Abstract

We study long range mixed site-bond percolation and the question of interest for us is whether, when there is percolation, the long edges are needed in order for the process to percolate. Meester and Steif (1996) showed, among other things, that for a long range bond percolation model with exponentially decaying connections, the limit of critical values of any sequence of long range percolation models approaching the original model from below is the critical value for the original long range percolation model. As a corollary they obtained, for the long range bond percolation model, the result that within the supercritical regime, the long edges can be removed and percolation still occurs. Here we extend these results to site-bond models. One of the reasons for interest in this is that it seems that this is a first step in analysing the analogous problem in continuum percolation.

Keywords: percolation, renormalization, continuum percolation

AMS 1991 subject classification: 60K35

Contents

1. Introduction

The problem to be discussed in the present work emanates from an article by Meester and Steif (1996). They study the continuity of the critical value for long range bond percolation with exponentially decaying connections. As a corollary to their main result, they obtain that if we are in the supercritical phase then there exists some k so that edges longer than k are not needed in order for the process to percolate.

We wanted to extend this result to continuum percolation. In these models points are distributed in space according to a Poisson process and there is an edge between any two pair of points with a probability that depends on the distance between these two points. In our case the connection function would be exponentially decaying. However, this problem turned out to be difficult.

Many problems in continuum percolation are studied by some sort of discretisation process, and it is then hoped that results for the discretisation process can be translated to the original continuum model. When this discretisation is done, typically site-bond models arise. Therefore in our case, a possible first step in showing the result for the continuum model, is to obtain the analogous result for the site-bond model. This is what we prove in this paper.

In section 2 we give a brief introduction to some aspects of percolation. Section 3 consists of a summary of the relevant parts of the two articles that compose the background to the current one. Section 4 includes a concise description of continuum percolation and a discussion of some of the problems involved in the extension of the discrete result to continuum percolation. Finally sections 5, 6 and 7 contain the actual work, namely the extension of the result of Meester and Steif for long range bond percolation, to an analogous result for long range mixed site-bond percolation.

2. Percolation

Percolation can be thought of as the study of connectivity properties of random graphs. The theory as it is known today originates from an article by Broadbent and Hammersley (1957). They study the spread of a fluid through a random medium. The fluid itself shows no random behaviour and it thereby differs from diffusion theory, where all randomness comes from the behaviour of the fluid and the medium is fixed. A couple of years earlier, Broadbent (1954) modelled the medium as a system of channels where some channels were narrow and some were wide. He assumed that the fluid could pass through all the wide channels, but not through any of the narrow ones. In an idealized form the system of channels was represented by the edges of $\mathbb{Z}^d$. Letting each edge between points of $\mathbb{Z}^d$ at distance 1 from each other (= a channel) be open (= wide) with probability p independent of all others, we arrive at ordinary bond percolation on $\mathbb{Z}^d$ with parameter p. One question of interest is if "fluid" is added to one point of $\mathbb{Z}^d$, whether the number of points that are then "wetted" is infinite or not.

Our mathematical formulation is as follows. Each element of the d-dimen-

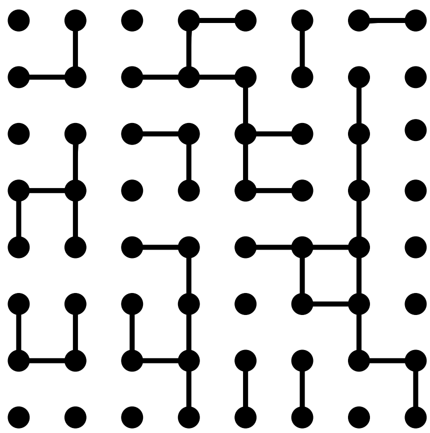

Figure 1: A realization of bond percolation with $p = 0.5$ on a part of $\mathbb{Z}^2$

sional integer lattice in $\mathbb{Z}^d$ is a vertex. Two vertices $x = (x_1, x_2, ..., x_d)$ and $y = (y_1, y_2, ..., y_d)$ are called nearest neighbours if their distance $|x - y| = \sum_{i=1}^{d} |x_i - y_i|$ equals 1. Between each pair of nearest neighbours there is an edge, also called a bond. The set of edges will be denoted by E. With each edge e of E there is associated a $\{0,1\}$-valued random variable $X(e)$. The edge e is said to be **open** if $X(e) = 1$ and **closed** if $X(e) = 0$. We consider the sample space $\Omega = \{0,1\}^E$ and we equip it with the usual σ-algebra $\mathcal{F}$. We study product measure with density p and we write P_p for it. A **path** is a sequence of distinct edges such that any two consecutive edges are connected to a common vertex, and such that there are no loops. An **open path** is a path consisting only of open edges. The **open cluster** of a vertex x, $C(x)$, is the set of vertices containing x and all other vertices that can be reached from x via open paths. Let $\mathbf{0}$ denote the origin and let $\mathbf{0} \leftrightarrow \infty$ be short for the event that there exists an infinite open path starting at the origin. Let $\theta(p) = P_p(0 \leftrightarrow \infty)$. $\theta(p)$ will be referred to as the percolation probability. It is obvious that $\theta(0) = 0$ and that $\theta(1) = 1$. It is also true that $\theta(p)$ is non-decreasing in p. This is intuitively obvious, and it can be shown by a simple coupling argument. There therefore exists a critical value $p_c = p_c(d)$ of p such that

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c \\ > 0 & \text{if } p > p_c. \end{cases}$$

The existence of an infinite path from the origin is equivalent to the event that the cluster of the origin is infinite, thus $\theta(p) = P_p(|C(0)| = \infty)$. When $\theta(p) > 0$ it has been shown that there a.s. is a unique infinite cluster. For general d this was proved by Aizenman, Kesten and Newman (1987). Harris (1960) had shown it for $d = 2$. Burton and Keane (1989) gave a more accessible proof in a more general case. As the existence of an infinite open cluster is not affected by the opening or closing of finitely many edges, Kolmogorov's 0-1 law tells us that the probability of that event is either 0 or 1 and we have

$$\theta(p) > 0 \Leftrightarrow P_p(\text{there is an infinite open cluster}) = 1.$$

For $d \geq 2$, it can be shown that p_c is strictly between 0 and 1. (In the case $d = 1$, there is a.s. no infinite open cluster unless $p = 1$.)

When $d = 2$, p_c equals $\frac{1}{2}$. This was not shown rigorously until 1980 by Kesten (1980), but it was generally believed to be true after Harris in 1960 had shown that $\theta(\frac{1}{2}) = 0$. As $\theta(\frac{1}{2}) = 0$, this means that there is no infinite cluster at the critical value in two dimension. This is thought to be true for all d, but so far has only been shown for $d = 2$ and for $d \geq 19$. See Hara and Slade (1994).

Another percolation process is obtained if we let the status of the vertices be random, instead of that of the edges. We declare a vertex open with probability p and closed otherwise, independently of all other vertices. The open clusters of this site percolation model are the subgraphs of the lattice induced by the set of open vertices. Every bond percolation process may be reformulated as a site percolation process on a different lattice. This is not true the other way around, and in that sense site percolation models are more general.

An even more general percolation process, is mixed percolation. Here both edges and vertices may be open or closed. The corresponding probabilities need not be the same.

Returning to bond percolation, one can also consider the more general situation where one allows edges between every pair of vertices in $\mathbb{Z}^d$. We call the obtained models long range bond percolation models and they can be constructed as follows. Let $\boldsymbol{p} = (p(x) : x \in \mathbb{Z}^d)$ be a collection of numbers in the interval $[0, 1)$. We suppose that $p(x) = p(-x)$ for all $x \in \mathbb{Z}^d$. Each pair of distinct points x and y is, independently of every other pair, joined together by an edge with probability $p(x - y)$. In order to ensure that every vertex a.s. has finitely many edges emanating from it, we assume that $\sum_x p(x) < \infty$. Ordinary bond percolation with edge probability p, can be recovered by letting

$$p(x) = \begin{cases} p & \text{if } x \text{ or } -x \text{ is a unit vector} \\ 0 & \text{otherwise.} \end{cases}$$

While nearest neighbour percolation on the line is obviously uninteresting, infinite-range percolation in one dimension has many interesting features. We will look at one of them here. Take $\boldsymbol{p}$ to be the vector $(p(n) : n \geq 1)$ of numbers in $[0, 1)$ and let each pair x and y of points in $\mathbb{Z}$ be joined by an edge with probability $p(|x - y|)$. Sometimes we will also use the notation p_n for $p(n)$. Here we restrict ourselves to a class of vectors for which $p(1) = p$ and

$$\frac{p(n)}{\beta n^{-\alpha}} \to 1 \text{ as } n \to \infty$$

for some positive constants α and β, meaning that for large n, $p(n)$ behaves more or less like $\beta n^{-\alpha}$. Since we assume that $\sum_{n=1}^{\infty} p(n) < \infty$, we must have α strictly greater than 1. We will see that if the $p(n)$'s are sufficiently small, there will a.s. not exist any infinite components. We look at two such conditions.

