A kinetic model for droplet coalescence in dense sprays

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ABSTRACT. – This paper is devoted to the matematical analysis of a kinetic equation describing the process of droplet coalescence in a spatially homogeneous spray. In the first part, existence of a positive, mass and momentum conserving, measure valued solution is proved, under very general assumptions on the collision kernel. In the second part, uniqueness and continuous dependence of the solution on the inital data is established under some more restrictive - but physically reasonable - hypotheses on the model. Lastly, the time-asymptotic behaviour of the solution is studied.

Key words: kinetic equation, spray modelling, coalescence.

1. Introduction

This paper is devoted to the mathematical analysis of the Cauchy Problem with unknown the (measure-valued) function $f(t, r, \mathbf{v})$ and reading

$$f_t = Q(f), \qquad f(0, r, \mathbf{v}) = f_0(r, \mathbf{v}) \tag{1.1}$$

where $Q(f)$ is a non linear integral operator defined as :

$$\begin{aligned} Q(f)(t, r, \mathbf{v}) \;=\; & -f(t, r, \mathbf{v}) \int_{\mathrm{R}_r^+ \times \mathrm{R}_v^3} B(r, r^*, \mathbf{v}, \mathbf{v}^*) f(t, r^*, \mathbf{v}^*) dr^* d\mathbf{v}^* \\ & + \frac{1}{2} \int_{]0,r[\times \mathrm{R}_v^3} \frac{r^{11}}{r^{\diamond 11}} B(r^\diamond, r^*, \mathbf{v}^\diamond, \mathbf{v}^*) f(t, r^\diamond, \mathbf{v}^\diamond) f(t, r^*, \mathbf{v}^*) dr^* d\mathbf{v}^* \end{aligned} \tag{1.2}$$

with the notations :

$$r^\diamond = (r^3 - r^{*3})^{1/3}, \qquad \mathbf{v}^\diamond = \frac{r^3\mathbf{v} - r^{*3}\mathbf{v}^{*3}}{r^3 - r^{*3}},$$

and

$$B(r, r^*, \mathbf{v}, \mathbf{v}^*) = E(r, r^*, \mathbf{v}, \mathbf{v}^*)\pi(r + r^*)^2|\mathbf{v} - \mathbf{v}^*|.$$

The function E is a positive function, and is usally called coalescence efficiency [10, 11]. We shall come back later to the physical meaning of this function. Equation (1.1) describes the process of droplet coalescence in a spatially homogeneous spray. The notations r and $\mathbf{v}$ describe the radius and the velocity, of a drop a; the function $f(t, r, \mathbf{v})$ represents the averaged density of droplets located, at time t, at a given point of the phase space $\mathbb{R}_r^+ \times \mathbb{R}_v^3$. The derivation of equation (1.1) can be found in [15, 11]. Here, for clarity, we just recall that the term $\dfrac{r^{11}}{r^{\diamond 11}}$ is nothing but the Jacobian of the mapping $(r, \mathbf{v}) \to (r^\diamond, \mathbf{v}^\diamond)$ for fixed values of $(r^*, \mathbf{v}^*)$.

Model (1.1) is closely related to the coagulation equation encountered in colloid chemistry, physics of the atmosphere or astrophysics (see for example [1, 5]). This equation writes :

$$(1.3) \qquad \begin{aligned} f_t(t, m) &= -f(t, m) \int_{\mathbb{R}_m^+} \Phi(m, m^*) f(t, m^*) dm^* \\ &\quad + \frac{1}{2} \int_0^m \Phi(m - m^*, m^*) f(t, m - m^*) f(t, m^*) dm^* \end{aligned}$$

where the kernel Φ is a positive regular function related to the collision frequency and determined from physical considerations. In this model, the variable m denotes the mass of a particle and $f(t, m)$ represents the average density of particles of mass m at time t. The main difference between equation (1.1) and (1.3) lies in the fact that the velocity variable, $\mathbf{v}$, has been eliminated in the last one. But in many applications involving dense spray of droplets, no information is known *a priori* on the shape of velocity distribution and therefore the dependency on $\mathbf{v}$ must be kept in the model.

In the two last decades, a lot of mathematical studies have been devoted to equation (1.3). The existence, uniqueness and time-asymptotic stabilization of a non negative solution were established by Aizenman and Bak [1] for bounded kernel. A similar result was first proved for a large class of unbounded kernels by Galkin in [7, 8] but with strong decay conditions on the initial data. In [13], Stewart proved the existence of a positive solution to (1.3) for initial data in $L^{1+}(\mathbb{R}^+)$ with finite mass. He only required Φ to be a positive continuous symmetric function such that :

$$(1.4) \qquad \forall (x, y) \in (\mathbb{R}^+)^2, \quad \Phi(x, y) \leq C[(1 + x)^\alpha + (1 + y)^\alpha]$$

for some given constants $\alpha \in [0,1[$ and $C > 0$. In [14], Stewart also proved the uniqueness of the solution, but under the following restrictive assumptions on the collision kernel :

$$\forall (x,y) \in (\mathbb{R}^+)^2, \quad \Phi(x,y) \leq C(1+x)^{1/2}(1+y)^{1/2} \tag{1.5}$$

for a given positive constant C. With the hypothesis (1.4) and no strong decay conditions on the initial data, uniqueness seems to be still an open problem.

For further references on coagulation-fragmentation models, see the monograph of Dubovskii [6] and the paper of Slemrod [12] for coagulation models with discrete velocities.

In this paper, we shall first consider the question of existence and uniqueness of solution to equation (1.1) in the class of nonnegative measure valued solution, compactly supported with respect to $\mathbf{v}$, and satisfying for all $t \geq 0$ the natural estimate :

$$\int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} (1+r^3) df(t,r,\mathbf{v}) \ < +\infty$$

Next we shall study the time-asymptotic behaviour of the solutions. Under an additional assumption on the coalescence efficiency E, we shall prove that, as t goes to infinity, the solution $f(t,.)$ converges, in the weak $\star$ sense, towards a measure of the form $\mu^\infty(r) \otimes \delta_{\mathbf{v}=\mathbf{v}^\infty}$. Physically this result is rather intuitive; it means that the droplets will keep colliding, until their respective velocities stabilize at a certain value.

