Negative definite functions. Integral representations independent of a Lévy function and related problems

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Abstract

In this paper we give a unitary method for obtaining Lévy Khinchin type formulas for negative definite functions.

We obtain integral representations, independent of a Lévy function, for negative definite functions, with real part bounded below, defined on a commutative involutive semigroup, and for continuous negative definite functions defined on the group $\mathbb{R}^n$.

We reobtain integral representations for continuous negative definite functions defined on the semigroup $\mathbb{R}_+^n$.

**Key words:** positive definite function, negative definite function, Radon measure, commutative involutive semigroup, quadratic form.
1 Introduction

The negative definite functions occur in probability theory and in potential theory. Their integral representation, known as Lévy-Khinchine formula, depends on a Lévy function (see [2], p. 108, Theorem 3.19 and [7], p. 316, Theorem 8).

The existence of Lévy function is proved in [8] for locally compact commutative groups and in [4] for commutative involutive semigroups.

We give, in Section 3 of this paper, integral representations, for negative definite functions with real part bounded below, defined on a commutative involutive semigroup, which characterize these functions and are independent of a Lévy function. These integral representations can also be obtained using [2], p. 108, Theorem 3.19, but the proof from this paper does not depend on a Lévy function and gives a new method for treating Lévy Khinchin type formulas.

To obtain the integral representations we give in Section 2 a result inspired of Choquet theory on adapted cones (see [5]).

We also use the result of Section 2 to reobtain, in Section 3, the quadratic forms on the semigroup \((\mathbb{N}^2, +)\) with the involution \((m, n)^* = (n, m)\) (see [2], p. 117, Lemma 4.13).

A function \(f : [0, \infty] \rightarrow \mathbb{R}\) defined by

\[
f(x) = C + ax + \int_{-\infty}^{0} (1 - e^{xy})d\mu(y)
\]

where \(C, a \in [0, \infty]\) and \(\mu\) is a positive Radon measure on \([-\infty, 0]\) such that the function \(x \rightarrow \frac{x}{1+x}\) is \(\mu\) integrable, is called a Bernstein function (cf. [3], p. 64, 9.8). In Section 4 of this paper, we give a generalization for Bernstein functions by completing a result of Berg ([1], p. 86, 3.2). Using the method of Section
3 we also obtain in Section 4 a new proof for the integral representation of the negative definite functions defined on the semigroup $\mathbb{R}_+^n$ ([1], p. 81, 3.1). This integral representation depends on a Lévy function.

In Section 5 we consider the continuous negative definite functions defined on the group $\mathbb{R}^n$ and we give integral representations for these functions which are also independent of a Lévy function. (see [6] for the classical Lévy-Khintchin formula on $\mathbb{R}^n$).

2 A representation theorem

Let $X$ be a locally compact Hausdorff space. We denote by $C(X)$ the set \{ $f : X \to \mathbb{R}$| $f$ continuous and with compact support \} and by $C_+(X)$ the set \{ $f \in C(X)$| $f \geq 0$\}.

**Theorem 1** Let $V$ be a linear space of real continuous functions on $X$, such that $V \supset C(X)$, and $L : V \to \mathbb{R}$ a linear functional, such that $L(f) \geq 0$ for every $f \in V_+$, where $V_+ = \{ f \in V$| $f \geq 0$\}. The restriction of $L$ to $C(X)$ is a positive Radon measure $\mu$ with the following properties:

(i) every function of $V_+$ is $\mu$ integrable and we have $L(f) \geq \int_X f(x)d\mu(x)$ for $f \in V_+$;

(ii) if we denote by $M$ the set \{(f, h) \in V \times V_+\} there is a compact $K \subset X$ such that $|f(x)| \leq h(x)$ for $x \in X - K$ we have $\int_X f(x)d\mu(x) = L(f)$ for every $f \in V$ which satisfies the following condition: for each $\epsilon > 0$ there is a function $h \in V_+$, with $L(h) < \epsilon$, such that $(f, h) \in M$. 
Proof. If \( f \in V_+, g \in C_+(X) \) and \( f \geq g \), then

\[
L(f) \geq L(g) = \mu(g).
\]

It results that \( f \) is \( \mu \) integrable and \( L(f) \geq \mu(f) \), which proves (i).

Take \( \epsilon > 0 \) and \((f, h) \in M\) such that \( L(h) < \epsilon \). There exists a compact set \( K \subset X \) such that

\[
|f(x)| \leq h(x), \ x \in X \setminus K.
\]

There also exists a compact \( K' \subset X \) such that \( \int_{X \setminus K'} |f|d\mu \leq \epsilon \).

We choose a continuous function \( \varphi : X \to [0,1] \) with compact support such that \( \varphi(x) = 1 \) for \( x \in K \cup K' \). We have

\[
-h \leq f - f\varphi \leq h.
\]

The positivity of \( L \) yields

\[
|L(f) - \int f\varphi d\mu| \leq \epsilon.
\]

We obtain

\[
|L(f) - \int f d\mu| \leq |L(f) - \int f\varphi d\mu| + |\int f\varphi d\mu - \int f d\mu| \leq 2\epsilon,
\]

which finishes the proof. \( \blacksquare \)

3 Integral representations for negative definite functions

Let \((S, +, \ast)\) be a commutative involutive semigroup with neutral element 0 (see [2], p. 86). We say that a function \( \varphi : S \to \mathbb{C} \) is positive definite on \( S \) if for
each natural number $n \geq 1$, each family $c_1, \ldots, c_n$ of complex numbers and each family $x_1, \ldots, x_n$ of elements of $S$, we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \geq 0.$$ 

A function $\varphi : S \to \mathbb{C}$ is hermitian if $\varphi(x^*) = \overline{\varphi(x)}$ for each $x \in S$.

We say that a hermitian function $\varphi : S \to \mathbb{C}$ is negative definite on $S$ if for each natural number $n \geq 2$, each family $c_1, \ldots, c_n$ of complex numbers, such that $c_1 + \ldots + c_n = 0$, and each family $x_1, \ldots, x_n$ of elements of $S$, we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \leq 0.$$ 

We denote by $\Lambda$ the set $\{ \rho : S \to \mathbb{C} | \rho(0) = 1; \rho(x^*) = \overline{\rho(x)}, x \in S; \rho(x+y) = \rho(x)\rho(y), x, y \in S; |\rho(x)| \leq 1, x \in S \}$ and by $\Omega$ the set $\{ \rho \in \Lambda | \rho \not= 1 \}$.

With the product topology $\Lambda$ is a compact space and $\Omega$ is a locally compact space.

**Theorem 2** For a function $\varphi : S \to \mathbb{C}$ the following conditions are equivalent:

(i) $\varphi$ is negative definite on $S$ and has real part bounded below;

(ii) there are a real number $C$, a function $q : S \to [0, \infty]$, such that 

$$q(x) + q(y) = \frac{1}{2} (q(x+y) + q(x^*+y^*)), x, y \in S,$$

and a positive Radon measure $\mu$ on $\Omega$, such that the functions

$$(\rho \mapsto (1 - \text{Re } \rho(x)))_{x \in S}$$

are $\mu$ integrable, which satisfy

$$\text{Re } \varphi(x) = C + q(x) + \int_{\Omega} (1 - \text{Re } \rho(x)) d\mu(\rho), x \in S.$$
and

\[- \text{Im} \, \varphi(x + y) + \text{Im} \, \varphi(x) + \text{Im} \, \varphi(y) = \int_{\Omega} (\text{Im} \, \rho(x + y) - \text{Im} \, \rho(x) - \text{Im} \, \rho(y)) d\mu(\rho).\]

\(C, q\) and \(\mu\) are uniquely determined by \(\varphi\).