For the first one, let us compare the component at the origin with a branching process. This can be found in Schulman (1983). There is no problem making the following argument rigorous. The mean number of vertices that share an edge with the origin is $\mu = 2\sum_{n=1}^{\infty} p(n)$. For each such vertex x the mean number of new, in the sense not directly connected to the origin, vertices that share an edge with x is at most μ. If we look at the cluster of the origin as a branching tree with the origin as the ancester or root, we see that each component has a family mean size not greater than μ. Therefore the size of a certain branching process with mean family size μ dominates the size of the component at the origin in the percolation process. Branching processes with mean family size less than or equal to 1 are known to be finite, and this would then imply that if $2\sum_{n=1}^{\infty} p(n) \leq 1$ then there a.s. does not exist an infinite cluster of the origin. (For branching processes with mean family size 1, we need the additional assumption that $P(\text{one child}) < 1$ for the total size to be finite. In the percolation case this corresponds to assuming that $P(\text{there is exactly one edge emanating from an arbitrary vertex}) < 1$, which of course holds.) In terms of our parameters, this describes the case that β is sufficiently small.

In order to discuss the second condition, we introduce some events. Let A_k be the event that no vertex x $(\leq k)$ is joined to any vertex y $(\geq k+1)$. Now study the sequence of indicator functions I_{A_k} for $-\infty < k < \infty$. The sequence is stationary with trivial tail σ-algebra and the random variables all have mean $P(A_0)$. If we can show that $P(A_0)$ is greater than 0, the ergodic theorem applied to the sequence of indicator functions will tell us that infinitely many of the A_k's occur. This in turn means that there are no infinite components. To see when $P(A_0)$ is strictly positive we introduce some new events. Let B_n, for $n \geq 1$, be the event that there is at least one edge of length n from a vertex $x \leq 0$ to a vertex $y \geq 1$. We have that $A_0 = \cap_{n=1}^{\infty} B_n^{\mathbf{c}}$. It follows immediately that if

1. $P(B_n) < 1 \quad \forall n$
2. $\sum_{n=1}^{\infty} P(B_n) < \infty$
3. B_n's are independent

then $P(\cap_{n=1}^{\infty} B_n^{\mathbf{c}}) > 0$. $P(B_n) \leq np(n)$, $P(B_n) < 1 \quad \forall n$ and the events B_n are all independent, so if $\sum_{n=1}^{\infty} np(n) < \infty$ we have that $P(A_0) > 0$ and as a consequence there are no infinite components. Assuming that $\sum_{n=1}^{\infty} np(n) < \infty$ is the same as assuming that $\alpha > 2$.

Now we will see what happens if $\boldsymbol{p}$ satisfies

$$\sum_{n=1}^{\infty} np(n) = \infty \quad \text{and} \quad \tfrac{1}{2} < \sum_{n=1}^{\infty} p(n) < \infty.$$

This implies that α belongs to $(1, 2]$. In our model p measures the strength of nearest neighbour interactions and β and α measure the long distance interactions. Both types of interactions are relevant to the occurence of a critical phenomenon. Let $\theta(p, \alpha, \beta)$ denote the probability that there is percolation. It can be shown that if $1 < \alpha < 2$ and $\beta > 0$ there exists a critical value $p_c(\alpha, \beta)$ strictly between 0 and 1 such that

$$\theta(p, \alpha, \beta) \begin{cases} = 0 & \text{if } p < p_c(\alpha, \beta) \\ > 0 & \text{if } p > p_c(\alpha, \beta). \end{cases}$$

When $\alpha = 2$, $\beta \leq 1$ and $p < 1$ there exist no infinite cluster a.s.. But if $\alpha = 2$ and $\beta > 1$ there exists a critical value $p_c(2, \beta)$ such that

$$\theta(p, 2, \beta) \begin{cases} = 0 & \text{if } p < p_c(2, \beta) \\ \geq \beta^{1/2} & \text{if } p > p_c(2, \beta). \end{cases}$$

Both these latter facts are shown by rigorous renormalization type arguments. The arguments are more sophisticated than those concerning the case $\alpha > 2$ and the case of sufficiently small β.

Summarizing we see that there is a critical phenomenon if $1 < \alpha < 2$ or if $\alpha = 2$ and $\beta > 1$. Note that in the latter case the percolation probability is discontinuous at $p_c(2, \beta)$, a very interesting phenomenon. These results are due to Newman and Schulman (1986) and Aizenman and Newman (1986).

For more details on particularly bond percolation, the reader is referred to the book by Grimmett (1989). Another review can be found in Kesten (1987).

3. Some background material

We just give a summary of some of the ideas and results in the articles by Grimmett and Marstrand (1990) and by Meester and Steif (1996). The reader is referred to the relevant article for details.

Grimmett and Marstrand study site percolation in the supercritical phase. They prove a general result concerning critical probabilities of subsets of $\mathbb{Z}^d$. To state their main theorem, we introduce some notation. Let $B(n)$ denote the box $\{x \in \mathbb{Z}^d : |x| \leq n\}$ and if A is a set, let $p_c(A)$ denote the critical probability for percolation on A.

Theorem A. *If F is an infinite connected subset of $\mathbb{Z}^d$, and $p_c(F) < 1$, then for each $\eta > 0$ there exists an integer $k > 0$ such that*

$$p_c(2kF + B(k)) \leq p_c(\mathbb{Z}^d) + \eta.$$

To see what this theorem says, consider a slice of thickness k which is given by $S(k) = \{x \in \mathbb{Z}^d : 0 \leq x_j \leq k, j > 2\}$. Note that $p_c(\mathbb{Z}^d) \leq p_c(S_k)$ and that $p_c(S_k)$ is a non-increasing function of k. As a consequence of Theorem A they obtain that, when $d \geq 3$, the limit of the critical probabilities of a slice of $\mathbb{Z}^d$ whose thickness tends to infinity, equals the critical probability of percolation on $\mathbb{Z}^d$. This result follows immediately from the theorem if we choose $F = \mathbb{Z}^2$. In this case $2kF + B(k)$ is a translation of the slice $S(2k)$ having thickness $2k$.

However it is not their result so much as the techniques they use to prove it, that is of interest for us at present. The idea of the proof is to consider a renormalization of the lattice, and then study site percolation on the renormalized lattice. Place a grid with side-length $2N$ in the original lattice, so that the lattice can be seen as consisting of translates of the box $B(N)$. Let each box in $\{4Nx + B(N) : x \in \mathbb{Z}^d\}$ correspond to a vertex in $\mathbb{Z}^d$, the latter being the renormalized lattice. They present a method to build a cluster of boxes, in a stepwise manner, in such a way that if there is an infinite cluster of boxes (= an infinite cluster of vertices in the renormalized lattice), then there deterministically also exists an infinite cluster of vertices in $2kF + B(k)$. Let there be a fixed ordering of the edges in F. The ordering gives rise to an ordering of the vertices of F. The corresponding boxes $\{4Nx + B(N) : x \in F\}$ are studied in that order to determine whether they are open or closed. The rule that is used has the above mentioned property. By this property, we know that if there is a cluster of boxes, then there must also be a cluster of vertices in $4NF + B(N)$. There are two key lemmas to this procedure. Lemma 1 more or less says that if the probability that the next box to be studied is open given the past of the procedure, is strictly greater than $p_c(F)$ and is independent of how far we have come and of what we have seen in the cluster building procedure, then the probability of there being an infinite cluster of boxes is strictly greater than zero. The other is Lemma 6 which ensures that there is a high probability for a certain long open path within a box, and it is used to control the probability of boxes to be open.

The setting in the article by Meester and Steif is long range bond percolation and they study the continuity of the critical value in the case of exponentially decaying connections. Their main result is a sufficient condition under which the critical probability is continuous from below for long range bond percolation models. To state their result they introduce a condition which the model is assumed to satisfy. They later show that models with exponentially decaying connections satisfy this condition, which they call Condition C. In fact they do not believe the condition to be much stronger than having exponential connections, but find it convenient to have their results stated using the condition. We will not go into detail about this here, but merely state their main result. Recall that p_n denotes the probability of there being an edge between two vertices at distance n from each other. As in the above example of long range percolation in one dimension, it is only p_1 that is allowed to vary, all others are kept fixed.

Theorem 1.1. *Let $d \geq 2$. Assume that p_2, p_3, ... satisfies Condition C, and that p_2^n, p_3^n, ... are such that for all $i \geq 2$ and $n \geq 1$, $p_i^n \leq p_i$ and that for all $i \geq 2$, $\lim_{n\to\infty} p_i^n = p_i$. Then*

$$\lim_{n\to\infty} p_c(p_2^n, p_3^n, ...) = p_c(p_2, p_3, ...).$$

The result of interest for us, is obtained as an immediate corollary of Theorem 1.1.

Corollary 1.2. *Let $d \geq 2$. Assume that p_2, p_3, ... satisfies Condition C and let $p_1 > p_c(p_2, p_3, ...)$. Then there exist some integer k so that there is percolation under $(p_1, p_2, p_3, ..., p_k, 0, 0, ...)$.*

The proof relies very much on the renormalization technique described in Grimmett and Marstrand. The idea is that when determining whether a certain box is open or not, you only look at edges within that box. The cluster that is found using this method cannot contain any edges longer than the distance between the two corners in the box furthest away from each other. To be able to use the techniques of Grimmett and Marstrand, some lemmas are needed to take care of the fact that Meester and Steif study long range bond models as opposed to the site models of Grimmett and Marstrand.