2. Main results

Let us come back to the physical meaning of the function E. Roughly speaking, E is the probability that a coalescence really occurs, given that two drops have the possibility to collide. It takes into account the fact that, when a droplet is approaching from another one, it does not follow a straightline trajectory but instead is deflected due to its interaction with the surrounding gaz flow; thus it is possible that the smallest droplet circumvent the largest one and that the collision does not occur. This mechanism was for example studied by Langmuir [10] and Davis et al [4]. The function E also takes into account the fact the collision between two droplets does not necessarilly result in a coalescence but may also leads to a rebound, especially for grazing collisions (see for example the experimental study of Brazier-Smith et al [2]).

According to this definition, the two following assumptions are not restrictive at all.

(H1) $E \in C_b((\mathbb{R}_r^+)^2 \times (\mathbb{R}_v^3)^2)$

(H2) $\forall (r,r^*) \in (\mathbb{R}_r^+)^2, \forall (\mathbf{v},\mathbf{v}^*) \in (\mathbb{R}_v^3)^2, \quad E(r,r^*,\mathbf{v},\mathbf{v}^*) = E(r^*,r,\mathbf{v}^*,\mathbf{v})$

with $C_b((\mathbb{R}_r^+)^2 \times (\mathbb{R}_v^3)^2)$ being the space of bounded continuous functions on $(\mathbb{R}_r^+)^2 \times (\mathbb{R}_v^3)^2$. Our first result is

Theorem 2.1 *Assume $f_0(r,\mathbf{v})$ to be a nonnegative bounded measure and that its third moment: $\int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} r^3 \, df_0$ is finite. Moreover assume that f_0 is compactly supported with respect to the velocities - i.e. there exists a compact set K of $\mathbb{R}_v^3$ such that $\mathrm{supp}_{\mathbf{v}} f_0 \subset K$.*
Then, if E satisfies the assumptions (H1)-(H2), Equation (1.1) has a global solution $f(t) \in \mathrm{Lip}(\mathbb{R}_t^+, \mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3))$. Moreover if $f_0 \in L^1(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$, then $f \in \mathrm{Lip}(\mathbb{R}_t^+, L^1(\mathbb{R}_r^+ \times \mathbb{R}_v^3))$.

We do not know whether uniqueness holds in this case, but we recall that, even for equation (1.3), uniqueness is still an open problem for general unbounded kernels and general initial data.

If we refine our assumptions on the function E, it is possible to obtain a stronger result. As mentionned above, two drops of very different sizes will generally not coalesce - unless their relative velocity is very high - because the small droplet will be carried by the gas flow arround the large one. This effect can be taken into account by assuming that E is a rapidly decreasing function of the ratio $\dfrac{\max(r,r^*)}{\min(r,r^*)}$. We shall assume that

(H3) there exists $e \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that $\forall (r,r^*) \in (\mathbb{R}_r^+)^2, \forall (\mathbf{v},\mathbf{v}^*) \in (\mathbb{R}_v^3)^2$,

$$E(r,r^*,\mathbf{v},\mathbf{v}^*) \leq e(|\mathbf{v}-\mathbf{v}^*|)(1+\min(r,r^*))\frac{\min(r,r^*)^2}{\max(r,r^*)^2}$$

This is in agreement with Langmuir's law [10], at least in the domain where it is valid. With the help of this assumption we get the

Theorem 2.2 *Let f_0 satisfy the assumptions of Theorem 2.1, and additionally assume the existence of $r_0 > 0$ such that*

$$\mathrm{supp}_r f_0 \in [r_0, +\infty[.$$

Under the hypothesis (H1)-(H2)-(H3) on E, Equation (1.1) has a unique global solution $f(t)$ such that $r^3 f \in \mathrm{Lip}([0,+\infty[, \mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3))$. Moreover the mass and impulsion of f are preserved in time, and the kinetic energy decreases.

The assumption on $\mathrm{supp}_r f_0$ is most certainly unnecessary, but it simplifies the considerations. Further, it is physically relevant.

Finally, under Hypothesis (H3), we study the time-asymptotic behaviour of the solutions. The asymptotic behaviour of the pure coagulation equation (1.3) is rather straightforward: once a suitable global existence theorem is established, one only has to notice that, for every $m > 0$ we have

$$\frac{d}{dt}\int_0^m m^* f(t, m^*)\ dm^* \leq 0.$$

On the other hand, one may see that there is no nontrivial equilibrium to (1.3); therefore we have

$$\lim_{t\to+\infty}\int_0^m m^* f(t, m^*)\ dm^* = 0.$$

This means that the mass of all the droplets keeps increasing. This is intuitively plausible, since there is no mechanism to oppose coagulation.

The situation is different in our model including velocities, as there are many equilibrium states: a cloud of particles travelling with the same speed is a steady solution, since no collision may occur. Said in more mathematical terms, all distribution of the form $\mu(r)\otimes\delta_{\mathbf{v}=\overline{\mathbf{v}}}$, where $\overline{\mathbf{v}}$ is any vector of $\mathbb{R}^3$, cancels the collision kernel. One may therefore believe that, if all the mass does not go to $+\infty$, the droplets will keep colliding, until their respective velocities stabilize at a certain value. This scenario is all the more believable that Theorem 2.2 above says that the kinetic energy is a Liapounov function for the equation.

Theorem 2.3 *Under the assumptions of Theorem 2.2, the following alternative holds.*
(i). Either all the mass concentrates to to $r = +\infty$, i.e.

$$\forall r > 0,\quad \lim_{t\to+\infty}\int_{[0,r]\times\mathrm{R}_v} r^3\ df(t) = 0, \tag{2.6}$$

(ii) or there exists a nonnegative measure $\mu^\infty \in \mathcal{M}_b(\mathbb{R}_r^+)$ and a vector $\mathbf{v}^\infty$ such that

$$0 < \int r^3\ d\mu^\infty(r) \leq \int_{\mathrm{R}_r^+\times\mathbf{R}_v^3} r^3 df_0 \tag{2.7}$$

and such that

$$\lim_{t\to+\infty} f(t, r, \mathbf{v}) = \mu^\infty(r)\otimes\delta_{\mathbf{v}=\mathbf{v}^\infty} \tag{2.8}$$

in the weak $*$ sense.

