The Index of Cone Mellin Operators

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Abstract

We study topological invariants for elliptic Mellin operators in the cone, in particular, obtain the index formula.

Introduction

In the study of pseudodifferential operators with singularities in symbols or on singular manifolds, one encounters a hierarchy of local operators expressed in the terms of Mellin transform. These Mellin pseudodifferential operators represent the 'large' operator near singular points; generally, they supply additional (to the usual symbol) symbolic information about the problem, and it is in their terms the asymptotics of solutions, Fredholm properties and C^*- algebra structure are described.

This hierarchy starts with cone Mellin operators. They give a local model for operators on manifolds with conic points or with isolated singularities in the symbol (see [P],[S]). In addition, if the singularity is of edge type, Mellin cone operators act as symbols in operator-valued pseudodifferential calculus along the edge. In this context the information on the Fredholm property of cone Mellin operators, their index and invertibility becomes vital for the study of the problem in the large.

In [FS] the index formula for Mellin cone operators was obtained. The proof of the main topological index formula was reached via the analytic-algebraic formula representing the index as a rather complicated expression involving the symbol, its parametrix and thir derivatives up to a certain order.

In the present paper we give another, more direct proof for both topological and analytical index formulas, based on the topological analysis of the algebra of elliptic Mellin symbols. The reasoning uses ideas from the K-theory for operator algebras and follows major lines of [PR] and [R]. We study homotopies of cone Mellin symbols and establish connection between Mellin cone operators and a special class of Toeplitz operators with operator-valued symbols. The author is indebted to professors B. Fedosov and B.-W. Schulze for presenting the preprint of [FS] and enlightening discussions.

1. Cone Mellin Operators and the Index Formula

Following [FS], we consider Mellin operators on the cone $X^\wedge = (X \times \overline{\mathbb{R}}_+)/(X \times 0)$ where the base X is a smooth compact manifold of dimension n without boundary. The operators have the form

$$Au(t) = 1/(2\pi i) \int_\Gamma dz \int_0^\infty (t/t_1)^z a(t,z) u(t_1) dt_1/t_1, \tag{1.1}$$

where

$$u(t) \in C_0^\infty(\mathbb{R}_+, C^\infty(X, \mathcal{E})),$$

which means that $u(t)$ is a function with compact support whose values are sections of a vector bundle $\mathcal{E}$ over X. The line Γ is any fixed vertical line $\Gamma_\beta = \{\operatorname{Re} z = \beta\}$ in the complex plane (the choice of Γ corresponds to the choice of the weighted Sobolev space where the operator is considered), and we suppose, without loss in generality that $\beta = 0$, so Γ coincides with the imaginary axis.

The operator-valued Mellin symbol $a(t,z)$ belongs to the class ML^μ if it satisfies the following conditions.

$$1. \qquad a(t,z) \in C^\infty(\overline{\mathbb{R}}_+, L^\mu_{cl}(X, \Gamma)).$$

This means that a is a smooth function in $t \in \overline{\mathbb{R}}_+$ whose values are parameter dependent pseudodifferential operators of order μ on X with a parameter $z = i\tau, \tau \in \mathbb{R}$. In more details,

$$||\partial_t^\alpha \partial_z^\nu a(t,z)||_{H^s \to H^{s-\mu}} \le C(1+|z|)^{\mu-|\nu|}, \tag{1.2}$$

$$||\partial_t^\alpha \partial_z^\nu a(t,z)||_{H^s \to H^{s-\mu+|\nu|}} \le C, \tag{1.3}$$

with some constants C (whose dependence on the order of differentiation and spaces we suppress.)

As in [FS], we impose two more conditions on the symbol a:

2. For $t \in [0,c]$ and for $t \in [C,\infty)$ the symbol does not depend on t: $a(t,z) = a(0,z), t \in (0,c); a(t,z) = a(\infty,z), t \in (C,\infty)$.

3. The symbol $a(0,z)$ admits an analytical extension to a strip $S = \{|\operatorname{Re} z| < \varepsilon\}$ for some $\varepsilon > 0$ with uniform estimates of the form (1.2), (1.3).

Mellin symbols satisfying these conditions with $\mu \le 0$ generate, according to (1.1) operators which are bounded in the weighted Sobolev spaces $H^{s,(n+1)/2} \to H^{s-\mu,(n+1)/2}$ on the cone $X^\wedge$ (see [S]). Additionally, if $\mu < 0$ and $a(0,z) = a(\infty,z) = 0$, the operator A is compact in $H^{s,(n+1)/2}$.