4. The continuum case

Here a brief description of the ideas and some of the problems encountered will be discussed. The model to be considered is an example of the Poisson random connection model. Instead of just considering points in the lattice, points are now distributed in space according to a Poisson process. Between each pair of points, there is an edge with a probability that depends on the distance between the two points.

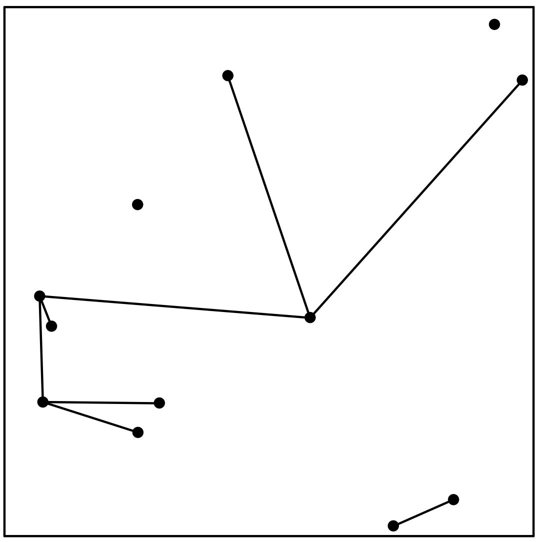

Figure 2: A realization of a random connection model

In order to be able to state our conjecture, we need some notation. First let λ be the intensity of the Poisson process and let g be the connection function, that is the probability of there being an edge between x and y in $\mathbb{R}^d$ is given by $g(x-y)$. Assume g satisfies $\int_{\mathbb{R}^d} g(x)\,dx < \infty$. Assume also that $g(x) = g(y)$ whenever $|x| = |y|$ and that g is non-increasing in the sense that $g(x) \leq g(y)$

whenever $|x| \geq |y|$. Denote by W the component containing the origin. A component is defined in the usual graph-theoretical way, namely it is a set of points such that any two points of the set are connected, and that the set is maximal in this respect. Let us write $\theta_g(\lambda) = P_{\lambda,g}(|W| = \infty)$, where $P_{\lambda,g}$ denotes the probability measure in the model. It is a fact that, when $d \geq 2$, under these conditions there exists a density $\lambda_H(g) \in (0, \infty)$ such that $\theta_g(\lambda) = 0$ for $\lambda < \lambda_H(g)$ and $\theta_g(\lambda) > 0$ for $\lambda > \lambda_H(g)$, see e.g. Meester and Roy (1995). We state our conjecture.

Conjecture 4.1. *Let $d \geq 2$. Assume the connection function $g(x)$ is $\alpha\gamma^{|x|}$, where $\alpha, \gamma \in (0,1)$, and let $\lambda > \lambda_H(g)$. Then there exists some $k > 0$ such that there still is percolation when $g(x)$ is replaced by*

$$g_k(x) = \begin{cases} g(x) & |x| \leq k \\ 0 & |x| > k. \end{cases}$$

How do we go about to prove this? One approach could be to find a continuum version of the main lemma in Meester and Steif (1996) that is needed for the renormalization. A problem that arises here is how to define the boundary of a set when there is no such thing as a nearest neighbour. Another approach could be to find a discrete translation of the process into a long range mixed percolation model. Let us discuss this latter approach.

We begin by placing a grid in space and letting each box correspond to a vertex in $\mathbb{Z}^d$. We choose the side-length of the boxes so small that the probability of having more than one point of the Poisson process within the same box is sufficiently small. A vertex in $\mathbb{Z}^d$ is said to be open if there is exactly one point of the Poisson process in the corresponding box in $\mathbb{R}^d$. One could of course instead have chosen a rule that declares a vertex in $\mathbb{Z}^d$ to be open if all Poisson points in the corresponding box are connected, but one would then run into problems with multiple edges between boxes. Let there be two ways to declare an edge open in the discrete case. If the two endpoints of the edge in $\mathbb{Z}^d$ are open, let there be an edge in $\mathbb{Z}^d$ if there is an edge between the corresponding boxes in $\mathbb{R}^d$. If on the other hand at most one of the edge's endpoints in $\mathbb{Z}^d$ is open, we let the edge be open with the following probability

$$\frac{\int_{B_1}\int_{B_2} g(x-y)dx\,dy}{(\mathrm{vol}(B_1))^2},$$

where B_1 and B_2 are the boxes in $\mathbb{R}^d$ corresponding to the endpoints of the edge. If we had chosen only the first rule, the process we would have ended up with would not have been a proper long range mixed percolation model, as there could never have been any edges without both their endpoints being open. To see that the process we get by using both edge rules is indeed an example of a long range mixed percolation model, we make some observations. First we observe that the status of a vertex is independent of the status of all other vertices. This is so because the number of Poisson points in disjoint parts of space are independent. Further, the probability of there being an open edge between any two pair of points in $\mathbb{Z}^d$ only depends on the distance between the two points since an easy computation shows that the probability we chose for edges to be open when at most one of their endpoints is open is exactly the probability to be open for an edge between two points that are open. The status of a given edge is independent of the status of every other edge. As the two conditional probabilities for the presence of open edges given the status of their endpoints are equal, it is also true that the status of edges and vertices are independent. These observations lead us to conclude that the process we obtain is a long range mixed percolation process.

As we defined our translation between continuous and discrete, we see that if the cluster of the origin in the discrete case is infinite so is the cluster of the origin in the continuum case. This would not have been the case if we had allowed vertices in $\mathbb{Z}^d$ to be open if there were more than one Poisson point in the corresponding box, not demanding them to be connected. We could then have had a path in the discrete model that had no counterpart in the continuum model if there were two edges incident to the same box, but not to the same Poisson point and these points were not connected. The extra edges that were added to the discrete process above can never be used in a path from the origin to infinity, as they do not have both their endpoints open, so they do not pose any problems.

Unfortunately, but as could be expected, knowing that there is percolation in the continuum model does not imply anything for the discrete model. *If* we could, e.g. by in some way altering the translation, ensure that the discrete translation was supercritical in the sense that if we lower both the probability for the existence of open vertices and the probability for the existence of open edges of length one, this new model would still percolate, then we could use the result to be described for long range mixed percolation to conclude that in the discrete model, the long edges could be removed. By the nature of our translation, we would then know that also in the continuum model the long edges could be removed without altering the fact that the process percolates. The only thing that remains, is to take care of the opening *if.* An attempt could be as follows.

Let us study the following process. Fix $\epsilon > 0$ such that $\lambda > \lambda - \epsilon > \lambda_H(g)$ and let there be points in space according to a Poisson process with intensity λ. Let there also be edges according to the edge law g. Now place a grid in space with side-length δ. If there is more than one Poisson point in a "δ-box", all points in the box are removed along with the corresponding edges. Let $P^{\delta}_{\lambda,g}$ be the measure of the resulting process. This continuum model percolates at the same time as the above described discrete translation of the original Poisson process with intensity λ and edge law g. How do we know when this new process percolates? If it were the case that it dominated a Poisson process with intensity $\lambda - \epsilon$ and edge law g, it would certainly percolate. If not, we have to find another way to ensure percolation. However, we end up with problems even if $P^{\delta}_{\lambda,g}$ should happen to dominate $P_{(\lambda-\epsilon),g}$. This is so because in order to use our result for long range mixed percolation we need our discrete model to be both "site-supercritical" and "bond-supercritical", and by comparing it with $P^{\delta}_{\lambda,g}$ we will only be able to establish whether or not it is "site-supercritical".

Not to our surprise, however, we find that $P^{\delta}_{\lambda,g}$ cannot dominate *any* Poisson process. One way to see this is to observe that the number of points in a ball with radius $<< \delta$ is deterministically at most 2^d in the $P^{\delta}_{\lambda,g}$-model, whereas in the Poisson case there is no such limit.

We now hope that we have made it clear that carrying over results from the discrete setting to the continuum setting is not an altogether simple task.

5. Preliminaries

We will now try to give a description of the long range mixed percolation model. We let $\mathbb{Z}^d$ denote the standard d-dimensional cubic lattice. Most of the time we will equip it with the L_1-norm given by $|(x_1, ..., x_d)| = \sum_{i=1}^{d} |x_i|$. If $|x - y| = 1$, x and y will be called nearest neighbours. Sometimes we will also need the L_∞-norm given by $||(x_1, ..., x_d)|| = \max_i |x_i|$. As we are in the world of mixed models, we will mostly study sets consisting of both vertices and edges. In fact, if not otherwise stated, a set is supposed to contain both vertices and edges.

Consider a set $R = R^v \cup R^e$, where R^v denotes the vertices of the set and R^e denotes the edges. We will often discuss what happens "just outside" R, and so we introduce the edge boundary of R and the vertex boundary of R. The **vertex boundary** of R, $\Delta^v R$, will consist of vertices that have at least one edge of R^e incident to it and that are not themselves a member of R^v. The edges contained in the **edge boundary**, $\Delta^e R$, will be those edges of length one which have one endpoint among the vertices in R and the other one outside both R and the vertex boundary of R. (Note $\Delta^e R \cap R^e = \emptyset$.) More formally we write;

$$\begin{aligned}
\Delta^v R &= \{x : x \notin R^v, \exists y \in R^v, \{x, y\} \in R^e\} \\
\Delta^e R &= \{\{x, y\} : \{x, y\} \notin R^e, x \in R^v, y \notin (R^v \cup \Delta^v R), |x - y| = 1\} \\
\Delta R &= \Delta^v R \cup \Delta^e R.
\end{aligned}$$

It may well be that for a particular R one of these two sets is empty, but they cannot both be, unless $R = \mathbb{Z}^d$ of course. We introduce the vertex boundary of a set S on a set T. If $R = S \cap T$ we let

$$\Delta^v_{S,T} R = (\Delta^v S) \cap T.$$

On T, $\Delta^v R$ is a subset of $\Delta^v_{S,T} R$. The need for this set is understood while studying the renormalization process.