**Proof.** (i) \(\Rightarrow\) (ii). For every \(t \in ]0, \infty[\) the function \(\psi_t : S \to \mathbb{C}\) defined by
\[\psi_t(x) = e^{-t\varphi(x)}\]
is positive definite (cf. [2], p. 74, Theorem 2.2) and bounded.

It follows from [2], p. 93 Theorem 2.5 that for each \(t \in ]0, \infty[\) there is a positive Radon measure \(\mu_t\) on \(\Lambda\) such that
\[e^{-t\varphi(x)} = \int_{\Lambda} \rho(x) d\mu_t(\rho), \ x \in S.\]

We denote by \(V\) the set
\[
\{ f : \Omega \to \mathbb{R} | f = F|_{\Omega}, \ F : \Lambda \to \mathbb{R}, \ F\ \text{continuous},
\ F(\theta) = 0, \ \lim_{t \to 0} \frac{1}{t} \int_{\Lambda} F(\rho) d\mu_t(\rho) \ \text{exists in} \ \mathbb{R}, \}
\]
where \(f = F|_{\Omega}\) means that \(f\) is the restriction of \(F\) to \(\Omega\) and \(\theta : S \to \mathbb{C}\) is defined by \(\theta(x) = 1\) for every \(x \in S\).

\(V\) is a real vector space and the function \(L : V \to \mathbb{R}\) defined by \(L(f) = \lim_{t \to 0} \frac{1}{t} \int_{\Lambda} F(\rho) d\mu_t(\rho)\) is a linear functional on \(V\) such that \(L(f) \geq 0\) for \(f \in V_+\).

Let \(\mathcal{F}\) denote the set of all families \((a_x)_{x \in S}\) of complex numbers such that \(a_x \neq 0\) only for finite number of \(x\) and which satisfy the relation \(\sum_{x \in S} a_x = 0\).

Let \(U\) denote the set
\[
\{ f : \Omega \to \mathbb{R} | f(\rho) = \sum_{x \in S} a_x \rho(x), \ (a_x)_{x \in S} \in \mathcal{F} \}.\]
$U$ is a real vector space. We shall prove that $U$ is a subspace of $V$. If we take 
$(a_x)_{x \in S} \in \mathcal{F}$ such that the function defined on $\Omega$ by

$$
\rho \mapsto \sum_{x \in S} a_x \rho(x)
$$

is in $U$ we have

$$
\sum_{x \in S} a_x \left( \frac{e^{-t\varphi(x)}}{t} - \frac{1}{t} \int_{\Lambda} \sum_{x \in S} a_x \rho(x) \, d\mu(\rho) \right).
$$

Letting $t$ tend to 0 we obtain that the function $(\rho \mapsto \sum_{x \in S} a_x \rho(x))$ is in $V$ and that

$$
L(\rho \mapsto \sum_{x \in S} a_x \rho(x)) = - \sum_{x \in S} a_x \varphi(x).
$$

Next we prove that $C(\Omega) \subset V$.

Let $f \in C(\Omega), f \neq 0$. We suppose that the compact support of $f$ is $A$.

We have $A \subset \Omega = \bigcup_{x \in S} \{ \rho \in \Omega | |1 - \rho(x)| > 0 \}$ and consequently we can find a natural number $n \geq 1$ and $a_1, \ldots, a_n$ elements of $S$ such that $\sum_{j=1}^{n} |1 - \rho(a_j)| > 0$ on $A$.

The function defined on $S$ by

$$
x \mapsto \frac{1}{t} \int_{\Lambda} \rho(x) \left( \sum_{j=1}^{n} |1 - \rho(a_j)|^2 \right) d\mu(\rho)
$$

is positive definite and it results from the inclusion $U \subset V$ that

$$
\lim_{t \to 0} \frac{1}{t} \int_{\Lambda} \rho(x) \left( \sum_{j=1}^{n} |1 - \rho(a_j)|^2 \right) d\mu(\rho)
$$

exists in $\mathbb{R}$. We denote by $u(x)$ this limit.

The function $u : S \to \mathbb{C}$ is positive definite and bounded. Using [2], p. 93, Theorem 2.5, we obtain a positive Radon measure $\nu$ on $\Lambda$ such that

$$
u(x) = \int_{\Lambda} \rho(x) d\nu(\rho), \quad x \in S
$$
The Theorem 2.11 from [2], p. 97 implies that we have

$$\lim_{t \to 0} \frac{1}{t} \int_{\Lambda} F(\rho) d\mu_t(\rho) = \lim_{t \to 0} \frac{1}{t} \int_{\Lambda} G(\rho) \left( \sum_{j=1}^{n} |1 - \rho(a_j)|^2 \right) d\mu_t(\rho)$$

$$= \int_{\Lambda} G(\rho) d\nu(\rho)$$

(where $G(\rho) = \frac{F(\rho)}{\sum_{j=1}^{n} |1 - \rho(a_j)|^2}$ for $\rho \in \Omega$ and $G(\theta) = 0$), which means that $f \in V$.

Let $x, y$ be elements of $S$ and $\epsilon$ a real number such that $0 < \epsilon < 1$. Let $K_{\epsilon,y}$ (resp. $K'_{\epsilon,y}$) be the compact $\{ \rho \in \Omega | \text{Re} \rho(y) \leq 1 - \epsilon \}$ (resp. $\{ \rho \in \Omega | |\text{Im} \rho(y) \geq \epsilon \}$).

If $x, y \in S$ we have

$$(1 - \text{Re} \rho(x))(1 - \text{Re} \rho(y)) \leq \epsilon(1 - \text{Re} \rho(x))$$

for $\rho \in \Omega - K_{\epsilon,y}$ and

$$|(1 - \text{Re} \rho(x))\text{Im} \rho(y)| \leq \epsilon(1 - \text{Re} \rho(x))$$

for $\rho \in \Omega - K'_{\epsilon,y}$.