The Mellin symbol $a \in ML^0$ is elliptic if, additionally, it is, for any t, an elliptic operator on X with parameter $z \in \Gamma$ (in other words, it is invertible for $|z|$ large enough, with estimates similar to (1.2), (1.3) for the inverse, with $\mu = 0$), the operator $a(0,z)$ is invertible for all z in the strip S and $a(\infty,z) = 1$. The class of such operators will be denoted by EML. According to [S], operators with symbols in EML are Fredholm in weighted Sobolev spaces. The following topological formula was proved in [FS].

For $a \in EML$, the operator family $a(t, z)$ parametrised by $(t, z) \in \overline{\mathbb{R}}_+ \times \Gamma$ consists of Fredholm operators and is invertible for (t, z) outside some compact set in $\mathbb{R}_+ \times \Gamma$. Thus, it defines an index bundle $\operatorname{ind} a \in K_c(\mathbb{R}_+ \times \Gamma)$, where K_c is K-functor with compact supports. The Chern character ch maps $K_c(\mathbb{R}_+ \times \Gamma)$ to $H_c^{2*}(\overline{\mathbb{R}}_+ \times \Gamma)$, and the formula

$$\operatorname{ind} A = \int_0^\infty \int_\Gamma ch(\operatorname{ind} a) \tag{1.4}$$

holds.

Having a parametrix $r_0(t, z)$ for the symbol a, so that $1 - r_0 a$ and $1 - a r_0$ are trace class and have compact support in $\mathbb{R}_+ \times \Gamma$, the general formula from [F] for the Chern character for Fredholm families gives

$$\operatorname{ind} A = \frac{1}{2\pi i} \int \operatorname{tr}((dr_0 + r_0 da r_0) \wedge da). \tag{1.5}$$

Now we describe our strategy of proving (1.4), (1.5). Consider the ideal $W = ML^{-1}$ in the algebra $\mathcal{A} = ML^0$. Symbols in W are compact for any (t, z) and are decaying for $z \to \infty$. If $a \in (1 + W) \cap EML$, the symbol $a(0, z)$, being an invertible operator family of the form $1 + T$, with compact T,$(T \in \mathcal{K})$, $T(z) \to 0$ at infinity, defines a class in the K- group in the sense of operator algebras:

$$\operatorname{ind}_z[a] \in \mathbf{K}_1(C_0(\Gamma) \otimes \mathcal{K}) = K^1(C_0(\Gamma)) = K_c^1(\Gamma) = \mathbb{Z}. \tag{1.6}$$

We will show that, for $a \in (1 + W) \cap EML$ the two K- theoretical indices just described, are connected by a homomorphism

$$\sigma^* : K_c^1(\Gamma) \to K_c^0(\mathbb{R}_+ \times \Gamma) \tag{1.7}$$

and this σ^* is nothing else than the connecting homomorphism ∂ in the exact K-theoretical sequence of the pair $(\mathbb{R}_+ \times \Gamma, \Gamma)$:

$$\ldots K^1(\mathbb{R}_+ \times \Gamma) \to K_c^1(\Gamma) \to K_c^0(\mathbb{R}_+ \times \Gamma) \to \tilde{K}^0(\mathbb{R}_+ \times \Gamma) \to \ldots$$

(the latter group is the reduced one). All of these groups are known: $K_c^1(\mathbb{R}_+ \times \Gamma) = 0, K_c^1(\Gamma) = \mathbb{Z}, K_c^0(\mathbb{R}_+ \times \Gamma) = \mathbb{Z}, \tilde{K}^0(\mathbb{R}_+ \times \Gamma) = 0$, so $\partial = \sigma^*$ is an isomorphism. Thus, for studying the index on $K_c^0(\mathbb{R}_+ \times \Gamma)$, it is sufficient to study it on $K_c^1(\Gamma)$. Here, the index map coincides with one for Toeplitz operators with operator-valued symbols $a(0, z)$, and can be described by a collection of formulas obtained in [PR].

This pattern, of course, requires certain detalisation which will be performed in the following sections. In particular, we have to justify the fact that the index of Mellin operator depends only on the topological index of its symbol as a Fredholm family. A similar fact for usual pseudodifferential operators is quite simple, however here, in the absence of appropriate

Garding inequality, we have to construct a special homotopy in the class of Mellin symbols. This homotopy, however, may turn out to be useful also in the further topological analysis of Mellin operators.

2. The ideal ML^{-1}

In this section we analyse the ideal of lower-order symbols.