It is not only the elements closest to R which we will need to consider, but also vertices "two steps outside" R. We will call them the two-step vertex boundary of R and denote them by $\Delta\Delta^e R$;

$$\begin{aligned}
\Delta\Delta^e R &= \{x : x \notin R^v, \exists y \in R^v, \{x, y\} \in \Delta^e R\} \\
\Delta\Delta R &= \Delta\Delta^e R.
\end{aligned}$$

So far nothing has been said about probabilities, but that will be taken care of now. The model which we will study will have exponentially decaying connection probabilities $p_n \in [0, 1)$, $n = 1, 2, ...$, meaning here that there exist some N_0 and $\alpha, \rho \in (0, 1)$ so that $p_n = \alpha\rho^n$ for $n \geq N_0$. (Hereafter writing exponentially decaying, we will always mean exponentially decaying in this strict sense.) We naturally have that

$$\sum_{x \in \mathbb{Z}^d, x \neq 0} p_{|x|} \quad < \quad \infty. \tag{5.1}$$

For each pair of vertices x and y, $x \neq y$, we declare the connecting edge to be on with probability $p_{|x-y|}$ independently of all other edges and of the vertices. By the Borel-Cantelli lemma and (5.1) we know that, a.s., every vertex will have a finite number of edges emanating from it. At the same time declare each vertex to be open, independently of all other vertices and all edges, with probability q.

An **open vertex-to-vertex path** is a finite alternating sequence of open vertices and open edges starting and ending with a vertex. There will also be need for vertex-to-edge paths, edge-to-vertex paths and edge-to-edge paths. They are defined in the obvious way. Writing $\boldsymbol{a} \leftrightarrow \boldsymbol{b}$ means that there is an open path from a to b, in that both a and b are open in addition to all other elements used for the path. Writing $\boldsymbol{A} \leftrightarrow \boldsymbol{B}$, where A and B are sets, means that there is an open path between some element of A and some element of B. By $\boldsymbol{A} \leftrightarrow \boldsymbol{B}$ **in** $\boldsymbol{C}$ we mean that the open path between A and B should use only elements of the set C. The first element in A and the last element in B are allowed to be outside C, but they must all be open. Let $\boldsymbol{A} \rightarrow \boldsymbol{\infty}$ denote the event that there is an open path from one element of A to ∞.

The cluster, C, of the origin, 0, consists of all vertices and edges that are reachable from the origin. We write

$$C = \begin{cases} \{x \in (\mathbb{Z}^d)^v : x \text{ is open and } \exists \text{ open vertex-to-vertex path form 0 to } x\} \\ \cup \{\{x,y\}, x,y \in (\mathbb{Z}^d)^v : \{x,y\} \text{ open and } \exists \text{ open vertex-to-edge path} \\ \quad \text{from 0 to } \{x,y\}\} \\ \hfill \text{if 0 is open} \\ \\ \emptyset \hfill \text{if 0 is closed.} \end{cases}$$

We let $\theta(q, p_1, p_2, ...) = P(|C| = \infty)$ and call it the percolation probability. It depends on q, p_1, p_2, ..., as well as on d. We say that there is percolation under q, p_1, p_2, ..., if $\theta(q, p_1, p_2, ...) > 0$. For a moment disregarding the vertices, the usual approach to long range percolation in one dimension is to consider p_1 as a parameter and to fix all the other connection probabilities. This is also the approach used by Meester and Steif (1996). Here we will consider both q and p_1 as parameters, while fixing the other connection probabilities.

In order to make some things more like the approach in Meester and Steif, so that we can build on their results, we introduce a condition for the connection probabilities. First we need some preliminary definitions. If V is a set of vertices, let $E(V) = \{\{x,y\} : x,y \in V\}$. Now $B(n) = B(n)^v \cup B(n)^e$, where

$$\begin{aligned} B(n)^v &= \{x \in (\mathbb{Z}^d)^v : ||x|| \leq n\} \quad \text{and} \\ B(n)^e &= E(B(n)^v). \end{aligned}$$

(Note that $\Delta^v(B(n)) = \emptyset$.)

We now introduce **Condition C**. We say that p_2, p_3,... satisfy Condition C if for all $p_1 \in (0,1)$ there exists $c > 0$ such that for all $n \geq 1$, if $R \equiv S \cap B(n)$ where $S \subseteq \mathbb{Z}^d$, $\gamma \in (B(n))^v$ and $\gamma \notin (R^v \cup \Delta^v_{S,B(n)} R \cup \Delta\Delta^e R)$, then

$$\sum_{x \in (\Delta^v_{S,B(n)} R \cup \Delta\Delta^e R) \cap B(n)} p_{|x-\gamma|} \geq c \sum_{x \in (R^v \cup \Delta^v_{S,B(n)} R \cup \Delta\Delta^e R) \cap B(n)} p_{|x-\gamma|}. \tag{5.2}$$

Note that nothing is required for the vertices in concern.

Remark. There exists $c > 0$ such that if $S \subseteq \mathbb{Z}^d$ and $\gamma \notin (S^v \cup \Delta^v S \cup \Delta\Delta^e S)$ then

$$\sum_{x \in (\Delta^v S \cup \Delta\Delta^e S)} p_{|x-\gamma|} \geq c \sum_{x \in (S^v \cup \Delta^v S \cup \Delta\Delta^e S)} p_{|x-\gamma|}. \tag{5.3}$$

If S is finite, we can take n large enough in Condition C and we are in the same situation. If we believe (5.3) to be true for a finite S, a simple limiting argument together with (5.1) will prove it for all S. □

Meester and Steif (1996) show that in the long range bond percolation model at least our form of exponentially decaying sequences of probabilities satisfies Condition C. We claim that this result is valid also in our setting. The first thing to note is that the condition only involves the status of edges and that these edges in both cases are independent. $\Delta^v S$ in the long range bond model "covers" the whole of the boundary of S. In the mixed model, the closest boundary is represented by $\Delta^v S \cup \Delta^e S$. For each edge in $\Delta^e S$ there is exactly one point in $\Delta\Delta^e S$, so $\Delta^v S \cup \Delta\Delta^e S$ are the points "covering" the boundary here. These reflections make clear the difference in appearance between the "infinite" case of Condition C in the both settings. In the "finite" form of Condition C there is yet another difference between the two formulations of the condition. In the mixed model $\Delta^v_{S,B(n)} R$ is used instead of $\Delta^v R$. That exponentially decaying sequences of probabilities satisfy our version of Condition C can be obtained by a trivial alteration of the proof of Meester and Steif.

We state the main theorem.

Theorem 5.1. *Let $d \geq 2$. Assume exponentially decaying connections in the strict sense described above and let p and q be supercritical, in the sense that $\theta(q, p, p_2, p_3, ...) > 0$ and that even if you lower both q and p this new model percolates, then there exists some integer k so that there is percolation under $(q, p, p_2, p_3, ..., p_k, 0, 0, ...)$.*

Here we just give a sketch of the proof. The actual proof is deferred to the end of the article.

Consider a rectangular grid of large disjoint cubes with side-length N. The boxes are ordered in some way and then investigated, according to a special rule, in that order. The rule declares a box to be either occupied or vacant. It will be important to find a good rule and there are at least three requirements for one to declare a box to be occupied;

(i) Conditioned on the past of the procedure, the probability for a new box to be occupied should always be larger than and uniformly bounded away from the critical probability for nearest neighbour independent site percolation. If this is the case we can use Lemma 1 of Grimmett and Marstrand (1990) to conclude that we have percolation in the renormalized model.

(ii) There should be a link between the boxes and the underlying long range mixed percolation model such that if there is a nearest neighbour path of occupied boxes, then there must be a corresponding open path in the underlying model visiting all boxes in this open path.

(iii) The event that a certain box is occupied should depend on the state of a uniformly bounded number of edges and vertices and the edges it depends on should have uniformly bounded lengths.

Having found such a rule, we can then find some large N so that if there is percolation in the long range mixed percolation model then the renormalized model percolates as a nearest neighbour model. But now we see, by (ii) and (iii), that the underlying model percolates using all the vertices but only edges up to a certain finite length and the proof is complete. It only remains to make this reasoning a bit more rigorous, which will be the aim of the next two sections.

6. The lemmas

This section will be devoted to stating and proving the main lemma needed for the renormalization. But first, we will have to introduce some more notation. Let $T(n)$ be the set

$$\{x = (x_1, x_2, ..., x_d) \in (\mathbb{Z}^d)^v : x_1 = n, 0 \leq x_j \leq n \text{ for } j = 2, ..., d\},$$

which is a subset of a face of $B(n)$. For positive m and n with $2m < n$, we let

$$T(m,n) = T(m,n)^v \cup E(T(m,n)^v)$$

where

$$T(m,n)^v = \bigcup_{j=1}^{2m+1} \{je_j + T(n)\}.$$

Here e_j is the vector which is 1 in the jth position and 0 in the other $d-1$. An **open m-pad** is a translate of $B(m)$ with all vertices on and such that each pair of vertices in this set is connected by an open path in this set. Finally we let $K(m,n)$ denote those open vertices in $T(n)$ that have an open edge to a nearest neighbour in $T(m,n)$ which is part of an open m-pad in $T(m,n)$.