Theorem 1 yields a positive Radon measure $\mu$ on $\Omega$ such that the elements of $V_+$ are $\mu$ integrable and we have

$$-\varphi(0) + \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x^*) \geq \int_{\Omega} (1 - \text{Re} \rho(x)) d\mu(\rho). \quad (1)$$

$$-\varphi(0) + \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x^*) + \frac{1}{2}\varphi(y) + \frac{1}{2}\varphi(y^*)$$

$$- \frac{1}{4}(\varphi(x + y) + \varphi(x^* + y) + \varphi(x + y^*) + \varphi(x^* + y^*)) \quad (2)$$

$$= \int_{\Omega} (1 - \text{Re} \rho(x))(1 - \text{Re} \rho(y)) d\mu(\rho).$$
\[
- \frac{1}{2i} (\varphi(y) - \varphi(y^*)) + \frac{1}{4i} (\varphi(x + y) + \varphi(x^* + y) - \varphi(x + y^*) - \varphi(x^* + y^*))
\]
\[
= \int_{\Omega} (1 - \Re \rho(x)) \Im \rho(y) d\mu(\rho). \tag{3}
\]

\[
- \frac{1}{2i} (\varphi(x) - \varphi(x^*)) + \frac{1}{4i} (\varphi(x + y) + \varphi(y^* + x) - \varphi(y + x^*) - \varphi(y^* + x^*))
\]
\[
= \int_{\Omega} (1 - \Re \rho(y)) \Im \rho(x) d\mu(\rho). \tag{4}
\]

If we denote by \( q : S \to \mathbb{R} \) the function defined by

\[ q(x) = -\varphi(0) + \Re \varphi(x) - \int_{\Omega} (1 - \Re \rho(x)) d\mu(\rho), \]

then the relation (1) gives \( q(x) \geq 0, \ x \in S \) and the formula (2) gives

\[ q(x) + q(y) = \frac{1}{2} (q(x + y) + q(x^* + y)). \]

From (3) and (4) we obtain

\[
- \frac{1}{2i} (\varphi(y) - \varphi(y^*)) - \frac{1}{2i} (\varphi(x) - \varphi(x^*)) + \frac{1}{2i} (\varphi(x + y) - \varphi(x^* + y^*))
\]
\[
= \int_{\Omega} (1 - \Re \rho(x)) \Im \rho(y) d\mu(\rho) + \int_{\Omega} (1 - \Re \rho(y)) \Im \rho(x) d\mu(\rho)
\]
\[
= \int_{\Omega} (\Im \rho(x) + \Im \rho(y) - \Im \rho(x + y)) d\mu(\rho)
\]

which gives the second integral formula from the theorem.

(ii) \( \Rightarrow \) (i). Let \( n \) be a natural number \( \geq 2, c_1, \ldots, c_n \) complex numbers such that \( c_1 + \ldots + c_n = 0 \) and \( x_1, \ldots, x_n \) elements of \( S \). If we have the integral representations of (ii) it follows that
\[
\sum_{j,k=1}^{n} c_j \tilde{c}_k \varphi(x_j + x^*_k) = \\
\sum_{j,k=1}^{n} c_j \tilde{c}_k \text{Re} \varphi(x_j + x^*_k) + \sum_{j,k=1}^{n} c_j \tilde{c}_k \text{Im} \varphi(x_j + x^*_k) = \\
\sum_{j,k=1}^{n} c_j \tilde{c}_k q(x_j + x^*_k) + \sum_{j,k=1}^{n} c_j \tilde{c}_k (\text{Im} \varphi(x_j) + \text{Im} \varphi(x^*_k)) \\
+ \int_{\Omega} (-|\sum_{j=1}^{n} c_j \rho(x_j)|^2 + \sum_{j,k=1}^{n} c_j \tilde{c}_k (\text{Im} \rho(x_j) + \text{Im} \rho(x^*_k))d\mu(\rho) \\
= \sum_{j,k=1}^{n} c_j \tilde{c}_k q(x_j + x^*_k) - \int_{\Omega} |\sum_{j=1}^{n} c_j \rho(x_j)|^2 d\mu(\rho) \leq 0
\]

because \( q \) is negative definite (cf. [2], p. 101, Theorem 3.9).

The unicity of \( \mu \) results from the equality:

\[-\varphi(x) + \varphi(x+y) + \varphi(x+y^*) = \varphi(x+2y) + 2\varphi(x+y+y^*) + \varphi(x+2y^*) = \int_{\Omega} \rho(x)(1 - \text{Re} \varphi(y))^2 d\mu(\rho), \ x, y \in S.\]

Unicity of \( q \) is a consequence of the unicity of \( \mu \) because \( C = \varphi(0) \). 

\[\blacksquare\]

**Remark 1** Choose a natural number \( n \geq 2, c_1, \ldots, c_n \) complex numbers and \( x_1, \ldots, x_n \) elements of \( S \). The function defined on \( \Omega \) by \( \rho \mapsto |\sum_{j=1}^{n} c_j \rho(x_j)|^2 \) is in \( V_+ \) and consequently if we have (i), we obtain

\[-\sum_{j,k=1}^{n} c_j \tilde{c}_k \varphi(x_j + x^*_k) \geq \int_{\Omega} |\sum_{j=1}^{n} c_j \rho(x_j)|^2 d\mu(\rho).\]

This proves that \( q \) is negative definite without using [2], p. 101, Theorem 3.9.

**Remark 2** It results from the proof of the theorem that we have \( \mu = 0 \) if and only if \( \varphi(x) = C + q(x) + i\ell(x) \), \( x \in S \), where \( C \) and \( q \) are as in the Theorem 2 and \( \ell : S \to \mathbb{R} \) is a function such that \( \ell(x+y) = \ell(x) + \ell(y) \), \( x, y \in S \), and \( \ell(x^*) = -\ell(x) \), \( x \in S \). This is Lemma 3.14 from [2], p. 105.
Remark 3 In the proof of Theorem 2 we have reobtained that \( \lim_{t \to 0} \frac{1}{t} \mu_t|_\Omega = \mu \) vaguely (cf. [2], p. 103, Lemma 3.12).

Proposition. Consider the semigroup \((\mathbb{N}^2, +)\) with the involution \((m, n)^* = (n, m)\). For a function \(\varphi : \mathbb{N}^2 \to \mathbb{C}\) the following conditions are equivalent:

(i) \(\varphi\) is negative definite and has real part bounded below;

(ii) there are real numbers \(C, \alpha, \beta\), such that \(\alpha, \beta \geq 0\), and a positive Radon measure \(\mu\) on \(\Omega = \{\rho \in \mathbb{C}| |\rho| \leq 1, \rho \neq 1\}\), such that the function \(\rho \mapsto 1 - \text{Re } \rho\) is \(\mu\) integrable, which satisfy

\[
\text{Re } \varphi(m, n) = C + (m + n)\alpha + (m - n)^2\beta + \int_{\Omega} (1 - \text{Re } \rho^m \bar{\rho}^n) d\mu(\rho).
\]

and

\[
\text{Im } \left(-\varphi(m + p, n + q) + \varphi(m, n) + \varphi(p, q)\right) = \int_{\Omega} \text{Im } (\rho^m \bar{\rho}^n + p^m \bar{p}^n - \rho^p \bar{\rho}^q) d\mu(\rho).
\]

\(C, \alpha, \beta\) and \(\mu\) are uniquely determined by \(\varphi\) and we have

\[
\alpha = -\varphi(0, 0) + \frac{1}{2}(\varphi(1, 0) + \varphi(0, 1)) - \frac{1}{8}(\varphi(2, 0) - 2\varphi(1, 1) + \varphi(0, 2)) - \int_{\Omega} (1 - \text{Re } \rho - \frac{1}{2}(\text{Im } \rho)^2) d\mu(\rho)
\]

and

\[
\beta = \frac{1}{4}(\varphi(2, 0) + \varphi(1, 1) + \varphi(0, 2)) - \int_{\Omega} (\text{Im } \rho)^2 d\mu(\rho).
\]

Proof. We denote by \(D\) the set \(\{t \in \mathbb{C}| |t| \leq 1\}\) and by \(\Lambda\) the set

\[
\{\rho : \mathbb{N}^2 \to \mathbb{C}| \rho(0, 0) = 1; \rho(m, n) = \overline{\rho(n, m)}; \\
\rho(m + p, n + q) = \rho(m, n) \cdot \rho(p, q); |\rho(m, n)| \leq 1, m, n, p, q \in \mathbb{N}\}
\]