A forthcoming paper will examine the dynamical connections between all these equilibrium solutions - and in particular when μ^∞ is nonzero.

The sequel of the paper is organized as follows. Section 3 is devoted to the proof of Theorem 2.1, and Section 4 to the proofs of Theorem 2.2 and 2.3.

Notation : In the sequel, the letter C will always denote a generic constant.

3. Weak formulation of the problem and general existence result

The whole paper will be based on the weak formulation of Equation (1.1). The duality pairing between $\mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$ and $C_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$ - the space of all bounded continuous functions from $\mathbb{R}_r^+ \times \mathbb{R}_v^3$ to $\mathbb{R}$ - will be denoted by

$$< f, \phi >= \int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} \phi(r, \mathbf{v}) df(r, \mathbf{v}). \tag{3.1}$$

It will also denote the pairing between $L^\infty(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$ and $L^1(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$; in that case, one only has to view the notation $df(r, \mathbf{v})$ as $f(r, \mathbf{v}) dr d\mathbf{v}$.

If $f(t) \in L^1(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$, it is easy to check that Equation (1.1) is formally equivalent to - see [15]

$$\forall \phi \in C_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3), \quad < f(t), \phi >=< f_0, \phi > + \int_0^t < Q(f(s)), \phi > \ ds$$

with

$$< Q(f), \phi >$$

$$= \frac{1}{2} \int_{(\mathbb{R}_r^+ \times \mathbb{R}_v^3)^2} B(r, r^*, |\mathbf{v} - \mathbf{v}^*|)(\phi^\# - \phi - \phi^*) \ f(t, r^*, \mathbf{v}^*) f(t, r, \mathbf{v}) dr^* d\mathbf{v}^* dr d\mathbf{v}$$

where we have set

$$r^\# = (r^3 + r^{*3})^{1/3}, \qquad \mathbf{v}^\# = \frac{r^3}{r^3 + r^{*3}} \mathbf{v} + \frac{r^{*3}}{r^3 + r^{*3}} \mathbf{v}^* \tag{3.2}$$

$$\phi = \phi(r, \mathbf{v}), \quad \phi^* = \phi(r^*, \mathbf{v}^*), \quad \phi^\# = \phi(r^\#, \mathbf{v}^\#)$$

Thus, a measure valued function f, belonging to $C(\mathbb{R}_t^+, \mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3))$, will be said to be a solution to Equation (1.1) if for all $t \geq 0$

$$\forall \phi \in C_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3), \quad < f(t), \phi >=< f_0, \phi > + \int_0^t < Q(f(s)), \phi > \ ds \tag{3.3}$$

with by definition

$$< Q(f), \phi >= \frac{1}{2} \int_{(\mathbb{R}_r^+ \times \mathbb{R}_v^3)^2} B(r, r^*, |\mathbf{v} - \mathbf{v}^*|)(\phi^\# - \phi - \phi^*) \ df^* \otimes df \tag{3.4}$$

where we have set

$$df^* = df(t, r^*, \mathbf{v}^*), \quad df = df(t, r, \mathbf{v})$$

The symmetry property of Q classically implies

$$
\begin{aligned}
(3.5) \qquad & < Q(f) - Q(g), \phi > \\
= & \ \frac{1}{2} \int_{(\mathbb{R}_r^+ \times \mathbb{R}_v^3)^2} B(r, r^*, |\mathbf{v} - \mathbf{v}^*|)(\phi^\# - \phi - \phi^*) \ d(f+g) d(f^* - g^*)
\end{aligned}
$$

The proof of Theorem 2.1 is obtained, as is usual, from a three-step process: *a priori* estimates, approximation by truncation, convergence estimates. Let us begin with the readily available *a priori* estimates.

Lemma 3.1 *Let f be a nonnegative solution of equation (3.3), belonging to $C(\mathbb{R}_t^+, \mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3))$. Then we have*

$$
\int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} df \leq \int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} df_0 \tag{3.6}
$$

$$
\int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} r^3 \ df \leq \int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} r^3 \ df_0 \tag{3.7}
$$

Moreover if every element $\mathbf{v}$ of $supp_{\mathbf{v}} f_0$ is such that $|\mathbf{v}| \leq R$, then for all $t \geq 0$, $\forall \mathbf{v} \in supp_{\mathbf{v}} f(t)$, we have $|\mathbf{v}| \leq R$.

PROOF. Let $(\chi_n)_n$ be the sequence of bounded continuous fonctions defined by : $\chi_n(r) = 1$ if $0 \leq r \leq n$, $\chi_n(r) = 1 + (r - n)$ if $n \leq r \leq n+1$ and $\chi_n(r) = 0$ if $r > n+1$. Using $\phi(r, v) = r^3 \chi_n(r)$ as a test function in (3.3), we get :

$$
< f(t), r^3 \chi_n(r) > = < f_0, r^3 \chi_n(r) > + Q
$$

with :

$$
Q = \frac{1}{2} \int_0^t \int_{(\mathbb{R}_r^+ \times \mathbb{R}_v^3)^2} B[r^3(\chi_n(r^\#) - \chi_n(r)) + r^{\star 3}(\chi_n(r^\#) - \chi_n(r^\star))] df^* \otimes df \ ds
$$

Due to the definition of $r^\#$ and χ_n, it is evident that $Q \leq 0$, hence for all n :

$$
< f(t), r^3 \chi_n(r) > \leq < f_0, r^3 \chi_n(r) >
$$

The estimate (3.7) follows easily by passing to the limit on n. The proof of estimate (3.6) is exactly the same, choosing $\phi(r, v) = \chi_n(r)$.