To be able to simultaneously couple all of the processes as q and p_1 varies and p_i with $i \geq 2$ are fixed, we introduce continuous uniform random variables for the vertices and edges of length 1. Let V denote $(\mathbb{Z}^d)^v$, E_1 all edges of length 1 and E_2 all edges of greater length. Now let $\Omega = [0,1]^V \times [0,1]^{E_1} \times \{0,1\}^{E_2}$ and let it be equipped with the usual σ-algebra and with product measure P

where each measure on $[0,1]$ is uniform and if $e \in E_2$ has length k (≥ 2) then the marginal for e is $p_k\delta_1 + (1-p_k)\delta_0$ (here 1 represents e being open). For $x \in V$, $y \in E_1$ and $\omega \in \Omega$ we let $\omega(x)$ and $\omega(y)$ denote the value at x and y respectively. $\omega(x)$ and $\omega(y)$ have uniform distribution. We say that a is ρ-open if $\omega(a) < \rho$ and that it is ρ-closed if $\omega(a) \geq \rho$. We will also talk about q,p-open m-pads and q,p-open paths. They are of course m-pads and paths consisting of q-open vertices which are connected by open paths using p-open edges of length one and which longer edges are all open.

We obtain a realization of a long range mixed percolation model with parameters γ, ρ, p_2, p_3, ... if we look at the set of γ-open vertices together with the ρ-open edges of length 1 and the open edges of length greater than 1.

In the rest of the section we have fixed p_2, p_3, ..., exponentially decaying, and $p, q \in (0,1)$ such that $\theta(q,p,p_2,p_3,...) > 0$. We begin by stating the main lemma.

Lemma A. *Let p_2, p_3, ... be exponentially decaying in the strict sense described above, p, $q \in (0,1)$ and $\theta(q,p,p_2,p_3,...) > 0$ and let ϵ, $\delta > 0$. Then there exist m and n such that $2m < n$ and such that if $S \supseteq B(m)$, $(S \cup \Delta S \cup \Delta\Delta S) \cap T(n) = \emptyset$ and $R = S \cap B(n)$ and if $\beta : \Delta^e R \cap B(n) \to [0, 1-\delta]$ and $\tilde{\beta} : \Delta^v_{S,B(n)} R \to [0,1-\delta]$ we have $P(G \mid H) > 1-\epsilon$ where*

$$H = \{\forall e \in \Delta^e R \cap B(n), e \text{ is } \beta(e)\text{-closed}\} \cap \{\forall v \in \Delta^v_{S,B(n)} R, v \text{ is } \tilde{\beta}(v)\text{-closed}\}$$

and G is the event that there is a path from R to $K(m,n)$ in $B(n)$ with one of the following characterizations;

- *It is a vertex-to-vertex path which second element e is a $(\beta(e)+\delta)$-open edge in $\Delta^e R$. All subsequent vertices and edges are open, meaning q-open for vertices and p-open for edges of length 1. Apart from the first edge of the path, no edges of $\Delta^e R$ and no vertices of $\Delta^v_{S,B(n)} R$ are used.*
- *It is an edge-to-vertex path which second element v is a $(\tilde{\beta}(v)+\delta)$-open vertex in $\Delta^v_{S,B(n)} R$. All subsequent vertices and edges are open, meaning q-open for vertices and p-open for edges of length 1. Apart from the first vertex, no edges of $\Delta^e R$ and no vertices of $\Delta^v_{S,B(n)} R$ are used.*

We first state three for us useful lemmas concerning $\{0,1\}$-valued random variables, without their proofs, from Meester and Steif (1996).

Lemma 6.1. *Let $\tilde{c} > 0$. Let $\{X_i\}_{i\in I}$ be independent $\{0,1\}$-valued random variables with $p_i = P(X_i = 1)$. Let $\{Y_j\}_{j\in J}$ be independent $\{0,1\}$-valued random variables with $p_j = P(Y_j = 1)$. If $\sum_{i\in I} p_i \geq \tilde{c} \sum_{j \in J} p_j$ then for all integers k and $\epsilon > 0$, either*

$$P(\sum_{i\in I} X_i \leq k) \leq \epsilon$$

or

$$P(\sum_{j\in J} Y_j \geq \frac{4}{\tilde{c}\epsilon^2} + \frac{2k}{\tilde{c}\epsilon}) \leq \epsilon.$$

Lemma 6.2. *Given M and ϵ there exists $L = L(M, \epsilon)$ such that if $X_1, X_2, ...$ are independent $\{0,1\}$-valued random variables with $p_i = P(X_i = 1)$ and $\sum_i p_i \le M$, then for all subsets $S \subseteq \{1, 2, ...\}$, we have*

$$P(\sum_{i=1}^{\infty} X_i \ge L \mid \sum_{i \in S} X_i \ge 1) < \epsilon.$$

Lemma 6.3. *Given $b, \gamma \in (0,1)$, and $N \ge 1$, there exists $\delta = \delta(b, \gamma, N)$ such that if $\{X_i\}_{i \in I}$ are independent Bernoulli random variables with $p_i = P(X_i = 1) \le b$ for all i and such that $P(\sum_{i \in I} X_i \ge 1) \ge 1 - \delta$, then $P(\sum_{i \in I} X_i \ge N) \ge 1 - \gamma$.*

The next lemma tells us that if there are many edges between a random set and the union of a fixed set together with its vertex boundary and its two-step vertex boundary, then there must also be a certain amount of edges between the random set and the union of the vertex boundary and the two-step vertex boundary of the fixed set.

Lemma 6.4. *Let $R \equiv S \cap B(n)$ with $(S \cup \Delta S \cup \Delta\Delta S) \cap T(n) = \emptyset$. Let T be a random subset of $(R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n)$ consisting of open edges and open vertices and which is measurable with respect to the edges and vertices contained in $(S \cup \Delta S \cup \Delta\Delta S)^{\mathbf{c}}$. Let E be the set of open edges between vertices in $(\Delta^v_{S,B(n)} R \cup \Delta\Delta^e R)$ and vertices in T and let F be the set of open edges between vertices in $(R \cup \Delta^v_{S,B(n)} R \cup \Delta\Delta^e R)$ and vertices in T. The vertices in R, $\Delta^v_{S,B(n)} R$ and $\Delta\Delta^e R$ can be either open or closed.*

Then for all integers k and all $\epsilon > 0$

$$P(|E| \le k, |F| \ge \frac{4}{c\epsilon^2} + \frac{2k}{c\epsilon}) \le \epsilon,$$

where c is as in (5.2).

Also if $S \subseteq \mathbb{Z}^d$, T is a random subset of $(S \cup \Delta S \cup \Delta\Delta S)^{\mathbf{c}}$ consisting of open edges and open vertices and which is measurable with respect to the edges and vertices contained in $(S \cup \Delta S \cup \Delta\Delta S)^{\mathbf{c}}$, E is the set of open edges between vertices in $(\Delta^v S \cup \Delta\Delta^e S)$ and vertices in T and F is the set of open edges between vertices in $(S \cup \Delta^v S \cup \Delta\Delta^e S)$ and vertices in T. The vertices in S, $\Delta^v S$ and $\Delta\Delta^e S$ can be either open or closed. Then for all integers k and all $\epsilon > 0$

$$P(|E| \le k, |F| \ge \frac{4}{c\epsilon^2} + \frac{2k}{c\epsilon}) \le \epsilon,$$

where c is as in (5.3).

Proof. It suffices to show that $P(|E| \le k, |F| \ge \frac{4}{c\epsilon^2} + \frac{2k}{c\epsilon} \mid T = \Gamma) \le \epsilon$ for all subsets Γ of $(R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n)$ consisting of open vertices and open edges and for which $P(T = \Gamma) > 0$. By the FKG inequality, see e.g. Grimmett (1989), and the measurability of T we need only show that

$$P(|E| \le k \mid T = \Gamma) P(|F| \ge \frac{4}{c\epsilon^2} + \frac{2k}{c\epsilon} \mid T = \Gamma) \le \epsilon$$

for all $\Gamma \subseteq (R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n)$ with $P(T = \Gamma) > 0$. Fix such a Γ. By Lemma 6.1 it is enough to show that

$$\sum_{\substack{x \in (\Delta^v_{S,B(n)} R \cup \Delta\Delta^e R) \cap B(n), \\ \gamma \in \Gamma}} p(x-\gamma) \;\geq\; c \sum_{\substack{x \in (R^v \cup \Delta^v_{S,B(n)} R \cup \Delta\Delta^e R) \cap B(n), \\ \gamma \in \Gamma}} p(x-\gamma),$$

but this follows from (5.2) if we sum over γ last.

The second case of the lemma is proved in a similar way using (5.3).

■

The following lemma demonstrates that the number of vertices in a fixed set that are endpoints to edges between that set and a random set, cannot be very small if there is a considerable amount of edges between the two sets.

