Interplay of elastic and inelastic scattering operators in extended kinetic models and their hydrodynamic limits – reference manual

J Banasiak

School of Mathematical and Statistical Sciences
University of Natal, Durban 4041
South Africa

Abstract

In this paper we consider the hydrodynamic/diffusive limits which can occur in the linear kinetic equation describing elastic and inelastic collisions when the involved collision operators act on different time-scales.

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1 Introduction

In recent years there has occurred a considerable development in the kinetic theory describing inelastic collisions. The interest in this field stems mainly from the fact that such collisions are important for electron transport even at low energies, such as in swarm propagation in gases and slowing down electron beams in solids, and thus play a major role in the semiconductor theory. However, they are also important in other branches of kinetic/transport theory describing interactions of point particles with composite systems, like the interaction of high-energy neutrons with nuclei, or the interchange of kinetic energy by low-energy neutrons propagating in gas media or solids.

There exists a vast literature devoted to such problems, both in semiconductor (see e.g. [18, 19, 23, 10]), and the extended kinetic theory frameworks (see e.g. [2, 3, 4, 5, 6, 8, 9, 12, 16, 14, 11]). These two ranges of applications share many similar features but the mathematical theories differ considerably due to the presence of the external electric field in the semiconductor models on one hand, and the inherent unboundedness of the inelastic collision operator (implied by the microreversibility principle) in the extended kinetic theory applications on the other hand.

In this paper we shall consider only the extended kinetic theory models, that is, models without the external force field. The evolution of the distribution function in these models is governed by a version of the Boltzmann equation containing all the collision operators responsible for the elastic and inelastic scattering. Adimensionalization of such an equation singles out typically three parameters: two Knudsen numbers which are proportional to the mean free path between elastic and inelastic collisions, respectively, and the Mach number. Depending on the relations between these three parameters we observe the domination of one (or more) processes, e.g. of inelastic or elastic collisions, in driving the system to the equilibrium. Mathematically such a dominance is accounted for by assuming that some parameters are much smaller than the others and considering the limiting case as the parameters tend to zero in various way.

The main objective of this paper is to describe all such limiting cases, which are referred hereafter to as hydrodynamic (or diffusive) limits, and to prove the corresponding convergence results. It is to some extent a survey which includes some earlier results of [3, 5, 9, 6]. However, here they are presented in a unified setting which allowed to fill some gaps left in earlier works. The new part is the formal analysis of Section 4 which facilitated finding all the hydrodynamic limits.

2 General description of the model

From the physical point of view we consider a gas of test particles having mass m propagating through a host medium of particles having mass M, and undergoing two types of collisions with them: elastic and inelastic ones. We shall not go into details regarding the mechanisms of such collisons, but it is not unreasonable to assume simply that a test particle can experience one of these two collisions with certain probabilities, depending e.g. on its velocity. This probability can be incorporated into the scattering cross-sections and therefore we shall not comment on this any further.

We assume moreover that the host medium is at thermodynamical equilibrium with temperature T and we consider the Lorentz gas limit ($m/M << 1$, in other words, the test particles collide with something like a rigid net – they can be deflected (elastic collisions) or exchange quanta of energy with the background (inelastic collisions), but a classical exchange of kinetic energy is ruled out). Furthermore, the background particles can occur in two energy states: a ground level and an excited level, spaced by an energy gap ΔE (which throughout the paper will be normalized to one). The number densities of the particles in the ground and in the excited states are assumed to be constant; we denote them by n_1 and n_2, respectively. They are related through the Boltzmann factor $a := n_1/n_2 = e^{\frac{\Delta E}{kT}}$, where k is the Boltzmann constant.

We are concerned with two scattering operators: inelastic and elastic ones. In general, the elastic scattering operator is given by

$$(C^e f)(\boldsymbol{x},\boldsymbol{v}) = -f(\boldsymbol{x},\boldsymbol{v})\int\limits_{S^2}\lambda(\boldsymbol{x},v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')d\boldsymbol{\omega}' + \int\limits_{S^2}\lambda(\boldsymbol{x},v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')f(\boldsymbol{x},v\boldsymbol{\omega}')d\boldsymbol{\omega}', \tag{2.1}$$

where S^2 denotes the unit sphere in $\mathbb{R}^3_v$, $\boldsymbol{v} = v\boldsymbol{\omega}$ with $v \in [0,\infty[$ and $\boldsymbol{\omega} \in S^2$. Since we will be working with Maxwell type molecules, we may assume that λ is a measurable function bounded almost everywhere

$$\lambda_{min} \leq \lambda(\boldsymbol{x}, v, z) \leq \lambda_{max}$$

for all $(\boldsymbol{x}, v, z) \in \mathbb{R}^3 \times [0,\infty[\times[-1,1]$ and some $0 < \lambda_{min} \leq \lambda_{max} < +\infty$.

The inelastic scattering operator is given by

$$\begin{aligned}(C^i f)(\boldsymbol{x},\boldsymbol{v}) &= -f(\boldsymbol{x},\boldsymbol{v})\frac{1}{v}\int\limits_{S^2}\left(n_2 v_+ \nu(\boldsymbol{x},v_+,\boldsymbol{\omega}\cdot\boldsymbol{\omega}') + n_1 H(v^2-1)v\nu(\boldsymbol{x},v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')\right)d\boldsymbol{\omega}' \\ &+\frac{n_1 v_+}{v}\int\limits_{S^2}\nu(\boldsymbol{x},v_+,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')f(\boldsymbol{x},v_+\boldsymbol{\omega}')d\boldsymbol{\omega}' + n_2\int\limits_{S^2}H(v^2-1)\nu(\boldsymbol{x},v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')f(\boldsymbol{x},v_-\boldsymbol{\omega}')d\boldsymbol{\omega}'.\end{aligned} \tag{2.2}$$

where H denotes the Heaviside function, and $v_\pm = \sqrt{v^2 \pm 1}$.

As before, the Maxwell molecule hypothesis yields the following requirement:

$$\nu_{min} \leq \nu(v,\boldsymbol{x},\boldsymbol{\omega}\cdot\boldsymbol{\omega}') \leq \nu_{max} \quad \text{for } v \in [1,\infty[. \tag{2.3}$$

This model is obtained from the general inelastic scattering model (see [16, 11, 2]).

Using the operators introduced above, we shall write down the general kinetic equation in the following non-dimensional form

$$\delta\frac{\partial u}{\partial t} = Su + \kappa_e^{-1}C^e u + \kappa_i^{-1}C^i u \tag{2.4}$$

where δ is the Mach number and κ_e, κ_i are proportional to the mean scattering path for elastic and inelastic collisions, respectively. In other words, we can say that κ_e, κ_i are Knudsen numbers for elastic and inelastic processes. We shall further simplify our considerations by requiring that these three parameters are functions of a single parameter ϵ. Restricting our attention to regular functions of ϵ we see that there is no harm in assuming that the parameters are power functions of ϵ. Thus, we shall consider (2.4) in the form:

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon^p}Su + \frac{1}{\epsilon^q}C^e u + \frac{1}{\epsilon^r}C^i u, \tag{2.5}$$

where p, q, r are integers. We are interested in finding all reasonable hydrodynamic limits for this equation when $\epsilon \to 0$. Firstly, we note that we have three, possibly different, hydrodynamic subspaces corresponding to the null-spaces of scattering operators C^e, C^i and $C^e + C^i$. In which space we will have the limiting evolution depends on the relations between powers of ϵ in (2.5) and determining these will be the subject of Section 4. In the next section we shall provide an explicit description of these subspaces.

3 Hydrodynamic subspaces for the equation with elastic and inelastic scattering

The easiest part is to find the null-space of the elastic scattering operator. The following theorem was proved in [4], but for self-consistence we shall provide the proof in Appendix A. Since in all the considerations of this section $\boldsymbol{x}$ plays a role of a parameter, we shall drop it from the notation.

Theorem 3.1 *Let all the assumptions of Section 2 be satisfied. Then the operator C^e is a bounded operator in $X = L_1(\mathbb{R}^3)$ with the following properties*

i. For any $f \in X$ and any non-decreasing function κ we have

$$\int_{\mathbb{R}^3} \kappa(f) C^e f d\boldsymbol{v} \leq 0. \tag{3.6}$$

In particular, C^e is dissipative.

ii. The null-space of C^e is given by

$$N(C^e) = \{f \in L_1(\mathbb{R}^3);\ f \text{ is independent on } \boldsymbol{\omega}\}. \tag{3.7}$$

iii. The range of C^e is given by

$$R(C^e) = W = \{f \in L_1(\mathbb{R}^3);\ \int_{S^2} f d\boldsymbol{\omega} = 0\}. \tag{3.8}$$

The spectral projection onto $N(C^e)$ (parallel to W) is given by

$$\mathbb{P}f = \frac{1}{4\pi} \int_{S^2} f d\boldsymbol{\omega}. \tag{3.9}$$

iv. For $f \in W$ we have

$$\int_{\mathbb{R}^3} sign(f) C^e f d\boldsymbol{v} \leq -4\pi\lambda_{min} \|f\|_X \tag{3.10}$$

and hence the spectral bound of C^e, $s(C^e)$, satisfies $s(C^e) < -4\pi\lambda_{min}$.

Analogous properties hold in any weighted space $L_2(\mathbb{R}^3, w(v)d\boldsymbol{v})$ where w is a measurable strictly positive (a.e.) function.

The analysis of the inelastic kernel is considerably more involved. The case of purely inelastic collision operator C^i has been thoroughly investigated in [2]. Analysis of the full operator

$C = C^i + C^e$ is similar but we provide it for the sake of completeness. Note that since C^e is a bounded operator, the domain of C equals the domain of C^i, that is, $D(C) = D(C^i)$.

It is worthwhile to analyse the operator C in two spaces: $X = L_1(\mathbb{R}^3)$ and $X^2 = L_2(\mathbb{R}^3, a^{v2} d\boldsymbol{v})$ where, as mentioned in Section 2, a is the Boltzmann factor

$$a := n_1/n_2 = e^{\frac{\Delta E}{kT}} > 1. \tag{3.11}$$

Here again $\boldsymbol{x}$ plays the role of a parameter and thus will be suppressed in the notation.

To avoid confusion we shall write down explicitly definitions of the operators which will appear in our considerations. For a continuous function f we introduce the operators

$$\begin{aligned}
(N_+ f)(\boldsymbol{v}) &= f(\boldsymbol{v}) \frac{n_2 v_+}{v} \int\limits_{S^2} \nu(v_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') d\boldsymbol{\omega}' \\
(N_- f)(\boldsymbol{v}) &= f(\boldsymbol{v}) n_1 H(v^2 - 1) \int\limits_{S^2} \nu(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')) d\boldsymbol{\omega}' \\
(K_- f)(\boldsymbol{v}) &= \frac{n_1 v_+}{v} \int\limits_{S^2} \nu(v_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') f(v_+ \boldsymbol{\omega}') d\boldsymbol{\omega}' \\
(K_+ f)(\boldsymbol{v}) &= n_2 H(v^2 - 1) \int\limits_{S^2} \nu(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') f(v_- \boldsymbol{\omega}') d\boldsymbol{\omega}'.
\end{aligned} \tag{3.12}$$

Moreover we define

$$Nf = N_+ f + N_- f, \qquad Kf = K_+ f + K_- f, \tag{3.13}$$

and finally

$$Cf = -Nf + Kf.$$

It follows that in X setting only the operators K_+ and N_+ are unbounded due to the singularity v^{-1} at 0. On the contrary, in X^2 setting the operator K_- is also unbounded with the singularity $1/\sqrt{v^2 - 1}$ at $v = 1$. The multiplication operator N_- is bounded in both settings [2].

To characterize the domain of C for an arbitrary positive measurable function g, we denote the weighted space $L_1(\mathbb{R}^3, g(v) d\boldsymbol{v})$ by X_g. It follows [2] that

$$D(C) = X_{1+v^{-1}}.$$

Similar considerations can be carried out in the space $X^2 = L_2(\mathbb{R}^3, a^{v^2} d\boldsymbol{v})$. Then the domain will be denoted by $D^2(C)$. Using analogous notation, we have

$$D^2(C) = X^2_{v^{-1}} \cap X^2_{H(v^2-1)v_-^{-1}} \cap X^2.$$

In the sequel we will find convenient to use the following notation: $\xi = v^2$ (ξ is the dimensionless kinetic energy) and

$$\phi(\xi) = f(\sqrt{\xi}), \tag{3.14}$$

so that $f(v_\pm\boldsymbol{\omega}) = \phi((\xi \pm 1)\boldsymbol{\omega})$. By [2] we may put $D(C) = D(K)$ and $D^2(C) = D^2(K)$.

The theorem below is an extension of Theorem 2.2 of [2] to the case that includes an elastic scattering term. Apart from one step, the proofs of both theorems are similar but to make this paper self-contained we provide the proof of Theorem 3.2 in Appendix B.

Theorem 3.2 *a) Let $f \in D(C)$. $Cf = 0$ if and only if ϕ, defined by Eq. (3.14), satisfies for $\xi \in [0,1[$ and $n \in \mathbb{N}$*

$$\phi(\xi + n) = a^{-n}\phi_0(\xi), \tag{3.15}$$

where $\phi_0 \in L_1([0,1], d\xi)$.

b) Let $f \in D^2(C)$. $Cf = 0$ if and only if Eq. (3.15) holds with $\phi_0 \in L_2([0,1], d\xi)$.

In the theory of hydrodynamic limits an important role is played by the conditions of solvability of $Cf = g$. A partial answer to this question is presented below in Theorem 3.3 and completed in Theorem 5.7. Again, Theorem 3.3 is an extension of the result of [2] to cover the case with elastic scattering and the proofs turn out to be quite similar. However, for the sake of completeness we provide the proof of Theorem 3.3 in Appendix C.

Theorem 3.3 *a) Let C^* be the adjoint to C in X and $f \in D(C^*)$. Then $C^*f = 0$ if and only if*

$$f(\sqrt{v^2 + n}) = f_0(v)$$

for all $v \in [0,1[$, $n \in \mathbb{N}$ and some $f_0 \in L_\infty([0,1])$.

b) The closure of the range $\overline{R(C)}$ is characterized by

$$g \in \overline{R(C)} \;\; if\ and\ only\ if \;\; \sum_{j=0}^{\infty} \sqrt{v^2 + j} \int_{S^2} g(\sqrt{v^2 + j}\boldsymbol{\omega})d\boldsymbol{\omega} = 0 \tag{3.16}$$

c) We have the "spectral" decomposition

$$X = N(C) \oplus \overline{R(C)} \tag{3.17}$$

and the "spectral" projection onto $N(C)$ along $\overline{R(C)}$ is given by

$$(Pf)(v^2 + n) = a^{-n}\psi_0(v^2) \tag{3.18}$$

where

$$\psi_0(v^2) = \frac{\sum\limits_{j=0}^{\infty} \sqrt{v^2 + j} \int\limits_{S^2} f(\sqrt{v^2 + j}\boldsymbol{\omega})d\boldsymbol{\omega}}{4\pi \sum\limits_{j=0}^{\infty} a^{-j}\sqrt{v^2 + j}}. \tag{3.19}$$

We can summarize Theorems 3.1, 3.2 and 3.3 in the full $L_1(\mathbb{R}^6_{x,v})$-setting as follows.

Corollary 3.1 *Under the adopted assumptions* *(i) The null-space of C^e is given by*

$$N(C^e) = \{f \in L_1(\mathbb{R}^6_{x,v});\ f \text{ is independent on } \boldsymbol{\omega}\}. \tag{3.20}$$

and the spectral projection onto $N(C^e)$ is given by

$$(\mathbb{P}f)(\boldsymbol{x}, v) = \frac{1}{4\pi}\int\limits_{S^2} f(\boldsymbol{x}, v\boldsymbol{\omega})d\boldsymbol{\omega}. \tag{3.21}$$

(ii) The null-spaces of C^i and $C = C^i + C^e$ coincide and are given by

$$\begin{aligned} N(C) \quad = \quad & N(C^i) = \{f \in L_1(\mathbb{R}^6_{x,v}, (v^{-1}+1)d\boldsymbol{v}d\boldsymbol{x});\ f \text{ is independent of } \boldsymbol{\omega} \\ & \text{and satisfies } n_1 f(\boldsymbol{x}, v_+) = n_2 f(\boldsymbol{x}, v) \text{ for a.a. } \boldsymbol{x} \in \mathbb{R}^3_x, v \in \mathbb{R}_+\}. \end{aligned} \tag{3.22}$$

The spectral projections onto $N(C)$ and $N(C^i)$ coincide and are given by

$$(Pf)(\boldsymbol{x}, v^2 + n) = a^{-n}\psi_0(\boldsymbol{x}, v^2), \quad \boldsymbol{x} \in \mathbb{R}^3_x, v \in [0,1[, n \in \mathbb{N} \cup \{0\}, \tag{3.23}$$

where for $v \in [0,1[$,

$$\psi_0(\boldsymbol{x}, v^2) = \frac{\sum\limits_{j=0}^{\infty} \sqrt{v^2+j} \int\limits_{S^2} f(\boldsymbol{x}, \sqrt{v^2+j}\boldsymbol{\omega})d\boldsymbol{\omega}}{4\pi \sum\limits_{j=0}^{\infty} a^{-j}\sqrt{v^2+j}}. \tag{3.24}$$

Thus

$$N(C^e) \supset N(C) = N(C^i),$$

and, in particular,

$$C^e P = 0.$$

4 Diffusive limits - formal derivations

Let us return to the scaled equation (2.5)

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon^p} S f_\epsilon + \frac{1}{\epsilon^q} C^e f_\epsilon + \frac{1}{\epsilon^r} C^i u. \tag{4.1}$$

We are looking for diffusive/hydrodynamic limits of this equation. According to the considerations of the previous section, there are two possible hydrodynamic spaces: $N(C^e)$ and $N(C) = N(C^i) = N(C^i + C^e)$. We can expect evolution in $N(C^e)$ if the elastic collisions are dominant, and in $N(C)$ if either inelastic collisions are dominant, or both elastic and inelastic collisions are much stronger than free-streaming.

To find the possible limit equations we use the modification of the Chapman-Enskog procedure, developed recently in [21]. The advantage of this method is that in this approach the projection of the solution to the Boltzmann equation onto the null-space of the collision operator, that is, the hydrodynamic part of the solution, is not expanded in ϵ and thus the whole information carried by this part is preserved throughout. This is in contrast to the Hilbert type expansion where, if applicable, only the zero order term of the expansion of the hydrodynamic part is recovered from the limit equation.

Hence the idea we shall pursue is to separate the hydrodynamic part of the solution to the Boltzmann equation by means of the appropriate spectral projection. Next, by expanding the kinetic part into a series of ϵ, we determine the other relevant terms of the asymptotic expansion, and in the last step we discard the terms of higher order in ϵ, getting thus (at least formally) the limit equation satisfied by the hydrodynamic part.

Accordingly, in the first case we will be looking for cases when the limit solution is the projection onto $N(C^e)$, and in the second when it is the projection onto $N(C)$.

