An Inequality in Kinetic Theory

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We discuss a kinetic inequality related to the Cauchy equation and with all solutions close to Maxwellians .

Consider a kinetic equation of the type

$$\frac{\partial}{\partial t} f + v \cdot \nabla_x f + E \cdot \nabla_v f = Q(f)$$

with f a nonnegative density, or equivalently

$$\frac{d}{dt} f^{\#} = Q(f)^{\#},$$

where # denotes evaluation along the characteristics. For simplicity we work in a 3D space domain, $x \in \mathbb{R}^3$, with velocities $v \in \mathbb{R}^3$ and with t the time variable. This equation models streaming (transport) of f driven by exterior forces (E) and collisions (Q). When there is mass conservation, formally $\int Q(f) dxdv = 0$. If an entropy function such as $\int f \log f dxdv$ makes sense (and E const.), then formally

$$\frac{d}{dt} \int f \log f dxdv = \int Q(f) \log f := D(f).$$

If the entropy dissipation term $D(f)$ is non-positive, then the entropy is decreasing. If, moreover, the mass and energy of the system are bounded in time, then the entropy has a lower bound and so $\int_0^\infty D(f) dt < \infty$. That estimate is fundamental in most proofs of convergence to equilibrium for such equations. In a number of cases the convergence results were pioneered by NSA approaches. In e.g. the Boltzmann equation case a consequence of this entropy dissipation bound is that the factor $f_1' f_2' - f_1 f_2$ in the D-integrand in a suitable sense converges to zero when $t \to \infty$. One is thus lead to the infinitesimal relation

$$f_1' f_2' - f_1 f_2 \approx 0 \quad \text{Loeb a.e. in } ns\,{}^*\text{domain.} \qquad \text{(IR)}$$

Here 1, 2 indicate two precollisional velocity variables (v_1, v_2), and the prime indicates the corresponding postcollisional velocity (v_1', v_2') in a binary collision.

What does the relation (IR) per se imply about f? I first studied that question already in the 1980's by a combination of geometric and Loeb measure techniques in order to better understand the time asymptotics of the Boltzmann equation. The problem got a renewed actuality by a question from C. Cercignani earlier this year "What can strictly be proved (in the standard context) of the type: $f_1'f_2' - f_1f_2$ small, implies f close to a Maxwellian". Here f belongs to a family of (say L^1_{loc}) functions, not necessarily having the extra structure of being a 1-parameter family $(f_t)_{t\in\mathbb{R}_+}$ representing the solution of some Cauchy problem for a Boltzmann equation with $\int_0^\infty D(f_t)dt < \infty$.

Starting from the above infinitesimal problem, I shall here give one type of answer to Cercignani's question and base the presentation on a recent approach due to [HA].

Theorem 1 ([HA]). *Suppose $f : {}^*\mathbb{R}^3 \to {}^*\mathbb{R}_+$ is *Lebesque measurable, S-integrable in $\{|v| \le \lambda\}$ for $\lambda \in ns{}^*\mathbb{R}_+$, and satisfying (IR) in $ns{}^*\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. Then for $n \in \mathbb{N}$, there exist internal functions $h_n : {}^*\mathbb{R}^3 \to {}^*\mathbb{R}_+$ such that $f - h_n \approx 0$ Loeb a.e. $ns{}^*\mathbb{R}^3$, h_n n times *differentiable in $ns{}^*\mathbb{R}^3$ with S-continuous derivatives and h_n satisfying (IR). Here $f_j = f(v_j)$, $f_j' = f(v_j')$, $j = 1, 2$,
$v_1' = v_1 - (\omega, v_2 - v_1)\omega$, $v_2' = v_2 + (\omega, v_2 - v_1)\omega$, $\omega \in {}^*S^2$, the unit sphere in ${}^*\mathbb{R}^3$.*

Corollary 2. *Under the hypotheses of Theorem 1, there is a standard function $g \in C^\infty$, such that $f - {}^*g \approx 0$ Loeb a.e. $ns{}^*\mathbb{R}^3, g_1' \cdot g_2' - g_1 \cdot g_2 \equiv 0$ in $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$.*

Proof of Corollary 2. If $g, \tilde{g}$ are standard continuous and ${}^*g - {}^*\tilde{g} \approx 0$ Loeb a.e. $ns{}^*\mathbb{R}^3$, then $g \equiv \tilde{g}$ in $\mathbb{R}^3$. Define $g_n(x) = {}^0h_n(x)$ for $x \in \mathbb{R}^3$. Then ${}^*g_n - h_n \approx 0$ in $ns{}^*\mathbb{R}^3$, g_n is n times differentiable and $g_n = g_1 := g$, $n \in \mathbb{N}$.