Lemma 6.5. *Given ϵ and k there exists $a(k,\epsilon)$ such that if S_1 is a subset of $\mathbb{Z}^d$, S_2 is a random subset of $\mathbb{Z}^d \setminus S_1$ consisting of open vertices and open edges and which is measurable with respect to edges and vertices contained in $\mathbb{Z}^d \setminus S_1$, E is the set of open edges between vertices in S_2 and vertices in S_1, and V is the set of vertices in S_1 which are endpoints of open edges to vertices in S_2 then*

$$P(|V| \leq k, |E| \geq a(k,\epsilon)) \leq \epsilon.$$

Proof. In the same way as in Meester and Steif (1996). ■

We introduce some more notation. Let $W_{m,n}$ be the set of open vertices and open edges in $B(n-1)$ that are connected to $B(m)$ inside the box $B(n-1)$. Let $F_{m,n}$ be the set of open edges from $W_{m,n}$ to vertices in $B(n-1)^c$. Further let $V_{m,n}$ be the set of vertices in $\Delta\Delta^e B(n-1)$ which have an open edge to $W_{m,n}$ and let $E_{m,n}$ be the set of open edges from $W_{m,n}$ to vertices in $\Delta\Delta^e B(n-1)$.

Lemma 6.6. *For all k and $m \geq 1$*

$$\lim_{n\to\infty} P(|F_{m,n}| \leq k, B(m) \to \infty) = 0.$$

Proof. Fix k and m. Then

$$\limsup_{n\to\infty} P(|F_{m,n}| \leq k, B(m) \to \infty) \leq P(\limsup_{n\to\infty}\{|F_{m,n}| \leq k, B(m) \to \infty\})$$

by Fatou's lemma. We will now show that the latter probability equals 0.

Let R_n be the, possibly empty, set of vertices in $B(n-1)^{\mathbf{c}}$ which are endpoints of, open, edges in $F_{m,n}$. Define integers n_1, n_2, ... inductively as follows. Let $n_1 = m+1$, $n_2 = \inf\{i > n_1 : R_{n_1} \subseteq B(i-1), |F_{m,i}| \leq k\}$, and $n_{r+1} = \inf\{i > n_r : R_{n_r} \subseteq B(i-1), |F_{m,i}| \leq k\}$. These will be random numbers. In addition, if $R_{n_i} = \emptyset$, let $n_{i+1} = \infty$ and if some $n_r = \infty$, let all subsequent n_i's be ∞. For $r \geq 1$, let

$$E_r = \{n_r < \infty, \text{ there is an open edge between } R_{n_r} \text{ and } W^{\mathbf{c}}_{m,n_r}\}.$$

It is now immediate that

$$\limsup_n \{|F_{m,n}| \geq k, B(m) \to \infty\} \subseteq \bigcap_{r=1}^{\infty} E_r,$$

so we can concentrate on this latter event. However, it is clear from independence and the fact that $|R_n| \leq |F_{m,n}|$ that there exists $\alpha = \alpha(k) < 1$ such that for all $r \geq 1$, $P(E_{r+1} \mid E_1 \cap E_2 \cap \cdots \cap E_r) \leq \alpha$. From this the above follows. ■

Lemma 6.7. *For all k and $m \geq 1$*

$$\lim_{n\to\infty} P(|V_{m,n}| \leq k, B(m) \to \infty) = 0.$$

Proof. Let k, $m \geq 1$ and let $\epsilon > 0$. Let c be as in (5.3) and $a(k,\epsilon)$ be as in Lemma 6.5. We have

$$\begin{aligned} P(|V_{m,n}| \leq k, B(m) \to \infty) \quad &\leq \quad P(|V_{m,n}| \leq k, |E_{m,n}| \geq a(k,\epsilon)) \\ &\quad + P(|E_{m,n}| < a(k,\epsilon), |F_{m,n}| \geq \frac{4}{c\epsilon^2} + \frac{2a(k,\epsilon)}{c\epsilon}) \\ &\quad + P(|F_{m,n}| < \frac{4}{c\epsilon^2} + \frac{2a(k,\epsilon)}{c\epsilon}, B(m) \to \infty). \end{aligned}$$

The first term is at most ϵ by Lemma 6.5 with $S_1 = \Delta\Delta^e B(n-1)$ and $S_2 = W_{m,n}$. If we let $S^v = \{z : d(\{z\}, B(n-1)) \geq 2\}$, $S = S^v \cup E(S^v)$ and $T = W_{m,n}$, then $\Delta^v S = \emptyset$, $(S \cup \Delta\Delta^e S) = B(n-1)^{\mathrm{c}}$, and $\Delta\Delta^e S = \Delta\Delta^e B(n-1)$. An application of the second case in Lemma 6.4 with these choices of S and T, gives that also the second term can be bounded from above by ϵ. The third term goes to zero as $n \to \infty$ as a consequence of Lemma 6.6. As $\epsilon > 0$ is arbitrary, the result follows. ■

Lemma 6.8. *Let $\alpha > 0$. Then there exist integers m and n such that $2m < n$ and*

$$P(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) > 1 - \alpha.$$

Proof. Since $\theta(q,p,p_2,p_3,\ldots) > 0$ there exists $m = m(d,q,p,\alpha)$ so that

$$P(B(m) \to \infty) > 1 - (\frac{1}{2}\alpha)^{d2^d}.$$

Let $q_{B(m)}$ be the probability that $B(m)$ is an open m-pad and that there is an open edge between each vertex of $\{x \in B(m) : x_1 = -m\}$ and its unique nearest neighbour whose first coordinate is $-(m+1)$ and that this neighbour also is open. Now choose M sufficiently large so that at least one of M independent events, each with probability $q_{B(m)}$, occur with probability greater than $1-\alpha/2$. Next, choose l so large that for any n and for any subset S of $T(n)$ of size l or more, S will have the property that there will be M disjoint translates of $B(m)$ in $T(m,n)$ each at distance 1 from some point of S.

Let us now show that

$$\liminf_n P(|V'_{m,n}| \geq l) \geq 1 - \frac{\alpha}{2},$$

where $V'_{m,n}$ denotes the set of vertices in $T(n)$ which have an open edge to a vertex in $W_{m,n}$. We note that there exists a group of symmetries of the cube of order $(2d)2^{d-1} = d2^d$, and which have the following property: if the elements of the group transform $T(n)$ into $T_1(n), ..., T_{d2^d}(n)$, then

$$\Delta\Delta^e B(n-1) \subseteq \bigcup_{i=1}^{d2^d} T_i(n).$$

If we let $V^i_{m,n}$ denote the set of vertices in $T_i(n)$ which have an open edge to $W_{m,n}$, it follows that

$$\bigcap_{i=1}^{d2^d} \{|V^i_{m,n}| < l\} \subseteq \{|V_{m,n}| < d2^d l\}.$$

The FKG inequality now gives

$$P(|V_{m,n}| < d2^d l) \geq \prod_{i=1}^{d2^d} P(|V^i_{m,n}| < l) = \left(P(|V'_{m,n}| < l)\right)^{d2^d},$$

which implies that

$$P(|V'_{m,n}| \geq l) \geq 1 - P(|V_{m,n}| < d2^d l)^{\frac{1}{d2^2}}.$$

By Lemma 6.7 we have

$$\begin{aligned} \limsup_n P(|V_{m,n}| < d2^d l) &= \limsup_n P(|V_{m,n}| < d2^d l, B(m) \not\leftrightarrow \infty) \\ &\leq P(B(m) \not\leftrightarrow \infty). \end{aligned}$$

As desired, we now obtain

$$\liminf_n P(|V'_{m,n}| \geq l) \geq 1 - (P(B(m) \not\leftrightarrow \infty))^{\frac{1}{d2^d}} \geq 1 - \frac{\alpha}{2}.$$

It is now time to reach the conclusion of the lemma. Let A be the event that at least l in $T(n)$ can reach $B(m)$ in $B(n)$, that is $A = \{|V'_{m,n}| \geq l\}$, and let B denote the event that at least one in $T(n)$ that can reach $B(m)$ in $B(n)$ also is open and has an open edge of length 1 to an open m-pad in $T(m, n)$.

$$\begin{aligned} P(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) &\geq P(A \cap B) \\ &= P(B \mid A)P(A) \\ &\geq (1 - \frac{\alpha}{2})(1 - \frac{\alpha}{2}) \\ &\geq 1 - \alpha \end{aligned}$$

■

Having read the statement of the above lemma one could think that this is all that is going to be needed in the renormalization process. However this is not so, since the events we are going to study are not independent. Lemma A will take care of this.

Lemma 6.9. *Let $\epsilon > 0$ and $l \geq 1$. Then there exist m and n such that $2m < n$ and such that if S is a connected set with $S \supseteq B(m)$ and with $(S \cup \Delta S \cup \Delta\Delta S) \cap T(n) = \emptyset$ and $R \equiv S \cap B(n)$, then if W is the union of the set of vertices*

$$\{v \in (R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n) : v \leftrightarrow K(m,n) \text{ in } (R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n)\}$$

and the set of edges

$$\{e = \{v_1, v_2\},\ v_1 \in (R \cup \Delta R)^{\mathbf{c}} \cap B(n) \text{ and } v_2 \in (R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n) :$$
$$e \leftrightarrow K(m,n) \text{ in } (R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n)\}$$

(which includes $K(m,n)$ by convention) and $F = F_1 \cup F_2$ where

$$F_1 = \{e = \{v_1, v_2\} : e \text{ open}, v_1 \in (R \cup \Delta^v_{S,B(n)} R) \text{ and } v_2 \in W\}$$

and

$$F_2 = \{v \in \Delta\Delta^e R : v \text{ open}, \exists y \in W \text{ such that } \{v, y\} \in W\},$$

then $P(|F| \geq l) \geq 1 - \epsilon$.

