Partially Ordered Generalized Patterns

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The Classical Definition of a Pattern

$\mathcal{S}_n$ is the symmetric group on $\{1, 2, \ldots, n\}$.
A *pattern* is a permutation on $\{1, 2, \ldots, r\}$.

A permutation $\sigma \in \mathcal{S}_n$ *has an occurrence* of the pattern $p \in \mathcal{S}_r$ if there is $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $p = [\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)]$ in reduced form.

The *reduced form* of a permutation σ on the set $\{j_1, j_2, \ldots, j_n\}$, where $j_1 < j_2 < \ldots < j_n$ is the permutation obtained by renaming j_1 by 1, j_2 by 2 etc.

If the numbers of permutations that avoid the patterns p and q are equal, then p and q are *equivalent* and we write $p \equiv q$.

The *reverse* $R(\pi)$ of a permutation $\pi = a_1 a_2 \ldots a_n$ is the permutation $a_n a_{n-1} \ldots a_1$. The *complement* $C(\pi)$ is the permutation $b_1 b_2 \ldots b_n$ where $b_i = n + 1 - a_i$. Also, $R \circ C$ is the composition of R and C.

For example, $R(13254) = 45231$, $C(13254) = 53412$ and $R \circ C(13254) = 21435$.

We call these bijections of $\mathcal{S}_n$ to itself *trivial*, and it is easy to see that any pattern p is equivalent to the patterns $R(p)$, $C(p)$ and $R \circ C(p)$.

There are many interesting results concerning the classical patterns. A short survey of the results on this theme can be found in the slides to the talk

“On Pattern Avoidance in Permutations”

located here:

`http://www.math.chalmers.se/~kitaev/reports.html`

A *generalised pattern* is a pattern that allow the requirement that two adjacent letters in the pattern must be adjacent in a permutation.

For example, the permutation 1342 has one occurrence of the pattern $p = 1-32$, namely 142, whereas it has two occurrences of the classical pattern 132.

Definition 1 *A partially ordered generalized pattern (POGP) is a generalized pattern where some of the letters can be incomparable.*

Example 2 *The permutation 3142 has two occurrences of the pattern* $p = 1' - 2 - 1''$. *There are* 2^{n-1} *permutations that avoid* p. *Moreover,* $1' - 2 - 1'' \equiv (1 - 3 - 2, 2 - 3 - 1)$, *where* (p, q) *denotes the number of permutations that avoid simultaneously the patterns* p *and* q.

Definition 3 *Suppose* $\{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ *is a set of generalized patterns without internal dashes and* $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ *where each letter of* σ_i *is incomparable with any letter of* σ_j *whenever* $i \neq j$*. We call such POGPs* multi-patterns*.*

There is a number of results on the distribution of several classes of patterns without internal dashes. In particular we know the *exponential generating functions (e.g.f.)* for the number of permutations that avoid these classes (Elizalde and Noy).

Definition 4 *Suppose* $\{\sigma_0, \sigma_1, \dots, \sigma_k\}$ *is a set of generalized patterns without internal dashes and* $a_1 a_2 \dots a_k$ *is a permutation of* k *letters. We define a* shuffle *pattern to be the pattern of the form*

$$\sigma_0 - a_1 - \sigma_1 - a_2 - \cdots - \sigma_{k-1} - a_k - \sigma_k,$$

where for any i *and* j, *the letter* a_i *is greater than any letter of* σ_j *and for any* i *and* j *each letter of* σ_i *is incomparable with any letter of* σ_j. *We also allow* σ_0 *and* σ_k *to be empty patterns.*

The pattern $p = 1' - 2 - 1''$ is an example of shuffle pattern.

Theorem 5 (Elizalde and Noy, 2001) *Let m and a be positive integers with $a \leq m$, let $\sigma = 12\ldots a\tau(a+1) \in \mathcal{S}_{m+2}$, where τ is any permutation of $\{a+2, a+3, \ldots, m+2\}$, and let $P(u,z) = \sum_{\pi} u^{\sigma(\pi)} \frac{z^{|\pi|}}{|\pi|!}$. Then $P(u,z) = 1/w(u,z)$, where w is the solution of*

$$w^{a+1} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}w' = 0$$

with $w(0) = 1$, $w'(0) = -1$ and $w^{(k)} = 0$ for $2 \leq k \leq a$. In particular, the distribution does not depend on τ.