Let $a_0(z) \in L^{-1}(X,\Gamma)$. We associate to a_0 a symbol $a(t,z) \in ML^{-1}$ in the following way. For a cut-off function $\rho(t) \in C_0^\infty(\overline{\mathbb{R}}_+), \rho = 1$ near 0, we set

$$a(t,z) = \rho(t) a_0(z). \tag{2.1}$$

This mapping generates a homomorphism

$$\sigma : GL(L^{-1}) \to EML,$$

where $GL(L^{-1})$ is as usual, the group of invertible matrices of the form $1 + a_0 \in 1 + L^{-1}(X,\Gamma)$ and for $1 + a_0 \in GL(L^{-1})$, we set $\sigma(1 + a_0) = 1 + \rho(t)a_0(z)$.

Proposition 2.1. *The mapping σ generates a homomorphism*

$$\sigma^* : K_c^1(\Gamma) \to K_c^0(\mathbb{R}_+ \times \Gamma), \tag{2.2}$$

so that $\operatorname{ind} \circ \sigma = \sigma^* \circ \operatorname{ind}_z$ *on* $GL(L^{-1})$.

PROOF. What we have to prove, is that the equivalences in $GL(L^{-1})$ and in EML defining the topological index, are respected by the mapping σ. First, directly from the definition, it follows that with the cut-off function ρ fixed, $\sigma(GL(L^{-1}(X,\Gamma)))$ is actually in EML. Next, if $a_0(z;\tau)$ is a stable homotopy of the symbol a_0 in $L^{-1}(X,\Gamma)$ so that $1 + a_0(z;\tau)$ is invertible, $1 + \rho(t)a_0(z,\tau)$ gives a stable homotopy of $\sigma(a_0)$ in the class of Mellin symbols on $\overline{\mathbb{R}}_+ \times \Gamma$. Thus, σ defines a homomorphism of K-groups in (2.2).■

Homomorphism σ does not depend on the choice of the cut-off function ρ: having two such functions ρ and ρ_1, we connect corresponding σ by a homotopy $\tau\rho + (1-\tau)r_1$. Moreover, the following property takes place.

Proposition 2.2. *Let $a(t,z) \in EML$, so that $a(t,z) - 1 \in ML^{-1}$. Denote by A and A_1 the Mellin cone operators corresponding to $a(t,z)$ and $a_1(t,z) = \sigma(a(0,z))$. Then*

$$a(t,z) - a_1(t,z) \in ML^{-1}, \tag{2.3}$$

$$\operatorname{ind}(a) = \operatorname{ind}(a_1), \tag{2.4}$$

$$\operatorname{Ind} A = \operatorname{Ind} A_1. \tag{2.5}$$

PROOF (2.3) follows from the fact that $a = a_1$ for small and large t, and, in addition, $a - a_1 = (a-1) - (a_1 - 1)$, i.e. the sum of two symbols in ML^{-1}. Then, $a_\tau = \tau a + (1-\tau)a_1$ is a homotopy of Fredholm families in the class EML, so it does not change the index of a family- this gives (2.4). As for (2.5), the difference $a - a_0$ is a symbol in $C_0^\infty(\mathbf{R}_+, L^{-1})$, and Mellin pseudodifferential operators with such symbols are compact.∎

Next we obtain the index formula for operators with symbols in $\sigma GL(L^{-1})$.

Proposition 2.3. *Let $2k+1 > n/N$, $a_0 \in GL(L^{-N})$, $a = \sigma a_0$. Then*

$$\operatorname{Ind} A = 1/(2\pi i) \int_\Gamma \operatorname{tr}((a_0 - 1)^{2k} a_0^{-1} da_0). \tag{2.6}$$

Formula (2.6) is a particular case of index formulas for Toeplitz operators with operator-valued symbols obtained in [PR, Sect.3]. To see this, note that operators of order $-N$ on the n-dimensional compact manifold X belong to the Shatten ideal Σ_n/N, with corresponding decay of the Shatten (quasi-)norm at infinity.