Proof. In the notation of Lemma 6.3 we let $b = \sup_{i \in \{0,1,2,\dots\}} p_i$ (where $p_0 = q$ and $p_1 = p$), $\gamma = \epsilon/2$, $N = l$ and we let δ be as in the conclusion of that lemma. Let us also require δ to be less than $\epsilon/2$. By Lemma 6.8 we can choose integers m and n such that $2m < n$ and

$$P(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) > 1 - \delta^2.$$

Now, as $B(m) \subseteq R \subseteq B(n)$, it must follow that

$$P(R \leftrightarrow K(m,n) \text{ in } B(n)) > 1 - \delta^2,$$

which of course implies that

$$P\left((R \cup \Delta R \cup \Delta\Delta R) \cap B(n) \leftrightarrow K(m,n) \text{ in } B(n)\right) > 1 - \delta^2.$$

Considering the geometrical meaning of F, we see that we must have

$$P(|F| \geq 1) > 1 - \delta^2.$$

It follows that there is a subset $\mathcal{S}'$ of {all subsets of vertices in $(R \cup \Delta R \cup \Delta\Delta R)^{\mathbf{c}} \cap B(n)$ and edges in $B(n)$ with no endpoint in $(R \cup \Delta R)$ and at most one in $\Delta\Delta R\} = \mathcal{S}$, such that $P(|F| \geq 1 \mid W = S) \geq 1 - \delta$ for all $S \in \mathcal{S}'$ and $P(W \in \mathcal{S}') \geq 1 - \delta$.

To see this we let

$$\mathcal{S}' = \{S : P(|F| \geq 1 \mid W = S) \geq 1 - \delta\},$$

and suppose $P(W \in \mathcal{S}') < 1-\delta \Leftrightarrow P(W \notin \mathcal{S}') > \delta$. Then

$$
\begin{aligned}
&P(|F| \geq 1)\\
&= \sum_{S \in \mathcal{S}'} P(|F| \geq 1 \mid W = S)P(W = S) + \sum_{S \notin \mathcal{S}'} P(|F| \geq 1 \mid W = S)P(W = S)\\
&\leq P(W \in \mathcal{S}') + (1-\delta)P(W \notin \mathcal{S}')\\
&= 1 - \delta P(W \notin \mathcal{S}')\\
&< 1-\delta^2,
\end{aligned}
$$

and we arrive at a contradiction. Therefore it must be the case that $P(W \in \mathcal{S}') \geq 1-\delta$.

Conditioned on the event $\{W = S\}$, —F— is a sum of independent random variables. An application of Lemma 6.3 tells us that $P(|F| \geq l|W = S) \geq 1-\epsilon/2$ if S belongs to $\mathcal{S}'$. This in turn leads to the conclusion of the current lemma as $P(|F| \geq l) \geq \sum_{S \in \mathcal{S}'} P(|F| \geq l \mid W = S)P(W = S) \geq (1-\epsilon/2)P(W \in \mathcal{S}') \geq (1-\epsilon/2)^2 \geq 1-\epsilon$.

■

Lemma 6.10. *Let $\epsilon > 0$ and $k \geq 1$. Then there exist m and n such that $2m < n$ and which have the following properties; If S is a connected set with $S \supseteq B(m)$ and with $(S \cup \Delta S \cup \Delta\Delta S) \cap T(n) = \emptyset$ and $R \equiv S \cap B(n)$, then if*

$$
\begin{aligned}
V &= \{x \in \Delta^v_{S,B(n)} R : \exists y \in ((R \cup \Delta R \cup \Delta\Delta R)^{\mathsf{c}} \cap B(n))^v \text{ such that}\\
&\qquad \{x,y\} \leftrightarrow K(m,n) \text{ in } (R \cup \Delta R \cup \Delta\Delta R)^{\mathsf{c}} \cap B(n)\}
\end{aligned}
$$

and

$$
\begin{aligned}
E &= \{e = \{x,y\} \in \Delta^e R : x \in R^v, y \notin R^v \text{ such that}\\
&\qquad y \leftrightarrow K(m,n) \text{ in } (R \cup \Delta R \cup \Delta\Delta R)^{\mathsf{c}} \cap B(n)\},
\end{aligned}
$$

then $P(\{|V| > k\} \cup \{|E| > k\}) > 1-\epsilon$.

Proof. Let b be so large that if X is a binomial random variable with parameters b and q then $P(X \leq k) \leq \epsilon/8$. Next let $a = 2a(b, \frac{\epsilon}{8})$, where the function $a()$ is as in Lemma 6.5. Let W, F_1 and F_2 be as in Lemma 6.9 and then choose m and n according to that lemma with ϵ replaced by $\epsilon/2$ and l replaced by $2(\frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}})$.

$$
\begin{aligned}
&P(\{|V| > k\} \cup \{|E| > k\})\\
&\geq P(\,(|V| > k \,\cup\, |E| > k),\, |F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}\,) \qquad (6.1)\\
&\quad + P((|V| > k \,\cup\, |E| > k),\, |F_1| < \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}},\, |F_2| > \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}})
\end{aligned}
$$

We start investigating the second of the above terms. Since each edge in $\Delta^e R$ has exactly one endpoint in $\Delta\Delta^e R$, it is a fact that $|E|$ is equal to $|F_2|$. This in turn implies that $P((|V| > k \cup |E| > k), |F_1| < \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}, |F_2| > \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}})$ is equal to $P(|F_1| < \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}, |F_2| > \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}})$, since we can take for granted that k is not greater than $\frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}$.

Let us now turn our interest to the first term in (6.1). Let E' be the set of open edges between $\Delta^v_{S,B(n)}R$ and W and let E'' be the set of open edges with one endpoint in $\Delta\Delta^e R$ and one in W. We study the complement of the event on $\{|F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}\}$.

$$\begin{aligned}
&P(|V| \leq k,\ |E| \leq k,\ |F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \\
&\leq\ P(|V| \leq k,\ |E| \leq k,\ |F_1| + |E''| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \\
&=\ P(|V| \leq k,\ |E| \leq k,\ |E'| + |E''| < a,\ |F_1| + |E''| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \\
&\quad + P(|V| \leq k,\ |E| \leq k,\ |E'| + |E''| \geq a,\ |F_1| + |E''| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \\
&\leq\ P(|E'| + |E''| < a,\ |F_1| + |E''| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \\
&\quad + P(|V| \leq k,\ |E| \leq k,\ |E'| \geq \frac{a}{2}) \\
&\quad + P(|V| \leq k,\ |E| \leq k,\ |E'| < \frac{a}{2},\ |E''| > \frac{a}{2}) \\
&\leq\ P(|E'| + |E''| \leq a,\ |F_1| + |E''| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \qquad (6.2) \\
&\quad + P(|V| \leq k,\ |E'| \geq \frac{a}{2}) + P(|E| \leq k,\ |E''| > \frac{a}{2})
\end{aligned}$$

Let us first look at the last of the above terms. We introduce yet another set, V'', which will consist of the endpoints in $\Delta\Delta^e R$ to the edges in E''. $|E|$ is equal to $|F_2|$ and F_2 consists of those vertices in V'' that are open. Conditioned on $|V''|$ being equal to b, $|E|$ is binomially distributed with parameters b and q.

$$P(|E| \leq k, |E''| > \frac{a}{2}) \leq P(|V''| < b, |E''| > \frac{a}{2}) + P(|E| \leq k, |V''| \geq b)$$

and

$$P(|E| \leq k, |V''| \geq b) \leq P(|E| \leq k \mid |V''| \geq b) \leq P(|E| \leq k \mid |V''| = b).$$

By the way b was chosen we know that $P(|E| \leq k \mid |V''| = b)$ can be bounded form above by $\frac{\epsilon}{8}$. We now turn to $P(|V''| < b, |E''| > \frac{a}{2})$. This term also can be be dominated by $\frac{\epsilon}{8}$, as we see if we apply Lemma 6.5 with $\Delta\Delta^e R$ as S_1 and W as S_2.

$P(|V| \leq k, |E'| \geq \frac{a}{2})$ must be less than $P(|V| \leq b, |E'| \geq \frac{a}{2})$ as b is greater than k. To find an upper bound for this term we once again apply Lemma 6.5, this time with $\Delta^v_{S,B(n)}R$ as S_1 and W as S_2. As $\frac{a}{2} = a(b, \frac{\epsilon}{8})$, we get $\frac{\epsilon}{8}$ as an upper bound for $P(|V| \leq k, |E'| \geq \frac{a}{2})$.

There is still one term left in (6.2) to be studied. It can be seen to be less than $\frac{\epsilon}{8}$ by an application of Lemma 6.4 with S as itself and W as T. Now we have shown that $P(|V| \leq k,\ |E| \leq k,\ |F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}})$ cannot be greater than $\frac{\epsilon}{8} + \frac{\epsilon}{8} + (\frac{\epsilon}{8} + \frac{\epsilon}{8}) = \frac{\epsilon}{2}$.

