(n,k-1)-A_k(n,k))-(A_k(n-1,k-1)-A_k(n-1,k)) = C_k(n,k-1)-C_k(n-1,k-1),$$

$$\Rightarrow\ A_k(n,k-1)-A_k(n-1,k-1) = A_k(n,k)-A_k(n-1,k)+C_k(n,k-1)-C_k(n-1,k-1).$$

By Proposition 15(b) and Proposition 16(e),

$$\begin{aligned} A_k(n,k-1) - A_k(n-1,k-1) &= A_{k-2}(n-3,k-2) + A_{k-1}(n-3,k-2) \\ &+ A_{k-2}(n-4,k-2) + A_{k-1}(n-4,k-2). \end{aligned} \qquad (5)$$

The result of Corollary 13 completes the proof. □

Corollary 18 *For $k \geq 3$,*

$$f_k(x) = \frac{(1+2x^2+x^3)U_{2k-3}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1+x)U_{2k-4}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}\left[(1+2x^2+x^3)U_{2k-2}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1+x)U_{2k-3}\left(\frac{1}{2\sqrt{x}}\right)\right]}.$$

Proof Let $1 \leq 2k-5 \leq s$. Since $f_k(n) = c_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n = 0, 1, \ldots, 2k-1$, we have

$$\sum_{n\geq 2k} L_n^k(s)x^n = \sum_{i=0}^{s}(-x)^i\binom{s-i}{i}\left(f_k(x) - \sum_{j=0}^{2k-i-1} x^j c_j\right).$$

Recall that the Chebyshev polynomials of the second kind satisfy the relation

$$x^{s/2}U_s\left(\frac{1}{2\sqrt{x}}\right) = \sum_{i=0}^{s}(-x)^i\binom{s-i}{i}$$

(see [Ri, page 75-76]), while the Catalan numbers satisfy the relation

$$\sum_{i=0}^{l}(-1)^i\binom{s-i}{i}c_{l-i} = (-1)^l\binom{s-1-l}{l}$$

for all $l \leq s-1$ (see [Ri, page 152-154]), and hence

$$\sum_{i=0}^{s}(-x)^i\binom{s-i}{i}\sum_{j=0}^{2k-i-1} x^j c_j = \sum_{l=0}^{s-1} x^l \sum_{i=0}^{l}(-1)^i\binom{s-i}{i}c_{l-i} = \sum_{l=0}^{s-1}(-x)^l\binom{s-1-l}{l}.$$

Therefore, for $1 \leq 2k-5 \leq s$,

$$\sum_{n\geq 2k} L_n^k(s)x^n = x^{s/2}U_s\left(\frac{1}{2\sqrt{x}}\right) f_k(x) - x^{(s-1)/2}U_{s-1}\left(\frac{1}{2\sqrt{x}}\right).$$

Finally, the Chebyshev polynomials of the second kind satisfy also the relation

$$U_m\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{\sqrt{x}}U_{m-1}\left(\frac{1}{2\sqrt{x}}\right) - U_{m-2}\left(\frac{1}{2\sqrt{x}}\right)$$

for all $m \geq 2$, hence by Theorem 5 we get the desired result. □

3 Further results

In this section we describe several directions which generalize Theorem 5 by use of the same arguments as in the proofs of Theorem 5 and Corollary 6.

3.1 First generalization

For any $1 \leq d \leq k-2$, let $X^1_{k,d}$ be the set of all patterns

$$(d+1, d+2, \ldots, k-1, 1, 2, \ldots, j, k, j+1, j+2, \ldots, d)$$

for $j = 0, 1, 2, ..., d$, plus the pattern 321. For example, $X^1_{3,1} = \{321, 213, 231\}$. Denote the number of permutations in $S_n(321, X^1_{k,d})$ by $x^1_{k,d}(n)$.

Theorem 19 *Let $k \geq 3$ and $1 \leq d \leq k-2$. For all $n \geq k$,*

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1-j}{j} x^1_{k,d}(n-j) = \binom{n-d-1}{k-d-2},$$

and for all $0 \leq n \leq k-1$, $x^1_{k,d}(n) = \frac{1}{n+1}\binom{2n}{n}$. Thus, the generating function for $\{x^1_{k,d}(n)\}_{n\geq 0}$ is

$$\sum_{n\geq 0} x^1_{k,d}(n)x^n = \frac{U_{k-2}\left(\frac{1}{2\sqrt{x}}\right) + \frac{x^{k/2}}{(1-x)^{k-d-1}}}{\sqrt{x}U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}.$$