The remaining part of the lemma is intuitively obvious: the droplets may only coalesce, and the speed of a droplet resulting from a coalescence is smaller in modulus than the speeds of the initial droplets - see (3.2) -. A rigorous proof is obtained by using in (3.3) any test function $\phi(r, \mathbf{v})$ whose support in $\mathbf{v}$ is disjoint from $B_{R,\mathbf{v}}(0)$ - the ball of $\mathbb{R}_v^3$ with centre 0 and radius R. In fact, we may just choose $\phi(r, \mathbf{v}) = \mathbf{1}_{\{r \geq 0, |\mathbf{v}| \geq R\}}$. Then (i). if $|\mathbf{v}| \geq R$ or $|\mathbf{v}^*| \geq R$, then $\phi = 1$ or $\phi^* = 1$; hence $\phi^\# - \phi - \phi^* \leq 0$;

(ii). if $|\mathbf{v}| \leq R$ or $|\mathbf{v}^*| \leq R$, equation (3.2) implies $|\mathbf{v}^\#| \leq R$. Hence $\phi^\# = \phi = \phi^* = 0$.

Therefore we always have $\phi^\# - \phi - \phi^* \leq 0$; then (3.3) implies

$$\int_{r \geq 0, |\mathbf{v}| \geq R} df(t, r, \mathbf{v}) \; = 0.$$

This is what we wanted to prove. •

Now let us truncate the collision kernel. For every $M > 0$ let $\Gamma_M(r, r^*)$ be any nonnegative, smooth function equal to 1 on $\{r^2 + r^{*2} \leq M\}$, and to 0 outside $\{r^2 + r^{*2} \leq M + 1\}$. Let us consider the approximate Cauchy Problem

$$(3.8) \qquad f_t = Q_M(f), \qquad f(0, r, \mathbf{v}) = f_0(r, \mathbf{v})$$

where Q_M is the collision kernel obtained when replacing $B(r, r^*, |\mathbf{v} - \mathbf{v}^*|)$ by $\Gamma_M(r, r^*) B(r, r^*, |\mathbf{v} - \mathbf{v}^*|)$.

Lemma 3.2 *If f_0 satisfies the assumption of Theorem 2.1, the approximate problem (3.8) has a unique global solution $f_M(t) \in C^1(\mathbb{R}_t^+, \mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3))$, satisfying the properties of Lemma 3.1.*

PROOF. It is a copy of [3], Chapter 6 (Homogeneous Boltzmann Equation). We set $f_M^0 = f_0$ and, for $\Lambda > 0$ large enough, perform the iteration

$$(3.9) \qquad \partial_t f_M^{n+1} + \Lambda f_M^{n+1} = Q_M(f_M^n) + \Lambda f_M^n,$$

which produces a sequence $(f_M^n)_n$ satisfying all the estimates of Lemma 3.1. The relaxation term Λf has been introduced to keep the positivity of the f_M^n: indeed, assume that $f_M^n \geq 0$. We have, for all test nonnegative function ϕ:

$$\begin{aligned} < Q_M(f_M^n), \phi > \; & \geq -\frac{1}{2} \int_{(\mathbb{R}_r^+ \times \mathbb{R}_v^3)^2} \Gamma_M B(\phi + \phi^*) \; df_M^{n*} \otimes df_M^n \\ & \geq CM \|f_M^n(t)\|_{\mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)} < f_M^n, \phi > \end{aligned}$$

Because the total mass of f_M^n is bounded, we end up with

$$\frac{d}{dt} < f_M^{n+1}, \phi > + \Lambda < f_M^{n+1}, \phi > \geq (\Lambda - CM) < f_M^n, \phi >$$

which implies the nonnegativity of f_M^n for $\Lambda \geq CM$. Set $g^n = f_M^{n+1} - f_M^n$; from the boundedness of $\Gamma_M B$ - for $supp_{\mathbf{v}} f_M(t) \subset supp_{\mathbf{v}} f_0$ - and from equation (3.5) we obtain the differential inequality

$$(3.10) \quad \frac{d}{dt} < g^{n+1}, \phi > + \Lambda < g^{n+1}, \phi > \leq (\Lambda + CM) \|g^n\|_{\mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)} \|\phi\|_\infty.$$

This is enough tho ensure the convergence of the sequence $(f^n)_M$. •

Of course, the last inequality contains a factor CM, which is unbounded as $M \to +\infty$. This forces us to introduce an additional argument, detailed now.

PROOF OF THEOREM 2.1. We wish to prove that a subsequence $(f_{M_n})_n$ converges to a solution of (1.1). First we notice that the total mass of Q_M is bounded independently of M: for all non negative measure f satisfying the estimates of lemma 3.1, and for all test function ϕ we have, for a generic constant C:

$$\begin{aligned} |< Q_M(f), \phi >| \ &\le C\|\phi\|_\infty \int (r^2 + r^{*2}) df \otimes df^* \\ &\le 2C\|\phi\|_\infty \Big(\int r^2 \ df\Big)\Big(\int df^*\Big) \end{aligned}$$

which implies our assertion. Hence, for all $T > 0$, the sequence $(f_M)_M$ is uniformly Lipschitz from $[0,T]$ to $\mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$. In particular there exists a subsequence - still denoted by $(f_M)_M$ - converging to $f \in \mathrm{Lip}([0,T] \times \mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3))$ in the following sense: for all $\phi \in C_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$, $< f_M(t), \phi > \to < f(t), \phi >$ uniformly in $t \in [0,T]$; this can be done exactly as in Godlewski-Raviart [9], p. 70. In fact, because of estimates (3.7), we may relax a little the boundedness assumption on ϕ: we can only require

$$r^2\phi \in C_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3). \tag{3.11}$$