4.1 Evolution in $N(C^e)$

To find possible limit evolutions in $N(C^e)$ we shall use the projection $\mathbb{P}$ defined by (3.9). It is important to note that since $N(C^e) \supset N(C)$, $\mathbb{P}C^i$ and $C^i\mathbb{P}$ are not equal to zero. Denoting $\mathbb{Q} = I - \mathbb{P}$, we operate with these projections onto (4.1) and putting $v_\epsilon = \mathbb{P}f_\epsilon$ and $w_\epsilon = \mathbb{Q}f_\epsilon$, we obtain

$$\begin{aligned} \partial_t v_\epsilon &= \frac{1}{\epsilon^p}\mathbb{P}S\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^r}\mathbb{P}C^i\mathbb{P}v_\epsilon + \frac{1}{\epsilon^r}\mathbb{P}C^i\mathbb{Q}w_\epsilon \\ \partial_t w_\epsilon &= \frac{1}{\epsilon^p}\mathbb{Q}S\mathbb{P}v_\epsilon + \frac{1}{\epsilon^p}\mathbb{Q}S\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^r}\mathbb{Q}C^i\mathbb{P}v_\epsilon + \frac{1}{\epsilon^r}\mathbb{Q}C^i\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^q}\mathbb{Q}C^e\mathbb{Q}w_\epsilon, \end{aligned} \tag{4.2}$$

where we already used the fact that $\mathbb{P}S\mathbb{P} = 0$ (see [4]). Since we assumed that the elastic collisions are dominant, we must take $q > max\{p, r\}$. We are looking for limit equations, that is, the equation for the approximation of v_ϵ cannot contain ϵ. This yields $r \leq 0$ (if $r > 0$, then we would have $\partial_t v_\epsilon$ multiplied by ϵ^r which would lead to an uninteresting case with no evolution in time), and shows that p must be less or equal to the index k of the first nonzero term in the expansion of $w_\epsilon = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots$. Let us consider first the case when $p = k$. Inserting this expansion into the second equation in (4.2) we obtain

$$\begin{aligned} \epsilon^q(\partial_t w_0 + \epsilon\partial_t w_1 + \ldots) &= \epsilon^{q-p}\mathbb{Q}S\mathbb{P}v_\epsilon + \epsilon^{q-p}\mathbb{Q}S\mathbb{Q}(w_0 + \epsilon w_1 + \ldots) \\ &\quad + \epsilon^{q-r}\mathbb{Q}C^i\mathbb{P}v_\epsilon + \epsilon^{q-r}\mathbb{Q}C^i\mathbb{Q}(w_0 + \epsilon w_1 \ldots) \\ &\quad + \mathbb{Q}C^e\mathbb{Q}(w_0 + \epsilon w_1 + \ldots), \end{aligned}$$

Since $q > r$ and $q > p$, we obtain

$$\mathbb{Q}C^e\mathbb{Q}w_0 = 0$$

which yields $w_0 = 0$, because $\mathbb{Q}$ is the complementary spectral projection. Clearly, the first nonzero term in the expansion of w will be w_k with k satisfying $k = \min\{q - p, q - r\}$. But if $q - p \geq q - r$, then $r \geq p$, but $r \leq 0$, thus $p \leq 0$. This however would contradict the assumption that $p = k$. Thus $k = q - p$ and $q = 2p$. In any such case we obtain $w_k = -(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}v_\epsilon$, where the inverse exists by Theorem 3.1. Changing now the notation from v_ϵ into ρ to emphasize the fact that the forthcoming equation is an approximating equation for v_ϵ, we obtain the limit equation, independent of ϵ, in the form

$$\partial_t \rho = -\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho + \mathbb{P}C^i\mathbb{P}\rho, \tag{4.3}$$

if $r = 0$, and

$$\partial_t \rho = -\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho, \tag{4.4}$$

if $r < 0$.

If $p < k$, then the power of the coefficient multiplying $\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho$ is positive and therefore this term is negligible when ϵ tends to zero. Then, the possible limit equations are

$$\partial_t \rho = \mathbb{P}C^i\mathbb{P}\rho, \tag{4.5}$$

if $r = 0$, and

$$\partial_t \rho = 0, \tag{4.6}$$

if $r < 0$.

To summarize, we list the generic (simplest) combinations of powers of ϵ giving particular limit equations discussed above. Thus, we see that (4.3) is the limit equation for the scaling

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon}Sf_\epsilon + \frac{1}{\epsilon^2}C^e f_\epsilon + C^i f_\epsilon$$

and (4.4) for the scaling

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon}Sf_\epsilon + \frac{1}{\epsilon^2}C^e f_\epsilon + \epsilon C^i f_\epsilon.$$

Further, (4.5) is the limit equation for the scaling

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon + C^i f_\epsilon$$

and (4.6) for the scaling

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon + \epsilon C^i f_\epsilon.$$

More generally, for the equation (4.1)

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon^p}Sf_\epsilon + \frac{1}{\epsilon^q}C^e f_\epsilon + \frac{1}{\epsilon^r}C^i u,$$

with $q > \max\{p, r\}$, we obtain the hydrodynamic limits:

- if $p = \min\{q-p, q-r\}$ (then $q = 2p$) and $r = 0$, then

$$\partial_t \rho = -\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho + \mathbb{P}C^i\mathbb{P}\rho,$$

- if $p = \min\{q-p, q-r\}$ (then $q = 2p$) and $r < 0$, then

$$\partial_t \rho = -\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho,$$

- if $p < \min\{q-p, q-r\}$ and $r = 0$, then

$$\partial_t \rho = \mathbb{P} C^i \mathbb{P} \rho,$$

  and finally,

- if $p < \min\{q-p, q-r\}$ and $r < 0$, then

$$\partial_t \rho = 0.$$

4.2 Evolution in $N(C)$

As we have seen in Section 3, the cases when either C^i or $C^i + C^e$ dominate have the same hydrodynamic subspace $N(C)$ and the same projectors onto it. Since $N(C^e) \supset N(C^i)$, we have $C^e P = P C^e = 0$, where P is the projector defined by Eqns. (3.18) and (3.19). Operating with P and $Q = I - P$ onto (4.1) and denoting $v_\epsilon = P f_\epsilon$, $w_\epsilon = Q f_\epsilon$ we obtain

$$\begin{aligned}
\partial_t v_\epsilon &= \frac{1}{\epsilon^p} PSQw_\epsilon \\
\partial_t w_\epsilon &= \frac{1}{\epsilon^p} QSPv_\epsilon + \frac{1}{\epsilon^p} QSQw_\epsilon + \frac{1}{\epsilon^r} QC^iQw_\epsilon + \frac{1}{\epsilon^q} QC^eQw_\epsilon,
\end{aligned} \tag{4.7}$$

where we used $PSP = 0$, [3]. As before, we observe that p must be less or equal to the index k of the first non-zero term w_k in the expansion of $w = w_0 + \epsilon w_1 + \ldots$. However, if it is strictly smaller, then the equation for ρ (the approximation to v_ϵ) will be trivially reduced to

$$\partial_t \rho = 0, \tag{4.8}$$

independently of what is happening in the second equation (though as we shall see in Subsection 5.4.3, the initial layer corrector complementing the hydrodynamic equation changes slightly depending on whether C^e and C^i are of the same or different magnitude). Hence, we can safely assume that $p = k > 0$. Let us then consider the expanded version of the second equation in (4.7)

$$\begin{aligned}
\partial_t(w_0 + \epsilon w_1 + \ldots) &= \frac{1}{\epsilon^p} QSPv_\epsilon + \frac{1}{\epsilon^p} QSQ(w_0 + \epsilon w_1 + \ldots) \\
&\quad + \frac{1}{\epsilon^r} QC^iQ(w_0 + \epsilon w_1 + \ldots) + \frac{1}{\epsilon^q} QC^eQ(w_0 + \epsilon w_1 + \ldots),
\end{aligned} \tag{4.9}$$

Here we have to distinguish two cases: $r > q$ and $r = q$ (in both cases we must have of course $r > p$).

Let us consider first the case $r > q$. Multiplying the last equation by ϵ^r we obtain

$$\begin{aligned}
\epsilon^r \partial_t(w_0 + \epsilon w_1 + \ldots) &= \epsilon^{r-p} QSPv_\epsilon + \epsilon^{r-p} QSQ(w_0 + \epsilon w_1 + \ldots) \\
&\quad + QC^iQ(w_0 + \epsilon w_1 + \ldots) + \epsilon^{r-q} QC^eQ(w_0 + \epsilon w_1 + \ldots),
\end{aligned}$$

Due to the definition of Q we have as before $w_0 = 0$. Let w_k be the first non-zero term of the expansion of w. Because $r - q > 0$, w_k at QC^eQ is multiplied by ϵ^{r-q+k} which is always of higher order than ϵ^k. The term w_k at QSQ is multiplied by $\epsilon^{r-p+k} = \epsilon^r$, and since QSP is multiplied by ϵ^{r-k} and $k > 0$, we see that w_k is to be determined from

$$QSP\varrho = -QC^iQw_k$$

when $r - p = k$, that is $r = 2p$. The index q seems to have no significance whatsoever on the final result. Consequently, the limit equation for the approximation ρ of v_ϵ is of the form

$$\partial_t\rho = -PSQ(QC^iQ)^{-1}QSP\rho, \tag{4.10}$$

provided the inverse exists (this follows from the results of Subsection 5.4).

The case $r = q$ is similar. Multiplying equation (4.9) by ϵ^r we obtain

$$\begin{aligned}\epsilon^r\partial_t(w_0 + \epsilon w_1 + \ldots) &= \epsilon^{r-p}QSPv_\epsilon + \epsilon^{r-p}QSQ(w_0 + \epsilon w_1 + \ldots) \\ &\quad + QC^iQ(w_0 + \epsilon w_1 + \ldots) + QC^eQ(w_0 + \epsilon w_1 + \ldots),\end{aligned}$$

Due to the definition of Q we have as before $w_0 = 0$. Let w_k be the first non-zero term of the expansion of w. Here $r - p > 0$, and therefore the term w_k at QSQ is multiplied by $\epsilon^{r-p+k} = \epsilon^r$ which is of higher order than ϵ^{r-p}, since $k > 0$, and consequently w_k is to be determined from

$$QSP\varrho = -Q(C^i + C^e)Qw_k$$

when $r - p = k$, that is $r = q = 2p$. Consequently, the limit equation is of the form

$$\partial_t\rho = -PSQ(Q(C^i + C^e)Q)^{-1}QSP\rho, \tag{4.11}$$

where inverse exists by virtue of the results of Subsection 5.4.

Thus the typical cases will be

(i)

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon^q}C^ef_\epsilon + \frac{1}{\epsilon}C^if_\epsilon, \tag{4.12}$$

with $q \leq 1$ (here $p = 0 < 1$ and 1 is the smallest possible index k for which $w_k \neq 0$), giving the hydrodynamic limit

$$\partial_t\rho = 0,$$

(ii)

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon}Sf_\epsilon + \frac{1}{\epsilon^q}C^ef_\epsilon + \frac{1}{\epsilon^2}C^if_\epsilon \tag{4.13}$$

with $q < 2$, giving the limit

$$\partial_t\rho = -PSQ(QC^iQ)^{-1}QSP\rho,$$

and

(iii)

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon}Sf_\epsilon + \frac{1}{\epsilon^2}C^ef_\epsilon + \frac{1}{\epsilon^2}C^if_\epsilon, \tag{4.14}$$

giving the limit

$$\partial_t\rho = -PSQ(Q(C^i + C^e)Q)^{-1}QSP\rho.$$

More generally, for the equation (4.1)

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon^p} S f_\epsilon + \frac{1}{\epsilon^q} C^e f_\epsilon + \frac{1}{\epsilon^r} C^i u,$$

with $r > \max\{p, q-1\}$, we will have the following hydrodynamic limits:

- if $p < \min\{r-p, r-q\}$, then
$$\partial_t \rho = 0,$$
- if $p = \min\{r-p, r-q\}$ (then $r = 2p$) and $r > q$, then
$$\partial_t \rho = -PSQ(QC^iQ)^{-1}QSP\rho,$$
- if $p = \min\{r-p, r-q\}$ (then $r = 2p$) and $r = q$, then
$$\partial_t \rho = -PSQ(Q(C^i + C^e)Q)^{-1}QSP\rho,$$

5 Rigorous analysis

In this section we shall give a survey of rigorous results concerning the hydrodynamic limits discussed in the previous section. By "rigorous results" we mean the results providing the estimate of the error between the solution to the kinetic equation (2.5) and the solution to the respective limit equation (4.3)-(4.6) or (4.8)-(4.11) (supplemented, if necessary, by an appropriate initial layer corrector). Such an estimate should be provided, if possible, in the space $L_1(\mathbb{R}^6_{x,v})$ which is a natural space from the physical point of view.

Such an analysis consists of two steps. At the first one we have to determine all the terms of the asymptotic expansion which would give the desired estimates of the error, provided they are well-defined (that is, the corresponding equations are solvable) and have sufficient regularity. The second step is to prove that all the equations obtained via the asymptotic procedure are solvable and that their solutions have sufficient regularity for the error estimates to be available.

It is to be noted that the last step seems to be purely of mathematical interest - the practitioner would be probably satisfied with the statement that the solution to the limit equation approximates the solution to the original equations provided all the terms are sufficiently regular, thus from this point of view the asymptotic analysis would be completed after the first step. However, even if such an argument leads very often to satisfactory results, one must remember that in many cases the equations defining the terms of the asymptotic expansion are rather artificial and don't reflect any physical reality; thus their solvability cannot be judged from the fact that "the Universe exists".

To present the asymptotic results in a unified framework we shall limit ourselves to the simplest case of isotropic scattering of Maxwell molecules. Thus we shall consider the following scattering operators:

$$(C^e f)(\boldsymbol{x}, \boldsymbol{v}) = -4\pi\lambda(v) f(\boldsymbol{x}, \boldsymbol{v}) + \lambda(v) \int_{S^2} f(\boldsymbol{x}, v\boldsymbol{\omega}') d\boldsymbol{\omega}', \tag{5.1}$$

with

$$0 < \lambda_{min} \le \lambda(v) \le \lambda_{max} < +\infty$$

for all $v \in [0, \infty[$, and

$$\begin{aligned}(C^i f)(\boldsymbol{x}, \boldsymbol{v}) &= -f(\boldsymbol{x}, \boldsymbol{v}) 4\pi \left(n_2 \frac{v_+}{v} \nu(v_+) + n_1 H(v^2 - 1) \nu(v) \right) \\ &\quad + n_1 \frac{v_+}{v} \nu(v_+) \int_{S^2} f(\boldsymbol{x}, v_+ \boldsymbol{\omega}') d\boldsymbol{\omega}' + n_2 \nu(v) H(v^2 - 1) \int_{S^2} f(\boldsymbol{x}, v_- \boldsymbol{\omega}') d\boldsymbol{\omega}' \end{aligned} \quad (5.2)$$

where H denotes the Heaviside function, $v_\pm = \sqrt{v^2 \pm 1}$ and $0 < \nu_{min} \le \nu(v) \le \nu_{max} < +\infty$ for $v \in [1, \infty[$. We note that the extension of the forthcoming results to the case of $\boldsymbol{x}$ dependent coefficients is straightforward if computationally unpleasant. To allow more general (but still isotropic in $\boldsymbol{v}$) scattering cross-sections is more demanding but can be accomplished in several cases (see e.g. [3, 9]). The extension to $\boldsymbol{\omega}$ dependent scattering cross-sections seems to present serious difficulties due to the adopted L_1 setting and the employed techniques.

5.1 Notation for operators and function spaces

In this subsection we introduce the function spaces relevant to the further considerations. Since from now on the dependence on the spatial variable will become important, in contrast to the previous section we must introduce notation which will distinguish L_1 spaces in $\boldsymbol{x}$ and $\boldsymbol{v}$ variables. The basic space is

$$\mathcal{X} = X_x \otimes_\pi X_v = L_1(\mathbb{R}^3_x, X_v) = L_1(\mathbb{R}^3_v, X_x) = L_1(\mathbb{R}^6_{x,v}),$$

where $X_\alpha = L_1(\mathbb{R}^3_\alpha)$ for $\alpha = x, v$. Most considerations will be carried out in X_v with fixed $\boldsymbol{x}$. Typically, if A_x is an operator in X_v (possibly depending on $\boldsymbol{x}$ as a parameter), then by A we will denote the extension of this operator to $\mathcal{X}$. If A_x is unbounded in X_v with domain $D(A_x)$, then A is considered with the natural domain

$$D(A) = \{f \in \mathcal{X}; f(\boldsymbol{x}, \cdot) \in D(A_x) \text{ for a.a } \boldsymbol{x}, \boldsymbol{x} \to (A_x f)(\boldsymbol{x}) \in \mathcal{X}\}.$$

If A_x does not depend on $\boldsymbol{x}$, and it is clear from the context in which space it acts, we will omit the subscript $\boldsymbol{x}$. The same convention will be applied to operators A_v acting in X_x.

Occasionally, if the above procedure can be reverted, for acting in $\mathcal{X}$ operator A we shall write A_x or A_v to denote this operator acting with $\boldsymbol{x}$ or $\boldsymbol{v}$, respectively, fixed as a parameter (that is, e.g. $A_x f = A(f \otimes 1)$ where $f \in X_v$, if the latter defines an element of X_v).

The asymptotic analysis requires some additional regularity of the data. Typically, the required regularity in $\boldsymbol{v}$ variable is related to the integrability with respect to a certain weight function and the required regularity in $\boldsymbol{x}$ variable is related to the differentiability. Accordingly, we introduce

$$X_{v,w} = L_1(\mathbb{R}^3_v, w(\boldsymbol{v}) d\boldsymbol{v})$$

and for the most typical "moment" weight $w(\boldsymbol{v}) = w(v) = 1 + v^k$, $k \in \mathbb{Z}$, we denote

$$X_{v,k} = L_1(\mathbb{R}^3_v, (1 + v^k)d\boldsymbol{v}).$$

Consequently, we denote

$$\mathcal{X}_w = L_1(\mathbb{R}^3_x, X_{v,w})$$

and

$$\mathcal{X}_k = L_1(\mathbb{R}^3_x, X_{v,k}).$$

If A is an operator in $\mathcal{X}$ with domain $D(A)$, then the domain of its part in $\mathcal{X}_k$ will be denoted by $D_k(A)$.

Combining the spatial and the velocity regularity, for a given operator A we introduce

$$\mathcal{X}_{lkm,A} = \{f \in \mathcal{X}_k;\ \partial^\beta_{x_\beta} f \in D_k(A^m),\ |\beta| \leq l\}, \tag{5.3}$$

where for the multi-index $\beta = (\beta_1, \beta_2, \beta_3)$ with $|\beta| = \beta_1 + \beta_2 + \beta_3$ we denoted $\partial^\beta_{x_\beta} = \partial^{\beta_1}_{x_1}\partial^{\beta_2}_{x_2}\partial^{\beta_3}_{x_3}$. This space can be normed by the natural graph norm.

Note that the space $\mathcal{X}_{lk0,A}$ is independent of the operator, thus it is sensible to denote it by $\mathcal{X}_{lk}$. Clearly we have then

$$\mathcal{X}_{lk0,A} = \mathcal{X}_{lk} = \{f \in \mathcal{X}_k;\ \partial^\beta_{x_\beta} f \in \mathcal{X}_k,\ |\beta| \leq l\}.$$

In many cases the operator under considerations is independent of $\boldsymbol{x}$. The most important example is rendered by the collision operator and to illustrate the notation discussed above we specify it for this particular case.

It follows from Section 3 that the domain of the operator $C = C^i + C^e = C^i_+ + C^i_- + C^e$ in X_v satisfies

$$D(C) = D(C^i_+) = X_{v,-1},$$

thus, if treated as an operator in $\mathcal{X}$, C has the domain $D(C) = \mathcal{X}_{-1}$.

Since the moment weight $1 + v^k$ and the weight defining the domain of C don't affect each other, it is easy to show that the domain of C in $X_{v,k}$ is given by

$$D_k(C) = X_{v,v^k+v^{-1}} \tag{5.4}$$

(the term 1 can be omitted as any function integrable with respect to v^k and v^{-1} is necessarily integrable). Considering again C as an operator acting in the full space $\mathcal{X}_k$, we can write (5.3) in the simpler form

$$\mathcal{X}_{lkm,C} = W^l_1(\mathbb{R}^3_x, D_k(C^m)). \tag{5.5}$$

Very often we will use the space

$$\mathcal{X}_{0km,C} = \mathcal{X}_{km,C} = L_1(\mathbb{R}^3_x, D_k(C^m)).$$

In particular, using the equivalent definition of $D_k(C)$, we get

$$\mathcal{X}_{0k1,C} = \mathcal{X}_{k1,C} = L_1(\mathbb{R}^3_x, X_{v,v^k+v^{-1}}) = \mathcal{X}_{v^k+v^{-1}}.$$

Summarizing, $f \in \mathcal{X}_{lkm,C}$ if $\partial^\beta_x f \in D_k(C^m)$ for $|\beta| \leq l$ and the norm $\boldsymbol{x} \to \|\partial^\beta_x f(\boldsymbol{x}, \cdot)\|_{D_k(C^m)}$ is integrable over $\mathbb{R}^3_x$. In particular, if $m = 1$, then it is sufficient that all the derivatives be integrable over $\mathbb{R}^6_{x,v}$ with weight function defined in (5.4).