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Let \( z \in D \). The function \( \rho_z : \mathbb{N}^2 \to \mathbb{C} \) given by \( \rho_z(m, n) = z^m \bar{z}^n \) is in \( \Lambda \) and the mapping \( z \mapsto \rho_z \) is a topological isomorphism of \( D \) onto \( \Lambda \). Using this isomorphism, the Proposition is a particular case of Theorem 2 and we only have to calculate \( q(m, n) \) where

\[
q(m, n) = -\varphi(0,0) + \text{Re} \varphi(m, n) - \int_{\Omega} (1 - \text{Re} \rho^m \bar{\rho}^n) d\mu(\rho).
\]

As in the proof of the Theorem 2, we notice that the function defined on \( \Omega \) by

\[
\rho \mapsto (1 - \text{Re} \rho)^m (\text{Im} \rho)^n
\]

is \( \mu \) integrable for \( m \geq 1 \) or \( n \geq 2 \) and we have

\[
L(\rho \mapsto (1 - \text{Re} \rho)^m (\text{Im} \rho)^n) = \int_{\Omega} (1 - \text{Re} \rho)^m (\text{Im} \rho)^n d\mu(\rho)
\]

for \( m \geq 2 \) or \( n \geq 3 \) or \( (m \geq 1 \) and \( n \geq 1 \)).

Using this and the binomial theorem, we obtain that

\[
L(\rho \mapsto 1 - \text{Re} \rho^m \bar{\rho}^n - (m+n)(1 - \text{Re} \rho - \frac{1}{2}(\text{Im} \rho)^2) - (m-n)^2(\text{Im} \rho)^2)
\]

\[
= \int_{\Omega} (1 - \text{Re} \rho^m \bar{\rho}^n - (m+n)(1 - \text{Re} \rho - \frac{1}{2}(\text{Im} \rho)^2) - (m-n)^2\frac{1}{2}(\text{Im} \rho)^2) d\mu(\rho)
\]

This is equivalent to \( q(m, n) = (m+n)\alpha + (m-n)\beta \), where

\[
\alpha = L(\rho \mapsto 1 - \text{Re} \rho - \frac{1}{2}(\text{Im} \rho)^2) - \int_{\Omega} (1 - \text{Re} \rho - \frac{1}{2}(\text{Im} \rho)^2) d\mu(\rho)
\]

and

\[
\beta = L(\rho \mapsto (\text{Im} \rho)^2) - \int_{\Omega} (\text{Im} \rho)^2 d\mu(\rho).
\]

We have \( 1 - \text{Re} \rho - \frac{1}{2}(\text{Im} \rho)^2 = \frac{1}{2}(1 - \text{Re} \rho)^2 + \frac{1}{2}(1 - (\text{Re} \rho)^2 - (\text{Im} \rho)^2) \geq 0 \),

which implies that \( \alpha \geq 0 \). That \( \beta \geq 0 \) is evident. This finishes the proof of the Proposition.

\[ \blacksquare \]

**Remark 4** The integral representation of the negative definite functions considered in the Proposition, which depends on a Lévy function, is in [2], p. 119, Proposition 4.15.
4 A generalization for Bernstein functions

In this section $\mathbb{R}^n_+ = ([0, \infty[)^n$ and $\langle , \rangle$ is the usual scalar product in $\mathbb{R}^n$. We will consider the semigroup $(\mathbb{R}^n_+ , +)$, and assume that this semigroup has identical involution.

**Theorem 3** For a function $\varphi : \mathbb{R}^n_+ \to \mathbb{R}$ the following conditions are equivalent:

(i) $\varphi$ is positive, continuous and negative definite on $\mathbb{R}^n_+$;

(ii) we have

$$\varphi(x) = C + \langle a, x \rangle + \int_{\Omega} (1 - e^{i\langle \rho, x \rangle}) d\mu(\rho), \; x \in \mathbb{R}^n_+$$

where $C \in [0, \infty[; a = (a_1, \ldots , a_n) \in \mathbb{R}^n_+$, $a_j \geq 0; \Omega = (-\infty, 0]^n \setminus (0, \ldots , 0)$ and $\mu$ is a positive Radon measure on $\Omega$ such that the function $\rho \mapsto \frac{\|\rho\|}{1+\|\rho\|}$ is $\mu$ integrable.

$C, a$ and $\mu$ are uniquely determined by $\varphi$.

**Proof.** (i) $\Rightarrow$ (ii) For every $t \in ]0, \infty]$ the function $\psi_t : \mathbb{R}^n_+ \to \mathbb{R}$ defined by $\psi_t(x) = e^{-t\varphi(x)}$ is continuous positive definite (cf. [2], p. 74, Theorem 2.2) and bounded. It follows from [2], p.115, Proposition 4.7, that for each $t \in ]0, \infty[$ there is a positive Radon measure $\mu_t$ on $]-\infty, 0]^n$ such that

$$e^{-t\varphi(x)} = \int_{]-\infty, 0]^n} e^{i\langle \rho, x \rangle} d\mu_t(\rho).$$

We denote by $V'$ the set

$$\{ f : \Omega \to \mathbb{R} | f = F|_\Omega, F : \mathbb{R} \setminus (0, \ldots , 0] \to \mathbb{R}, F \text{ continuous},$$

$$F(0, \ldots , 0) = 0, \lim_{t \to 0} \frac{1}{t} \int_{]-\infty, 0[^n} F(\rho) d\mu_t(\rho) \text{ exists in } \mathbb{R} \}. $$
Let \( L' : V' \to \mathbb{R} \) be the function defined by
\[
L'(f) = \lim_{t \to 0} \frac{1}{t} \int_{[-\infty,0]^n} F(\rho) \, d\mu_t(\rho).
\]
Let \( \mathcal{F} \) denote the set of all families \((a_x)_{x \in \mathbb{R}^+_0}\) of real numbers such that \( a_x \neq 0 \) only for a finite number of \( x \) and which satisfy the relation
\[
\sum_{x \in \mathbb{R}^+_0} a_x = 0.
\]
Let \( U \) denote the set
\[
\{ f : \Omega \to \mathbb{R} | f(\rho) = \sum_{x \in \mathbb{R}^+_0} a_x e^{\langle \rho,x \rangle}, (a_x)_{x \in \mathbb{R}^+_0} \in \mathcal{F} \}.
\]
We obtain as in the proof of Theorem 2 that \( U \) is a subspace of \( V \) and that
\[
L'(\rho \mapsto \sum_{x \in \mathbb{R}^+_0} a_x e^{\langle \rho,x \rangle}) = -\sum_{x \in \mathbb{R}^+_0} a_x \varphi(x),
\]
if the function \( \rho \mapsto \sum_{x \in \mathbb{R}^+_0} a_x e^{\langle \rho,x \rangle} \) is an element of \( U \).