Let us prove that the sequence $(f_M)_M$ converges in fact strongly in $C([0,T],$ $\mathcal{M}_b([0,R] \times \mathbb{R}_v^3))$ for all $R > 0$. Let $\delta > 0$. Choose $\phi(r, \mathbf{v}) \in C_c([0, R+\delta[\times \mathbb{R}_v^3)$; extend it on the whole $\mathbb{R}_r^+ \times \mathbb{R}_v^3$ to a continuous function that is 0 outside $[0, R+\delta] \times \mathbb{R}_v^3)$; consider then the quantity $< Q_M(f_M) - Q(f), \phi >$. We have, using (3.5)

$$\begin{aligned} & < Q_M(f_M) - Q(f), \phi > \\ = \ & < Q_M(f_M) - Q(f_M), \phi > + < Q(f_M) - Q(f), \phi > \\ = \ & \frac{1}{2}\int (\Gamma_M - 1) B(\phi^\# - \phi - \phi^*) \ df_M^* \otimes df_M \\ & + \frac{1}{2}\int B(\phi^\# - \phi - \phi^*) \ d(f_M + f) \otimes d(f_M^* - f^*) \\ = \ & I + II \end{aligned} \tag{3.12}$$

We have immediately

$$\begin{aligned} I \ &\le C\|\phi\|_\infty \int_{r^2 + r^{*2} \ge M} (r^2 + r^{*2}) df_M \otimes f_M^* \\ &\le CM^{-1}\|\phi\|_\infty \int (r^3 + r^{*3}) df_M \otimes f_M^* \\ &\le CM^{-1}\|\phi\|_\infty \end{aligned}$$

Let us investigate II; the term factoring $\phi^{\#}$ is trivial: indeed, due to the definition of $r^{\#}$ in (3.2), $\phi^{\#}=0$ as soon as $r^3+r^{*3}\geq(R+\delta)^3$, hence as soon as $r\geq R+\delta$ or $r^*\geq R+\delta$, which implies the bound

$$\left|\int\Gamma_M B\phi^{\#}\ d(f_M+f)\otimes d(f_M^*-f^*)\right|\leq CR^2\|\phi\|_\infty\|f-f_M\|_{\mathcal{M}_b([0,R+\delta]\times\mathrm{R}_v^3)}.$$

Consider then the terms factoring ϕ. Let us set

$$h_M(t,r,\mathbf{v})=\int r^{*2}E(r,r^*,\mathbf{v},\mathbf{v}^*)|\mathbf{v}-\mathbf{v}^*|d(f_M^*-f^*)$$

and

$$g_M(t,r,\mathbf{v})=\int E(r,r^*,\mathbf{v},\mathbf{v}^*)|\mathbf{v}-\mathbf{v}^*|d(f_M^*-f^*)$$

Due to the fact that $supp_{\mathbf{v}}f_M$ and $supp_{\mathbf{v}}f$ are uniformly bounded, we have

$$\begin{aligned}
&\left|\int B\phi d(f_M+f)\otimes d(f_M^*-f^*)\right|\\
=\ & 2\left|\int_{r\leq R+\delta}Bdf\otimes d(f_M^*-f^*)\phi\right|+\left|\int_{r\leq R+\delta}Bd|f_M-f|\otimes d|f_M^*-f^*|\right|\|\phi\|_\infty\\
\leq\ & CR^2\Big(\int_{r\leq R+\delta}d|f_M-f|\Big)\Big(\int d|f_M^*-f^*|\Big)\|\phi\|_\infty\\
&+C\Big(\int_{r\leq R+\delta}d|f_M-f|\Big)\Big(\int r^{*2}d|f_M^*-f^*|\Big)\|\phi\|_\infty\\
&+R^2\|\phi\|_\infty\int_{r\leq R+\delta}|g_M(t,r,\mathbf{v})|\ df(t,r,\mathbf{v})\\
&+\|\phi\|_\infty\int_{r\leq R+\delta}|h_M(t,r,\mathbf{v})|\ df(t,r,\mathbf{v})\\
\leq\ & C\|f_M-f\|_{\mathcal{M}_b([0,R+\delta]\times\mathrm{R}_v^3)}\|\phi\|_\infty+(1+R^2)\|\phi\|_\infty\int(|g_M|+|h_M|)df
\end{aligned}$$

Let us recall that, thanks to the above-mentioned weak convergence result for the sequence $(f_M)_M$ - and assertion (3.11) - h_M and g_M converge pointwise to 0 when M goes to infinity and moreover they remain uniformly bounded; therefore Lebesgue's dominated convergence Theorem ensures that, for almost all $t\in[0,T]$, the quantity

$$\varepsilon_M(t)=\int_{r\leq R+\delta}(|h_M(t,r,\mathbf{v})|\ +|g_M(t,r,\mathbf{v})|)df(t,r,\mathbf{v}) \tag{3.13}$$

converges to 0.

Finally, let us investigate the term factoring ϕ^*. Here we easily obtain

$$\begin{aligned} & \|\phi\|_\infty{}^{-1} \Big| \int_{r^* \leq R+\delta} B d|f_M + f| \otimes d|f_M^* - f^*| \\ \leq \; & CR^2 \|f_M - f\|_{\mathcal{M}_b([0,R+\delta]\times \mathbb{R}_v^3)} \end{aligned}$$

Gathering all the information, we have for all $\delta > 0$:

$$\|f_M(t) - f(t)\|_{\mathcal{M}_b([0,R]\times \mathbb{R}_v^3)}$$

$$\leq C(1+R^2) \int_0^t (\frac{1}{M} + \varepsilon_M(s) + \|f_M(s) - f(s)\|_{\mathcal{M}_b([0,R+\delta]\times \mathbb{R}_v^3)})\; ds,$$

Passing to the limit when δ goes to 0 - *via* the monotone convergence theorem -, we obtain

$$\|f_M(t) - f(t)\|_{\mathcal{M}_b([0,R]\times \mathbb{R}_v^3)}$$

$$\leq C(1+R^2) \int_0^t (\frac{1}{M} + \varepsilon_M(s) + \|f_M(s) - f(s)\|_{\mathcal{M}_b([0,R]\times \mathbb{R}_v^3)})\; ds,$$

hence applying Gronwall's Lemma and passing to the limit when M goes to infinity, we deduce the strong convergence of the sequence $(f_M)_M$ towards f in $C(0, T, \mathcal{M}_b([0,R] \times \mathbb{R}_v^3)$ for all $R > 0$. It remains to prove that this implies the convergence of $\int_0^t < Q_M(f_M(s)), \phi > \;\; ds$ to $\int_0^t < Q(f(s)), \phi > \;\; ds$. We start with :

$$\begin{aligned} & | < Q_M(f_M(t)), \phi > - < Q(f(t)), \phi > | \\ & \leq | < Q_M(f_M(t)), \phi > - < Q_M(f(t)), \phi > | + | < Q_M(f(t)), \phi > - < Q(f(t)), \phi > | \\ & \leq I + II \end{aligned}$$