Clearly g satisfies ${}^*g_1'\,{}^*g_2' - {}^*g_1^*g_2 \approx 0$ Loeb a.e $ns{}^*\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. Since g is standard, this implies

$$g_1'g_2' - g_1g_2 \equiv 0. \tag{FE}$$

□

For the proof of Theorem 1 we notice that if $\int f^*dv \approx 0$, then the theorem holds with $h \equiv 0$. Otherwise the proof will depend on the following lemma

Lemma 3 ([PLL], [BW]). *Suppose $b(|v|, |w{\cdot}v|) \in C^\infty(\mathbb{R}^3 \times S^2)$, $g \in L^1(\mathbb{R}^3_v)$, $f \in L^2(\mathbb{R}^3_v)$ (or f, g conversely), b vanishes for v near 0 and for v large, uniformly in w, as well as for $|w \cdot v|$ near 0 and near $|v|$ $(w \in S^2)$. Set*

$$Q^+(f,g)(v_1) = \int_{\mathbb{R}^3} dv_2 \int_{S^2} dwb(|v_1 - v_2|)|w \cdot (v_1 - v_2)|)f(v_1')g(v_2'), v_1 \in \mathbb{R}^3.$$

Then

$$\|Q^+(f,g)\|_{H^1} \le C\|f\|_{L^2}\|g\|_{L^1} \quad \textit{(or } f,g \textit{ conversely)}$$

for some C *independent of* f,g.

Lemma 4 ([LA1], [PLL]). *Solutions* $0 \le f \in L^1_{\rm loc}(\mathbb{R}^2)$ *of the functional equation (FE) are smooth.*

Proof. We give the short proof of [PLL]. Clearly $g := \sqrt{f} \in L^2_{\rm loc}(\mathbb{R}^3)$ satisfies (FE). There is nothing to prove if $g \equiv 0$. Otherwise, introduce

$$b_\epsilon(v_1,v_2,\omega) = \varphi^{(1)}_\epsilon(v_1^2+v_2^2)\varphi^2_\epsilon(|v_1-v_2| - |\omega\cdot(v_1-v_2)|)\cdot\varphi^{(3)}_\epsilon(|\omega\cdot|v_1-v_2|),$$

where $0 \le \varphi^{(j)}$, $j = 1,2,3$,

$$\begin{aligned}&\varphi^{(1)}_\epsilon \in C^\infty_0(\mathbb{R}),\ \varphi^{(1)}_\epsilon(t) \equiv 1 \quad \text{for } 0 \le t \le \epsilon^{-1}, \quad \text{and}\\ &\varphi^{(2,3)}_\epsilon \in C^\infty(\mathbb{R}),\ \varphi^{(2,3)}_\epsilon(t) = 0 \quad \text{for } t \le \frac{\epsilon}{2}, \varphi^{(2,3)}_\epsilon(t) = 1 \quad \text{for } t \ge \epsilon.\end{aligned}$$

Given b_ϵ, define Q^+_ϵ as in Lemma 3, and set

$$\ell_\epsilon(v_1) = \int_{\mathbb{R}^3\times S^2} b_\epsilon(v_1,v_2,\omega)g(v_2)dv_2d\omega \in C^\infty_0(\mathbb{R}^3).$$

For given $C > 0$, we have $\ell_\epsilon(v_1) > 0$ where $|v_1| \le C$ and ϵ is small enough. From $g'_1g'_2 \equiv g_1g_2$ it follows

$$g(v_1)\ell_\epsilon(v_1) = Q^+_\epsilon(g,g)(v_1).$$

So using Lemma 3, $g \in H^1_{\rm loc}(\mathbb{R}^3)$. The proof of Lemma 3 actually implies that if $g \in L^1(\mathbb{R}^3)$, $f \in H^s(\mathbb{R}^3)$, then $\|Q^+(f,g)\|_{H^{s+1}} \le C\|f\|_{H^s}\|g\|_{L^1}$. We conclude that in our case $g \in H^k_{\rm loc}(\mathbb{R}^3)$, $k \in \mathbb{N}$, hence that $f,g \in C^\infty(\mathbb{R}^3)$. □