Theorem 2.4. *Let $a \in (1 + ML^{-1}) \cap EML$. Then for the index of A the formulas* (1.4), (1.5)*hold.*

PROOF. The equivalence of (1.4), and (1.5) is established in [F]. Take first $a \in (1 + ML^{-N}) \cap EML, N > n$. Then $a - 1$ is trace class, with trace norm decaying at infinity. Both parts of (1.5) are homotopy invariant, so, according to Propositions 2.1, 2.2 , we can take $a = \sigma(a(0,z))$ and $r_0 = \sigma(a(0,z)^{-1})$. Now, since both da and dr_0 are trace class forms, we can write the integrand in (1.5) as the sum $\operatorname{tr}(dr_0 \wedge da) + \operatorname{tr}(r_0 da \wedge r_0 da)$. The last term vanishes due to symmetrycity of trace and anti-symmetrycity of the exterior product under permutation. Then we have $\operatorname{tr}(dr_0 \wedge da) = d(\operatorname{tr}(r_0 \wedge da))$, and integration by parts gives that the right-hand side in (1.5) equals

$$1/(2\pi i) \int \operatorname{tr}(a(0,z)^{-1} d_z a(0,z)).$$

This exactly is the index of A, according to (2.6)(with $k = 0$).

The case of a general a follows from the homotopy

$$1 + b_\tau = (1+b)\exp(\tau[-b + b^2/2 - ... + (-b)^N/N]),$$

(the starting segment of the Taylor series for logarithm stands in brackets),, connecting $a = 1 + b = a_0 \in (1 + ML^{-1})$ with $a_1 = 1 + b_1 \in (1 + ML^{-N})$ in EML.∎

Proposition 2.5. *The homomorphism σ^* is the connecting homomorphism in the K-theoretical sequence* (1.8).

PROOF. If we were dealing with finite-dimensional operator-functions a, the assertion would directly follow from the definition of K-groups and of the connecting homomorphism. In fact, $K_c^1(\Gamma)$ is the group of equivalence classes of invertible matrices $h_0(z)$, coinciding with 1 for large $|z|$; $K_c^0(\mathbf{R}_+ \times \Gamma)$ is the group of equivalence classes of triples $(\mathcal{E}_1, \mathcal{E}_2, h)$, where $\mathcal{E}_1, \mathcal{E}_2$ are some vector bundles over $K_c^0(\mathbf{R}_+ \times \Gamma)$ and h is an isomorphism of these bundles over the complement of some compact set. The connecting mapping ∂ associates to h_0 the triple $(\mathcal{E}, \mathcal{E}, h)$, where $\mathcal{E}$ is the trivial bundle and h is the continuation of h_0 by 1 outside a compact. One can take $h(t,z) = 1 + \rho(t) h_0(z)$ as such continuation. For the general case of operator-valued symbols $a_0 \in GL(L^{-1})$ the theorem follows from the possibility of approximating any such a_0 by $1 \oplus h_0(z)$ with a finite-dimensional matrix h_0.■

Corollary 2.6. *The mapping σ^* in (2.2) is an isomorphism.*

Since $\sigma^* = \partial$, the property in question follows from the exact sequence (1.8), as explained in Sect.1.

3. The index formula on EML

The results of Sect.2 show that the index of Mellin cone operators with symbols in $EML \cap (1 + ML^{-1})$ is determined by the topological index of the symbol of the symbol in $K_c^0(\mathbb{R}_+ \times \Gamma)$. The general index formula (1.4), (1.5) will be established if we prove the same property for all symbols in EML.

Proposition 3.1. *The index of Mellin operator with symbol $a \in EML$ depends only on* $\operatorname{ind} a \in K_c^0(\mathbb{R}_+ \times \Gamma)$.

Granted this Proposition, we can prove

Theorem 3.2. *For operators with symbols in EML the index formulas (1.4), (1.5) hold.*

PROOF. Formulas (1.4), (1.5) are equivalent. Since σ^* in (2.2) is an isomorphism, for any given $a \in EML$ there exists $a_0 \in EML \cap (1 + ML^{-1})$ so that $\operatorname{ind} a = \operatorname{ind} a_0 \in K_c^0(\mathbb{R}_+ \times \Gamma)$. Proposition 3.1 then implies that $\operatorname{Ind} A = \operatorname{Ind} A_0$, and since (1.4) holds for A_0, it holds for A.■

To PROVE Proposition 3.1, we construct a symbol in $(1 + ML{-}1) \cap EML$ having the same topological and analytical indices, as the symbol a.

First, we apply Proposition 2.1.4/9 in [S], which gives, for any fixed t, an analytical symbol $a'(t,z)$ so than $a(t,\cdot) - a'(t,\cdot) \in L^{-\infty}(X,\Gamma)$ and $a(t,\cdot) - a'(t,\cdot + \beta) \in L^{-1}(X,\Gamma)$, for any β small enough. The construction in [S] implies, in particular, that this a' can be chosen depending smoothly on t. The symbols a and a' define the same class in $K_c^0(\mathbb{R}_+ \times \Gamma)$; the corresponding Mellin cone operators differ by a compact one. Thus, it is sufficient to prove our assertion for the symbol a' -and therefore we omit prime in the sequel.