We get easily that :

$$I \leq C\|\phi\|_\infty \int (r^2 + r^{*2}) d|f_M - f| \otimes d(f_M^* + f^*),$$

and

$$II \leq C\|\phi\|_\infty \int |1 - \Gamma_M| (r^2 + r^{*2}) df \otimes df^*.$$

Using the uniform estimates (3.6) and (3.7), which are valid for both f_M and f, we obtain :

$$I \leq C\|\phi\|_\infty \int (1 + r^2) d|f_M - f|,$$

and

$$II \leq \frac{C}{M}.$$

By virtue again of (3.6) and (3.7), we note that

$$\begin{aligned} I &\leq C\|\phi\|_\infty \int_{[0,R]\times\mathbb{R}^3} (1+r^2)d|f_M - f| + \frac{C}{R}, \\ &\leq C(1+R^2)\|\phi\|_\infty \|f_M(t) - f(t)\|_{\mathcal{M}_b([0,R]\times\mathbb{R}^3_v)} + \frac{C}{R}. \end{aligned}$$

The convergence, for all $t > 0$, of $< Q_M(f_M(t)), \phi >$ to $< Q(f(t)), \phi >$ follows easily. Finally, noticing that $| < Q_M(f_M(t)), \phi > |$ is uniformly bounded and applying Lebesgue's dominated convergence theorem, we conclude that :

$$\int_0^t < Q_M(f_M(s)), \phi > \ ds \to \int_0^t < Q(f(s)), \phi > \ ds$$

This completes the proof of the existence part of Theorem 2.1.

To prove that, if f_0 is a function, then $f(t)$ remains a function for all time, we only have to notice that f_M is a function, and that $L^1(\mathbb{R}^+_r \times \mathbb{R}^3_v)$ is closed in $\mathcal{M}_b(\mathbb{R}^+_r \times \mathbb{R}^3_v)$. •

4. Circumvention effect: uniqueness and time-asymptotic stabilization

Scanning back the computations leading to the proof of Theorem 2.1, we see that what really prevents uniqueness is the term II in equation (3.12); more precisely it is the part of II that does not contain any $\phi^\#$. Indeed we are not able to bound it by $\|f_M - f\|_{\mathcal{M}_b}$; if we were able to do so, a classical Gronwall argument would work. This is exactly what happens under the hypothesis (H3) on the collision kernel.

Lemma 4.1 *There holds*

$$r^2 E(r, r^*, \mathbf{v}, \mathbf{v}^*) \leq e(|\mathbf{v} - \mathbf{v}^*|) \min((1+r)r^2, (1+r^*){r^*}^2)$$

and, symmetrically, the same inequalities for ${r^*}^2 E$.

PROOF. Use the inequality

$$\frac{\min(r, r^*)}{\max(r, r^*)} \leq \min(1, \frac{{r^*}^2}{r^2}). \bullet$$

Let us consider the space $\tilde{\mathbb{M}}(\mathbb{R}^+_r \times \mathbb{R}^3_v)$, which is the space of all bounded measures f on $\mathbb{R}^+_r \times \mathbb{R}^3_v$ such that $r^3 f$ also belongs to $\mathcal{M}_b(\mathbb{R}^+_r \times \mathbb{R}^3_v)$. Setting

$g := r^3 f$ we may compute that we have, for any continuous compactly supported ϕ and all $f \in \tilde{\mathbb{M}}(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$:

$$
\begin{aligned}
&< Q(f), \phi) \\
(4.1) \qquad &= \Big(\int (r^2 + r^{*2}) E(r, r^*, \mathbf{v}, \mathbf{v}^*) \Big((\frac{1}{r^3} + \frac{1}{r^{*3}}) \phi^\# - \frac{\phi}{r^{*3}} - \frac{\phi^*}{r^3} \Big) \; dg \otimes dg^*
\end{aligned}
$$

The right handside of (4.1) defines a new kernel denoted by $\tilde{Q}(g)$. It will be convenient for us to give some of the arguments in terms of $\tilde{Q}$ or in terms of the - equivalent to (1.1) - equation

$$
(4.2) \qquad g_t = \tilde{Q}(g), \quad g(0, x, \mathbf{v}) = g_0(x, \mathbf{v}).
$$

Lemma 4.2 *Any solution g of the new problem (4.2) such that there exists $r_0 > 0$ for which*

$$
\mathrm{supp_r g}(0, .) \in [r_0, +\infty[
$$

satisfies the same property for all times. Furthermore, under this assumption the following estimates are true.

$$
(4.3) \qquad
\begin{aligned}
\int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} dg &= \int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} dg_0 \\
\int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} \mathbf{v} \; dg &= \int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} \mathbf{v} \; g_0 \\
\int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} \mathbf{v}^2 \; dg &\leq \int_{\mathbb{R}_r^+ \times \mathbb{R}_v^3} \mathbf{v}^2 \; dg_0
\end{aligned}
$$

PROOF. The first fact is intuitively obvious - if all radii are larger than r_0, coalescence will only increase them - and can be proved by using the same technique as for the velocity support in Lemma 3.1. Let us for instance prove the mass conservation, the impulsion conservation being similar. To prove the mass conservation, we have to prove that we may integrate 1 against $d|\tilde{Q}(g)|$, for $g \in \mathbb{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$. Namely, we have to prove that

$$
(4.4) \qquad \Big[\int (r^2 + r^{*2}) E(\frac{1}{r^3} + \frac{1}{r^{*3}}) \Big] \; dg \otimes dg^* < +\infty.
$$