Corollary 5. *Solutions* $0 \le f \in L^1_{\rm loc}(\mathbb{R}^3)$ *of (FE) are Maxwellians,* $f \equiv a\exp(b(v-c)^2)$ *for some* $a \in \mathbb{R}_+, b \in \mathbb{R}, c \in \mathbb{R}^3$.

Proof. This is a well known result for smooth f's. The proof is there reduced to solving the Cauchy equation

$$\varphi(x) + \varphi(y) = \varphi(x+y), \quad x,y \in \mathbb{R}, \tag{CE}$$

for which it is easy to see that any continuous solution φ is of the type $\varphi =$ constant $\cdot\, x$. □

Proof of Theorem 1. We consider the case ${}^0\!\int f^*dv > 0$. Set $g = \sqrt{f}$. Similarly to the proof of Lemma 4

$$g(v_1) \approx Q^+_\epsilon(g,g)(v_1)/\ell_\epsilon(v_1).$$

(Here ϵ is chosen depending on the set $|v| \leq \lambda$, so that $\inf_{|v|\leq\lambda} {}^0\ell_\epsilon(v) > 0$, which is possible since f is S-integrable with ${}^0\!\int f^*dv > 0$.)

The first derivatives of the right-hand side in the above relation are in ${}^*L^2(|v| \leq \lambda)$. Since f is finite Loeb a.e. $ns{}^*\mathbb{R}^3$, by overspill there is a function $q^1 \in {}^*L^1(|v| \leq \lambda)$ with its first derivatives having finite norms in ${}^*L^1(|v| \leq \lambda)$ for λ finite, $q^1 \approx f$and (IR) holding for q^1, Loeb a.e. $ns{}^*\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. By iteration, for $n \in \mathbb{N}$ there is a function q^n with all derivatives of order $\leq n$ finite in ${}^*L^1(|v| \leq \lambda)$, when λ is finite, $q^n \approx f$ and (IR) holding for q^n, Loeb a.e. $ns{}^*\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$.

It follows that a subsequence $(h^n)_{n\in\mathbb{N}}$ of (q^n) has all derivatives up to order n S-continuous in $ns{}^*\mathbb{R}$. □

It follows from Corollary 2 and Corollary 5 that

Theorem 6 ([LA2]). *Suppose $f : {}^*\mathbb{R}^3 \to {}^*\mathbb{R}_+$ is *Lebesgue measurable, is S-integrable on $\{|v| \leq \lambda\}$ for λ finite, satisfies (IR) Loeb a.e. $ns{}^*\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. Then there is a standard Maxwellian M_f such that*

$$f - {}^*M_f \approx 0 \quad \textit{Loeb a.e. } ns{}^*\mathbb{R}^3.$$

One implication of Theorem 6 in the standard context, is the following result.

Theorem 7. *Given $C > 0$, consider the set of non-negative functions with $\int f(1 + |\log f|)dv \leq C$. Set*

$$S_\delta = \{(v_1, v_2, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2; |v_1| \leq \delta^{-1},\ |v_2| \leq \delta^{-1}\}.$$

Given $\epsilon > 0$, there is $\delta > 0$ such that if $|f'_1f'_2 - f_1f_2| < \delta$ in S_δ outside of some (f-dependent) subset of measure bounded by δ, then there is a Maxwellian M_f with

$$|f - M_f| < \epsilon \quad \textit{for } |v| \leq \epsilon^{-1},$$

outside of some (f-dependent) subset of measure bounded by ϵ.

Remark. The condition $f \log f \in L^1(\mathbb{R}^3)$ can be replaced by L^1-conditions involving weaker, strictly convex functions of f than $f \log f$. In fact, the theorem holds for any locally weakly precompact set of positive L^1-functions.