The operator $a(t,z)$ is invertible for all (t,z) with $|z|$ big enough. Fix such z_0 and consider the symbol $\tilde{a}(t,z) = a(t,z)a(t,z_0)^{-1}$. This symbol belongs to the same class as a, has the same analytical and topological index, and, in addition, $1-\tilde{a}$ is a compact operator. It is sufficient to prove our Proposition for $\tilde{a}$, and we proceed with this, omitting tilde in notations.

For any fixed $s \in [c,C]$, the operator function $a(s,z)^{-1}$ has only a finite number of poles on the line Γ. So, we can find a $\beta = \beta(s)$ and an interval Δ_s around s, such that $a(t,z+\beta)$ is invertible for all $t \in \Delta_s, z \in \Gamma$. There exists a finite covering of $[c,C]$ by intervals $\Delta_{s_j} = \Delta_j, j \leq p$, with corresponding $\beta = \beta_j$. We chose the points $t_0 = c, ..., t_p = C$ so that $[t_{j-1}, t_j] \subset \Delta_j$. Set $\Delta_0 = [0, c+\delta), \Delta_{p+1} = (C-\delta, \infty)$. On each of Δ_j we construct the symbol $\tilde{b}_j$.

Define $h_j(z) = a(t_j, z+\beta_{j+1})a(t_j, z+\beta_j)^{-1}, 1 \leq j \leq p$. We set $\tilde{b}_0(t,z) = a(0,z)$ for $t \in \Delta_0$, $\tilde{b}(t,z) = a(t, z+\beta_j)a(t_j, z+\beta_j)^{-1}h_{j-1}(z)...h_1(z)a(0,z)$ for $j \leq p$ and $\tilde{b}_{p+1}(t,z) = 1 + \rho(t-C-\delta)(\tilde{b}_p(C,z)-1), t \in \Delta_{p+1}$. For any $t \leq C$, the symbols $\tilde{b}_j$ are invertible and for all t they differ only by a symbol in a class L^{-1} from $a(t,z)$. Additionally, at the point $t = t_j$, we have $\tilde{b}_j(t_j,z) = \tilde{b}_{j+1}(t_j,z)$. We will glue together $\tilde{b}_j$, to get a smooth symbol. Take a smooth partition of unity $\chi_j(t)$ corresponding to the covering $\{\Delta_0, \Delta_1, ...\Delta_{p+1}\}$ of $[0,\infty)$. Set

$$b(t,z) = \sum_{j=0}^{p+1} \tilde{b}_j(t,z)\chi_j(t).$$

Let us consider properties of $b(t,z)$. It is a symbol in ML^0, with

$$b - a \in C_0^\infty(\mathbb{R}_+ \times \Gamma, L^{-1}). \tag{3.1}$$

Additionally, if the supports of $\partial\chi_j$ are taken small enough, the norm of $b(t,z) - \tilde{b}_j(t,z), t \in \Delta_j, j \leq p$ can be made arbitrary small, so $b(t,z)$ is invertible for $t \leq C$. For $t \geq C$, the symbol $b(t,z)$ belongs to $1 + L^{-1}(X,\Gamma)$, and $b(t,z) = 1$ for t big enough. Thus, $b \in EML$ and due to (3.1), we have both $\operatorname{Ind} A = \operatorname{Ind} B$ and $\operatorname{ind} a = \operatorname{ind} b$, where B is the Mellin operator with symbol b.

Finally, we construct a homotopy in EML of the symbol b to a symbol $b' \in 1 + ML^{-1}$. We set

$$b_y(t,z) = b(y,z)(1-\chi(t-y)) + b(t,z)\chi(t-y), 0 \leq y \leq C,$$

where $\chi \in C^\infty(\mathbb{R})$ equals 1 on $[\delta,\infty)$ and 0 on $(-\infty,-\delta]$. If δ is taken small enough, b_y is invertible for all $t \leq C$. Thus, our homotopy is in EML and it connects b with the symbol $b' = b_C \in (1 + ML^{-1})$. This homotopy conserves both topological and analytical index. Applying the results of

Sect.2 to b', we see that $\operatorname{Ind} B'$ depends only on the $\operatorname{ind} b'$, therefore the same takes place for original symbol a.■

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