If \( \rho = (\rho_1, \ldots, \rho_n) \), we have
\[
\lim_{t \to 0} \frac{1}{t} \int_{[-\infty,0]^n} \sum_{j=1}^n (1 - e^{\rho_j}) \, d\mu_t(\rho) = -n \varphi(0, \ldots, 0) + \varphi(1, 0, \ldots, 0) + \cdots + \varphi(0, \ldots, 0, 1)
\]
It results from [2], p. 52, Proposition 4.6 that there is a sequence \((t_k)_{k \in \mathbb{N}} \subseteq [0, 1]\),
with \( \lim_{k \to \infty} t_k = 0 \), such that the sequence
\[
\left( \frac{1}{t_k}(\rho \mapsto \sum_{j=1}^n (1 - e^{\rho_j})) \mu_{t_k} \right)_{k \in \mathbb{N}}
\]
converges vaguely.

We define
\[
V = \{ f : \Omega \to \mathbb{R} | f = F|_\Omega, F : [-\infty, 0]^n \to \mathbb{R},
\]
\( F \) continuous, \( F(0, \ldots, 0) = 0 \), \( \lim_{k \to \infty} \frac{1}{t_k} \int_{[-\infty,0]^n} F(\rho) \, d\mu_{t_k} \) exists in \( \mathbb{R} \).
We also define $L : V \to \mathbb{R}$ by

$$L(f) = \lim_{k \to \infty} \frac{1}{t_k} \int_{[-\infty,0]^n} F(\rho) d\mu_k(\rho).$$

We have $V' \subset V$ and $L|_{V'} = L'$. If we take $f \in C(\Omega)$, we obtain that

$$\lim_{k \to \infty} \frac{1}{t_k} \int_{[-\infty,0]^n} F(\rho) d\mu_k(\rho) = \lim_{k \to \infty} \frac{1}{t_k} \int_{[-\infty,0]^n} G(\rho) \sum_{j=1}^n (1 - e^{\rho_j}) d\mu_k(\rho)$$

(where $G(\rho) = \frac{F(\rho)}{\sum_{j=1}^n (1 - e^{\rho_j})}$ for $\rho \in \Omega$ and $G(0, \ldots, 0) = 0$) exists in $\mathbb{R}$. This proves that $C(\Omega) \subset V$.

We have, using Taylor’s formula,

$$\lim_{\rho \to (0, \ldots, 0)} \frac{1 - e^{\rho(x)} - \sum_{j=1}^n (1 - e^{\rho_j}) x_j}{\sum_{j=1}^n (1 - e^{\rho_j})} = 0,$$

where $x \in \mathbb{R}_+^n$ and $\rho = (\rho_1, \ldots, \rho_n)$.

If we take $x, \alpha \in \mathbb{R}_+^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$, then it is easy to see that for each $\epsilon > 0$ there exists a compact set $K \subset \Omega$ such that

$$|e^{\rho(\alpha)} (1 - e^{\rho(x)}) - \sum_{j=1}^n (1 - e^{\rho_j}) x_j| \leq \epsilon \sum_{j=1}^n (1 - e^{\rho_j}) \quad \text{for } \rho \in \Omega \setminus K.$$

Theorem 1 yields a positive Radon measure on $\Omega$ such that the functions $(\rho \mapsto (1 - e^{\rho(x)}))_{x \in \mathbb{R}_+^n}$ are $\mu$ integrable and we have

$$-\varphi(0) + \varphi(x) \geq \int_{\Omega} (1 - e^{\rho(x)}) d\mu(\rho), \quad x \in \mathbb{R}_+^n.$$  \hfill (5)

$$-\varphi(\alpha) + \varphi(\alpha + x) - \sum_{j=1}^n x_j (-\varphi(\alpha) + \varphi(\alpha + e_j))$$

$$= \int_{\Omega} e^{\rho(\alpha)} (1 - e^{\rho(x)} - \sum_{j=1}^n x_j (1 - e^{\rho_j})) d\mu(\rho).$$  \hfill (6)
$(x, \alpha \in \mathbb{R}^n_0, \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j > 0, (e_j)_{1 \leq j \leq n}$ the canonical base in $\mathbb{R}^n$).

If in (6) we let $\alpha$ tend to $(0, \ldots, 0)$, we obtain

$$\varphi(x) = \varphi(0) + \sum_{j=1}^{n} x_j a_j + \int_{\Omega} (1 - e^{(\rho, x)}) d\mu(\rho),$$

(7)

where $a_j = (-\varphi(0) + \varphi(e_j) - \int_{\Omega} (1 - e^{(\rho, x)}) d\mu(\rho)).$ Using (5), we obtain $a_j \geq 0$.

If we show that the function, $\rho \mapsto \frac{|\rho|}{1 + |\rho|}$ is $\mu$ integrable the formula (7) is the integral representation from (ii).

To this end, it is enough to prove that for a compact neighbourhood $O$ of the origin in $\mathbb{R}^n$, we have $\mu(\Omega \setminus O) < 0$.

Take $\rho \in \Omega$. We have

$$\int_{[0,1]^n} e^{(\rho, x)} dx = \prod_{\rho_j \neq 0} \left( \frac{1}{e^{\rho_j}} - 1 \right)$$

where $dx$ is Lebesgue measure in $\mathbb{R}^n$.

Consider the function $\psi : \Omega \to \mathbb{R}$ defined by

$$\psi(\rho) = 1 - \int_{[0,1]^n} e^{(\rho, x)} dx = \int_{[0,1]^n} (1 - e^{(\rho, x)}) dx$$

Using Fubini’s theorem and relation (5), we obtain

$$\int_{[0,1]^n} (-\varphi(0) + \varphi(x)) dx \geq \int_{[0,1]^n} \left( \int_{\Omega} (1 - e^{(\rho, x)}) d\mu(\rho) \right) dx = \int_{\Omega} \psi(\rho) d\rho$$

We have

$$\mu(\{ \rho \in \Omega | \rho_j \geq -2, 1 \leq j \leq n\}) \leq \mu(\{ \rho \in \Omega \mid \psi(\rho) \geq \frac{1}{2}\})$$

$$\leq 2 \int_{[0,1]^n} (-\varphi(0) + \varphi(x)) dx < \infty.$$  

This finishes the proof of the implication (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (i) is trivial.

Next we prove the assertion concerned with unicity.
We note that it is enough to prove the unicity of \( \mu \). We have the relation

\[
\int_{\Omega} e^{(\rho x + \alpha)} (1 - e^{(\rho y)})^2 d\mu(\rho) = -\varphi(x + \alpha) + 2\varphi(x + \alpha + y) - \varphi(x + \alpha + 2y)
\]

where \( x, y, \alpha \in \mathbb{R}^n_+ \), \( \alpha = (\alpha_1, \ldots, \alpha) \), \( \alpha_j > 0 \). Letting \( \alpha \to (0, \ldots, 0) \), we obtain

\[
\int_{\Omega} e^{(\rho x)} (1 - e^{(\rho y)})^2 d\mu(\rho) = -\varphi(x) + 2\varphi(x + y) - \varphi(2y).
\]

Now the unicity of \( \mu \) is a consequence of the unicity of the measure in [2], p. 115, Proposition 4.7. This completes the proof.

\[\square\]

**Theorem 4** For a function \( \varphi : \mathbb{R}^n_+ \to \mathbb{R} \) the following conditions are equivalent

(i) \( \varphi \) is continuous and negative definite on \( \mathbb{R}^n_+ \);

(ii) we have

\[
\varphi(x) = C + \langle \alpha, x \rangle - q(x) + \int_{\Omega} (1 - e^{(\rho x)}) - \sum_{j=1}^{n} x_j (1 - e^{(\rho_j)}) d\mu(\rho) ,
\]

where \( C \in \mathbb{R} \), \( \alpha \in \mathbb{R}^n \), \( q(x) = \sum_{j,k=1}^{n} p_{jk} x_j x_k \) is a positive quadratic form on \( \mathbb{R}^n \), \( \Omega = \mathbb{R}^n \setminus (0, \ldots, 0) \) and \( \mu \) is a positive Radon measure on \( \Omega \) such that the functions \( \langle \rho \to (1 - e^{(\rho x)})^2 \rangle_{x \in \mathbb{R}^n_+} \) are \( \mu \) integrable.