Using an inclusion-exclusion argument we get this:

Theorem 6 (Goulden and Jackson, 1983) *Let* $A_k(x) = A_0 + A_1 x + \frac{A_2}{2!}x^2 + \cdots$ *be the e.g.f. for the number of permutations avoiding the pattern* $p = 123\ldots k$. *Then*

$$A_k(x) = \frac{1}{\displaystyle\sum_{i\geq 0} \frac{x^{ki}}{(ki)!} - \sum_{i\geq 0} \frac{x^{ki+1}}{(ki+1)!}}.$$

For given k it is possible to simplify $A_k(x)$ in the theorem above. If

$$F_k(x) = \sum_{i \geq 0} \frac{x^{ki}}{(ki)!} - \sum_{i \geq 0} \frac{x^{ki+1}}{(ki+1)!}$$

then $F_k(x)$ satisfies the differential equation $F_k^{(k)}(x) = F_k(x)$. The k initial conditions are $F_k(0) = 1$, $F_k'(0) = -1$ and $F_k^{(i)}(0) = 0$ for all $i = 2, 3, \ldots, k-1$. For instance, if $k = 4$ then

$$A_4(x) = \frac{2}{\cos x - \sin x + e^{-x}}.$$

Let $p = \sigma - k$, where σ is an arbitrary pattern without internal dashes on the elements $1, 2, \dots, k-1$. So the last letter of p is greater than any other letter. Let $A(x)$ (resp. $B(x)$) be the e.g.f. for the number of permutations that avoid σ (resp. p).

Theorem 7 *We have* $B(x) = e^{F(x,A(y))}$*, where*

$$F(x, A(y)) = \int_0^x A(y)\ dy.$$

Example 8 *Let $p = 1 - 2$. Here $\sigma = 1$, whence $A(x) = 1$ since $A_n = 0$ for all $n \geq 1$. So*

$$B(x) = e^{F(x,1)} = e^x.$$

Example 9 *Suppose $p = 12 - 3$. Here $\sigma = 12$, whence $A(x) = e^x$, since there is only one permutation that avoids σ. So*

$$B(x) = e^{F(x,e^y)} = e^{e^x - 1}.$$

It is known [Claesson, 2001] that the number of n-permutations that avoid p is the n-th Bell number whose e.g.f. is $B(x)$.

Let $p = \sigma - k - \tau$ be a shuffle pattern, where σ and τ are patterns without internal dashes. So k is the greatest letter of the pattern, and each letter of σ is incomparable with any letter of τ. Let $A(x)$, $B(x)$ and $C(x)$ be the e.g.f. for the number of permutations that avoid σ, τ and p respectively.

Theorem 10 *$C(x)$ is the solution of the differential equation*

$$C'(x) = (A(x) + B(x))C(x) - A(x)B(x)$$

with $C(0) = 1$.

Example 11 *Let* $p = 1' - 2 - 1''$*. That is* $\sigma = \tau = 1$*. So* $A(x) = B(x) = 1$ *and we solve the equation*

$$C'(x) = 2C(x) - 1$$

with $C(0) = 1$*. The solution is* $C(x) = \frac{1}{2}(e^{2x} + 1)$*, so* $C_n = 2^{n-1}$*.*

σ	τ	**initial values for C_n**
1	12	1, 2, 6, 21, 82, 354, 1671, 8536, 46814
1	132	1, 2, 6, 24, 116, 652, 4178, 30070, 240164
1	123	1, 2, 6, 24, 116, 657, 4260, 31144, 253400
1	1234	1, 2, 6, 24, 120, 715, 4946, 38963, 344350
12	12	1, 2, 6, 24, 114, 608, 3554, 22480, 152546
12	132	1, 2, 6, 24, 120, 710, 4800, 36298, 302780
12	123	1, 2, 6, 24, 120, 710, 4815, 36650, 308778
12	1234	1, 2, 6, 24, 120, 720, 5025, 39926, 355538
123	123	1, 2, 6, 24, 120, 720, 5020, 39790, 352470
123	132	1, 2, 6, 24, 120, 720, 5020, 39755, 351518
132	132	1, 2, 6, 24, 120, 720, 5020, 39720, 350496