5.2 Full asymptotic expansion - preliminary comments

The preliminary considerations leading the to the hydrodynamic equations have been carried out in Section 4. To be able to obtain the desired error estimates, in most cases we have to supplement these equations with bulk and initial layer correctors. This is done by an extension of the Chapman-Enskog asymptotic procedure of Section 4, which is sketched below.

The asymptotic solution is sought in the form

$$\begin{aligned} f(t,\tau) &= \bar{f}(t) + \tilde{f}(\tau) = \rho(t) + \bar{w}_0(t) + \epsilon\bar{w}_1(t) + \ldots \\ &\quad + \tilde{\rho}_0(\tau) + \epsilon\tilde{\rho}_1(\tau) + \ldots + \tilde{w}_0(\tau) + \epsilon\tilde{w}_1(\tau) + \ldots . \end{aligned} \tag{5.6}$$

where $\tau = t/\epsilon^n$ for some $n \in \mathbb{N}$. The terms

$$\rho, \tilde{\rho}_0, \tilde{\rho}_1 \ldots \in N(C^e) \text{ (resp. } N(C)),$$

are called the *hydrodynamic* part of the expansion, whereas

$$\bar{w}_0, \bar{w}_1, \ldots, \tilde{w}_0, \tilde{w}_1, \ldots \in N(C^e)^\perp \text{ (resp. } N(C)^\perp),$$

and are called the *kinetic* part of the expansion.

Moreover, the terms depending on t are referred to as the *bulk* part of the asymptotic expansion and the terms depending on τ are known as the *initial layer*; they are to be determined independently of each other. Note that in accordance with the Chapman-Enskog (compressed) procedure, [21], the hydrodynamic term of the bulk part of the expansion, ρ, is not expanded.

The number of the terms in each expansion and the value of n in the definition of τ are determined in some sense *a posteriori* after having written the formal equations for the error, so that the error could be conjectured to be of required order.

In the following subsections we shall provide a mathematical analysis showing that the presented formal asymptotic analysis is correct, that is, that the terms of the formal asymptotic expansion approximate the solution to the Boltzmann equation in $L_1(\mathbb{R}^6_{x,v})$ norm, uniformly in t on bounded intervals.

For simplicity we shall focus our attention on generic scalings introduced in Subsections 4.1 and 4.2.

5.3 Dominant elastic scattering

In this subsection we shall provide mathematical analysis of the procedures leading to the limit equations (4.3)-(4.6) in the hydrodynamic space $N(C^e)$. We start with the kinetic-diffusion equation (4.3) which mathematically is the most interesting case; the results presented here are taken from [4, 6].

Next we shall briefly discuss the case (4.5) which has been recently solved in [5]. Equation (4.5) can be viewed as a simplified (spatially homogeneous) version of (4.3) and no surprisingly the basic ideas of [5] play an important role in the analysis of the latter equation.

Two other equations of this subsection, (4.4) and (4.6) are much simpler and though they haven't been analysed before, we shall limit ourselves to some general comments.

5.3.1 The kinetic-diffusion equation

Let us consider the Boltzmann equation

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon} S f_\epsilon + \frac{1}{\epsilon^2} C^e f_\epsilon + C^i f_\epsilon. \tag{5.7}$$

The preliminary considerations leading to the limit equation (4.3) has been carried out in Subsection 4.1. Here we have to supplement the hydrodynamic equations with bulk and initial layer correctors which will enable us to obtain the desired error estimates.

We start with the system (4.2) where we put $p = 1, q = 2, r = 0$. The assumed isotropic form of the scattering operators allows us to simplify (4.2) even further. To do this we note that by Corollary 3.1 the operator C^i reduces $N(C^e)$ and since $W = N(C^e)^\perp$, we get $\mathbb{P}C^i\mathbb{Q} = \mathbb{Q}C^i\mathbb{P} = 0$. The system (4.2) takes the form

$$\begin{aligned} \partial_t v_\epsilon &= \frac{1}{\epsilon}\mathbb{P}S\mathbb{Q}w_\epsilon + \mathbb{P}C^i\mathbb{P}v_\epsilon \\ \partial_t w_\epsilon &= \frac{1}{\epsilon}\mathbb{Q}S\mathbb{P}v_\epsilon + \frac{1}{\epsilon}\mathbb{Q}S\mathbb{Q}w_\epsilon + \mathbb{Q}C^i\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^2}\mathbb{Q}C^e\mathbb{Q}w_\epsilon, \end{aligned} \tag{5.8}$$

with the initial conditions

$$\begin{aligned} v_\epsilon(0) &= \mathbb{P}\overset{\circ}{f} = \overset{\circ}{v} \\ w_\epsilon(0) &= \mathbb{Q}\overset{\circ}{f} = \overset{\circ}{w}. \end{aligned}$$

Inserting (5.6) into (5.8) and equating the terms at the same powers of ϵ we find that we have to take $\tau = t/\epsilon^2$ and we obtain the following system

$$\begin{aligned} \partial_t \rho &= -\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho + \mathbb{P}C^i\mathbb{P}\rho, & (5.9)\\ \mathbb{Q}C^e\mathbb{Q}\bar{w}_0 &= 0, \\ \mathbb{Q}C^e\mathbb{Q}\bar{w}_1 + \mathbb{Q}S\mathbb{P}\rho &= 0, \\ \mathbb{Q}C^e\mathbb{Q}\bar{w}_2 + \mathbb{Q}S\mathbb{Q}\bar{w}_1 &= 0, \\ \partial_\tau \tilde{\rho}_0 &= 0, & (5.10)\\ \partial_\tau \tilde{w}_0 &= \mathbb{Q}C^e\mathbb{Q}\tilde{w}_0, & (5.11) \end{aligned}$$

which, as we shall see, defines enough terms of the asymptotic expansion to obtain, at least formally, the convergence of the difference $f_\epsilon - f(t,\tau)$ to zero as $\epsilon \to 0$.

In fact, let us assume for a time being that all the equations above can be solved and that the solutions are sufficiently regular to make the manipulations to follow available. It can be proved that on this level of approximation the correct initial values for (5.9) and (5.11) are

$$\rho(0) = \overset{\circ}{v}, \qquad \tilde{w}_1(0) = \overset{\circ}{w}.$$

Note that the equations for $\bar{w}_1$ and $\bar{w}_2$ do not require any side conditions, and the solution to (5.10) is determined by the stipulated decay to zero as $\tau \to \infty$. Thus we obtain $\bar{w}_0 = \tilde{\rho}_0 = 0$ and

$$\begin{aligned}
\bar{w}_1 &= -(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho, \\
\bar{w}_2 &= -(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}\bar{w}_1 = (\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho, \\
\tilde{w}_0 &= e^{\tau\mathbb{Q}C^e\mathbb{Q}}\,\overset{\circ}{w}\,.
\end{aligned} \tag{5.12}$$

Hence, we take the pair $(\rho, \tilde{w}_0 + \epsilon\bar{w}_1 + \epsilon^2\bar{w}_2)$ as the approximation of $f_\epsilon = (v_\epsilon, w_\epsilon)$; the error of this approximation is given by

$$\begin{aligned}
y_\epsilon &= v_\epsilon - \rho \\
z_\epsilon &= w_\epsilon - \tilde{w}_0 - \epsilon\bar{w}_1 - \epsilon^2\bar{w}_2.
\end{aligned} \tag{5.13}$$

Assuming that the solution and the terms of the asymptotic expansion are regular enough (we require that they belong to the domains of all the operators involved in (5.8)), and taking into account Eqs. (5.9)-(5.11) we obtain the following system of equations for the error

$$\partial_t y_\epsilon - \frac{1}{\epsilon}\mathbb{P}S\mathbb{Q}z_\epsilon - \mathbb{P}C^i\mathbb{P}y_\epsilon = \epsilon\mathbb{P}S\mathbb{Q}\bar{w}_2 + \frac{1}{\epsilon}\mathbb{P}S\mathbb{Q}\tilde{w}_0 \tag{5.14}$$

$$\begin{aligned}
\partial_t z_\epsilon - \frac{1}{\epsilon}\mathbb{Q}S\mathbb{P}y_\epsilon - \frac{1}{\epsilon}\mathbb{Q}S\mathbb{Q}z_\epsilon - \mathbb{Q}C^i\mathbb{Q}z_\epsilon - \frac{1}{\epsilon^2}\mathbb{Q}C^e\mathbb{Q}z_\epsilon &= \epsilon\mathbb{Q}S\mathbb{Q}\bar{w}_2 + \epsilon^2\mathbb{Q}C^i\mathbb{Q}\bar{w}_2 + \frac{1}{\epsilon}\mathbb{Q}S\mathbb{Q}\tilde{w}_0 \\
&\quad +\epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_1 + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0 \\
&\quad -\epsilon\partial_t\bar{w}_1 - \epsilon^2\partial_t\bar{w}_2,
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
y_\epsilon(0) &= 0 \\
z_\epsilon(0) &= \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\,\overset{\circ}{v} - \epsilon^2(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\,\overset{\circ}{v}\,.
\end{aligned}$$

Keeping in mind our assumption that all the terms of the asymptotic equation are sufficiently regular, we see that the error $e_\epsilon = y_\epsilon + z_\epsilon$ is a classical solution of the problem

$$\begin{aligned}
\partial_t e_\epsilon - \frac{1}{\epsilon}Se_\epsilon - C^i e_\epsilon - \frac{1}{\epsilon^2}C^e e_\epsilon &= \epsilon(S\mathbb{Q}\bar{w}_2 + \mathbb{Q}C^i\mathbb{Q}\bar{w}_1 + \epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_2 - \partial_t\bar{w}_1 - \epsilon\partial_t\bar{w}_2) \\
&\quad + \frac{1}{\epsilon}S\mathbb{Q}\tilde{w}_0 + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0, \\
e_\epsilon(0) &= \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\,\overset{\circ}{v} - \epsilon^2(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\,\overset{\circ}{v},
\end{aligned} \tag{5.15}$$

The semigroup solving this equation is contractive in $\mathcal{X}$, [2], thus using the Duhamel formula we obtain the estimate

$$\begin{aligned}
\|e_\epsilon(t)\| &\le \epsilon\|(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\,\overset{\circ}{v} - \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\,\overset{\circ}{v}\,\| \\
&\quad + \epsilon\int_0^t \|S\mathbb{Q}\bar{w}_2(s) + \mathbb{Q}C^i\mathbb{Q}\bar{w}_1(s) + \epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_2 - \partial_s\bar{w}_1(s) - \epsilon\partial_s\bar{w}_2(s)\|ds \\
&\quad + \frac{1}{\epsilon}\int_0^t \|S\mathbb{Q}\tilde{w}_0(s/\epsilon^2) + \epsilon\mathbb{Q}C^i\mathbb{Q}\tilde{w}_0(s/\epsilon^2)\|ds.
\end{aligned} \tag{5.16}$$

From the above inequality we see that if all the expressions in the first two terms exist and are locally bounded in t on $[0,\infty[$, then the contribution of this integral is of order of ϵ. As far as the second integral is concerned, the initial layer is assumed to be exponentially decaying with $\tau \to \infty$, that is, to be of order of $e^{-\omega t/\epsilon^2}$ for some $\omega > 0$. If this property is preserved after having operated on $\tilde{w}_0$ with the operators $S\mathbb{Q}$ and $\mathbb{Q}C^i\mathbb{Q}$, then upon integration we obtain that also the contribution of this term is of order of ϵ, thus $\|e_\epsilon\| = O(\epsilon)$ and the convergence is proved.

So we see that to complete the analysis we must prove that all the terms exist and have the desired regularity. For instance, for $\mathbb{Q}S\mathbb{Q}\bar{w}_2$ to be well-defined we need the existence of $S^3\rho$ or, in other words, the solvability of (4.3) in the moment space $\mathcal{X}_3$ together with three-fold differentiability with respect to $\boldsymbol{x}$; we also need certain regularity of the moments with respect to the operator C^i: the existence of $\mathbb{Q}C^i\mathbb{Q}\bar{w}_2$ requires that $S^2\rho$ be in $D(C^i)$, etc.

The first step is to establish the equivalence of the forms (5.8) and (4.1) (with relevant scaling). This requires the projections of the solution to (4.1) to belong to the domain of S. In this respect we have the following lemma [5] which also applies to any other case with dominant elastic scattering.

Lemma 5.1 *If $\overset{\circ}{f} \in D(S) \cap D(C^i)$, then $\mathbb{P}f_\epsilon, \mathbb{Q}f_\epsilon \in D(S) \cap D(C^i)$ and therefore (4.1) and (5.8) are equivalent.*

The explicit form and the solvability of (4.3) have been investigated in [4] for much more general models. In the theorem below we shall summarize the main results of [4], specified to the case in hand.

Theorem 5.1 *1. Let us denote $\mathsf{D} = -\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}$. Then defining $d = d(v) = \frac{4\pi}{3\lambda(v)} \geq \frac{4\pi}{3\lambda_{max}}$ we have*

$$(\mathsf{D}\rho)(\boldsymbol{x},\boldsymbol{v}) = dv^2(\Delta\rho)(\boldsymbol{x},\boldsymbol{v}) \tag{5.17}$$

where Δ is the Laplace operator in $\boldsymbol{x}$-variable.

2. *The operators $\mathsf{D}_v = dv^2\Delta$ (v fixed) defined on the domains $D(\mathsf{D}_v) = L_{1,2}(\mathbb{R}^3) \subset X_x$ (Bessel potential space, see e.g. [17]) for $v > 0$ and $D(\mathsf{D}_0) = X_x$, generate positive semigroups of contractions in X_x, denoted hereafter by $(G_{\mathsf{D}_v}(t))_{t\geq 0}$.*

3. *The operator D with the domain*

$$D(\mathsf{D}) = \{f \in \mathcal{X}; f(\cdot,v) \in D(\mathsf{D}_v), (\boldsymbol{x},v) \to (\mathsf{D}_v f)(\boldsymbol{x},v) \in \mathcal{X}\}$$

generates a semigroup of contractions $(G_{\mathsf{D}}(t))_{t\geq 0}$ in $\mathcal{X}$. This semigroup is conservative for nonnegative initial data.

4. *The operator $T = \mathsf{D} + \mathbb{P}C^i\mathbb{P}$ defined on the domain $D(T) = D(\mathsf{D}) \cap D(C)$ generates a positive semigroup of contractions in $\mathcal{X}$, denoted by $(G_T(t))_{t\geq 0}$, which is conservative for nonnegative initial data.*

In further considerations it turns out to be more convenient to replace the speed v by the energy related variable according to $v^2 = \xi$, as it was done in Section 3. To avoid the introduction of yet another set of definitions, we shall not change the symbols of the functions appearing in the problem. With this convention, the Cauchy problem for the limit hydrodynamic equation (4.3) takes the form

$$\begin{aligned} \partial_t \rho &= \xi d\Delta\rho - \left(n_2\sqrt{\frac{\xi+1}{\xi}}m(\xi+1) + n_1 H(\xi-1)m(\xi)\right)\rho \\ &\quad + n_1\sqrt{\frac{\xi+1}{\xi}}m(\xi+1)\rho(\xi+1) + n_2 H(\xi-1)m(\xi)\rho(\xi-1), \\ \rho(0) &= \overset{\circ}{\rho}, \end{aligned} \tag{5.18}$$

where $m(\xi) = 4\pi\nu(\xi)$.

The problem is posed in $\mathcal{X}$. Since in the velocity space X_v all the terms have been averaged with respect to $\boldsymbol{\omega}$, this space is reduced to $X_v = L_1(\mathbb{R}_+, v^2 dv) = L_1(\mathbb{R}_+, \sqrt{\xi}d\xi)$.

The crucial result which allows us to prove the asymptotic estimate is the existence of the moments of the solution to (4.3). In this regard we have the following theorem the proof of which was referred to the Appendix D.

Theorem 5.2 *For any $k \geq 0$, the operator T generates a positive semigroup in $\mathcal{X}_k$, denoted by $(G_T(t))_{t\geq 0}$. Thus, if $\overset{\circ}{\rho} \in D_k(T)$, then the Cauchy problem (5.18):*

$$\partial_t \rho = T\rho \tag{5.19}$$

with the initial condition

$$\rho(0) = \overset{\circ}{\rho}\,. \tag{5.20}$$

has a unique solution $\rho \in C^1([0,\infty[, \mathcal{X}_k)$. ■

Now we are ready to provide the error estimates. Let us recall that according to (5.5) we have

$$\mathcal{X}_{lkm,T} = \{f \in \mathcal{X}_k;\ \partial^\beta_{x^\beta} f \in D(T^m),\ |\beta| \leq l\} \tag{5.21}$$

We have the following lemma.

Lemma 5.2 *Let $\overset{\circ}{v} = \mathbb{P}\overset{\circ}{f} \in \mathcal{X}_{331,T}$. Then for each interval $[0,t_0]$, $0 < t_0 < +\infty$, there exists a constant M such that*

$$\begin{aligned} &\|(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v} - \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v}\|_{\mathcal{X}} \\ &+ \max_{t\in[0,t_0]}\{\|S\mathbb{Q}\bar{w}_2(t) + \mathbb{Q}C^i\mathbb{Q}\bar{w}_1(t) + \epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_2(t) - \partial_t\bar{w}_1(t) - \epsilon\partial_t\bar{w}_2(t)\|_{\mathcal{X}}\} \leq M \end{aligned} \tag{5.22}$$

Proof. Let us fix $0 < t_0 < \infty$ and note that as in [5] we can prove that the distributional derivatives commute with T and therefore commute with the semigroup $(G_T(t))_{t\geq 0}$. This shows in particular that $(G_T(t))_{t\geq 0}$ is also a semigroup in $\mathcal{X}_{lk} = W_1^l(\mathbb{R}^3_x, X_{v,k})$ for any $l, k \geq 0$, generated by the part of T in this space. We also note that in our isotropic case we have $(\mathbb{Q}C^e\mathbb{Q})^{-1}f = -\lambda^{-1}f$. First we consider

$$S\mathbb{Q}\bar{w}_2(t) = S(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho.$$

Is enough to analyze $t \to S^3\rho(t)$. By the discussion above, the assumptions are sufficient for the differentiation with respect to $\boldsymbol{x}$ to commute with the semigroup $(G_T(t))_{t\geq 0}$. Using Theorem 5.2 we have

$$\|S^3\rho(t)\|_{\mathcal{X}} \leq M_1 \sum_{|\beta|=0}^{3} \|G_T(t)\partial^{\beta}_{\boldsymbol{x}_\beta} \overset{\circ}{v}\|_{\mathcal{X}_3} \leq M_1'\|\overset{\circ}{v}\|_{\mathcal{X}_{330,T}},$$

for some constant M_1'.