\( C, \alpha, q \) and \( \mu \) are uniquely determined by \( \varphi \).

**Proof.** For each \( t \in [0, \infty] \) the function \( \psi_t : \mathbb{R}^n_+ \to \mathbb{R} \) defined by \( \psi_t(x) = e^{-t\varphi(x)} \) is continuous positive definite. It follows from [2], p. 214, Theorem 5.8 that for each \( t \in [0, \infty] \) there is a positive Radon measure \( \mu_t \) on \( \mathbb{R}^n \) such that

\[
e^{-t\varphi(x)} = \int_{\mathbb{R}^n} e^{(\rho, x)} d\mu_t(x), \ x \in \mathbb{R}^n .
\]
We define $V', \mathcal{F}, U, L'$ as in the proof of Theorem 3.

We have
\[
\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^n} \sum_{j=1}^n (1 - e^{\rho})^2 d\mu_\epsilon(\rho) = \\
- n \varphi(0, \ldots, 0) + 2(\varphi(1, 0, \ldots, 0) + \ldots + \varphi(0, \ldots, 0, 1)) \\
- (\varphi(2, 0, \ldots, 0) + \ldots + \varphi(0, \ldots, 0, 2))
\]

There is a sequence $(t_k)_{k \in \mathbb{N}} \subset [0, 1]$, with $\lim_{k \to \infty} t_k = 0$, such that the sequence
\[
\left( \frac{1}{t_k}(\rho \mapsto \sum_{j=1}^n (1 - e^{\rho})^2)\mu_\epsilon \right)_{k \in \mathbb{N}}
\]
converges vaguely.

Now we define $V$ and $L$, as in the proof of Theorem 3 and obtain $U \subset V$ and $\mathcal{C}(\Omega) \subset V$.

We denote by $T : \mathbb{R}^n_+ \times \Omega \to \mathbb{R}$ the function defined by
\[
T(x, \rho) = 1 - e^{\rho x} - \sum_{j=1}^n x_j (1 - e^{\rho_j}) + \sum_{j=1}^n \frac{x_j (x_j - 1)}{2} (1 - e^{\rho_j})^2 + \sum_{j, k=1}^n \sum_{j \neq k} x_j x_k (1 - e^{\rho_j})(1 - e^{\rho_k}).
\]

Using Taylor’s formula, we have for each $x \in \mathbb{R}^n_+$
\[
\lim_{\rho \in \Omega, \rho \to (0, \ldots, 0)} \frac{T(x, \rho)}{\sum_{j=1}^n (1 - e^{\rho_j})^2} = 0.
\]

If we take $x, \alpha, \beta \in \mathbb{R}^n_+, \alpha_j > 0$ and $\beta_j = 1$, it is easy to see that for each $\epsilon > 0$ there is a compact $K \subset \Omega$ such that
\[
|e^{\rho x} T(x, \rho)| \leq \epsilon (\sum_{j=1}^n (1 - e^{\rho_j})^2 + (1 - e^{\rho x + \alpha + \beta})^2), \rho \in \Omega \setminus K.
\]

Theorem 1 yields a positive Radon measure $\mu$ on $\Omega$ such that
\[
- \sum_{x \in \mathbb{R}^n_+} a_x \varphi(x) \geq \int_{\Omega} \sum_{x \in \mathbb{R}^n_+} a_x e^{\rho x} d\mu(\rho),
\]
where

\[(a_x)_{x \in \mathbb{R}_+^n} \in \mathcal{F}, \text{ with } \sum_{x \in \mathbb{R}_+^n} a_x e^{(\rho,x)} \geq 0, \text{ for } \rho \in \Omega.\]

We also have \(L(\rho \mapsto e^{(\rho,x)}T(x,\rho)) = \int_{\Omega} e^{(\rho,x)}T(x,\rho)d\mu(y), x, \alpha \in \mathbb{R}_+^n, \alpha_j > 0.\)

Letting \(\alpha\) tend to \((0,\ldots,0)\), we obtain

\[L(T(x,\rho)) = \int_{\Omega} T(x,\rho)d\mu(\rho), x \in \mathbb{R}_+^n\]

which can be written

\[\varphi(x) = \varphi(0, \ldots, 0) + \sum_{j=1}^{n} x_j a_j - q(x) + \int_{\Omega} (1 - e^{(\rho,x)}) - \sum_{j=1}^{n} x_j (1 - e^{\rho_j})d\mu(\rho)\]

where \(q(x) = \frac{1}{2}(L(\rho \mapsto (\sum_{j=1}^{n} x_j \rho_j)^2) - \int_{\Omega} (\sum_{j=1}^{n} x_j (1 - e^{\rho_j})^2 d\mu(\rho)), a_j = -\varphi(0) + \varphi(e_j) + \frac{1}{2}(L(\rho \mapsto (1 - e^{\rho_j})^2) - \int_{\Omega} (1 - e^{\rho_j})^2 d\mu(\rho)).\) The function \(g : \Omega \to \mathbb{R}\) defined by \(g(\rho) = (\sum_{j=1}^{n} x_j (1 - e^{\rho_j})^2\) is in \(U\) and consequently the inequality (8) implies \(q(x) \geq 0, x \in \mathbb{R}_+^n.\) This completes the proof of implication \((i) \Rightarrow (ii).\) The implication \((ii) \Rightarrow (i)\) is immediate.

The unicity of \(\mu\) results from the relation

\[L(\rho \mapsto e^{(\rho,x)}(1 - e^{(\rho,y)})^4) = \int_{\Omega} e^{(\rho,x)}(1 - e^{(\rho,y)})^4 d\mu(\rho), x, y \in \mathbb{R}_+^n.\]

If \(q(x) = \sum_{j,k=1}^{n} p_{jk} x_j x_k,\) we have

\[p_{jk} = \frac{1}{2}(-\varphi(0) + \varphi(e_j) + \varphi(e_k) - \varphi(e_j + e_k) - \int_{\Omega} (1 - e^{\rho_j})(1 - e^{\rho_k})d\mu(\rho)).\]

This proves the unicity of \(q\) and finishes the proof because \(C = \varphi(0)\) and the unicity of \(a\) is also a consequence of the unicity of \(\mu.\)
5 Integral representations for continuous negative definite function on the groupe $\mathbb{R}^n$ 

**Theorem 5** For a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ the following conditions are equivalent:

(i) $\varphi$ is continuous and negative definite on $\mathbb{R}^n$;