Corollary 12 *Suppose* $p = \sigma - k - \tau$*. We consider the pattern* $\varphi(p) = \varphi_1(\sigma) - k - \varphi_2(\tau)$*, where* φ_1 *and* φ_2 *are any trivial bijections. Then* $p \equiv \varphi(p)$*.*

Corollary 13 *We have* $\sigma - k - \tau \equiv \tau - k - \sigma$*.*

Proposition 14 *Let the number of letters in* σ *(resp.* τ*) be* $|\sigma| = k_1$ *(resp.* $|\tau| = k_2$*). Then to avoid* p *from Theorem 16 is the same as to avoid simultaneously* $\binom{k_1+k_2}{k_1}$ *ordinary generalized patterns.*

Example 15 *Let* $p = 1'2' - 3 - 1''$*. That is* $\sigma = 12$ *and* $\tau = 1$*. To avoid* p *is the same as to avoid* $\binom{3}{2} = 3$ *generalized patterns simultaneously, namely* $12 - 4 - 3$*,* $13 - 4 - 2$ *and* $23 - 4 - 1$*.*

Theorem 16 *Let* $p = \sigma_0 - a_1 - \sigma_1 - a_2 - \cdots - \sigma_{k-1} - a_k - \sigma_k$ *be an arbitrary shuffle pattern with* $|\sigma_i| = \ell_i$ *for each* i. *Then to avoid the pattern* p *is the same as to avoid*

$$\prod_{i=1}^{k} \binom{\ell_0 + \ell_1 + \ldots + \ell_i}{\ell_i} = \binom{\ell_0 + \ell_1}{\ell_1} \cdots \binom{\ell_0 + \ell_1 + \ldots + \ell_k}{\ell_k}$$

ordinary generalized patterns.

Theorem 17 *Suppose we have the multi-patterns* $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ *and* $q = \tau_1 - \tau_2 - \cdots - \tau_k$*, where* $\tau_1\tau_2\ldots\tau_k$ *is a permutation of* $\sigma_1\sigma_2\ldots\sigma_k$*. Then* $p \equiv q$*.*

Corollary 18 *Suppose we have the multi-patterns* $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ *and* $q = \varphi_1(\sigma_1) - \varphi_2(\sigma_2) - \cdots - \varphi_k(\sigma_k)$*, where each* φ_i *is a trivial bijection. Then* $p \equiv q$*.*

The presence of the dash in the patterns is essential here. For instance, the pattern $\sigma\tau$ is not necessarily equivalent to the pattern $\varphi_1(\sigma)\varphi_2(\tau)$. Indeed, in $\mathcal{S}_5$, there are 66 permutations that avoid the pattern $122'1'$ whereas there are only 61 permutations that avoid the pattern $121'2'$.

Theorem 19 *There are* $(n-2)2^{n-1}+2$ *permutations in* $\mathcal{S}_n$ *that avoid the pattern* $p = 12-1'2'$.

Theorem 20 *Let* $A(x)$ *be the e.g.f. for the number of permutations that avoid the pattern* $p = 122'1'$. *Then*

$$A(x) = \frac{1}{2} + \frac{1}{4}\tan x(1 + e^{2x} + 2e^x \sin x) + \frac{1}{2}e^x \cos x.$$

Theorem 21 *Let* $p = \sigma_1 - \sigma_2 - \sigma_3 - \cdots - \sigma_k$ *be a multi-pattern with* $|\sigma_i| = \ell_i$*. Then to avoid* p *is the same as to avoid* $\prod_{i=1}^{k} \binom{\ell_1 + \ldots + \ell_i}{\ell_i}$ *ordinary generalised patterns.*

Definition 22 *A permutation* π quasi-avoids *a pattern* p *if* π *has exactly one occurrence of* p *and this occurrence is the* $|p|$ *rightmost letters of* π*.*