Next we consider

$$\mathbb{Q}C^i\mathbb{Q}\bar{w}_1(t) = -\lambda^{-1}C^iS\rho(t).$$

Using (5.4), Theorem 5.2, and the commutativity of C^i and $G_T(t)$, we obtain

$$\begin{aligned}\|\mathbb{Q}C^i\mathbb{Q}\bar{w}_1(t)\|_{\mathcal{X}} \leq M_2'\|S\rho(t)\|_{\mathcal{X}_{001,C^i}} &\leq M_2''\|(1+v^{-1})S\rho(t)\|_{\mathcal{X}}\\ \leq \sum_{|\beta|=0}^{1}\|v(1+v^{-1})G_T(t)\partial^{\beta}_{\boldsymbol{x}_\beta}\overset{\circ}{v}\|_{\mathcal{X}} &\leq \sum_{|\beta|=0}^{1}\|(1+v)G_T(t)\partial^{\beta}_{\boldsymbol{x}_\beta}\overset{\circ}{v}\|_{\mathcal{X}}\\ = \sum_{|\beta|=0}^{1}\|(G_T(t)T+I)\partial^{\beta}_{\boldsymbol{x}_\beta}\overset{\circ}{v}\|_{\mathcal{X}_1} &\leq M_2'''\|\overset{\circ}{v}\|_{\mathcal{X}_{110,T}},\end{aligned}$$

uniformly on $[0, t_0]$. The next term to consider is $\mathbb{Q}C^i\mathbb{Q}\bar{w}_2$. We have as above

$$\begin{aligned}\|\mathbb{Q}C^i\mathbb{Q}\bar{w}_2(t)\|_{\mathcal{X}} \leq M_3'\|S^2\rho(t)\|_{\mathcal{X}_{001,C^i}} &\leq M_3''\|(1+v^{-1})S^2\rho(t)\|_{\mathcal{X}}\\ \leq \sum_{|\beta|=0}^{2}\|v^2(1+v^{-1})(G_T(t)T+I)\partial^{\beta}_{\boldsymbol{x}_\beta}\overset{\circ}{v}\|_{\mathcal{X}} &\leq \sum_{|\beta|=0}^{2}\|(v^2+v)G_T(t)\partial^{\beta}_{\boldsymbol{x}_\beta}\overset{\circ}{v}\|_{\mathcal{X}}\\ &\leq M_3'''\|\overset{\circ}{v}\|_{\mathcal{X}_{220,T}},\end{aligned}$$

To estimate the next term we observe first that ρ is differentiable in the norm of $\mathcal{X}_{11}$, thus we have

$$\partial_t\bar{w}_1(t) = -\lambda^{-1}\mathbb{Q}S\mathbb{P}\partial_t\rho(t) = -\lambda^{-1}\mathbb{Q}ST\rho(t),$$

therefore

$$\|\partial_t\bar{w}_1(t)\|_{\mathcal{X}} \leq M_4' \sum_{|\beta|=0}^{1}\|G_T(t)T\partial^{\beta}_{\boldsymbol{x}_\beta}\overset{\circ}{v}\|_{\mathcal{X}_1} \leq M_4\|\overset{\circ}{v}\|_{\mathcal{X}_{111,T}}.$$

Similarly, since $\bar{w}_2(t) = (\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho$ and ρ is differentiable in the norm of $\mathcal{X}_{22}$, we have

$$\partial_t\bar{w}_2(t) = \lambda^{-2}\mathbb{Q}S^2\partial_t\rho(t) = \lambda^{-2}\mathbb{Q}S^2T\rho(t),$$

and as above

$$\|\partial_t \bar{w}_2(t)\|_{\mathcal{X}} \leq M_5' \sum_{|\beta|=0}^{2} \|G_T(t) T \partial^{\beta}_{\boldsymbol{x}_\beta} \overset{\circ}{v} \|_{\mathcal{X}_2} \leq M_5 \| \overset{\circ}{v} \|_{\mathcal{X}_{221,T}}.$$

The estimate of the first term in (5.22) follows easily as the assumptions of the lemma ensure that the function $t \to S^2\rho$ is continuous. Combining all the estimates we complete the proof of the lemma. ■

The regularity of the initial layer is dealt with in the next lemma.

Lemma 5.3 *Let* $\overset{\circ}{w} = \mathbb{Q} \overset{\circ}{f} \in \mathcal{X}_{111,C}$. *Then there exist a positive constant* L *such that for any* $t \geq 0$

$$\|S\mathbb{Q}\tilde{w}_0(t/\epsilon^2)\|_{\mathcal{X}} + \|\mathbb{Q}C^i\mathbb{Q}\tilde{w}_0(t/\epsilon^2)\|_{\mathcal{X}} \leq Le^{-\lambda_{min}t/\epsilon^2}. \tag{5.23}$$

Proof. Since the terms of the initial layer expansion are exactly the same as in Subsection 5.3.3, this lemma coincides with Lemma 4.3 of [5]. ■

Thus we have the theorem.

Theorem 5.3 *Assume that* $\mathbb{P} \overset{\circ}{f} \in \mathcal{X}_{331,T}$ *and* $\mathbb{Q} \overset{\circ}{f} \in \mathcal{X}_{111,C}$. *Let* f_ϵ *be the solution of (5.7) with the initial datum* $\overset{\circ}{f}$, *and* ρ *be the solution to (4.3) with the initial value* $\mathbb{P}\overset{\circ}{f}$. *Then for each interval* $[0, t_0]$, $0 < t_0 < +\infty$, *there exists a constant* K *depending only on the initial data, the coefficients of the equation, and* t_0, *such that*

$$\|f_\epsilon(t) - \rho(t) - e^{-\lambda t/\epsilon^2}\mathbb{Q}\overset{\circ}{f}\|_{\mathcal{X}} \leq K\epsilon \tag{5.24}$$

uniformly on $[0, t_0]$.

Proof. For the proof we note that the assumptions on the initial data adopted here are not weaker than that of any lemma (in particular, $D(T) \subset D(S) \cap D(\mathsf{D})$) so that all the steps of this subsection are justified. Hence using Lemmas 5.2 and 5.3 we have by (5.16)

$$\begin{aligned}
\|e_\epsilon(t)\|_{\mathcal{X}} &\leq \epsilon\|(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v} - \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v}\|_{\mathcal{X}} \\
&\quad + \epsilon\int_0^t \|S\mathbb{Q}\bar{w}_2(s) + \mathbb{Q}C^i\mathbb{Q}\bar{w}_1(s) + \epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_2(s) - \partial_s\bar{w}_1(s) - \epsilon\partial_s\bar{w}_2(s)\|_{\mathcal{X}} ds \\
&\quad + \frac{1}{\epsilon}\int_0^t \|S\mathbb{Q}\tilde{w}_0(s/\epsilon^2) + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0(s/\epsilon^2)\|_{\mathcal{X}} ds \\
&= \epsilon M t_0 + \epsilon L \int_0^{t/\epsilon^2} e^{-\lambda_{min} r} dr \leq K\epsilon
\end{aligned}$$

The only difference now is that in (5.13) and (5.16) we had $e_\epsilon = f_\epsilon - \rho - \tilde{w}_0 - \epsilon\bar{w}_1 - \epsilon^2\bar{w}_2$, whereas in (5.24) the last two terms are missing. However, the estimates of Lemma 5.2 can be carried also for $\bar{w}_1$ and $\bar{w}_2$ alone, showing that they are bounded on $[0, t_0]$. Since they are multiplied by ϵ and ϵ^2 respectively, they can be moved to the right-hand side of the inequality (5.24) without changing it. ■

5.3.2 Purely diffusive hydrodynamical limit

In this subsection we shall describe the steps leading asymptotically from

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon} S f_\epsilon + \frac{1}{\epsilon^2} C^e f_\epsilon + \epsilon C^i f_\epsilon \tag{5.25}$$

to Eq. (4.4) which by Theorem 5.1 is given by

$$\partial_t \rho = v^2 d \Delta \rho. \tag{5.26}$$

Since the limit equation is simpler than in the previous case, we shall skip most technical details.

Using the isotropy of the scattering we arrive at the following counterpart to (5.8)

$$\begin{aligned} \partial_t v_\epsilon &= \frac{1}{\epsilon}\mathbb{P}S\mathbb{Q}w_\epsilon + \epsilon \mathbb{P}C^i\mathbb{P}v_\epsilon, \\ \partial_t w_\epsilon &= \frac{1}{\epsilon}\mathbb{Q}S\mathbb{P}v_\epsilon + \frac{1}{\epsilon}\mathbb{Q}S\mathbb{Q}w_\epsilon + \epsilon \mathbb{Q}C^i\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^2}\mathbb{Q}C^e\mathbb{Q}w_\epsilon. \end{aligned} \tag{5.27}$$

Apart from (4.4), all the other terms of the asymptotic expansion coincide with those given by (5.12). Defining the approximation and the error as in (5.13), we obtain the error equations in the form

$$\begin{aligned} \partial_t y_\epsilon - \frac{1}{\epsilon}\mathbb{P}S\mathbb{Q}z_\epsilon - \epsilon\mathbb{P}C^i\mathbb{P}y_\epsilon &= \epsilon\mathbb{P}S\mathbb{Q}\bar{w}_2 + \epsilon\mathbb{P}C^i\mathbb{P}\rho + \frac{1}{\epsilon}\mathbb{P}S\mathbb{Q}\tilde{w}_0 \\ \partial_t z_\epsilon - \frac{1}{\epsilon}\mathbb{Q}S\mathbb{P}y_\epsilon - \frac{1}{\epsilon}\mathbb{Q}S\mathbb{Q}z_\epsilon - \epsilon\mathbb{Q}C^i\mathbb{Q}z_\epsilon - \frac{1}{\epsilon^2}\mathbb{Q}C^e\mathbb{Q}z &= \epsilon\mathbb{Q}S\mathbb{Q}\bar{w}_2 + \epsilon^3\mathbb{Q}C^i\mathbb{Q}\bar{w}_2 + \frac{1}{\epsilon}\mathbb{Q}S\mathbb{Q}\tilde{w}_0 \\ &\quad + \epsilon^2\mathbb{Q}C^i\mathbb{Q}\bar{w}_1 + \epsilon\mathbb{Q}C^i\mathbb{Q}\tilde{w}_0 \\ &\quad - \epsilon\partial_t\bar{w}_1 - \epsilon^2\partial_t\bar{w}_2, \end{aligned} \tag{5.28}$$

with the initial conditions

$$\begin{aligned} y_\epsilon(0) &= 0 \\ z_\epsilon(0) &= \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v} - \epsilon^2(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v}. \end{aligned}$$

Clearly the estimates are analogous with the only difference that this time ρ is a solution of the diffusion equation in $\boldsymbol{x}$ multiplied by v^2, as seen from Eq. (5.26) and the solution to this equation must have the regularity required in Lemma 5.2. Since the operators of differentiation and multiplication by v^k commute with D, and D generates a C_0-semigroup, the assumptions will be much milder here and the proof of the counterpart of this lemma is much easier, hence we shall only sketch it. Recalling the notation

$$\mathcal{X}_{lk} = W_1^l(\mathbb{R}^3_x, X_k),$$

we have state the following lemma.

Lemma 5.4 *Let $\overset{\circ}{v} = \mathbb{P}\overset{\circ}{f} \in \mathcal{X}_{44}$. Then for each interval $[0,t_0]$, $0 < t_0 < +\infty$, there exists a constant M such that*

$$\|(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v} - \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v}\|_{\mathcal{X}} \tag{5.29}$$
$$+ \max_{t\in[0,t_0]}\{\|S\mathbb{Q}\bar{w}_2(t) + \epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_1(t) + \epsilon^2\mathbb{Q}C^i\mathbb{Q}\bar{w}_2(t) - \partial_t\bar{w}_1(t) - \epsilon\partial_t\bar{w}_2(t)\|_{\mathcal{X}}\} \leq M$$

Proof. Following the approach of Lemma 5.2 in the same order we see that the estimate (5.29) is satisfied if: $v^3\partial^\beta_{x_\beta}\overset{\circ}{v} \in \mathcal{X}$ for $|\beta| = 3$, $(v+1)\partial^\beta_{x_\beta}\overset{\circ}{v} \in \mathcal{X}$ for $|\beta| = 1$, $(v^2+v)\partial^\beta_{x_\beta}\overset{\circ}{v} \in \mathcal{X}$ for $|\beta| = 2$, and $v^k\partial^\beta_{x_\beta}\overset{\circ}{v} \in D(\mathsf{D})$ for $|\beta| = k, k = 1,2$. Recalling the definition of $D(\mathsf{D})$ we see that if $\overset{\circ}{v} \in \mathcal{X}_{44}$, then all the above requirements are satisfied. ■

The initial layer terms are the same so that for the estimates we can use Lemma 5.3.

To make the statement of the final theorem more clear, we note that the space $\mathcal{X}_{111,C}$ used in Lemma 5.3 is given by

$$\mathcal{X}_{111,C} = W^1_1(\mathbb{R}^3_x, X_{v,v+v^{-1}}).$$

Also to ensure that the f_ϵ is the classical solution we must ensure that $\mathbb{P}\overset{\circ}{f} \in D(C)$ in addition to the assumptions of Lemma 5.4. With these we have the following counterpart of

Theorem 5.4 *Assume that $\mathbb{P}\overset{\circ}{f} \in \mathcal{X}_{441,C}$ and $\mathbb{Q}\overset{\circ}{f} \in \mathcal{X}_{111,C}$. Let f_ϵ be the solution of (5.25) with the initial datum $\overset{\circ}{f}$, and ρ be the solution to (5.26) with the initial value $\mathbb{P}\overset{\circ}{f}$. Then for each interval $[0,t_0]$, $0 < t_0 < +\infty$, there exists a constant K depending only on the initial data and the coefficients of the equation and t_0, such that*

$$\|f_\epsilon(t) - \rho(t) - e^{-\lambda t/\epsilon^2}\mathbb{Q}\overset{\circ}{f}\|_{\mathcal{X}} \leq K\epsilon \tag{5.30}$$

uniformly on $[0,t_0]$.

5.3.3 Purely kinetic hydrodynamic limit

Now we shall present the counterpart of Theorems 5.3 and 5.4 for the scaling

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon + C^i f_\epsilon \tag{5.31}$$

resulting in the hydrodynamic limit (4.5)

$$\partial_t\rho = \mathbb{P}C^i\mathbb{P}\rho, \tag{5.32}$$

where, as before,

$$\begin{aligned}\mathbb{P}C^i\mathbb{P}\rho &= -\left(n_2\sqrt{\frac{\xi+1}{\xi}}m(\xi+1) + n_1H(\xi-1)m(\xi)\right)\rho \\ &\quad + n_1\sqrt{\frac{\xi+1}{\xi}}m(\xi+1)\rho(\xi+1) + n_2H(\xi-1)m(\xi)\rho(\xi-1),\end{aligned}$$

with $m(\xi) = 4\pi\nu(\xi)$.

This problem was thoroughly investigated in [5] so that we mention here only that the solvability of (5.32) in the moment spaces $\mathcal{X}_k$ presents similar (though technically less involved) difficulties to those encountered for (4.3).

It follows that in this case there is no need to go to $\bar{w}_2$ as the terms

$$\rho, \quad \bar{w}_1 = -(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho, \quad \tilde{w}_0 = e^{\tau\mathbb{Q}C^e\mathbb{Q}}\overset{\circ}{w},$$

where $\tau = t/\epsilon$ and ρ is the solution to (5.32) with the initial condition $\rho(0) = \overset{\circ}{v}$, suffice to obtain the desired estimates. This follows as the error of the approximation

$$\begin{aligned} y_\epsilon &= v_\epsilon - \rho \\ z_\epsilon &= w_\epsilon - \tilde{w}_0 - \epsilon\bar{w}_1. \end{aligned}$$

formally satisfies

$$\begin{aligned} \partial_t y_\epsilon - \mathbb{P}S\mathbb{Q}z_\epsilon - \mathbb{P}C^i\mathbb{P}y_\epsilon &= \epsilon\mathbb{P}S\mathbb{Q}\bar{w}_1 + \mathbb{P}S\mathbb{Q}\tilde{w}_0 \\ \partial_t z_\epsilon - \mathbb{Q}S\mathbb{P}y_\epsilon - \mathbb{Q}S\mathbb{Q}z_\epsilon - \mathbb{Q}C^i\mathbb{Q}z_\epsilon - \frac{1}{\epsilon}\mathbb{Q}C^e\mathbb{Q}z &= \epsilon\mathbb{Q}S\mathbb{Q}\bar{w}_1 + \mathbb{Q}S\mathbb{Q}\tilde{w}_0 \\ &\quad +\epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_1 + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0 - \epsilon\partial_t\bar{w}_1, \end{aligned} \tag{5.33}$$

and the initial conditions

$$\begin{aligned} y_\epsilon(0) &= 0 \\ z_\epsilon(0) &= -\epsilon\bar{w}_1(0) = \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v}. \end{aligned}$$

The conditions under which the error is of order of ϵ are given in the following theorem, which was originally proved in [5].

Theorem 5.5 *Assume that $\mathbb{P}\overset{\circ}{f} \in \mathcal{X}_{221,C^i}$ and $\mathbb{Q}\overset{\circ}{f} \in \mathcal{X}_{111,C^i}$. Let f_ϵ be the solution of (5.31) with the initial datum $\overset{\circ}{f}$, and ρ be the solution to (4.5) with the initial value $\mathbb{P}\overset{\circ}{f}$. Then for each interval $[0,t_0]$, $0 < t_0 < +\infty$, there exists a constant K depending only on the initial data, the coefficients of the equation, and t_0, such that*

$$\|f_\epsilon(t) - \rho(t) - e^{-\lambda t/\epsilon}\mathbb{Q}\overset{\circ}{f}\|_{\mathcal{X}} \leq K\epsilon \tag{5.34}$$

uniformly on $[0,t_0]$.

5.3.4 Continuity equation as the hydrodynamic limit

The last case of the limit evolution in the hydrodynamic space $N(C^e)$ is given by the scaling

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \epsilon C^i f_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon, \tag{5.35}$$

which produces formally the trivial hydrodynamic limit

$$\partial_t \rho = 0.$$

For the sake of completeness we note that the standard asymptotic procedure (with the initial layer time $\tau = t/\epsilon$) gives the same terms of the expansion as in the case of purely kinetic hydrodynamic limit. The error equation takes form

$$\begin{aligned}
\partial_t y_\epsilon - \mathbb{P}S\mathbb{Q}z_\epsilon - \epsilon\mathbb{P}C^i\mathbb{P}y_\epsilon &= \epsilon\mathbb{P}S\mathbb{Q}\bar{w}_1 + \mathbb{P}S\mathbb{Q}\tilde{w}_0 + \epsilon\mathbb{P}C^i\mathbb{P}\rho, \\
\partial_t z_\epsilon - \mathbb{Q}S\mathbb{P}y_\epsilon - \mathbb{Q}S\mathbb{Q}z_\epsilon - \epsilon\mathbb{Q}C^i\mathbb{Q}z_\epsilon - \frac{1}{\epsilon}\mathbb{Q}C^e\mathbb{Q}z &= \epsilon\mathbb{Q}S\mathbb{Q}\bar{w}_2 + \epsilon^2\mathbb{Q}C^i\mathbb{Q}\bar{w}_1 + \mathbb{Q}S\mathbb{Q}\tilde{w}_0 \\
&\quad + \epsilon\mathbb{Q}C^i\mathbb{Q}\tilde{w}_0 - \epsilon\partial_t\bar{w}_1, \qquad (5.36)
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
y_\epsilon(0) &= 0 \\
z_\epsilon(0) &= \epsilon(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\overset{\circ}{v}
\end{aligned}$$

and we see that the only difference with (5.33) (apart from possibly higher powers of ϵ in some places) is the presence of the term $\epsilon\mathbb{P}C^i\mathbb{P}\rho$ in the first equation of (5.36). However, the solution to the limit equation is constant in time

$$\rho(t) = \rho(0) = \overset{\circ}{v}, \qquad (5.37)$$

so that all the regularity requirements for ρ will be satisfied provided they are imposed on the initial value $\overset{\circ}{v}$.