Proof As in Subsection 2.1, we define

$$\mathcal{A}_s(n,m) = \sum_{s\leq i_1<i_2<\cdots<i_m\leq n} f^1_{k,d}(n; i_1, \ldots, i_m)$$

where $f^1_{k,d}(n; i_1, \ldots, i_m)$ is the number of permutations $\pi \in S_n(321, X^1_{k,d})$ which start with $i_1 i_2 \cdots i_m$: $\pi_1\pi_2\cdots\pi_m = i_1 i_2 \cdots i_m$. Using the same arguments in the proof of Lemma 12 and Corollary 13 we obtain that

$$\mathcal{A}_{d+1}(n, k-d-1) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1-j}{j} x^1_{k,d}(n-j) \tag{6}$$

On the other hand, it is easy to verify that

$$\mathcal{A}_{d+1}(n, k-d-1) = \sum_{d+1\leq i_1<i_2<\cdots<i_{k-d-1}\leq n} x^1_{k,d}(n; i_1, \ldots, i_{k-d-1}, 1).$$

Since our permutations avoid $X^1_{d,k}$, we set $i_{k-d-1} = n$. Thus, if $\pi \in X^1_{k,d}(n; i_1, \ldots, i_{k-d-2}, n, 1)$ with $d+1 \leq i_1 < \cdots < i_{k-d-2} \leq n-1$, then $\pi_{k-d+1} < \cdots < \pi_n$. This means

$$\mathcal{A}_{d+1}(n, k-d-1) = \binom{n-d-1}{k-d-2} \tag{7}$$

Combining (6) and (7) yields the desired recursive relation. The rest of the theorem is easy to derive by use of the same argument as in the proofs of Corollary 6. □

Example 20 For $d = 1$ and $k = 3$, Theorem 19 yields

$$x^1_{3,1}(n) - x^1_{3,1}(n-1) = 1, \text{ and } x^1_{3,1}(0) = x^1_{3,1}(1) = 1,\ x^1_{3,1}(2) = 2.$$

Hence, $x^1_{3,1}(n) = n$ for all $n \geq 1$ (see [SS]). For $d = 1$, $k = 4$ and $n \geq 0$, Theorem 19 yields

$$x^1_{4,1}(n) = |S_n(321, 2341, 2314)| = 2^n - n,$$

while for $d = 2$, $k = 4$ and $n \geq 2$:

$$x^1_{4,2}(n) = |S_n(321, 3412, 3142, 3124)| = 3 \cdot 2^{n-2} - 1.$$

3.2 Second generalization

Let $X^2_{k,d}$ consist of the three patterns 321, $(d+1)(d+2)\cdots(k-1)1k23\cdots d$, and $(d+1)(d+2)\cdots(k-1)k12\cdots d$. The number of $X^2_{k,d}$-avoiding permutations in S_n is denoted by $x^2_{k,d}(n)$.

Theorem 21 *Let $k \geq 4$ and $2 \leq d \leq k-2$. Then for all $n \geq k$,*

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j\binom{k-1-j}{j}x^2_{k,d}(n-j) &= \sum_{j=0}^{k-1}(-1)^j\binom{k-1-j}{j}x^2_{k,d}(n-1-j)\\ &+\sum_{j=0}^{k-4}(-1)^j\binom{k-4-j}{j}x^2_{k,d}(n-3-j),\end{aligned}$$

and $x^2_{k,d}(n) = \frac{1}{n+1}\binom{2n}{n}$ for all $0 \leq n \leq k-1$. Thus, the generating function for $\{x^2_{k,d}(n)\}_{n\geq 0}$ is

$$\sum_{n\geq 0} x^2_{k,d}(n)x^n = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right) + xU_{k-3}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}\left[U_k\left(\frac{1}{2\sqrt{x}}\right) + xU_{k-2}\left(\frac{1}{2\sqrt{x}}\right)\right]}.$$

Proof As in Subsection 2.1, we define

$$\mathcal{A}_s(n,m) = \sum_{s\leq i_1<i_2<\cdots<i_m\leq n} f^2_{k,d}(n; i_1, \ldots, i_m)$$

where $f^2_{k,d}(n; i_1, \ldots, i_m)$ is the number of permutations $\pi \in S_n(321, X^2_{k,d})$ which start with $i_1 i_2 \cdots i_m$: $\pi_1\pi_2\cdots\pi_m = i_1 i_2 \cdots i_m$. Using the same arguments as in the proof of Corollary 13 we arrive at

$$\mathcal{A}_{d+1}(n,m) = \sum_{j=0}^{d+m-1}(-1)^j\binom{d+m-1-j}{j}x^2_{k,d}(n-j),$$

for all $m \leq k-d-1$. On the other hand, the same arguments as in the proof of Proposition 16 yield

$$A_{d+1}(n, k-d-1) = A_{d+1}(n-1, k-d-1) + A_{d-1}(n-3, k-d-2).$$

Hence,

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j\tbinom{k-1-j}{j}x_{k,d}^2(n-j) &= \sum_{j=0}^{k-1}(-1)^j\tbinom{k-1-j}{j}x_{k,d}^2(n-1-j)\\ &+\sum_{j=0}^{k-4}(-1)^j\tbinom{k-4-j}{j}x_{k,d}^2(n-3-j).\end{aligned}$$