(ii) there is a real number $C$, a positive quadratic form $q : \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$q(x) = \sum_{j,k=1}^{n} a_{jk} x_j x_k$$

with $a_{jk} \in \mathbb{R}$ and $a_{jk} = a_{kj}$, and a positive Radon measure $\mu$ on $\mathbb{R}^n \setminus \{0, \ldots, 0\} = \Omega$, such that the function $g : \Omega \rightarrow \mathbb{R}$ defined by

$$g(\rho) = \frac{||\rho||^2}{1 + ||\rho||^2}$$

is $\mu$ integrable, which satisfy

$$Re \varphi(x) = C + q(x) + \int_{\Omega} (1 - \cos(\rho, x))d\mu(\rho)$$

and

$$-Im \varphi(x + y) + Im \varphi(x) + Im \varphi(y) = \int_{\Omega} (\sin(\rho, x + y) - \sin(\rho, x) - \sin(\rho, y))d\mu(\rho)$$

$C, q$ and $\mu$ are uniquely determined by $\varphi$ and we have

$$a_{jj} = -\varphi(0) + \frac{1}{2}(\varphi(e_j) + \varphi(-e_j)) - \int_{\Omega} (1 - \cos \rho_j)d\mu(\rho)$$

and

$$a_{jk} = \frac{1}{8}(\varphi(e_j + e_k) + \varphi(-e_j - e_k) - \varphi(e_j - e_k) - \varphi(e_k - e_j))$$

$$- \frac{1}{2} \int_{\Omega} \sin \rho_j \sin \rho_k d\mu(\rho), \quad \text{for} \ j \neq k.$$

**Proof.** Using Bochner’s theorem in $\mathbb{R}^n$, we define the measures $(\mu_t)_{t \in [0, \infty[}$ as in Section 3. In this section $\mathcal{F}$ will be the set of all families $(a_x)_{x \in \mathbb{R}^n}$ of complex
numbers such that \( a_x \neq 0 \) only for a finite number of \( x \), which satisfy the relation 
\[
\sum_{x \in \mathbb{R}^n} a_x = 0.
\]
If we denote by \( V \) the set
\[
\{ f : \Omega \to \mathbb{R} \mid f = F|_\Omega, \; F : \mathbb{R}^n \to \mathbb{R}, \; F \text{ continuous}, \; 
F(0, \ldots, 0) = 0, \lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^n} F(\rho) d\mu_t(\rho) \text{ exists in } \mathbb{R} \},
\]
we obtain, as in Section 3, that the set:
\[
U = \{ f : \Omega \to \mathbb{R} \mid f(\rho) = \sum_{x \in \mathbb{R}} a_x e^{i\langle \rho, x \rangle}, (a_x)_{x \in \mathbb{R}^n} \in \mathcal{F} \}
\]
is included in \( V \). Using the classical Lévy's theorem, we also obtain that \( \mathcal{C}(\Omega) \subset V \). We will show that for each \( \beta > 0 \) the function \( h_\beta : \Omega \to [0, \infty[ \) defined by
\[
h_\beta(\rho) = \frac{1}{\beta^n} \int_{[0,\beta]^n} (1 - \cos(\rho, x)) dx. \text{ (} dx \text{ Lebesgue measure in } \mathbb{R}^n ) \text{ is in } V.
\]
First it is clear that
\[
\lim_{\rho \in \Omega, \rho \to (0, \ldots, 0)} h_\beta(\rho) = 0.
\]
If we take \( h_\beta(0, \ldots, 0) = 0 \), we have
\[
\frac{1}{t} \int_{\mathbb{R}^n} h_\beta(\rho) d\mu_t(\rho) = \frac{1}{\beta^n t} \int_{\mathbb{R}^n} \left( \int_{[0,\beta]^n} (1 - \cos(\rho, x)) dx \right) d\mu_t(\rho)
\]
\[
= \frac{1}{\beta^n t} \int_{[0,\beta]^n} \left( \int_{\mathbb{R}^n} (1 - \cos(\rho, x)) d\mu_t(\rho) \right) dx
\]
\[
= \frac{1}{\beta^n} \int_{[0,\beta]^n} (e^{-t\varphi(0)} - \frac{1}{2}e^{-t\varphi(x)} - \frac{1}{2}e^{-t\varphi(-x)}) dx
\]
Consequently,
\[
\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^n} h_\beta(\rho) d\mu_t(\rho) = \frac{1}{\beta^n} \int_{[0,\beta]^n} (\text{Re } \varphi(x) - \varphi(0)) dx
\]
and therefore \( h_\beta \in V \).

We define \( L \) as in Section 3. It results, from the continuity of \( \varphi \) in 0, that for \( \varepsilon > 0 \) there is a \( \beta_\varepsilon > 0 \) such that \( L(h_{\beta_\varepsilon}) \leq \varepsilon \). An elementary calculus shows that there is a real number \( M > 0 \), such that \( h_{\beta_\varepsilon}(\rho) \geq \frac{1}{2} \) for \( \| \rho \| \geq M \).
Choose $x$ and $y$ in $\mathbb{R}^n$. If we take $\gamma > 0$ such that $\|\rho\| \leq \gamma$ implies $1 - \cos\langle \rho, y \rangle \leq \epsilon$ we have

$$(1 - \cos\langle \rho, x \rangle)(1 - \cos\langle \rho, y \rangle) \leq \epsilon(1 - \cos\langle \rho, x \rangle) + 4h_{\beta}(\rho)$$

for $\|\rho\| \leq \gamma$ or $\|\rho\| \geq M$.

If we take $\delta > 0$ such that $\|\rho\| \leq \delta$ implies $|\sin\langle \rho, y \rangle| \leq \epsilon$ we have

$$|(1 - \cos\langle \rho, x \rangle) \sin\langle \rho, y \rangle| \leq \epsilon(1 - \cos\langle \rho, x \rangle) + 2h_{\beta}(\rho)$$

for $\|\rho\| \leq \delta$ or $\|\rho\| \geq M$.

Using the preceding inequalities and Theorem 1 we can obtain as in the proof of Theorem 2, the measure $\mu$ on $\Omega$ and the integral representations of Theorem 5.

We denote by $T : \mathbb{R}^n \times \Omega \to \mathbb{R}$ the function defined by

$$T(x, \rho) = 1 - \cos\langle \rho, x \rangle - \sum_{j=1}^{n} x_j^2 (1 - \cos \rho_j) - \sum_{j,k=1}^{n} x_j x_k \sin \rho_j \sin \rho_k,$$

and by $Q : \mathbb{R}^n \times \Omega$ the function defined by

$$Q(x, \rho) = \sum_{j=1}^{n} x_j^2 (1 - \cos \rho_j) + \sum_{j,k=1}^{n} x_j x_k \sin \rho_j \sin \rho_k.$$

The Taylor's formula implies that

$$\lim_{\rho \in \Omega, \rho \to \{0, \ldots, 0\}} \frac{T(x, \rho)}{\sum_{j=1}^{n} (1 - \cos \rho_j)} = 0, \quad x \in \mathbb{R}^n.$$