Thus we can state the theorem

Theorem 5.6 *Assume that $\mathbb{P}\overset{\circ}{f} \in \mathcal{X}_{221,C^i}$ and $\mathbb{Q}\overset{\circ}{f} \in \mathcal{X}_{111,C^i}$. Let f_ϵ be the solution of (5.35) with the initial datum $\overset{\circ}{f}$. Then for each interval $[0, t_0]$, $0 < t_0 < +\infty$, there exists a constant K depending only on the initial data, the coefficients of the equation and t_0 such that*

$$\|f_\epsilon(t) - \mathbb{P}\overset{\circ}{f} - e^{-\lambda t/\epsilon}\mathbb{Q}\overset{\circ}{f}\|_{\mathcal{X}} \leq K\epsilon \qquad (5.38)$$

uniformly on $[0, t_0]$.

5.4 Dominant inelastic scattering

In this subsection we shall discuss the hydrodynamic limits in the cases when either inelastic, or both inelastic and elastic scattering dominate. A simplified version of these models (without C^e) was thoroughly investigated in [3, 9], and from the numerical point of view in [7].

It follows that taking C^e into account does not change the overall picture – the differences are rather cosmetic as was already noted in Section 3. Thus in this section we shall present formal considerations to show that the presence of C^e does not necessitate any new assumptions. Recall also that by Corollary 3.1 the null-spaces and the projection operators P and Q coincide in all the cases discussed in this section.

5.4.1 The case of C^i and C^e being of the same magnitude

First we shall consider the Boltzmann equation with the scaling

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon} S f_\epsilon + \frac{1}{\epsilon^2} C^e f_\epsilon + \frac{1}{\epsilon^2} C^i f_\epsilon, \tag{5.39}$$

which as we know has the hydrodynamic limit

$$\partial_t \rho = -PSQ(Q(C^i + C^e)Q)^{-1} QSP\rho.$$

Let us first note that the question of the invertibility of $Q(C^i + C^e)Q$ is much more complicated than in the cases with dominant elastic scattering. The case with $C^e = 0$ was investigated in [3, 9]. Here we shall show that the addition of C^e, as defined in (5.1), introduces only insignificant changes.

To solve the equation $(C^i + C^e)f = g$ we use the component notation as in Appendix D (with the single change that $p_i(\xi) = \frac{\xi_{i+1}}{\xi_i}\nu_{i+1}(\xi), \nu_i(\xi) = \nu(\xi + i)$) to obtain

$$\begin{aligned}
g_0(\boldsymbol{x}, \xi, \boldsymbol{\omega}) &= -4\pi(\lambda_0(\xi) + n_2 p_0(\xi)) f_0(\boldsymbol{x}, \xi, \boldsymbol{\omega}) + n_1 p_0 \int_{S^2} f_1(\boldsymbol{x}, \xi, \boldsymbol{\omega}') d\boldsymbol{\omega}' \\
&\quad + \lambda_0(\xi) \int_{S^2} f_0(\boldsymbol{x}, \xi, \boldsymbol{\omega}') d\boldsymbol{\omega}' \\
\vdots\ & \vdots\ \vdots \\
g_n(\boldsymbol{x}, \xi, \boldsymbol{\omega}) &= -4\pi(\lambda_n + n_2 p_n(\xi) + n_1 \nu_n(\xi)) f_n(\boldsymbol{x}, \xi, \boldsymbol{\omega}) \\
&\quad + n_1 p_n(\xi) \int_{S^2} f_{n+1}(\boldsymbol{x}, \xi, \boldsymbol{\omega}') d\boldsymbol{\omega}' + n_2 \nu_n(\xi) \int_{S^2} f_{n-1}(\boldsymbol{x}, \xi, \boldsymbol{\omega}') d\boldsymbol{\omega}' \\
&\quad + \lambda_n(\xi) \int_{S^2} f_n(\boldsymbol{x}, \xi, \boldsymbol{\omega}') d\boldsymbol{\omega}' \\
\vdots\ & \vdots\ \vdots,
\end{aligned} \tag{5.40}$$

where $n > 0$. If we decompose $g_n = g_{n0} + g_{n1}$, where

$$g_{n0}(\boldsymbol{x}, \xi) = (\mathbb{P} g_n)(\boldsymbol{x}, \xi) = \frac{1}{4\pi} \int_{S^2} g_n(\boldsymbol{x}, \xi, \boldsymbol{\omega}') d\boldsymbol{\omega}',$$

and $g_{n1} = g_n - g_{n0}$, and we introduce the similar notation for the unknown function f, we see that due to (5.1) we must have

$$\begin{aligned}
g_{00}(\boldsymbol{x}, \xi) &= -4\pi n_2 p_0(\xi) f_{00}(\boldsymbol{x}, \xi) + 4\pi n_1 p_0(\xi) f_{10}(\boldsymbol{x}, \xi) \\
\vdots\ & \vdots\ \vdots \\
g_{n0}(\boldsymbol{x}, \xi) &= -4\pi(n_2 p_n(\xi) + n_1) f_{n0}(\boldsymbol{x}, \xi) + 4\pi n_1 p_n(\xi) f_{n+1,0}(\boldsymbol{x}, \xi) + 4\pi n_2 f_{n-1,0}(\boldsymbol{x}, \xi) \\
\vdots\ & \vdots\ \vdots
\end{aligned} \tag{5.41}$$

and in the same time

$$\begin{array}{rcl} g_{01}(\boldsymbol{x},\xi,\boldsymbol{\omega}) & = & -4\pi(\lambda_0(\xi)+n_2p_0(\xi)f_{01}(\boldsymbol{x},\xi,\boldsymbol{\omega}) \\ \vdots & \vdots & \vdots \\ g_{n1}(\boldsymbol{x},\xi,\boldsymbol{\omega}) & = & -4\pi(\lambda_n(\xi)+n_2p_n(\xi)+n_1\nu_n(\xi))f_{n1}(\boldsymbol{x},\xi,\boldsymbol{\omega}) \\ \vdots & \vdots & \ddots \end{array} \tag{5.42}$$

Thus it is clear that the presence of the elastic scattering operator C^e affects only the easy diagonal part of the equation, wheres the troublesome part is the same as in the case $C^e = 0$ and we can repeat the proof of Theorem 2.1 of [3].

Denoting by $\mathcal{X}^0$ the subspace of $\mathcal{X}$ of functions satisfying the compatibility condition of Theorem 3.3:

$$\mathcal{X}^0 = \{f \in L_1(\mathbb{R}^6_{x,v});\ \sum_{j=0}^{\infty}\sqrt{\xi+j}\int\limits_{S^2} f(\boldsymbol{x},\sqrt{\xi+j},\boldsymbol{\omega})d\boldsymbol{\omega} = 0 \text{ for a.a. } \xi\in[0,1], \boldsymbol{x}\in\mathbb{R}^3_x\},$$

and decomposing as above $g = g_0 + g_1$, where $g_0 = \mathbb{P}g$ and $g_1 = g - g_0$, we obtain the following theorem.

Theorem 5.7 *Let $k \geq 0$. If $g \in \mathcal{X}^0$ is such that $g_0 \in \mathcal{X}_{k+2}$, then there exists a unique solution $f \in D_k(C)\cap\mathcal{X}^0$ to the equation $(C^i + C^e)f = g$, and there exists a constant M such that for any such g*

$$\|f\|_{\mathcal{X}_{0k1,C^i}} \leq M(\|g_0\|_{\mathcal{X}_{k+2}} + \|g_1\|_{\mathcal{X}_k}). \tag{5.43}$$

■

Remark 5.1 It is worthwhile to note that Theorem 5.7 gives only sufficient conditions for the solvability of $(C^i + C^e)f = g$ and the condition (5.43) is not necessary as can be checked by considering the sequence $(f_{n0})_{n\geq 1} = ((-1)^n\xi_n^{-1}n^{-2})_{n\geq 1}$ with f_{00} defined so that the compatibility condition is satisfied. It is straightforward to check that $g = C^i f$ does not satisfy (5.43).

On the other hand it can be proved that if g is of constant sign at least for large velocities, then (5.43) gives also the necessary condition for it to belong to the range of C, hence $C(D(C)\cap\mathcal{X}^0) \neq \mathcal{X}^0$. The details of these considerations can be found in [9]. ■

Denote

$$\mathcal{W} = \{f \in \mathcal{X};\ \int\limits_{S^2} f(\boldsymbol{x},\xi,\boldsymbol{\omega})d\boldsymbol{\omega} = 0 \text{ for a.a } \boldsymbol{x}\in\mathbb{R}^3_x, \xi\in[0,\infty[\}.$$

Let us observe that $\mathcal{W} \subset \mathcal{X}^0$, that is, any function annihilating constants on S^2 automatically satisfies the compatibility condition (3.16). This makes $\mathcal{W}$ a very useful space as follows from the following corollary to Theorem 5.7.

Corollary 5.1 *The operator $C^i + C^e$ is continuously invertible on $\mathcal{W}$ with the inverse given by*

$$\begin{array}{rcl} f_0 & = & -(\lambda_0 + n_2 p_0)^{-1} g_0 \\ \vdots & \vdots & \vdots \\ f_n & = & -(\lambda_n + n_2 p_n + n_1 \nu_n)^{-1} g_n \\ \vdots & \vdots & \vdots. \end{array} \tag{5.44}$$

Moreover, $C^i + C^e$ is coercive in $\mathcal{W}$ and generates there a semigroup of negative type $-\gamma = -(\lambda_{min} + n_2 \nu_{min})$.

Proof. The first statement follows immediately from (5.42). The proof of the second statement is similar to the proof of Proposition 4.2 of [3].■

This corollary allows the construction of the diffusion operator. Before doing this we should however note that the structure of the diffusion operator here is completely different than of that for the dominant elastic scattering. In the latter case the diffusion reflected the feature of the elastic scattering that the particles could be only deflected without changing their velocity, thus the kinetic energy $\xi \in [0, \infty[$ entered the diffusion only as a parameter labeling particles coming from classes of different energy. Here the inelastic collisions could change the energy of particles, however, for a given particle this change could only occur between countably many states. In other words, the particles are divided into non-overlapping classes with energies taking values $\xi + \mathbb{N}$ where $\xi \in [0, 1[$, and these classes remain separate throughout the evolution. As we shall see these feature is preserved in the diffusion operator where the energy parameter is free only in the interval $[0, 1[$ and for higher energies the values of the solution are reproduced in a periodical manner.

Let us recall that by Theorem 3.3 the spectral projection onto $N(C^i + C^e)$ is given by

$$(Pf)_n(\boldsymbol{x}, \xi) = a^{-n} H^{-1} \sum_{j=0}^{\infty} \xi_j (\mathbb{P}f)(\boldsymbol{x}, \xi), \tag{5.45}$$

where $a = n_1/n_2 > 1$ and $H(\xi) = \sum\limits_{j=0}^{\infty} a^{-j} \xi_j$. Here $\xi \in [0, 1[$ and $\boldsymbol{x} \in \mathbb{R}^3_x$. It can be proved, [9], that

$$\varrho(f) = \sum_{j=0}^{\infty} \xi_j \mathbb{P}f$$

is the proper hydrodynamical quantity, that is, it remains constant throughout the evolution. Thus, our limiting diffusion equation will determine the approximation of ϱ. This will be defined only for $\xi \in [0, 1[$ and the extension to the approximation valid for all ξ, according to (5.45), is given by $\rho = (\rho_n)_{n \geq 0}$ where

$$\rho_n = a^{-n} H^{-1} \varrho. \tag{5.46}$$

To find ϱ we have the following result which is a straightforward generalization of Proposition 3.1 of [3].

Proposition 5.1 *The diffusion equation (4.11) is of the form*

$$\partial_t \varrho = \frac{B(\xi)}{3H(\xi)}\Delta_x \varrho, \tag{5.47}$$

where

$$B(\xi) = \frac{\xi_0^4}{\lambda_0(\xi)\xi_0 + n_2\xi_1\nu_1(\xi)} + \sum_{j=1}^{\infty} \frac{a^{-j}\xi_j^4}{\lambda_j(\xi)\xi_j + n_2\xi_{j+1}\nu_{j+1}(\xi) + n_1\xi_j\nu_j(\xi)}.$$

■

The existence of the initial layer is also not automatic as the operator $Q(C^i + C^e)Q$ is not continuously invertible on $Q\mathcal{X}$. Therefore even if $Q(C^i + C^e)Q$ generates a semigroup, this semigroup cannot be of negative type and we will not have the exponentially decaying initial layer. To circumvent this difficulty we use Corollary 5.1 and restrict the initial values to those satisfying

$$\overset{\circ}{f} = \overset{\circ}{v} + \overset{\circ}{w} = P\overset{\circ}{f} + \mathbb{Q}\overset{\circ}{f}, \tag{5.48}$$

that is we assume that $\overset{\circ}{w} \in \mathcal{W}$. It follows from (5.45) that $P\mathbb{Q} = 0$, hence for the initial values satisfying (5.48) we have $Q\overset{\circ}{f} = \mathbb{Q}\overset{\circ}{f}$.

Let us write down formal expressions for the remaining terms of the asymptotic expansion and for the error. The projected system takes the form

$$\begin{aligned} \partial_t v_\epsilon &= \frac{1}{\epsilon} PSQw_\epsilon, \\ \partial_t w_\epsilon &= \frac{1}{\epsilon} QSQw_\epsilon + \frac{1}{\epsilon} QSPv_\epsilon + \frac{1}{\epsilon^2} Q(C^i + C^e)Qw_\epsilon, \\ u(0) &= \overset{\circ}{u} := P\overset{\circ}{f}, \quad w(0) = \overset{\circ}{w} := Q\overset{\circ}{f}, \end{aligned} \tag{5.49}$$

where we have taken into account that $PC = CP = 0$, and that $PSP = 0$, which follows from (5.45) (see also [3]). As before we obtain that for the bulk part $\bar{w}_0 = 0$ and

$$Q(C^i + C^e)Q\bar{w}_1 = -QSP\rho, \qquad Q(C^i + C^e)Q\bar{w}_2 = -QSQ\bar{w}_1. \tag{5.50}$$

where ρ is defined by (5.46) and (5.47). Fortunately, $QSP\rho$ is linear in $\boldsymbol{\omega}$ and thus belongs to $\mathcal{W}$, hence the equation for $\bar{w}_1$ is solvable, defining

$$\bar{w}_1 = -(Q(C^i + C^e)Q)^{-1} QSP\rho.$$

Moreover, due to the presence of the exponential factor in (5.46), $QSQ\bar{w}_1 \in \mathcal{X}^0 \cap \mathcal{X}_k$ for any k (at least for sufficiently smooth in $\boldsymbol{x}$ function ϱ). Thus by Theorem 5.7 the term $\bar{w}_2$ is also well-defined.

The initial layer time is given by $\tau = t/\epsilon^2$ and the standard procedure produces $\tilde{\rho}_0 = 0$ and

$$\partial_\tau \tilde{w}_0 = Q(C^i + C^e)Q\tilde{w}_0. \tag{5.51}$$

Due to the assumption (5.48) we restrict this equation to the subspace $\mathcal{W}$ where, due to Corollary 5.1, we have the generation of a semigroup of negative type.

Finally, we obtain that the proper initial values for (5.47) and (5.51) are given by, respectively

$$\varrho(0) = \overset{\circ}{v} = P\overset{\circ}{f}, \qquad \tilde{w}_0(0) = \overset{\circ}{w} = \mathbb{Q}\overset{\circ}{f}. \tag{5.52}$$

Combining these results we see that the error $(z_\epsilon, y_\epsilon) = (v_\epsilon - \rho, w_\epsilon - \tilde{w}_0 - \epsilon\bar{w}_1 - \epsilon^2\bar{w}_2)$ is (formally) the solution to

$$\begin{aligned} \partial_t y_\epsilon - \frac{1}{\epsilon}PSQz_\epsilon &= \frac{1}{\epsilon}PSQ\tilde{w}_0 + \epsilon PSQ\bar{w}_2 \\ \partial_t z_\epsilon - \frac{1}{\epsilon}QSPy_\epsilon - \frac{1}{\epsilon}QSQz_\epsilon - \frac{1}{\epsilon^2}Q(C^i + C^e)Qz_\epsilon &= \frac{1}{\epsilon}QSQ\tilde{w}_0 + \epsilon QSQ\bar{w}_2 - \epsilon\partial_t\bar{w}_1 - \epsilon^2\partial_t\bar{w}_2, \\ y(0) &= 0, \quad z(0) = -\epsilon\bar{w}_1(0) - \epsilon^2\bar{w}_2(0). \end{aligned} \tag{5.53}$$

The rigorous analysis of this system is no different from that provided in [3]. However, due to a simpler form of the diffusion operator (instead of general second order elliptic we have the Laplacian which commutes with the differentiation with respect to $\boldsymbol{x}$) we can ease the assumptions on the spatial regularity of the initial value. It follows that in this case we have the following theorem.

Theorem 5.8 *Let us assume that* $\overset{\circ}{u} = P\overset{\circ}{f} \in \mathcal{X}_{411,C^i}$ *and* $Q\overset{\circ}{f} \in \mathcal{X}_{111,C^i} \cap \mathcal{W}$. *For any* $t_0 < +\infty$ *there exists a constant* M *(depending on* t_0 *in an affine way) such that*

$$\|f_\epsilon(t) - \rho(t) - \tilde{w}_0(t/\epsilon^2)\|_{\mathcal{X}} \leq \epsilon M. \tag{5.54}$$

5.4.2 Strictly dominant inelastic scattering

Here we shall consider the equation (4.13) with $q = 1$ which presents the most involved case. Thus we have

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon}Sf_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon + \frac{1}{\epsilon^2}C^i f_\epsilon, \tag{5.55}$$

which, as we know, has the hydrodynamic limit

$$\partial_t\rho = -PSQ(QC^iQ)^{-1}QSP\rho. \tag{5.56}$$

It is clear that due to Theorems 3.2, 3.3, 5.7 and Corollary 3.1 the asymptotic analysis in this case will not differ much from that for (5.39).

The hydrodynamic part ρ is again given by (5.46) with the only difference that the diffusion coefficient B in (5.47) should be calculated with $\lambda_n = 0$ for any n.

It follows that $\bar{w}_0, \bar{w}_1, \tilde{w}_0$ are the same and $\bar{w}_2$ is the solution of

$$QSQ\bar{w}_1 + QC^eQ\bar{w}_1 = QC^iQ\bar{w}_2. \tag{5.57}$$

Note that the operation with QC^eQ on $\bar{w}_1$ does not change the properties of $\bar{w}_1$ which are relevant to the solvability of (5.57) (integrability with weight $1 + v^k$) so that the term $\bar{w}_2$ is well-defined as in (5.50).

Defining the approximation as in the previous case we obtain the error equations in the form

$$\begin{aligned}
\partial_t y_\epsilon - \frac{1}{\epsilon} PSQz_\epsilon &= \frac{1}{\epsilon} PSQ\tilde{w}_0 + \epsilon PSQ\bar{w}_2 \\
\partial_t z_\epsilon - \frac{1}{\epsilon} QSPy_\epsilon - \frac{1}{\epsilon} QSQz_\epsilon - \frac{1}{\epsilon} QC^eQz_\epsilon - \frac{1}{\epsilon^2} QC^iQz_\epsilon &= \frac{1}{\epsilon} QSQ\tilde{w}_0 + \frac{1}{\epsilon} QC^eQ\tilde{w}_0 + \epsilon QC^eQ\bar{w}_2 \\
&\quad + \epsilon QSQ\bar{w}_2 - \epsilon\partial_t\bar{w}_1 - \epsilon^2\partial_t\bar{w}_2, \\
y(0) = 0, \qquad & z(0) = -\epsilon\bar{w}_1(0) - \epsilon^2\bar{w}_2(0).
\end{aligned} \tag{5.58}$$

We see that but for the terms $\frac{1}{\epsilon} QC^eQ\tilde{w}_0$ and $\epsilon QC^eQ\bar{w}_2$, Eq. (5.58) is the same as Eq. (5.53). However, since the operator QC^eQ is bounded, the addition of these terms will not affect the error estimates and we see that the main theorem will not change here.