Proof. Consider the set of *Lebesgue measurable functions f with $\int f(1+|\log f|)^* dv \le C$. Such function are S-integrable on $\{|v| \le \lambda\}$ for λ finite. Let F_δ be the set of such functions with $|f_1'f_2' - f_1 f_2| < \delta$ on S_δ outside a subset of measure $\le \delta$. If this holds for $\delta \approx 0$, then (IR) holds Loeb a.e. $ns^*\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. (A converse is also true.) It follows by Theorem 6 that there is a Maxwellian M_f such that $f - M_f \approx 0$, Loeb a.e. $ns^*\mathbb{R}^3$. In particular given $\epsilon > 0$ and standard, $|f - M_f| < \epsilon$ for $|v| \le \epsilon^{-1}$ outside of a (f dependent) set of *Lebesgue measure bounded by ϵ. For each infinitesimal δ this holds for all $f \in F_\delta$. But the set of δ for which the ϵ-property holds for all $f \in F_\delta$, is internal. Hence there is also a standard $\delta > 0$ such that it holds for all $f \in F_\delta$. For ϵ, δ standard, by transfer the statement holds in the standard context. □

Corollary 8. *Given $C > 0$ consider the set of non-negative functions f with $\int f(1+v^2+|\log f|)dv \le C$. Given $\epsilon > 0$ there is $\delta > 0$ (only depending on C, ϵ) such that if $|f_1'f_2' - f_1 f_2| < \delta$ in S_δ outside of some subset of measure bounded by δ, then for some Maxwellian M_f (depending on f) $\int |f - M_f| dv < \epsilon$.*

Remark. The corollary holds for any weakly precompact set of positive L^1-functions.

Proof. In the class of non-negative functions f with

$$\int f(1+v^2+|\log f|)dv < C (< \infty),$$

evidently

$$\int_{|v| \ge \lambda} f dv \le \frac{C}{1+\lambda^2} < \epsilon \quad \text{for} \quad 1+\lambda^2 > \frac{C}{\epsilon}.$$

There is $(\epsilon >>)\epsilon_1 > 0$ (and depending on C) such that for any f in the class and any set S of measure bounded by ϵ_1,

$$\int_S f dv < \epsilon 10^{-3} \quad \text{(equi-integrability)}.$$

Given λ take ϵ_1 so that, moreover, $\int \epsilon_1 dv < \epsilon$.

Recall that by Theorem 7, there is $\delta > 0$ such that the following holds. If in S_δ outside of some subset of measure bounded by δ

$$|f_1'f_2' - f_1 f_2| < \delta,$$

then there is a Maxwellian M_f such that outside of some set S_1 of measure bounded by ϵ_1,

$$|f - M_f| < \epsilon_1 10^{-3} \quad \text{for} \quad |v| \le \epsilon_1^{-1},$$

in particular for $|v| \leq \lambda$.

If $\int f dv < \epsilon$, then the corollary holds with $M_f \equiv 0$. Otherwise, take λ so that

$$\int_{|v|>\lambda 10^{-3}} f dv < \epsilon 10^{-3}.$$

If M_f attains its maximum for $|v| \geq \frac{\lambda}{2} - 1$, then

$$\int f dv \leq \int_{A_1} f dv + \int_{A_2} |f - M_f| dv + \int_{A_2} M_f dv + \int_{S_1} f dv \leq$$
$$\leq 3\epsilon 10^{-3} + \int_{A_2} M_f dv.$$

There

$$A_1 = \{v; |v| \geq \lambda 10^{-3}\},$$
$$A_2 = \{v; |v| \leq \lambda 10^{-3}, v \notin S_1\}.$$

But in this case

$$\int_{A_2} M_f dv \leq \int_{A_3} M_f dv \leq \int_{A_3} |M_f - f| dv + \int_{A_3} f dv \leq 2\epsilon 10^{-3}$$

with $A_3 = \{v; \lambda \geq |v| \geq \lambda 10^{-3},\ v \notin S_1\}$, and so

$$\int f dv \leq 5\epsilon 10^{-3}.$$

That contradicts the present assumption $\int f dv > \epsilon$, and so M_f attains its maximum for $|v| \leq \frac{\lambda}{2} - 1$.