Now, using the same arguments as in the proof of Corollary 6 we obtain the desired result. □

Example 22 For $d=2$, $k=4$ and $n \geq 3$, Theorem 21 yields

$$x_{4,2}^2(n) = 3x_{4,2}^2(n-1) - 2x_{4,2}^2(n-2) + x_{4,2}^2(n-3), \text{ and } x_{4,2}^2(0) = x_{4,2}^2(1) = 1,\ x_{4,2}^2(2) = 2.$$

A comparison of Theorem 21 for different cases suggests that there should exist a bijection between the sets $S_n(X_{k,2}^2)$ and $S_n(132, X_{k,d}^2)$ for any d such that $2 \leq d \leq k-2$. Producing such a bijection remains yet an open question.

3.3 Third generalization

Let $X_{k,d}^3$ consist of the two patterns 321 and $(d+1)(d+2)\cdots(k-1)1k23\cdots d$. The number of $X_{k,d}^3$-avoiding permutations in S_n is denoted by $x_{k,d}^3(n)$. Similarly as in the argument proofs of the main theorem in [MV3] and Theorem 5, we obtain

Theorem 23 *Let $k \geq 4$ and $2 \leq d \leq k-2$. Then for all $n \geq k$,*

$$\sum_{i=0}^{k}(-1)^j\binom{k-j}{j}x_{k,d}^3(n-j) = 0,$$

and $x_{k,d}^3(n) = \frac{1}{n+1}\binom{2n}{n}$ for all $0 \leq n \leq k-1$. Thus, the generating function for $\{x_{k,d}^3(n)\}_{n\geq 0}$ is

$$\sum_{n\geq 0} x_{k,d}^3(n)x^n = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}.$$

Proof Again, we define

$$\begin{aligned}\mathcal{A}_s(n,m) &= \sum_{s\le i_1<i_2<\cdots<i_m\le n} f^3_{k,d}(n;i_1,\ldots,i_m),\\ \mathcal{C}_s(n,m) &= \sum_{s\le i_1<i_2<\cdots<i_m\le n} f^3_{k,d}(n;i_1,\ldots,i_m,1)\end{aligned}$$

where $f^3_{k,d}(n;j_1,\ldots,j_p)$ is the number of permutations $\pi \in S_n(321, X^3_{k,d})$ which start with $j_1j_2\cdots j_p$: $\pi_1\pi_2\cdots\pi_p = j_1j_2\cdots j_p$. Using the same arguments as in the proof of Proposition 16 we obtain

$$\mathcal{A}_{d+1}(n,m) - \mathcal{A}_{d+1}(n,m+1) = \mathcal{C}_{d+1}(n,m) \text{ and } \mathcal{C}_{d+1}(n,m) = \mathcal{C}_d(n-1,m),$$

for all $m \ge k-d-1$. So, for all $m \ge k-d-1$:

$$\mathcal{A}_{d+1}(n,m) - \mathcal{A}_{d+1}(n,m+1) = \mathcal{A}_d(n-1,m) - \mathcal{A}_d(n-1,m+1).$$

Adding over all $m \ge k-d-1$ yields

$$\mathcal{A}_{d+1}(n,k-d-1) = \mathcal{A}_d(n-1,k-d-1).$$

On the other hand, using the same arguments as in the proof of Corollary 13 we arrive at

$$\mathcal{A}_{d+1}(n,m) = \sum_{j=0}^{d+m-1} (-1)^j \binom{d+m-1-j}{j} x^2_{k,d}(n-j),$$

for all $m \le k-d-1$. Hence,

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-1-j}{j}x^3_{k,d}(n-j) = \sum_{j=0}^{k-2}(-1)^j\binom{k-2-j}{j}x^3_{k,d}(n-1-j),$$

or equivalently,

$$\sum_{j=0}^{k}(-1)^j\binom{k-j}{j}x^3_{k,d}(n-j) = 0.$$

Now, using the same arguments as in the proof of Corollary 6 we obtain the desired result. □

Again, finding a direct bijection between the sets $S_n(X^3_{k,d})$ and $S_n(132,(d{+}1)\cdots k1\cdots d)$ for any $k \ge 4$ and $2 \le d \le k-2$ is, as of now, an open question.

Acknowledgments. The authors are grateful to the referees for the careful reading of the manuscript and to M. Atkinson for his remarks. The final version of the paper was written during the first author's (T.M.) stay at Chalmers University of Technology in Göteborg, Sweden. T.M. want to express his gratitude to Chalmers University of Technology for the support.

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