Theorem 5.9 *Let us assume that* $\overset{\circ}{u} = P\overset{\circ}{f} \in \mathcal{X}_{411,C^i}$ *and* $Q\overset{\circ}{f} \in \mathcal{X}_{111,C^i} \cap \mathcal{W}$. *For any* $t_0 < +\infty$ *there exists a constant* M *such that*

$$\|f_\epsilon(t) - \rho(t) - \tilde{w}_0(t/\epsilon^2)\|_{\mathcal{X}} \leq \epsilon M. \tag{5.59}$$

5.4.3 Continuity equation as the hydrodynamic limit

The last case is when we have the scaling

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon^q} C^e f_\epsilon + \frac{1}{\epsilon} C^i f_\epsilon, \tag{5.60}$$

with $q = 1$ or $q = 0$, both giving the hydrodynamic limit

$$\partial_t \rho = 0.$$

Let us consider first the case with $q = 0$. Clearly, then $\rho = \overset{\circ}{\rho} = P\overset{\circ}{f}_\epsilon$ and

$$\bar{w}_1 = -(QC^iQ)^{-1} QSP\overset{\circ}{\rho}.$$

It follows that the proper initial layer time is $\tau = t/\epsilon$; apart from this the initial layer corrector is defined as above, by

$$\partial_\tau \tilde{w}_0 = QC^iQ\tilde{w}_0, \quad \tilde{w}_0(0) = \overset{\circ}{w}. \tag{5.61}$$

Defining the error by $y = v_\epsilon - \rho$, $z = w - \epsilon\bar{w}_1 - \tilde{w}_0$ we obtain the following equation

$$\begin{aligned}
\partial_t y_\epsilon - PSQz_\epsilon &= PSQ\tilde{w}_0 + \epsilon PSQ\bar{w}_1 \qquad (5.62) \\
\partial_t z_\epsilon - QSPy_\epsilon - QSQz_\epsilon - QC^eQz_\epsilon + \frac{1}{\epsilon} QC^iQz_\epsilon &= QSQ\tilde{w}_0 + QC^eQ\tilde{w}_0 + \epsilon QC^eQ\bar{w}_1 \\
&\quad + \epsilon QSQ\bar{w}_1, \\
y(0) &= 0, \quad z(0) = -\epsilon\bar{w}_1(0).
\end{aligned}$$

where we used the fact that $\partial_t \bar{w}_1 = 0$.

If we take $q = 1$, then the only change will be that

$$\bar{w}_1 = -(Q(C^e + C^i)Q)^{-1} QSP \overset{\circ}{\rho},$$

and, with the same τ and the initial value $\tilde{w}_0(0)$,

$$\partial_\tau \tilde{w}_0 = Q(C^e + C^i)Q\tilde{w}_0. \tag{5.63}$$

We know that the addition of C^e doesn't affect relevant properties of C^i, so the above equations are solvable as in the previous case. The only difference will be that in the error system (5.63) we won't have terms involving QC^eQ on the right hand side of the limit equation but this doesn't affect the error estimates. Thus, both cases can be treated as one.

The regularity assumptions can be relaxed even further here. We note that for the estimates we require the existence of only two spatial derivatives of $\overset{\circ}{\rho}$ and the regularity with respect to v is ensured by the exponential factor appearing in the definition of $\overset{\circ}{\rho} = P\overset{\circ}{f}$. All the other assumptions on $P\overset{\circ}{f}$ come from the requirement that the error be the classical solution of (5.63). The assumptions on $Q\overset{\circ}{f}$ are the same as in both previous cases and are dictated by the solvability of (5.61) and (5.63). Thus we can summarize this discussion by stating the theorem.

Theorem 5.10 *Let us assume that* $\overset{\circ}{v} = P\overset{\circ}{f} \in \mathcal{X}_{211,C^i}$ *and* $\overset{\circ}{w} = Q\overset{\circ}{f} \in \mathcal{X}_{111,C^i} \cap \mathcal{W}$*. For any* $t_0 < +\infty$ *there exists a constant* M *such that*

$$\|f_\epsilon(t) - P\overset{\circ}{f} - \tilde{w}_0(t/\epsilon)\|_{\mathcal{X}} \leq \epsilon M. \tag{5.64}$$

A Proof of Theorem 3.1

Proof. For any $g \in L_\infty(\mathbb{R}^3)$ we have

$$\int\limits_{S^2} (gC^e f)(v\boldsymbol{\omega})d\boldsymbol{\omega} = -\int\limits_{S^2}\int\limits_{S^2} \lambda(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\left(g(v\boldsymbol{\omega})(f(v\boldsymbol{\omega}) - f(v\boldsymbol{\omega}'))\right) d\boldsymbol{\omega}d\boldsymbol{\omega}'.$$

Since λ is symmetric in $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$, we have also

$$\begin{aligned}
&\int\limits_{S^2} (gC^e f)(\boldsymbol{\omega})d\boldsymbol{\omega} \\
&= -\int\limits_{S^2}\int\limits_{S^2} \lambda(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')g(v\boldsymbol{\omega}')f(v\boldsymbol{\omega}')d\boldsymbol{\omega}d\boldsymbol{\omega}' + \int\limits_{S^2} \lambda(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')f(v\boldsymbol{\omega})g(v\boldsymbol{\omega}')d\boldsymbol{\omega}d\boldsymbol{\omega}' \\
&= -\int\limits_{S^2}\int\limits_{S^2} \lambda(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\left(g(v\boldsymbol{\omega}')(f(v\boldsymbol{\omega}') - f(v\boldsymbol{\omega}))\right) d\boldsymbol{\omega}d\boldsymbol{\omega}',
\end{aligned}$$

hence

$$2\int\limits_{S^2}(gC^ef)(v\boldsymbol{\omega})d\boldsymbol{\omega} = -\int\limits_{S^2}\int\limits_{S^2}\lambda(v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')(g(v\boldsymbol{\omega})-g(v\boldsymbol{\omega}'))(f(v\boldsymbol{\omega})-f(v\boldsymbol{\omega}'))d\boldsymbol{\omega}d\boldsymbol{\omega}'.$$

Taking any bounded strictly increasing function $\kappa:\mathbb{R}\to\mathbb{R}$ we obtain

$$\begin{aligned}&2\int\limits_{S^2}(\kappa(f)C^ef)(v\boldsymbol{\omega})d\boldsymbol{\omega}\\ &= -\int\limits_{S^2}\int\limits_{S^2}\lambda(v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')(\kappa(f(v\boldsymbol{\omega}))-\kappa(f(v\boldsymbol{\omega}')))(f(v\boldsymbol{\omega})-f(v\boldsymbol{\omega}'))d\boldsymbol{\omega}d\boldsymbol{\omega}'\leq 0. \qquad (A.1)\end{aligned}$$

Integrating over the remaining variables we get the H-theorem.

If, in particular, we take $\kappa(t)=sign(t)$ we obtain

$$\int\limits_{\mathbb{R}^3} sign(f)C^ef d\boldsymbol{v}\leq 0,$$

which gives the dissipativity of C^e. This proves (i).

Let $C^ef=0$; then the left hand side of (A.1) is zero, but due to strict monotonicity of κ this is possible only if

$$f(v\boldsymbol{\omega}) = f(v\boldsymbol{\omega}'), \qquad (A.2)$$

for almost all v, that is, to the kernel of C^e may belong only functions independent of $\boldsymbol{\omega}$. On the other hand, such functions clearly belong to $N(C^e)$, therefore the kernel of C^e is given by (3.7).

Next we turn our attention to the solvability of

$$\alpha f(\boldsymbol{v}) + f(\boldsymbol{v})\int\limits_{S^2}\lambda(v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')d\boldsymbol{\omega}' - \int\limits_{S^2}\lambda(v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')f(v\boldsymbol{\omega}\cdot\boldsymbol{\omega}')d\boldsymbol{\omega}' = g(\boldsymbol{v}) \qquad (A.3)$$

for $g\in L_1(\mathbb{R}^3)$. Denote

$$W=\{f\in L_1(\mathbb{R}^3);\ \int\limits_{S^2} fd\boldsymbol{\omega}=0\}.$$

For $f\in W$ we get

$$\begin{aligned}&\int\limits_{S^2}(sign(f)C^ef(v\boldsymbol{\omega})d\boldsymbol{\omega}\\ &= -\frac{1}{2}\int\limits_{S^2}\int\limits_{S^2}\lambda(v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')(sign(f(v\boldsymbol{\omega}))-sign(f(v\boldsymbol{\omega}')))(f(v\boldsymbol{\omega})-f(v\boldsymbol{\omega}'))d\boldsymbol{\omega}d\boldsymbol{\omega}'\end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{1}{2}\lambda_{min}\left(\int_{S^2}\int_{S^2} sign(f(v\boldsymbol{\omega}))f(v\boldsymbol{\omega})d\boldsymbol{\omega}d\boldsymbol{\omega}' + \int_{S^2}\int_{S^2} sign(f(v\boldsymbol{\omega}'))f(v\boldsymbol{\omega}')d\boldsymbol{\omega}d\boldsymbol{\omega}'\right.\\
&\qquad \left. -\int_{S^2}\int_{S^2} sign(f(v\boldsymbol{\omega}))f(v\boldsymbol{\omega}')d\boldsymbol{\omega}d\boldsymbol{\omega}' - \int_{S^2}\int_{S^2} sign(f(v\boldsymbol{\omega}'))f(v\boldsymbol{\omega})d\boldsymbol{\omega}d\boldsymbol{\omega}'\right)\\
&= -4\pi\lambda_{min}\|f\|_{L_1(S^2)}
\end{aligned}
$$

where, upon integration with respect to v, we obtain that $C^e|_W - \alpha I$ is dissipative for $\alpha > -4\pi\lambda_{min}$. Since $C^e|_W$ is bounded, $C^e|_W - \alpha I$ must be m-dissipative, therefore if $\alpha > -4\pi\lambda_{min}$, then $\alpha \in \rho(C^e|_W)$.

It is also clear that the spectral projection onto $N(C^e)$ is given by (3.9) which ends the proof of (iii)-(iv).

The statement for the L_2 space follows in the same way as all the operation are first performed on the unit sphere and only later integrated with respect to v to get estimates valid on $\mathbb{R}^3$.

■

B Proof of Theorem 3.2

Proof. a) Firstly, we consider the case with bounded C^i in X. This can be done by introducing the regularized ν by

$$
\nu_n(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') = \begin{cases} \nu(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') & \text{for } 1 + n^{-1} < v \\ 0 & \text{for } v < 1 + n^{-1} \end{cases} \tag{B.1}
$$

Following some ideas from [18] (see also [2]) we obtain for $g \in X^* = L_\infty(\mathbb{R}^3)$

$$
\begin{aligned}
\langle g, C_n f\rangle &= -\int_{S^2}\int_{S^2}\int_0^{+\infty} v_+\nu_n(v_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\left(n_1 f(v_+\boldsymbol{\omega}')g(v\boldsymbol{\omega}) + n_2 f(v\boldsymbol{\omega})g(v_+\boldsymbol{\omega}')\right.\\
&\qquad \left.-n_2 g(v\boldsymbol{\omega})f(v\boldsymbol{\omega}) - n_1 g(v_+\boldsymbol{\omega}')f(v_+\boldsymbol{\omega}')\right) v dv d\boldsymbol{\omega}d\boldsymbol{\omega}' \qquad \text{(B.2)}\\
&\quad -\frac{1}{2}\int_0^{\infty}\int_{S^2}\int_{S^2} \lambda(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')(g(v\boldsymbol{\omega}) - g(v\boldsymbol{\omega}'))(f(v\boldsymbol{\omega}) - f(v\boldsymbol{\omega}'))d\boldsymbol{\omega}d\boldsymbol{\omega}'\\
&= -\int_{\mathbb{R}^3}\int_{S^2} \frac{v_+}{v}\nu_n(v_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')(n_1 f(v_+\boldsymbol{\omega}') - n_2 f(v\boldsymbol{\omega}))(g(v_+\boldsymbol{\omega}') - g(v\boldsymbol{\omega}))d\boldsymbol{v}d\boldsymbol{\omega}'\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^3}\int_{S^2} \lambda(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')a^{-v^2}(g(v\boldsymbol{\omega}) - g(v\boldsymbol{\omega}'))(a^{v^2} f(v\boldsymbol{\omega}) - a^{v^2} f(v\boldsymbol{\omega}'))d\boldsymbol{v}d\boldsymbol{\omega}',
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L_1(\mathbb{R}^3)$ and $L_\infty(\mathbb{R}^3)$, and C_n is the collision operator corresponding to the regularized ν_n. In particular, if κ is any bounded nondecreasing function, then we obtain

$$\langle \kappa(a^{v^2}f), C_n f\rangle = -n_2 \int\limits_{\mathbb{R}^3}\int\limits_{S^2} \frac{v_+}{v}\nu_n(v_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')a^{-v^2}\left(\frac{n_1}{n_2}a^{v^2}f(v_+\boldsymbol{\omega}') - a^{v^2}f(v\boldsymbol{\omega})\right)$$

$$\cdot\left(\kappa\left(\frac{n_1}{n_2}a^{v^2}f(v_+\boldsymbol{\omega}')\right) - \kappa(a^{v^2}f(v\boldsymbol{\omega}))\right)d\boldsymbol{v}d\boldsymbol{\omega}' \quad \text{(B.3)}$$

$$-\frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{S^2}\lambda(v,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')a^{-v^2}(\kappa(a^{v^2}f(v\boldsymbol{\omega}) - \kappa(a^{v^2}f(v\boldsymbol{\omega}'))(a^{v^2}f(v\boldsymbol{\omega}) - a^{v^2}f(v\boldsymbol{\omega}'))d\boldsymbol{v}d\boldsymbol{\omega}',$$

hence for any $f \in L_1(\mathbb{R}^3)$ we have the H - theorem

$$\langle \kappa(a^{v^2}f), C_n f\rangle \leq 0. \tag{B.4}$$

If we take as κ any bounded strictly increasing function, then we see that if for some f we have $C_n f = 0$, then

$$\langle \kappa(a^{v^2}f), C_n f\rangle = 0,$$

and since each term in (B.4) is nonpositive, this is only possible when we simultaneously have

$$\begin{aligned} n_1 f(v_+\boldsymbol{\omega}') - n_2 f(v\boldsymbol{\omega}) &= 0\\ f(v\boldsymbol{\omega}') - f(v\boldsymbol{\omega}) &= 0 \end{aligned} \tag{B.5}$$

for all $\boldsymbol{\omega}, \boldsymbol{\omega}' \in S^2$ and $v \in \mathbb{R}_+$. On the other hand, functions satisfying Eq. (B.5) belong to the null-space of C_n, $N(C_n)$, by (B.2), hence it is described fully by (B.5). To characterize these functions explicitly we note that by (A.2), the second equation in (B.5) shows that the elements of the kernel of C_n are independent of the angular variable. Hence, using notation (3.14) for (B.5), by recurrence and Eq. (3.14) we get

$$\phi(\xi + n) = a^{-n}\phi_0(\xi)$$

where $\xi \in [0, 1]$, which gives (3.15), showing that $N(C_n) = N(C_n^i)$.

Returning now to the original unbounded C, we pass to the limit, exactly as it was done in [2], Theorem 2.2, obtaining the same statement.

b) Let us consider now the problem in X^2. Again we start with ν_n defined by (B.1). For arbitrary $g \in X^2$, similar calculations show that by $\nu_n(\cdot, \boldsymbol{\omega}\cdot\boldsymbol{\omega}') = \nu_n(\cdot, \boldsymbol{\omega}'\cdot\boldsymbol{\omega})$ we have

$$(g, C_n f)_{X^2} = (C_n g, f)_{X^2}$$

that is C_n is self-adjoint in X^2. Let us take arbitrary $f \in X^2$; then

$$\begin{aligned}(f, C_n f)_{X^2} &= -\frac{1}{n_2}\int\limits_{\mathbb{R}^3}\int\limits_{S^2}\frac{v_+}{v}\nu_n(v_+, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')\left(n_1 f(v_+\boldsymbol{\omega}') - n_2 f(v\boldsymbol{\omega})\right)^2 a^{v^2}d\boldsymbol{v}d\boldsymbol{\omega}'\\ &\quad -\frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{S^2}\lambda(v, \boldsymbol{\omega}\cdot\boldsymbol{\omega}')(f(v\boldsymbol{\omega}) - f(v\boldsymbol{\omega}'))^2 d\boldsymbol{v}d\boldsymbol{\omega}'\end{aligned} \tag{B.6}$$

Eq. (B.6) shows that, as in X-setting, $C_n f = 0$ if and only if $\phi(\xi) = f(\sqrt{\xi})$ satisfies

$$\phi(\xi + n) = a^{-n}\phi_0(\xi), \tag{B.7}$$

where $\xi \in [0,1]$ and $\phi \in L_2([0,1], \xi d\xi)$. The extension of this result to unbounded ν is performed as in X_1 case. ■

C Proof of Theorem 3.3

Proof.
a) Let $g \in N(C^*)$. Since then $g \in D(C^*)$, in particular it is a bounded function, thus $a^{-v^2}g \in X^2$. Moreover, if χ is the weight function defining $D^2(Q)$, we obtain

$$\int\limits_{\mathbb{R}^3} \chi(\boldsymbol{v})g^2(\boldsymbol{v})a^{-2v^2}a^{v^2}d\boldsymbol{v} = \int\limits_0^{+\infty}\int\limits_{S^2} \chi(v\boldsymbol{\omega})g^2(v\boldsymbol{\omega})a^{-v^2}dvd\boldsymbol{\omega} < +\infty,$$

where the integrability in a neighbourhood of $v = 0$ and $v = 1$ is due to the boundedness of g and integrability of v^{-1} and v_-^{-1} over compact subsets of $\mathbb{R}^3$. Hence $a^{-v^2}g \in D^2(C)$. Since $X^2 \subset X$, if $f \in X^2$ is bounded at $v = 0$, then $f \in D(C)$. Moreover, the set of such functions is dense in X_2. Thus, by self-adjointness of C in X^2 we have

$$0 = \langle C^*g, f\rangle_X = \langle g, Cf\rangle_X = (a^{-v^2}g, Cf)_{X^2} = (C(a^{-v^2}g), f)_{X^2}, \tag{C.8}$$

and from density we see that $g \in D(C^*)$ solves $C^*g = 0$ if and only if $a^{-v^2}g \in N(C)$ in X^2. This shows that g is a 1-periodic function of the variable ξ, independent of the angle.

b) Since $\overline{R(C)} = N(C^*)^{\perp}$, $g \in \overline{R(C)}$ if and only if for any bounded 1-periodic function f of $\xi = v^2$

$$0 = \int\limits_{\mathbb{R}^3} g(\boldsymbol{v})f(\boldsymbol{v})d\boldsymbol{v} = \frac{1}{2}\int\limits_0^1 \phi_0(\xi)\left(\sum_{j=0}^{\infty}\sqrt{\xi + j}\int\limits_{S^2} g(\sqrt{\xi + j}\boldsymbol{\omega})d\boldsymbol{\omega}\right)d\xi.$$

which, since ϕ_0 is arbitrary, gives

$$\sum_{j=0}^{\infty}\sqrt{v^2 + j}\int\limits_{S^2} g(\sqrt{v^2 + j}\boldsymbol{\omega})d\boldsymbol{\omega} = 0. \tag{C.9}$$

c) The projection P onto $N(C)$ along $\overline{R(C)}$ must satisfy $\langle f - Pf, g\rangle_X = 0$ for any $g \in N(C^*) = \overline{R(C)}^{\perp}$, that is

$$\begin{aligned} 0 &= \int\limits_{\mathbb{R}^3}(f - Pf)(\boldsymbol{v})g(\boldsymbol{v})d\boldsymbol{v} \qquad \text{(C.10)} \\ &= \frac{1}{2}\int\limits_0^1 g_0(\xi)\left(\sum_{j=0}^{\infty}\sqrt{\xi + j}\int\limits_{S^2} f(\sqrt{\xi + j}\boldsymbol{\omega})d\boldsymbol{\omega} - 4\pi\psi_0(\xi)\sum_{j=0}^{\infty}a^{-j}\sqrt{\xi + j}\right)d\xi, \end{aligned}$$

and since g_0 is arbitrary we obtain Eqs. (3.18) and (3.19).