With $A_4 = \{v; \lambda \geq |v| \geq \frac{\lambda}{2},\ v \in S_1\}$

$$\int_{A_4} M_f dv \leq \int_{A_4} |M_f - f| dv + \int_{A_4} f dv < 2\epsilon 10^{-3}.$$

Hence

$$\int_{|v|\geq \lambda/2} M_f dv \leq 4\epsilon 10^{-3}.$$

If $\int_{|v|\leq \lambda/2} M_f dv \leq \frac{\epsilon}{2}$, then

$$\int f dv \leq \int_{|v|\geq \lambda/2} f dv + \int_{A_5} |f - M_f| dv + \int_{A_5} M_f dv + \int_{S_1} f dv < \epsilon.$$

Here

$$A_5 = \{v; |v| \leq \frac{\lambda}{2},\ v \notin S_1\}.$$

Since $\int f dv \geq \epsilon$, we conclude that $\int_{|v|\leq\lambda/2} M_f dv > \frac{\epsilon}{2}$.

Consider now the case when S_1 is a sphere with centre at the maximum of M_f. If $\int_{S_1} M_f dv \leq \frac{\epsilon}{2}$, then the corollary holds. If $\int_{S_1} M_f dv > \frac{\epsilon}{2}$, set S_2 as the sphere concentric with S_1 and with ten times its radius. If

$$\int_{S_2\setminus S_1} M_f dv > \epsilon 10^{-2}$$

then

$$\int_{S_2\setminus S_1} f dv \geq \int_{S_2\setminus S_1} M_f dv + \int_{S_2\setminus S_1} |f - M_f| dv > \epsilon 10^{-3}.$$

This contradiciton implies that $\int_{R^3\setminus S_1} M_f dv \leq \epsilon 10^{-1}$, hence that

$$\int f dv \leq \int_{|v|\geq\lambda} f dv + \int_{|v|\leq\lambda} f dv \leq \epsilon 10^{-3} + \int_{A_6} |f - M_f| dv + \int_{A_6} M_f dv$$
$$+ \int_{S_1} f dv \leq 2\epsilon 10^{-1}, \quad A_6 = \{v; |v| \leq \lambda,\, v \notin S_1\},$$

which again contradicts our assumption.

Hence the corollary holds provided S_1 is a sphere with centre at the maximum of M_f. Finally if all or part of the bad set S_1 lies outside of the above sphere, then the previous argument still holds with minor changes. □

There are corresponding results in the space-dependent case.

Theorem 9. *Given $C > 0$ and $\Omega \subset \mathbb{R}^3$ measurable. Consider the set of non-negative functions f with $\int_{\Omega\times\mathbb{R}^3} f(1 + |\log f|) dx dv \leq C$. Given $\epsilon > 0$, there is $\delta > 0$ such that if for $x \in \Omega, |x| \leq \epsilon^{-1}$, outside of some ($f$ dependent) subset $S(f,\epsilon)$ of measure $< \frac{\epsilon}{2}$, it holds that $|f_1' f_2' - f_1 f_2| < \delta$ in S_δ outside some (x, f dependent) subset of measure bounded by δ, then there is a (local) Maxwellian M_f such that outside of the subset $S(f,\epsilon)$ of measure bounded by $\frac{\epsilon}{2}$ in x, $|f - M_f| < \epsilon$ for $|v| \leq \epsilon^{-1}$ outside of a v-subset of measure bounded by ϵ.*

Proof. This is similar to the proof of Theorem 7. □

Corollary 10. *Given $C > 0$ and a bounded measurable set Ω in x-space. Consider the set of non-negative functions f with $\int f(1 + v^2 + |\log f|) dx dv \leq C$. Given $\epsilon > 0$, there is $\epsilon_1 > 0$ and $\delta > 0$, such that if $|f_1' f_2' - f_1 f_2| < \delta$ in S_δ outside of some subset of measure $< \delta$ for all x in Ω outside a subset of measure $< \epsilon_1$, then for some local Maxwellian M_f (depending on f)*

$$\int |f - M_f| dx dv < \epsilon.$$

Remark. The Corollary also holds for unbounded measurable sets in x-space when $\int f(1+v^2+x^2+|\log f|)dxdv < C$.

Remark. The integral bounds in Theorem 9 and Corollary 10 can be replaced by conditions of weak L^1 precompactness.

References

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