Theorem 23 *Suppose* $p = \sigma - p'$*, where* p' *is an arbitrary POGP, and the letters of* σ *are incomparable to the letters of* p'*. Let* $C(x)$ *(resp.* $A(x)$*,* $B(x)$*) be the e.g.f. for the number of permutations that avoid* p *(resp.* σ*,* p'*). Moreover let* $A^{\star}(x)$ *be the e.g.f. for the number of permutations that quasi-avoid* σ*. Then*

$$C(x) = B(x)A^{\star}(x) + A(x).$$

Theorem 24 *Let $A_k^\star(x)$ be the e.g.f. for the number of permutations that quasi-avoid the pattern $p = 123\ldots k$. Then*

$$A_k^\star(x) = 1 + \frac{x-1}{\displaystyle\sum_{i\geq 0} \frac{x^{ki}}{(ki)!} - \sum_{i\geq 0} \frac{x^{ki+1}}{(ki+1)!}}.$$

Theorem 25 *Let k and a be positive integers with $a < k$, let $p = 12\dots a\tau(a+1) \in \mathcal{S}_{k+1}$, where τ is any permutation of the elements $\{a+2, a+3, \dots, k+1\}$, and let $A_{k,a}(x)$(resp. $A^{\star}_{k,a}(x)$) be the e.g.f. for the number of permutations that avoid (resp. quasi-avoid) p. Let $F_{k,a}(x) = \sum_{i\geq 1} \frac{(-1)^{i+1}}{(ik+1)!} \prod_{j=2}^{i} \binom{jk-a}{k-a} x^{ik+1}$. Then*

$$A_{k,a}(x) = 1/(1-x+F_{k,a}(x)) \text{ and } A^{\star}_{k,a}(x) = F_{k,a}(x)/(1-x+F_{k,a}(x)).$$

If $k = 2$ and $a = 1$, that is $p = 132$, then

$$F_{2,1}(x) = \sum_{i\geq 1} \frac{(-1)^{i+1} x^{ik+1}}{i!(k!)^i(ik+1)} = x - \int_0^x e^{-t^2/2}\,dt.$$

Theorem 26 *Let $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ be a multi-pattern and let $A_i(x)$ be the number of permutations that avoid σ_i. Also assume that we know all e.g.f. $A_i^\star(x)$ for the number of permutations that quasi-avoid σ_i, except perhaps one of them. By Theorem 19, without loosing generality, we assume that we know all but $A_k^\star(x)$. Then the e.g.f. $B(x)$ for the number of permutations that avoid p is*

$$B(x) = \sum_{i=1}^{k} A_i \prod_{j=1}^{i-1} A_j^\star.$$

Corollary 27 *Let $p = \sigma_1 - \sigma_2 - \cdots - \sigma_k$ be a multi-pattern, where $|\sigma_i| = 2$ for all i. That is σ_i is equal to either 12 or 21. If $B(x)$ is the e.g.f. for the number of permutations that avoid p then*

$$B(x) = \frac{1 - (1 + (x-1)e^x)^k}{1 - x}.$$

A *descent* in a permutation $\pi = a_1 a_2 \ldots a_n$ is an i such that $a_i > a_{i+1}$. The number of descents (des π) is a *statistic* for a permutation π.

Any statistic that is *equidistributed* with des is said to be *Eulerian*. The *Eulerian numbers* $A(n,k)$ count permutations in the symmetric group $\mathcal{S}_n$ with k descents and they are the coefficients of the *Eulerian polynomials* $A_n(t)$ defined by $A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{1+\mathrm{d}es\pi}$. The Eulerian polynomials satisfy the identity

$$\sum_{k \geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Two descents i and j *overlap* if $j = i + 1$.

We define a new statistic, namely the *maximum number of non-overlapping descents* in a permutation.

We find the distribution of this new statistic by using the results for multi-patterns.

Theorem 28 *Let p be a pattern without internal dashes. Let $B(x)$ (resp. $B^\star(x)$) be the e.g.f. for the number of permutations that avoid (resp. quasi-avoid) p. Let $D(x,y) = \sum_\pi y^{p(\pi)} \frac{x^{|\pi|}}{|\pi|!}$ where y corresponds to the maximum number of occurrences of non-overlapping p. Then*

$$D(x,y) = B(x)/(1 - yB^\star(x)).$$

If we consider descents then $B(x) = e^x$ and $B^\star(x) = 1 + (x-1)e^x$, hence the distribution of the maximum number of non-overlapping descents is given by the formula:

$$D(x,y) = \frac{e^x}{1 - y(1 + (x-1)e^x)}.$$

Referenser

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