From equation (C.9) we obtain that $N(C) \cap \overline{R(C)} = \{0\}$, and since $Pu \in N(C)$ and by (C.11), $(I-P)u \in N(C^*)^{\perp} = \overline{R(C)}$ and we have

$$X_1 = N(C) \oplus \overline{R(C)}. \tag{C.11}$$

■

D The proof of the existence of moments

In this Appendix we prove Theorem 5.2. For greater generality we shall allow some spatial inhomogeneity, that is we consider the Cauchy problem

$$\begin{aligned}
\partial_t \rho &= \xi \partial_{\boldsymbol{x}} \left(d(\boldsymbol{x}, \xi) \partial_{\boldsymbol{x}} \rho \right) - \left(n_2 \sqrt{\frac{\xi+1}{\xi}} m(\xi+1) + n_1 H(\xi-1) m(\xi) \right) \rho \\
&\quad + n_1 \sqrt{\frac{\xi+1}{\xi}} m(\xi+1) \rho(\xi+1) + n_2 H(\xi-1) m(\xi) \rho(\xi-1), \\
\rho(0) &= \mathring{\rho}
\end{aligned} \tag{D.1}$$

in $\mathcal{X}_k$, $k \geq 0$. Using the definition of m we introduce m_{min} and m_{max} by

$$m_{min} = 4\pi \nu_{min} \qquad m_{max} = 4\pi \nu_{max}. \tag{D.2}$$

Let us recall that by [3] for each fixed $\xi > 0$ the operator

$$\mathsf{D}_\xi = \xi \partial_{\boldsymbol{x}} \left(d(\boldsymbol{x}, \xi) \partial_{\boldsymbol{x}} \rho \right)$$

generates a positive semigroup of contractions in X_x with domain $L_{1,2}(\mathbb{R}^3_x)$ (the Bessel potential space, see e.g. [17]). Also for $\xi = 0$, D_ξ with domain X_x generates a (uniformly) continuous semigroup. We denote these semigroups by $(G_{\mathsf{D}_\xi}(t))_{t \geq 0}$.

Let us denote by A the operator of multiplication by an arbitrary non-negative measurable and almost everywhere finite function $a(\xi)$ of $\xi \in \mathbb{R}_+$, and define $\mathsf{D}_A = \mathsf{D} - A$ in $\mathcal{X}_k$. Consider now the Cauchy problem

$$\partial_t \rho = \mathsf{D}_A \rho, \quad \rho(0) = \mathring{\rho} \tag{D.3}$$

in the space $\mathcal{X}_k$.

Lemma D.1 *The operator* D_A *defined on*

$$D_k(\mathsf{D}_A) = D_k(\mathsf{D}) \cap D_k(A).$$

is the generator of a semigroup of contractions in $\mathcal{X}_k$.

Proof. Since A and D commute with each other, and with the weight function $(1+\xi^{\frac{k}{2}})$, D_A generates a semigroup (of contractions) in $\mathcal{X}_k$ such that $D_k(\mathsf{D}) \cap D_k(A)$ is a core of D_A, [13]. Since ξ here is a parameter, we can consider the family of operators $\mathsf{D}_{A,\xi} = \mathsf{D}_\xi - a(\xi)I$ in X_x. Clearly, for any ξ such that $a(\xi)$ is finite, the operator $\mathsf{D}_{A,\xi}$, defined on $D(\mathsf{D}_{A,\xi}) = D(D_\xi)$ generates a semigroup of contractions, say $(G_{\mathsf{D}_{A,\xi}}(t))_{t\geq 0}$, which satisfies

$$\|G_{\mathsf{D}_{A,v}}(t)\rho(\cdot,\xi)\|_{X_x} \leq \exp(-a(\xi)t)\|\rho(\cdot,\xi\|_{X_x}, \tag{D.4}$$

with the equality sign if ρ is positive. We shall identify $D_k(\mathsf{D}_A)$. Since for $s>0$ the resolvent of $\mathsf{D}_{A,\xi}$ is given by

$$(R_\xi(s,\mathsf{D}_{A,\xi})f)(\cdot,\xi) = \int_0^\infty e^{-st}(G_{\mathsf{D}_{A,\xi}}(t)f)(\cdot,\xi)dt,$$

we obtain by (D.4)

$$\|(R_\xi(s,\mathsf{D}_{A,\xi})f)(\cdot,\xi)\|_{X_x} \leq \int_0^\infty e^{-(s+a(\xi))t}\|f(\cdot,\xi)\|_{X_x}dt \leq \frac{1}{s+a(\xi)}\|f(\cdot,\xi)\|_{X_x}. \tag{D.5}$$

Since $a \geq 0$ almost everywhere, and by (D.5), we obtain for any $f \in \mathcal{X}_k$ and almost any ξ

$$\|a(\xi)(R_\xi(s,\mathsf{D}_{A,\xi})f)(\cdot,\xi)\|_{X_x} \leq \frac{a(\xi)}{s+a(\xi)}\|f(\cdot,\xi)\|_{X_x} \leq \|f(\cdot,\xi)\|_{X_x}, \tag{D.6}$$

hence

$$\|AR(s,\mathsf{D}_A)f\|_{\mathcal{X}_k} \leq \|f\|_{\mathcal{X}_k} \tag{D.7}$$

and $D_k(\mathsf{D}_A) \subset D_k(A)$.

Using the fact that for a fixed ξ the multiplication by $a(\xi)$ is bounded, we obtain the following relation for resolvents in X_x:

$$R_\xi(s,\mathsf{D}_{A,\xi}) = R_\xi(s,\mathsf{D}_\xi)[I - a(\xi)R_\xi(s,\mathsf{D}_{A,\xi})].$$

Since the operator $AR(s,\mathsf{D}_A)$ is bounded in $\mathcal{X}_k$ by Eq. (D.7), extending the above equation to $\mathcal{X}_k$ shows that $D_k(\mathsf{D}_A) \subset D_k(\mathsf{D})$ and consequently $D_k(\mathsf{D}_A) \subset D_k(\mathsf{D}) \cap D_k(A)$ which yields

$$D_k(\mathsf{D}_A) = D_k(\mathsf{D}) \cap D(A.) \tag{D.8}$$

■

Since it will not create any misunderstanding, in this appendix we shall denote $\mathbb{P}C^i\mathbb{P}$ by C^i; thus also $\mathbb{P}C^i\mathbb{P}$ acting in $\mathcal{X}$ will be denoted by C^i.

Let us consider the operator

$$\begin{aligned}((\mathsf{D}+C^i)f)(\boldsymbol{x},\xi) \;=\;& \xi\partial_{\boldsymbol{x}}\left(d(\boldsymbol{x},\xi)\partial_{\boldsymbol{x}}f(\boldsymbol{x},\xi)\right)\\ &-\left(n_2\sqrt{\frac{\xi+1}{\xi}}m(\xi+1)+n_1H(\xi-1)m(\xi)\right)f(\boldsymbol{x},\xi)\\ &+n_1\sqrt{\frac{\xi+1}{\xi}}m(\xi+1)f(\boldsymbol{x},\xi+1)+n_2H(\xi-1)m(\xi)f(\boldsymbol{x},\xi-1).\end{aligned}$$

Let us recall the definitions (3.12) and (3.13) of the parts of the operator C^i specified to the present case.

$$\begin{aligned}
(Nf)(\boldsymbol{x},\xi) &= \left(n_2\sqrt{\frac{\xi+1}{\xi}}m(\xi+1)+n_1H(\xi-1)m(\xi)\right)f(\boldsymbol{x},\xi),\\
(K_+f)(\boldsymbol{x},\xi) &= n_2H(\xi-1)m(\xi)f(\boldsymbol{x},\xi-1),\\
(K_-f)(\boldsymbol{x},\xi) &= n_1\sqrt{\frac{\xi+1}{\xi}}m(\xi+1)f(\xi+1),
\end{aligned}$$

so that in $\mathcal{X}$ we have the decomposition

$$\mathsf{D}+C^i=\mathsf{D}-N+K_++K_-.$$

Due to the translational character of the operator C^i it is convenient to introduce the reduced energy $\xi\in[0,1[$ and to re-define all the functions as functional sequences in the following way: for $n=0,1,2\ldots$ and fixed $\xi\in[0,1[$

$$\begin{aligned}
\rho_n(\boldsymbol{x},\xi) &= \rho(\boldsymbol{x},\xi+n),\\
d_n(\xi) &= d(\xi+n),\\
m_n(\xi) &= m(\xi+n),\\
\xi_n &= \sqrt{\xi+n}\\
p_n(\xi) &= \frac{\xi_{n+1}}{\xi_n}m_{n+1}(\xi);
\end{aligned}$$

similarly we define $\mathsf{D}_n=\xi_n^2\partial_{\boldsymbol{x}}\left(d_n(\boldsymbol{x},\xi)\partial_{\boldsymbol{x}}\right)$ and $\mathsf{D}_{\xi,n}$ will denote this operator acting in X_x (with fixed ξ).

It follows that using this notation we can introduce the following equivalent norm in $\mathcal{X}_k$

$$\|f\|_k=\int_0^1\xi_0\|f_0(\cdot,\xi)\|_{X_x}d\xi+\sum_{j=1}^{\infty}j^{\frac{k}{2}}\int_0^1\xi_j\|f_j(\cdot,\xi)\|_{X_x}d\xi<\infty. \tag{D.9}$$

and the domain $D_k(C^i)$ can be identified by the finiteness of the following expression

$$\int_0^1\|f_0(\cdot,\xi)\|_{X_x}d\xi+\sum_{j=1}^{\infty}j^{\frac{k}{2}}\int_0^1\xi_j\|f_j(\xi,\boldsymbol{\omega})\|_{X_x}d\xi<\infty. \tag{D.10}$$

It will simplify the notation if we introduce, for any $\xi\in[0,1[$ and $f=(f_j)_{j\in\mathbb{N}}$, the functional

$$\|f\|_{X_x,k}=\xi_0\|f_0\|_{X_x}+\sum_{j=1}^{\infty}j^{\frac{k}{2}}\xi_j\|f_j\|_{X_x}. \tag{D.11}$$

Thus, the norm in $\mathcal{X}_k$ can be expressed as

$$\|f\|_k=\int_0^1\|f\|_{X_x,k}d\xi. \tag{D.12}$$

With this notation the equation (D.1) can be written in the recursive form

$$\begin{array}{rcl}\partial_t\rho_0 & = & \mathsf{D}_0\rho_0 - n_2p_0\rho_0 + n_1p_0\rho_1,\\ \vdots & & \vdots\\ \partial_t\rho_n & = & \mathsf{D}_n\rho_n - (n_2p_n + n_1m_n)\rho_n + n_1p_n\rho_{n+1} + n_2m_n\rho_{n-1}\\ \vdots & & \vdots,\end{array} \tag{D.13}$$

where $n = 1, 2, \ldots$.

First we shall prove the following lemma.

Lemma D.2 *The resolvent of $R(s, \mathsf{D} - N + K_+)$ exists and is a positive operator in $\mathcal{X}_k$ for any $s > 0, k \geq 0$.*

Proof. To find $R(s, \mathsf{D}-N+K_+)$ we shall use the representation (D.13) to obtain the system of equations

$$\begin{array}{rcl}g_0 & = & (s + n_2p_0 - \mathsf{D}_0)\rho_0\\ \vdots & & \vdots\\ g_n & = & (s + n_1m_n + n_2p_n - \mathsf{D}_n)\rho_n - n_2m_n\rho_{n-1}\\ \vdots & & \vdots,\end{array} \tag{D.14}$$

for $n = 1, 2, \ldots$.

Let us denote $R_{\xi,0}(s) = (s + n_2p_0 - \mathsf{D}_{\xi,0})^{-1}$ and $R_{\xi,n}(s) = (s + n_1m_n + n_2p_n - \mathsf{D}_{\xi,n})^{-1}$ for $n \geq 1$ in X_x. Note that all the assertions of Lemma D.1 are valid also for $R_{\xi,n}(s)$, as they were independent of the coefficient d (as long as it remains positive). Also, these resolvent operators commute with any function of $\xi \in [0, 1[$.

We claim that for any fixed $\xi \in [0, 1[$, $n = 0, 1\ldots$ and $s > 0$

$$\rho_n = \frac{1}{\xi_n}R_{\xi,n}(s)\left(\sum_{i=0}^{n}\xi_i\left(\prod_{l=0}^{n-i-1}n_2p_{n-l-1}R_{\xi,n-l-1}(s)\right)g_i\right) \tag{D.15}$$

with the convention that the product denotes the composition of operators and, if $n-i-1 < 0$, then it is equal to the identity operator I on X_x. In fact, for $n = 0$ the formula (D.15) gives

$$\rho_0 = \frac{1}{\xi_0}R_{\xi,0}(s)\left(\sum_{i=0}^{0}\xi_i\left(\prod_{l=0}^{-i-1}n_2p_{n-l-1}R_{\xi,n-l-1}(s)\right)g_i\right) = R_{\xi,0}(s)g_0$$

which agrees with the direct solution to the first equation of (D.13). The basic recurrence formula which will enable us to prove (D.15) is

$$\frac{\xi_n}{\xi_{n+1}}p_n = m_{n+1}, \qquad n = 0, 1\ldots. \tag{D.16}$$

Assuming that (D.15) is correct for $n-1$, we find from (D.13) that

$$\begin{aligned}\rho_n &= R_{\xi,n}(s)(g_{n0}+n_2 m_n \rho_{n-1})\\ &= \frac{1}{\xi_n}R_{\xi,n}(s)\left(\xi_n g_n + n_2 p_{n-1}R_{\xi,n-1}(s)\left(\sum_{i=0}^{n-1}\xi_i\left(\prod_{l=0}^{n-i-2} n_2 p_{n-l-2}R_{\xi,n-l-2}(s)\right)g_i\right)\right)\\ &= \frac{1}{\xi_n}R_{\xi,n}(s)\left(\xi_n g_n + \sum_{i=0}^{n-1}\xi_i\left(\prod_{l=0}^{n-i-1} n_2 p_{n-l-1}R_{\xi,n-l-1}(s)\right)g_i\right)\\ &= \frac{1}{\xi_n}R_{\xi,n}(s)\left(\sum_{i=0}^{n}\xi_i\left(\prod_{l=0}^{n-i-1} n_2 p_{n-l-1}R_{\xi,n-l-1}(s)\right)g_i\right)\end{aligned}$$

By (D.5) we obtain for almost any $\xi \in [0,1[$

$$\begin{aligned}|||R_{\xi,0}(s)||| &\leq \frac{1}{n_2 p_0(\xi)+s},\\ |||R_{\xi,i}(s)||| &\leq \frac{1}{n_1 m_i(\xi)+n_2 p_i(\xi)+s}, \quad i\geq 1,\end{aligned} \tag{D.17}$$

where $|||\cdot|||$ here and further on will denote the operator norm in the relevant space. By (D.16) we have $m_{min} \leq p_0 \leq \infty$ and $m_{min} \leq p_i \leq \sqrt{2}m_{max}$ uniformly for $\xi \in [0,1[$ and $i \geq 1$. Since the function $\mathbb{R}_+ \ni r \to r/(r+s)$ is increasing for any $s>0$, we obtain

$$\begin{aligned}|||n_2 p_0 R_{\xi,0}(s)||| &\leq \frac{n_2 p_0(\xi)}{n_2 p_0(\xi)+s} \leq 1\\ \vdots\quad &\quad \vdots\\ |||n_2 p_i R_{\xi,i}(s)||| &\leq \frac{n_2 p_i(\xi)}{n_2 p_i(\xi)+n_1 m_i(\xi)+s} \leq \frac{\sqrt{2}n_2 m_{max}}{\sqrt{2}n_2 m_{max}+n_1 m_{min}+s}\\ \vdots\quad &\quad \ddots\end{aligned}$$

For $s>0$ we denote $\beta = \frac{\sqrt{2}n_2 m_{max}}{\sqrt{2}n_2 m_{max}+n_1 m_{min}+s} < 1$. Thus

$$|||\prod_{l=0}^{n-i-1} n_2 p_{n-l-1}R_{\xi,n-l-1}(s)||| \leq \beta^{n-i} \tag{D.18}$$

for $i=0,\ldots,n-1$, and for $i=n$, $|||\prod_{l=0}^{-1} n_2 p_{n-l-1}R_{\xi,n-l-1}(s)||| = |||I||| = 1$ by definition. Hence, using the fact that the resolvents $R_{\xi,n}(s)$ are positive, we obtain

$$\xi_n|\rho_n| \leq R_{\xi,n}(s)\left(\sum_{i=0}^{n-1}\xi_i\prod_{l=0}^{n-i-1} n_2 p_{n-l-1}R_{\xi,n-l-1}(s)|g_i| + \xi_n|g_n|\right)$$

which, upon integration with respect to $\boldsymbol{x}$, gives the X_x norm of the solution

$$\xi_n||\rho_n||_{X_x} \leq \left\|R_{\xi,n}(s)\left(\sum_{i=0}^{n-1}\xi_i\prod_{l=0}^{n-i-1} n_2 p_{n-l-1}R_{\xi,n-l-1}(s)|g_i| + \xi_n|g_n|\right)\right\|$$

$$\leq \frac{1}{s}\sum_{i=0}^{n-1}\xi_i\|g_i\|_{X_x}\beta^{n-i}+\xi_n\|g_n\|_{X_x}. \tag{D.19}$$

By (D.9) we have to prove that the series

$$\xi_0\|\rho_0\|_{X_x}+\sum_{n=1}^{\infty}n^{k/2}\xi_n\|\rho_n\|_{X_x} \tag{D.20}$$

is integrable over $[0,1]$. To do this, it is enough to prove the statement for any remainder of the series. Let us fix $n_0>1$ (which will be determined later) and consider

$$\begin{aligned}\sum_{n=n_0}^{\infty}n^{k/2}\xi_n\|\rho_n\|_{X_x} &\leq \frac{1}{s}\sum_{n=n_0}^{\infty}\left(\sum_{i=0}^{n-1}n^{k/2}\xi_i\|g_i\|_{X_x}\beta^{n-i}+n^{k/2}\xi_n\|g_n\|_{X_x}\right)\\ &= \frac{1}{s}\left(\sum_{i=0}^{\infty}\xi_i\|g_i\|_{X_x}\beta^{-i}\left(\sum_{n=n_0+i}^{\infty}n^{k/2}\beta^n\right)+\sum_{i=n_0}^{\infty}i^{k/2}\xi_i\|g_i\|_{X_x}\right),\end{aligned}$$

where the change of the order of summation is justified by the positivity of the terms.