The function $\rho \mapsto T(x, \rho)$ is bounded and therefore using the preceding limit and a $h_{\beta}$ function we obtain, as before, that

$$L(\rho \mapsto T(x, \rho)) = \int_{\Omega} T(x, \rho) d\mu(\rho)$$
The function \( \rho \mapsto \sin \rho_j \sin \rho_k \) is \( \mu \) integrable, because the functions \( 1 - \cos 2 \rho_j \) and \( 1 - \cos 2 \rho_k \) are \( \mu \) integrable, and consequently the function \( \rho \mapsto Q(x, \rho) \) is \( \mu \) integrable for every \( x \in \mathbb{R}^n \). We have

\[
q(x) = -\varphi(0) + \frac{1}{2} \varphi(x) + \frac{1}{2} \varphi(-x) - \int_{\Omega} (1 - \cos \langle \rho, x \rangle) d\mu(\rho)
\]

\[
= L(\rho \mapsto T(x, \rho)) - \int_{\Omega} T(x, \rho) d\mu(\rho) + L(\rho \mapsto Q(x, \rho)) - \int_{\Omega} Q(x, \rho) d\mu(\rho)
\]

\[
= \sum_{j, k=1}^n a_{jk} x_j x_k
\]

where \( a_{jj} = L(\rho \mapsto (1 - \cos \rho_j)) - \int_{\Omega} (1 - \cos \rho_j) d\mu(\rho) \) and

\[
a_{jk} = a_{kj} = \frac{1}{2} \left( L(\rho \mapsto \sin \rho_j \sin \rho_k) - \int_{\Omega} \sin \rho_j \sin \rho_k d\mu(\rho) \right) \text{ for } j \neq k.
\]

Next we prove that the function \( \rho \mapsto \frac{\|\rho\|^2}{1 + \|\rho\|^2} \) is \( \mu \) integrable.

We have

\[
\int_{[0,1]^n} (-\varphi(0) + \frac{1}{2} \varphi(x) + \frac{1}{2} \varphi(-x)) dx
\]

\[
\geq \int_{[0,1]^n} \left( \int_{\Omega} (1 - \cos \langle \rho, x \rangle) d\mu(\rho) \right) dx
\]

\[
= \int_{\Omega} \left( \int_{[0,1]^n} (1 - \cos \langle \rho, x \rangle) dx \right) d\mu(\rho)
\]

\[
= \int_{\Omega} h_1(\rho) d\mu(\rho).
\]

Choosing a real number \( M \), such that \( h_1(\rho) \geq \frac{1}{2} \) for \( \|\rho\| \geq M \), the preceding inequality gives

\[
\mu(\{ \rho \in \Omega \| \rho \| \geq M \}) \leq 2 \int_{[0,1]^n} (-\varphi(0) + \frac{1}{2} \varphi(x) + \frac{1}{2} \varphi(-x)) dx.
\]

The limits \( \lim_{\rho_j \rightarrow 0} \frac{1 - \cos \rho_j}{\rho_j^2} = 1, j = 1, \ldots, n \) prove that the function \( \rho \mapsto \|\rho\|^2 \) is \( \mu \) integrable on a set of the form \( O \setminus (0, \ldots, 0) \), where \( O \) is a neighbourhood of the origin.
We also have proved that the function $\rho \mapsto \frac{||\rho||^2}{1+||\rho||^2}$ is $\mu$ integrable. If we notice that the unicity of $C$, $q$ and $\mu$ and the implication (ii) $\Rightarrow$ (i) can be proved as in Section 3 we finish the proof. □

**Remark 5** The inclusion $C(\Omega) \subset V$ results also from [3], p. 172, Proposition 18.2.
References


6 Negative definite functions on $\mathbb{N}^*$

For a function $\varphi : \mathbb{N}^* \to \mathbb{R}$ the following conditions are equivalent

1. $\varphi$ is negative definite on $\mathbb{N}^*$

2. there is a positive Radon measure $\mu$ on $\mathbb{R} \setminus \{0, 1\}$, such that every polynomial divisible by $x^2(1-x)^2$ is $\mu$ integrable, and real number $a, b, c$ such that $c \leq 0$, which satisfy

$$\varphi(2) \geq t(2) \quad \text{and} \quad \varphi(n) = E(n), \text{ for } n \geq 3,$$

where $E(n) = a + bn + cn^2 + \int_{\mathbb{R} \setminus \{0, 1\}} (x^4 - x^n + (n - 4)x^4(x - 1))dx$.

Proof. It is clear that $(ii) \Rightarrow (i)$. We will prove $(i) \Rightarrow (ii)$.

The set

$$V = \{P : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}| P \text{ polynomial function}, P(0) = P'(0) = P(1) = P'(1) = 0\}$$

is an adapted space.

The function $L : V \to \mathbb{R}$ defined by

$$L_{\varphi}(a_2x^2 + \ldots + a_nx^n) = -a_2\varphi(2) - \ldots - a_n\varphi(n)$$

is positive on $V_+$ because every element of $V_+$ can be expressed as a sum of the form $P_1^2 + P_2^2$ where $P_1$ and $P_2$ are polynomial functions (cf. [3]).

Let $P$ be polynomial function of degree $m$. We notice that for every $\varepsilon$ there is a compact $K \subset \mathbb{R} \setminus \{0, 1\}$ such that if $\varphi : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ is a continuous function with compact support which is 1 on $K$ the following inequality holds

$$|x^3(1-x)^3P(x)(1 - \varphi(x))| \leq \varepsilon(x^2(1-x)^2 + x^{2m+2})$$
for $x \in \mathbb{R} \setminus \{0, 1\} \setminus K$.

Theorem 1 and Proposition 1 yield a positive Radon measure on $\mathbb{R} \setminus \{0, 1\}$ such that

$$L(x \mapsto x^2(1 - x)^2Q(x)) \geq \int_{\mathbb{R} \setminus \{0, 1\}} x^2(1 - x)^2Q(x)d(x),$$

(9)

for every positive polynomial function $Q$, and

$$L(x \mapsto x^3(1 - x)^3P(x)) = \int_{\mathbb{R} \setminus \{0, 1\}} x^3(1 - x)^3P(x)d\mu(x),$$

(10)

for every polynomial function $P$.

The relation (10) gives for $n \geq 3$

$$= \frac{(n - 4)(n - 5)}{2}(\varphi(6) - 2\varphi(5) + \varphi(4)) = \int_{\mathbb{R} \setminus \{0, 1\}} (x^4 - x^n + (n - 4)(x - 1)x^4 + \frac{(n - 4)(n - 5)}{2}(x - 1)^2x^4)d\mu(x)$$

Consequently, we have for $n \geq 3$

$$\varphi(n) = a + bn + cn^2 + \int_{\mathbb{R} \setminus \{0, 1\}} (x - 1)^2x^4d\mu(x)).$$

Using (9) we obtain $c \leq 0$.

For $n = 2$ the polynomial

$$x^4 - x^n + (n - 4)(x - 1)x^4 + \frac{(n - 4)(n - 5)}{2}(x - 1)^2x^4$$

becomes

$$x^2(1 - x)^2(-1 - 2x + 3x^2).$$

The relation (10) implies that

$$\int_{\mathbb{R} \setminus \{0, 1\}} (x^2 - x)^3 = -\varphi(6) + 3\varphi(5) - 3\varphi(4) + \varphi(3)$$

(11)
Using (11) and the identity

\[ x^2(1 - x)^2(-1 - 2x + 3x^2) + 4x^3(1 - x)^3 = -x^2(1 - x)^4, \]

we obtain as a consequence of (9) the following relation

\[ \varphi(2) - E(2) = -L(x \mapsto x^2(1 - x)^4) + \int_{\mathbb{R}} (x^2(1 - x)^4) d\mu(x) \leq 0 \]

which completes the proof.