Due to the fact that $r \geq r_0$, there is no problem with $r \to 0$. Thanks to Lemma 4.1, we have

$$
\frac{E}{r} \leq C, \qquad \frac{r^{*2} E}{r^3} \leq C.
$$

Hence the integral (4.4) is finite. Finally, the kinetic energy decay results from the convexity inequality : $r^{\#^3} \mathbf{v}^{\#^2} \leq r^3 \mathbf{v}^2 + r^{*3} \mathbf{v}^{*2}$. $\bullet$

For all positive numbers M and V let us consider the set $\mathcal{E}_{r_0,M,V}$ of all bounded measures g on $\mathbb{R}_r^+ \times \mathbb{R}_v^3$ such that

$$
(4.5) \qquad \mathrm{supp}_r g \in [r_0, +\infty[, \quad < g, 1 > \leq M \quad \text{and} \quad \forall \mathbf{v} \in \mathrm{supp}_{\mathbf{v}} g, \; |\mathbf{v}| \leq V.
$$

Proposition 4.1 *Given $r_0 > 0$, $V > 0$ and $M > 0$, the kernel $\tilde{Q}$ is globally Lipschitz in $\mathcal{E}_{r_0,M,V}$.*

PROOF. In order to bound $| < \tilde{Q}(h) - \tilde{Q}(g), \phi > |$ we use once again equation 4.1 and we have three terms to deal with, respectively containing $\phi^{\#}$, ϕ and ϕ^*. The idea is to factor out $\|\phi\|_\infty$ and to use Lemma 4.1. Examining, for example, the term containing ϕ we obtain

$$\begin{aligned} & \|\phi\|_\infty{}^{-1}\left|\int (r^2 + r^{*2})\frac{E}{r^{*3}} d(g+h) \otimes d(g^* - h^*)\phi\right| \\ \leq\ & 2C\left(\int r^{*-1} d(g+h) \otimes d|g^* - h^*|\right) \quad \text{by Lemma 4.1} \\ \leq\ & 2C\|g-h\|_{\mathcal{M}_b}\|g+h\|_{\mathcal{M}_b}. \end{aligned}$$

The other terms in $< \tilde{Q}(h) - \tilde{Q}(g), \phi >$ may also be bounded by similar quantities. The mass conservation implies the result. $\bullet$

PROOF OF THEOREM 2.2. It is now a mere consequence of Cauchy-Lipschitz Theorem. $\bullet$

Corollary 4.1 *Let g be a solution of (4.2). Then $t \mapsto \tilde{Q}(g(t))$ is globally Lipschitz in $\mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$.*

The Cauchy Problem (4.2) defines a nonlinear semigroup $\mathcal{T}(t)$. Because of Lemma 4.1, the family $\{\mathcal{T}(t)g_0,\ t \geq 0\}$ is relatively compact in $\mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$ weak $*$. Call $\omega(g_0)$ the set of all limiting values of $f(t)$ - although it is not an ω-limit set in the usual of dynamical systems sense. Our aim is to prove the existence of a measure $\mu^\infty(r)$ and a vector $\mathbf{v}^\infty$ such that $\omega(f_0)$ is reduced to $\mu(r) \otimes \delta_{\mathbf{v}=\mathbf{v}^\infty}$. This will imply Theorem 2.3. As an introduction to Theorem 2.3 we study the equilibrium states of Equation (4.2).

Lemma 4.3 *If $g \in \mathcal{M}_b(\mathbb{R}_r^+ \times \mathbb{R}_v^3)$, with support in r bounded away from 0 and bounded support in $\mathbf{v}$, is such that $\tilde{Q}(g) = 0$, then g has the form $\mu(r) \otimes \delta_{\mathbf{v}=\mathbf{v}^0}$.*

PROOF. If $\tilde{Q}(g) = 0$, this is the same for $< \tilde{Q}(g), \mathbf{v}^2 >$ - recall that $\mathrm{supp}_v\tilde{Q}(g)$ is also bounded - which reads

$$\int B\left(\left(\frac{1}{r^3} + \frac{1}{r^{*3}}\right)\mathbf{v}^{\#2} - \frac{\mathbf{v}^2}{r^{*3}} - \frac{\mathbf{v}^{*2}}{r^3}\right) dg \otimes dg^* = 0$$

and notice that, by convexity, the integrand is nonpositive. Assume $supp_{\mathbf{v}}g$ not to be reduced to one point. Then we may find two compact sets K_1 and K_2 such that their projections on $\mathbb{R}_v^3$ are disjoint, and two constants $\delta_1 > 0$, $\delta_2 > 0$, such that

$$\int_{K_1} dg \geq \delta_1, \quad \int_{K_2} dg \geq \delta_2;$$

the strict convexity of $\mathbf{v} \mapsto \mathbf{v}^2$ implies the existence of $\delta > 0$ such that:

$$(4.6) \qquad \forall ((r,\mathbf{v}),(r^*,\mathbf{v}^*)) \in K_1 \times K_2, \quad (\frac{1}{r^3}+\frac{1}{r^{*3}})\mathbf{v}^{\#2} - \frac{\mathbf{v}^2}{r^{*3}} - \frac{\mathbf{v}^{*2}}{r^3} \leq -\delta.$$