Consider now the function $r\to r^{k/2}\beta^r$. This function is monotonically decreasing for $r>r_0=-k(2\ln\beta)^{-1}$. Taking $n_0\geq\max\{r_0,1-(\ln\beta)^{-1}\}$ we have the estimate

$$\sum_{n=n_0+i}^{\infty}n^{k/2}\beta^n\leq\int_{n_0+i-1}^{\infty}r^{k/2}\beta^r dr.$$

To shorten notation, let $\gamma=\beta^{-1}$. Then

$$\int_{n_0+i-1}^{\infty}r^{k/2}\beta^r dr=\int_{n_0+i-1}^{\infty}r^{k/2}\gamma^{-r}dr=(\ln\gamma)^{-(1+k/2)}\int_{\ln\gamma(n_0+i-1)}^{\infty}z^{k/2}e^{-z}dz$$

and the last integral is the well-known incomplete Γ function. From [1], formula 6.5.32, we infer that for some M_k which is independent of the lower limit of integration $z\geq 1$ (but dependent on k) we have

$$\int_{z}^{\infty}z^{k/2}e^{-z}dz\leq M_k z^{k/2}e^{-z}. \tag{D.21}$$

Because of the definition of n_0, for any γ and $i\geq 0$ the lower limit of integration satisfies $(n_0+i-1)\ln\gamma\geq 1$, thus we can use the estimate (D.21) to obtain

$$\begin{aligned}\sum_{n=n_0+i}^{\infty}n^{k/2}\beta^n &\leq (\ln\gamma)^{-(1+k/2)}\int_{\ln\gamma(n_0+i-1)}^{\infty}z^{k/2}e^{-z}dz\leq M_k(\ln\gamma)^{-1}(n_0+i-1)^{k/2}\beta^{n_0+i-1}\\ &\leq M_k(\ln\gamma)^{-1}(n_0+i-1)^{k/2}\beta^{i},\end{aligned} \tag{D.22}$$

where we used the fact that $n_0 - 1 > 0$ and $\beta < 1$, so that $\beta^{n_0-1} < 1$. Since $i^{-k/2}(n_0 + i - 1)^{k/2} \leq (n_0+1)^{k/2}$ for $i \geq 1$ and also $(n_0-1)^{k/2} \leq (n_0+1)^{k/2}$, we obtain

$$\begin{aligned}
\sum_{n=n_0}^{\infty} n^{k/2}\xi_n\|\rho_n\|_{X_x} &\leq \frac{M_k}{s\ln\gamma}\left(\xi_0\|g_0\|_{X_x}(n_0-1)^{k/2} + \sum_{i=1}^{\infty}\xi_i(n_0+i-1)^{k/2}\|g_i\|_{X_x}\right.\\
&\qquad \left. + \sum_{i=n_0}^{\infty} i^{k/2}\xi_i\|g_i\|_{X_x}\right)\\
&\leq \frac{M_k(n_0+1)^{k/2}}{s\ln\gamma}\left(\xi_0\|g_0\|_{X_x} + \sum_{i=1}^{\infty}\xi_i i^{k/2}\|g_i\|_{X_x} + \sum_{i=1}^{\infty} i^{k/2}\xi_i\|g_i\|_{X_x}\right)\\
&= \frac{2M_k(n_0+1)^{k/2}}{s\ln\gamma}\|g\|_{X_x,k}
\end{aligned}$$

where $g = \{g_i\}_{i\in\mathbb{N}_0}$. For the initial part of the series (D.20) we have

$$\begin{aligned}
\sum_{n=1}^{n_0-1} n^{k/2}\xi_n\|\rho_n|_{X_x} &\leq \frac{1}{s}\sum_{n=1}^{n_0-1}\left(\sum_{i=0}^{n-1} n^{k/2}\xi_i\|g_i\|_{X_x}\beta^{n-i} + n^{k/2}\xi_n\|g_n\|_{X_x}\right)\\
&\leq \frac{1}{s}\|g\|_{X_x,k}\left(\sum_{n=1}^{n_0-1} n^{k/2}(n-1)\right)\\
&\leq \frac{M_k' n_0^{2+k/2}}{s}\|g\|_{X_x,k},
\end{aligned}$$

where M_k' is independent of β and n_0 and where we used the fact that $n - i > 0$ for $0 \leq i \leq n-1$, so that $\beta^{n-i} < 1$ in the above series. The final inequality was obtained by integration. Moreover

$$\rho_0 = R_{\xi,0}(s)g_0 = (s + n_2p_0 - \mathsf{D}_0)^{-1}g_0 = \xi_0(\xi_0 s + n_2\xi_1 m_1 - \xi_0\mathsf{D}_0)^{-1}g_0, \tag{D.23}$$

which, on one hand, shows that

$$\rho_0 \in D_k(K_+) = D_k(N),$$

and on the other gives the estimate

$$\|\xi_0\rho_0\|_{X_x} \leq \frac{1}{s}\xi_0\|g_0\|_{X_x}.$$

Combining all the estimates we see that for any $s > 0$ there exists a constant

$$M_{k,s} = 2M_k(\ln\gamma)^{-1}(n_0+1)^{k/2} + M_k' n_0^{2+k/2} + 1 \tag{D.24}$$

(depending on s and k) such that

$$\|\rho\|_k \leq \frac{M_{k,s}}{s}\|g\|_k. \tag{D.25}$$

Next we shall prove that, as constructed, $R(s, \mathsf{D}_N + K_+) : \mathcal{X}_k \to D_k(\mathsf{D}_N) = D_k(\mathsf{D}) \cap D_k(N)$. To do this, we have to show that

$$(s - \mathsf{D} + N)\rho = (s - \mathsf{D} + N)R(s, \mathsf{D} - N + K_+)g \in \mathcal{X}_k$$

or equivalently that the sequence $(s - \mathsf{D}_n + N_n)\rho_n)_{n\geq 0}$ has finite the norm (D.9). Here $N_0 = n_2 p_0$ and $N_n = n_1 m_n + n_2 p_n$ for $n \geq 1$. However, from (D.15) we have for any $n \geq 0$

$$(s - \mathsf{D}_n + N_n)\xi_n\rho_n = \sum_{i=0}^{n} \xi_i \left(\prod_{l=0}^{n-i-1} n_2 p_{n-l-1} R_{\xi,n-l-1}(s) \right) g_i$$

hence we see that we can repeat all the estimates leading to (D.25) with the single difference that the factor $1/s$ will be missing. This factor however does not affect the convergence of the series, thus $\rho \in D_k(\mathsf{D}) \cap D_k(N)$. Since $D_k(K_+) = D_k(N)$ we see that $D_k(\mathsf{D} - N + K_+) = D_k(\mathsf{D}) \cap D_k(N) = D_k(\mathsf{D}) \cap D_k(C^i)$. This shows that the resolvent exists for any $s > 0$ and, by (D.15) and Lemma D.1, it is a positive operator. ■

Let us note that the constant $M_{k,s}$ in (D.25) is larger than 1 so that this estimate alone is not sufficient for the generation of a semigroup.

Proof of Theorem 5.2. Since the operator $\mathsf{D}_N = \mathsf{D} - N$ generates a positive semigroup and K_+ is a positive operator bounded from $D_k(\mathsf{D}_N)$ to $\mathcal{X}_k$ such that the resolvent of $\mathsf{D}_N + K_+$ exists and is positive, we can use Desch's theorem (see e.g. [22], Theorem 8.1) to claim that $\mathsf{D}_N + K_+$ generates a positive semigroup. Since K_- is a bounded positive operator in $\mathcal{X}_k$, the application of the Bounded Perturbation Theorem ends the proof. ■

E Reference manual

In this appendix we have collected the main formulas of this paper for easy reference.

We avoid here making any regularity assumptions - it is enough to state that all the results are valid if the initial data are smooth with respect to $\boldsymbol{x}$ and decay sufficiently fast to zero as $v \to \infty$. The only additional assumption (adopted for technical reasons) is that the kinetic part of the initial datum annihilates constants over S^2 in the case of dominant inelastic or elastic and inelastic scattering.

We are dealing with the following Boltzmann equation

$$\begin{aligned}
\partial_t f_\epsilon &= \frac{1}{\epsilon^p} S f_\epsilon + \frac{1}{\epsilon^q} C^e f_\epsilon + \frac{1}{\epsilon^r} C^i f_\epsilon \\
&= \frac{1}{\epsilon^p} v\boldsymbol{\omega} \cdot \partial_{\boldsymbol{x}} f_\epsilon + \frac{1}{\epsilon^q} \left(-4\pi\lambda f + \lambda \int_{S^2} f d\boldsymbol{\omega}' \right) \\
&+ \frac{1}{\epsilon^r} \left(-4\pi \left(n_2 \frac{v_+}{v} \nu_+ + n_1 H\nu \right) f + n_1 \frac{v_+}{v} \nu_+ \int_{S^2} f_+ d\omega' + n_2 \nu H \int_{S^2} f_- d\omega' \right), \\
f_\epsilon(0) &= \overset{\circ}{f}
\end{aligned} \tag{E.1}$$

where $v_\pm = \sqrt{v^2 \pm 1}$, for any function g we denoted $g_\pm = g(\sqrt{v^2 \pm 1})$ and $H = H(v^2-1)$ is the Heaviside function. Functions λ and ν are functions of v variable only.

In what follows we shall mainly use the energy variable $\xi = v^2$.

In all the cases the $O(\epsilon)$ approximation in $L_1(\mathbb{R}^6_{x,v})$, uniform on finite intervals $[0, t_0]$, is given by

$$f_\epsilon(t, \boldsymbol{x}, \boldsymbol{v}) = \rho(t, \boldsymbol{x}, \boldsymbol{v}) + \tilde{w}_0(t/\epsilon^k, \boldsymbol{x}, \boldsymbol{v}) + O(\epsilon), \tag{E.2}$$

where k is equal to the highest power of ϵ in (E.1).

E.1 Dominant elastic scattering: $q > \max\{p, r\}$

We define

$$\mathbb{P}f = \frac{1}{4\pi}\int\limits_{S^2} f d\boldsymbol{\omega}. \tag{E.3}$$

and $\mathbb{Q} = I - \mathbb{P}$

E.1.1 The case $p = \min\{q-p, q-r\}$ (thus $q = 2p$) and $r = 0$

The details (for the simplest case $p = 1, q = 2, r = 0$) are in Subsection 5.3.1.

Hydrodynamic limit.

$$\begin{aligned}
\partial_t \rho &= \xi d \Delta_x \rho - \left(n_2\sqrt{\frac{\xi+1}{\xi}} m(\xi+1) + n_1 H(\xi-1) m(\xi)\right)\rho \\
&\quad + n_1\sqrt{\frac{\xi+1}{\xi}} m(\xi+1)\rho(\xi+1) + n_2 H(\xi-1)m(\xi)\rho(\xi-1), \\
\rho(0) &= \mathbb{P}\,\mathring{f},
\end{aligned} \tag{E.4}$$

where $m(\xi) = 4\pi\nu(\xi)$ and the diffusion coefficient d is given by

$$d = \frac{4\pi}{3\lambda(v)}. \tag{E.5}$$

Initial layer corrector.

$$\tilde{w}_0(t/\epsilon^q, \boldsymbol{x}, \boldsymbol{v}) = e^{-\lambda(v)t/\epsilon^q}\mathbb{Q}\,\mathring{f}\,(\boldsymbol{x}, \boldsymbol{v}). \tag{E.6}$$

E.1.2 The case $p = \min\{q-p, q-r\}$ (thus $q = 2p$), and $r < 0$

The details (for the simplest case $p = 1, q = 2, r = -1$) are in Subsection 5.3.2.

Hydrodynamic limit.

$$
\begin{aligned}
\partial_t \rho &= \xi d \Delta_x \rho, && \text{(E.7)}\\
\rho(0) &= \mathbb{P} \overset{\circ}{f}, && \text{(E.8)}
\end{aligned}
$$

where the diffusion coefficient d is given by

$$
d = \frac{4\pi}{3\lambda(v)}.
$$

Initial layer corrector.

$$
\tilde{w}_0(t/\epsilon^q, \boldsymbol{x}, \boldsymbol{v}) = e^{-\lambda(v)t/\epsilon^q} \mathbb{Q} \overset{\circ}{f} (\boldsymbol{v}, \boldsymbol{x}). \tag{E.9}
$$

E.1.3 The case $p < \min\{q-p, q-r\}$ and $r = 0$

The details (for the simplest case $p = 0, q = 1, r = 0$) are in Subsection 5.3.3.

Hydrodynamic limit.

$$
\begin{aligned}
\partial_t \rho &= -\left(n_2 \sqrt{\frac{\xi+1}{\xi}} m(\xi+1) + n_1 H(\xi-1) m(\xi) \right) \rho \\
&\quad + n_1 \sqrt{\frac{\xi+1}{\xi}} m(\xi+1) \rho(\xi+1) + n_2 H(\xi-1) m(\xi) \rho(\xi-1), \\
\rho(0) &= \mathbb{P} \overset{\circ}{f}, && \text{(E.10)}
\end{aligned}
$$

where $m(\xi) = 4\pi\nu(\xi)$.

Initial layer corrector.

$$
\tilde{w}_0(t/\epsilon^q, \boldsymbol{x}, \boldsymbol{v}) = e^{-\lambda(v)t/\epsilon^q} \mathbb{Q} \overset{\circ}{f} (\boldsymbol{x}, \boldsymbol{v}). \tag{E.11}
$$

E.1.4 The case $p < \min\{q-p, q-r\}$ and $r < 0$

The details (for the simplest case $p = 0, q = 1, r = -1$) are in Subsection 5.3.4.

Hydrodynamic limit.

$$\begin{aligned} \partial_t \rho &= 0, \\ \rho(0) &= \mathbb{P}\, \overset{\circ}{f}, \end{aligned} \tag{E.12}$$

thus

$$\rho(t) = \mathbb{P}\, \overset{\circ}{f}\ . \tag{E.13}$$

Initial layer corrector.

$$\tilde{w}_0(t/\epsilon^q, \boldsymbol{x}, \boldsymbol{v}) = e^{-\lambda(v)t/\epsilon^q} \mathbb{Q}\, \overset{\circ}{f}\, (\boldsymbol{x}, \boldsymbol{v}). \tag{E.14}$$

E.2 Dominant inelastic scattering: $r > \max\{p, q-1\}$

In this subsection it will be convenient to use the sequential notation for functions: $f = (f_n)_{n\geq 0}$, where $f(\xi + n) = f_n(\xi)$ for $\xi \in [0,1[$ and $n = 0, 1, \ldots$.
The spectral projection on either $N(C^i)$ or $N(C^i + C^e)$ is given by

$$(Pf)_n(\boldsymbol{x}, \xi) = a^{-n} H^{-1} \sum_{j=0}^{\infty} \xi_j (\mathbb{P}f)(\boldsymbol{x}, \xi), \tag{E.15}$$

where $a = n_1/n_2 > 1$, $\xi_j = \sqrt{\xi + j}$, $H(\xi) = \sum_{j=0}^{\infty} a^{-j}\xi_j$ and $\mathbb{P}$ is defined by (E.3).

E.2.1 The case $p = \min\{r-p, r-q\}$ (thus $r = 2p$) and $r = q$

The details (for the special case $p = 1, q = r = 2$) are in Subsection 5.4.1.

Hydrodynamic limit.

We have $\rho = (\rho_n)_{n\geq 0}$ where

$$\rho_n = a^{-n} H^{-1} \varrho,$$

and ϱ is the solution to

$$\begin{aligned} \partial_t \varrho &= \frac{B(\xi)}{3H(\xi)} \Delta_x \varrho, \\ \varrho(0) &= P\overset{\circ}{f}, \end{aligned} \tag{E.16}$$

where

$$B(\xi) = \frac{\xi_0^4}{\lambda_0(\xi)\xi_0 + n_2 \xi_1 \nu_1(\xi)} + \sum_{j=1}^{\infty} \frac{a^{-j}\xi_j^4}{\lambda_j(\xi)\xi_j + n_2\xi_{j+1}\nu_{j+1}(\xi) + n_1 \xi_j \nu_j(\xi)}.$$

Initial layer.

In all cases below we require that $\overset{\circ}{w} = \mathbb{Q} \overset{\circ}{f}$ satisfy $\mathbb{P} \overset{\circ}{w} = 0$, and we use the standard notation

$$p_n(\xi) = \sqrt{\frac{\xi + n + 1}{\xi + n}} \nu(\xi + n + 1).$$

With these

$$\tilde{w}_{0,n}(t/\epsilon^r, \boldsymbol{x}, \xi \boldsymbol{\omega}) = \begin{cases} e^{-(\lambda_0(\xi) + n_2 p_0(\xi))t/\epsilon^r} \overset{\circ}{w}_0 (\boldsymbol{x}, \xi, \boldsymbol{\omega}), \\ e^{-(\lambda_n(\xi) + n_2 p_n(\xi) + n_1 \nu_n(\xi))t/\epsilon^r} \overset{\circ}{w}_n (\boldsymbol{x}, \xi, \boldsymbol{\omega}) \quad \text{for} \quad n \geq 1. \end{cases} \tag{E.17}$$

E.2.2 The case $p = \min\{r - p, r - q\}$ (then $r = 2p$) and $r > q$

The details (for the simplest case $p = 1, q = 1, r = 2$) are in Subsection 5.4.2.

Hydrodynamic limit.

We have $\rho = (\rho_n)_{n \geq 0}$ where

$$\rho_n = a^{-n} H^{-1} \varrho,$$

and ϱ is the solution to

$$\begin{aligned} \partial_t \varrho &= \frac{B(\xi)}{3H(\xi)} \Delta_x \varrho, \\ \varrho(0) &= P \overset{\circ}{f}, \end{aligned} \tag{E.18}$$

where

$$B(\xi) = \frac{\xi_0^4}{n_2 \xi_1 \nu_1(\xi)} + \sum_{j=1}^{\infty} \frac{a^{-j} \xi_j^4}{n_2 \xi_{j+1} \nu_{j+1}(\xi) + n_1 \xi_j \nu_j(\xi)}.$$

Initial layer.

$$\tilde{w}_{0,n}(t/\epsilon^r, \boldsymbol{x}, \xi \boldsymbol{\omega}) = \begin{cases} e^{-n_2 p_0(\xi)t/\epsilon^r} \overset{\circ}{w}_0 (\boldsymbol{x}, \xi, \boldsymbol{\omega}), \\ e^{-(n_2 p_n(\xi) + n_1 \nu_n(\xi))t/\epsilon^r} \overset{\circ}{w}_n (\boldsymbol{x}, \xi, \boldsymbol{\omega}) \quad \text{for} \quad n \geq 1. \end{cases} \tag{E.19}$$

E.2.3 The case $p < \min\{r - p, r - q\}$, $q = r$

The details (for the simplest case $p = 0, q = r = 1$) are in Subsection 5.4.3.

Hydrodynamic limit.

We have

$$\begin{aligned} \partial_t \rho &= 0, \\ \rho(0) &= P \overset{\circ}{f}, \end{aligned} \tag{E.20}$$

thus

$$\rho(t, \boldsymbol{x}, \boldsymbol{v}) = P \overset{\circ}{f}.$$

Initial layer.

$$\tilde{w}_{0,n}(t/\epsilon^r, \boldsymbol{x}, \xi\boldsymbol{\omega}) = \begin{cases} e^{-(\lambda_0(\xi)+n_2p_0(\xi))t/\epsilon^r} \overset{\circ}{w}_0 (\boldsymbol{x}, \xi, \boldsymbol{\omega}), \\ e^{-(\lambda_n(\xi)+n_2p_n(\xi)+n_1\nu_n(\xi))t/\epsilon^r} \overset{\circ}{w}_n (\boldsymbol{x}, \xi, \boldsymbol{\omega}) \quad \text{for} \quad n \geq 1. \end{cases} \tag{E.21}$$

E.2.4 The case $p < \min\{r-p, r-q\}$, $q < r$

The details (for the simplest case $p = 0, q = 0, r = 1$) are also in Subsection 5.4.3.

Hydrodynamic limit.

We have

$$\begin{aligned} \partial_t \rho &= 0, \\ \rho(0) &= P\overset{\circ}{f}, \end{aligned} \tag{E.22}$$

thus

$$\rho(t, \boldsymbol{x}, \boldsymbol{v}) = P\overset{\circ}{f}.$$

Initial layer

$$\tilde{w}_{0,n}(t/\epsilon^r, \boldsymbol{x}, \xi\boldsymbol{\omega}) = \begin{cases} e^{-n_2p_0(\xi)t/\epsilon^r} \overset{\circ}{w}_0 (\boldsymbol{x}, \xi, \boldsymbol{\omega}), \\ e^{-(n_2p_n(\xi)+n_1\nu_n(\xi))t/\epsilon^r} \overset{\circ}{w}_n (\boldsymbol{x}, \xi, \boldsymbol{\omega}) \quad \text{for} \quad n \geq 1. \end{cases} \tag{E.23}$$

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