Hence there holds:

$$\begin{aligned} < \tilde{Q}(g), \mathbf{v}^2 > &\leq \int_{\substack{(r,\mathbf{v}) \in K_1 \\ (r^*,\mathbf{v}^*) \in K_2}} B((\frac{1}{r^3}+\frac{1}{r^{*3}})\mathbf{v}^{\#2} - \frac{\mathbf{v}^2}{r^{*3}} - \frac{\mathbf{v}^{*2}}{r^3}) dg \otimes dg^* \\ &\leq -C\delta \end{aligned}$$

which is a contradiction. •

We may now proceed to the

PROOF OF THEOREM 2.3. Recall that, for all $t \geq 0$, the set $supp_{\mathbf{v}} g(t)$ is included in a fixed compact of $\mathbb{R}^3_v$. We start with the formula

$$< g(t), \mathbf{v}^2 > - < g_0, \mathbf{v}^2 > = \int_0^t < \tilde{Q}(g(s)), \mathbf{v}^2 > \; ds \leq 0;$$

this implies the convergence of the integral $\int_0^{+\infty} < \tilde{Q}(gs)), \mathbf{v}^2 > \; ds$. Because $t \mapsto < \tilde{Q}(g(t)), \mathbf{v}^2 >$ is globally Lipschitz and nonpositive we have

$$(4.7) \qquad \lim_{t \to +\infty} < \tilde{Q}(g(t)), \mathbf{v}^2 > = 0.$$

Let us also notice that

$$(4.8) \qquad \forall \rho > 0, \quad \frac{d}{dt} < g(t), \mathbf{1}_{[0,\rho]} > \leq 0.$$

This is derived from formula (3.5): if $\phi(r,\mathbf{v}) = \mathbf{1}_{[0,\rho]}(r,\mathbf{v})$, then $\phi^{\#} - \phi - \phi^* \leq 0$ as soon as we have $r \in [0,\rho]$ or $r^* \in [0,\rho]$. If none of them is in $[0,\rho]$, neither is $r^{\#}$ and $\phi^{\#} - \phi - \phi^* = 0$.

Let us now assume that not all the mass goes to $r = +\infty$**.** Then $\omega(g_0)$ - taken in the weak $*$ sense - is non void. Let us first prove that $\mathrm{supp}_v g(t)$ shrinks to a single vector as $t \to +\infty$: because of Lemma 4.2, we have $\mathrm{supp}_r f \in [r_0, +\infty[$ for all element f of $\omega(g_0)$. Therefore, $supp_{\mathbf{v}} g(t)$ did not shrink to a single vector as $t \to \infty$, there would exist:

- a subsequence $(t_n)_n$ going to $+\infty$,

- two sequences of compact subsets K_1^n and K_2^n - possibly reduced to one point each,

- two positive constants $\delta > 0$ and $\varepsilon > 0$

such that

- there holds $\int_{K_i^n} dg(t_n) \geq \delta$,

- the distance between the projections of K_1^n and K_2^n on $\mathbb{R}^3_{\mathbf{v}}$ is larger than ε. Arguing as in Lemma 4.3, we would prove that $< \tilde{Q}(g(t_n)), \mathbf{v}^2 >$ is bounded from above by a negative constant: a contradiction with (4.7).

As a consequence, every element f of $\omega(g_0)$ has the form $\mu_f(r) \otimes \delta_{\mathbf{v}_f}$. Consider $f \in \omega(g_0)$ and a sequence $(t_n)_n$ such that $\mathcal{T}(t_n) g_0$ converges to f weak $*$. There exists a sequence $(\nu_n)_n$ going to 0 such that

$$\lim_{n \to +\infty} \int_{\mathbb{R}_+ \times \{|\mathbf{v} - \mathbf{v}_f| \geq \nu_n\}} dg(t_n) = 0. \tag{4.9}$$

We have

$$\begin{aligned}
& |< g(t_n), \mathbf{v} > - < g(t_n), 1 > \mathbf{v}_f| \\
= \ & |\int (\mathbf{v} - \mathbf{v}_f) \ dg(t_n)| \\
\leq \ & 2 \max_{\mathbf{v} \in \mathrm{supp}_{\mathbf{v}} g_0} |\mathbf{v}| \int_{\mathbb{R}_+ \times \{|\mathbf{v} - \mathbf{v}_f| \geq \nu_n\}} dg(t_n) + \int_{\mathbb{R}_+ \times \{|\mathbf{v} - \mathbf{v}_f| \leq \nu_n\}} |\mathbf{v} - \mathbf{v}_f| \ dg(t_n) \\
\leq \ & < g(t_n), 1 > \nu_n + o(1) \quad \text{because of (4.9)}
\end{aligned}$$

On the other hand, we have $< g(t_n), \mathbf{v} > = < g_0, \mathbf{v} >$ and $< g(t_n), 1 > = < g_0, 1 >$. This implies

$$\forall f \in \omega(g_0), \qquad \mathbf{v}_f = \frac{< g_0, \mathbf{v} >}{< g_0, 1 >} := \mathbf{v}^\infty.$$

To conclude the proof, we set

$$F_t(\rho) = \int_{[0,\rho] \times \mathbb{R}^3_v} dg(t).$$

For all $t > 0$ we have $F_t \in L^\infty(\mathbb{R}^+_\rho)$. From (4.8), it converges pointwise. Let $F^\infty(\rho)$ denote the limiting function. Note that the function F_t has bounded total variation, uniformly with respect to t. This implies automatically that F^∞ belongs to $BV(\mathbb{R}^+_\rho)$.

Let now $\phi(r)$ belong to $C^1_c(\mathbb{R}^+)$; by definition we have

$$< g(t), \phi > = \int_0^{+\infty} \phi \ dF_t = - \int_0^{+\infty} F_t \phi' \ dr.$$

Since F_t is uniformly bounded, we may pass to the limit $t \to +\infty$ and obtain

$$\lim_{t \to +\infty} < g(t), \phi > = - \int_0^{+\infty} F_\infty \phi' \ dr. \tag{4.10}$$

On the other hand, let us consider $f \in \omega(g_0)$; we have $f = \mu_f \otimes \delta_{\mathbf{v}=\mathbf{v}^\infty}$. Let us consider a sequence $(t_n)_n$ such that $< g(t_n), \phi > \to < f, \phi >$; we have for all $\phi \in C_c^1(\mathbb{R}^+)$

$$< \mu_f, \phi > = -\int_0^{+\infty} F_\infty \phi' \, dr. \tag{4.11}$$

Set

$$\mu^\infty(\rho) = \frac{dF^\infty}{d\rho}(\rho), \tag{4.12}$$

the derivative being meant in the distributional sense. Equation (4.11) implies $\mu_f = \mu^\infty$ for all $f \in \omega(g_0)$. Hence $\omega(g_0)$ is reduced to one element. •

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