Let us go back to the very beginning of the proof and take up where we left off:

$$
\begin{aligned}
&P(\{|V| > k\} \cup \{|E| > k\}) \\
&\quad \geq \; P(\,(|V| > k \;\cup\; |E| > k),\ |F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}\,) \\
&\qquad + P(\,(|V| > k \;\cup\; |E| > k),\ |F_1| < \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}},\ |F_2| > \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}\,) \\
&\quad \geq \; P(|F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) - P(|V| \leq k,\ |E| \leq k,\ |F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \\
&\qquad + P(|F_1| < \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}},\ |F_2| > \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \qquad (\text{as } |E| = |F_2|\,) \\
&\quad \geq \; P(|F_1| + |F_2| \geq 2(\frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}})) \\
&\qquad - P(|V| \leq k,\ |E| \leq k,\ |F_1| \geq \frac{4}{c(\frac{\epsilon}{8})^2} + \frac{2a}{c\frac{\epsilon}{8}}) \\
&\quad \geq \; 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\
&\quad = \; 1 - \epsilon.
\end{aligned}
$$

■

Now we are ready for the highlight of this section, namely the proof of the main lemma.

Proof of Lemma A. Choose t such that $(1-\delta)^t < \epsilon/2$. Choose m and n according to Lemma 6.10 with t and $\epsilon/2$ playing the role of k and ϵ. Let V and E be as in Lemma 6.10.

Let $A' = \{e \text{ is } (\beta(e)+\delta)\text{-open for some } e \in E\}$ and let $A'' = \{v \text{ is } (\tilde{\beta}(v)+\delta)\text{-open for some } v \in V\}$.

$$
\begin{aligned}
&P(G \mid H) \\
&\quad \geq \; P(\,(A' \cap \{|E| \geq t\}) \;\cup\; (A'' \cap \{|V| \geq t\}) \mid H) \\
&\quad \geq \; P(\,(A' \cap \{|E| \geq t\}) \;\cup\; (A'' \cap (\{|V| \geq t\} \setminus \{|E| \geq t\})) \mid H) \\
&\quad = \; P(A' \cap \{|E| \geq t\} \mid H) \\
&\qquad + P(A'' \cap (\{|V| \geq t\} \setminus \{|E| \geq t\}) \mid H) \\
&\quad = \; P(A' \mid \{|E| \geq t\} \cap H) P(\{|E| \geq t\} \mid H) \\
&\qquad + P(A'' \mid (\{|V| \geq t\} \setminus \{|E| \geq t\}) \cap H) P(\{|V| \geq t\} \setminus \{|E| \geq t\} \mid H)
\end{aligned}
$$

In each pair of terms in this last equation, the first term is at least $1-(1-\delta)^t$ which is greater than $1-\epsilon/2$. For the second "first term" this is seen by noting that the fact that knowing that $\{|E| \geq t\}$ does not occur, does not tell us anything about the status of the vertices in V, which we know to be at least t in number. This means that the equation can be bounded from below by

$$\begin{aligned}
&(1-\frac{\epsilon}{2})\ (\ P(\{|E| \geq t\} \mid H) + P((\{|V| \geq t\} \setminus \{|E| \geq t\}) \mid H)\ ) \\
&= \ (1-\frac{\epsilon}{2})\ P(\{|E| \geq t\} \cup (\{|V| \geq t\} \setminus \{|E| \geq t\}) \mid H) \\
&= \ (1-\frac{\epsilon}{2})\ P(\{|E| \geq t\} \cup \{|V| \geq t\} \mid H) \\
&\geq \ (1-\frac{\epsilon}{2})^2 \\
&\geq \ 1-\epsilon.
\end{aligned}$$

■

7. The renormalization

Let $N = n + m + 1$, where $2m < n$. Take d to be at least 2 and order the edges between nearest neighbours of $\mathbb{Z}^d$ in some arbitrary way. Consider boxes of the form $\{4Nz + B(N) : z \in \mathbb{Z}^d\}$ and call them **site boxes**. Each one of them will correspond to a site in the lattice $\mathbb{Z}^d$, the latter will be referred to as the renormalized lattice. Between two site boxes there is room for exactly one translate of $B(N)$. These boxes sitting in between, we will call **halfway boxes**. Write $T_j(n)$ for the image of $T(n)$ under the "earliest" isometry that preserves the origin and maps the first coordinate direction onto the jth, for $j = 1, ..., 2d$. Define $T_j(m,n)$ and $K_j(m,n)$ as isometries of respectively $T(m,n)$ and $K(m,n)$ in the same manner.

We will examine each site box and determine if it is occupied or vacant, according to rules which are to be described below. See Grimmett and Marstrand (1990) for details.

Now use the following rules. We say that the site box of the origin, $B(N)$, is occupied if the following three things occur;

(*1*) $B(m)$ is a q,p-open m-pad.

(*2*) $B(m)$ is connected in $B(n)$ to $K_j(m,n)$ for $j = 1, ..., 2d$. If there is more than one q,p-open m-pad in $T_j(m,n)$, we choose one of them according to some nonrandom rule decided upon in advance. We call this q,p-open m-pad the **target** open m-pad in the jth direction.

(*3*) For all j, the target open m-pad in (*2*) in the jth direction, $(b_j + B(m))$, is connected in $(b_j + B(n))$ to a vertex next and connected to a q,p-open m-pad in $(b_j + T_j^*(m,n))$, which is in the halfway box next to $B(N)$ in the jth direction. Call this q,p-open m-pad a target open m-pad. $T_j^*(m,n)$ is the image of $T_j(m,n)$ under the symmetry which fixes e_j and for $k \neq j$ sends

e_k to $-e_k$. Here e_i, $i = 1, ..., 2d$, are the unit vectors. This reflection is necessary to make sure that the target open m-pads are in the appropriate halfway boxes, and we call this a **steering action**.

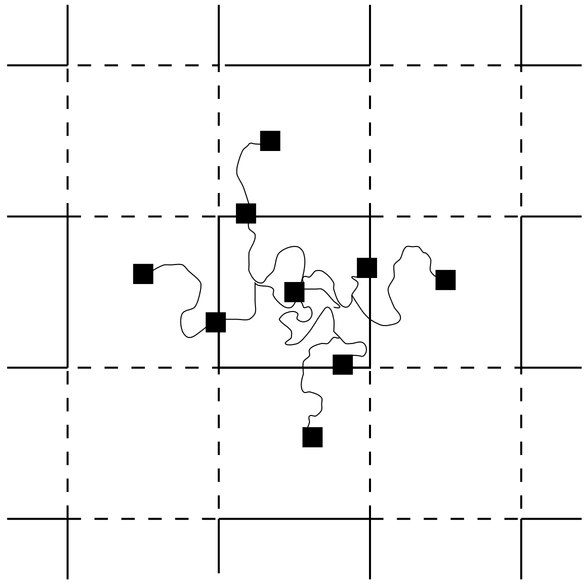

Figure 3: An illustration of the event that the origin in the renormalized lattice is occupied. The black boxes are q,p-open m-pads, the central one is $B(m)$. The larger boxes are translates of $B(N)$. The boxes drawn with all non-dashed lines are site boxes.

If all this happens, $B(m)$ is now connected in two steps to a target q,p-open m-pad in all its neighbouring halfway boxes, see Figure 3. To determine which site box to examine next, we consider the above mentioned ordering of the edges starting at the origin. We can suppose, without loss of generality, that the first edge is e_1. We call the site box $4Ne_1 + B(N)$ occupied if the following occur;

(*1'*) The target open m-pad of (*3*) above in the first direction is connected in two steps, which are analogous with the two steps described above, to a point next and connected to a q,p-open m-pad in $4Ne_1 + B(N)$. This will again be called a target open m-pad. Here we will have to use a steering action to make sure we end up in the correct site box.

(*2'*) The target q,p-open m-pad of (*1'*) is in two steps connected to a point next and connected to a q,p-open m-pad in all halfway boxes next to $4Ne_1 + B(N)$, which are not next to any site box that has yet been examined.

This procedure is continued. At each step we look for the first edge according to the ordering, which connects an occupied site box to a site box which has not yet been examined, and we decide on the status of the latter site box.

We will now show that if η is a small positive number and q and p are such that $\theta(q, p, p_2, p_3, ...) > 0$, i.e. the underlying long range mixed model percolates, then the renormalized nearest neighbour model percolates under $q + \eta$ and $p + \eta$ for a suitable choice of m and n. (By the expression "under q and p" we mean "using only q-open vertices, p-open edges of length one and open edges of length greater than one".) Consider a number Q which is strictly larger than the critical probability for independent nearest neighbour site percolation on $\mathbb{Z}^d$, $p_c(\text{site})$, and strictly less than 1. If we can show that at each step, the conditional probability given the past of the procedure that a new site box is occupied is at least Q and independent of the site box being examined and of what we have seen up to this point, then the renormalized model percolates with positive probability. This is due to Lemma 1 in Grimmett and Marstrand (1990).

Before we begin calculating probabilities, we fix p and q so that $\theta(q, p, p_2, p_3, ...) > 0$. Let $\eta > 0$, $\delta = \frac{\eta}{2d}$ and $\epsilon = \frac{(1-p_c(\text{site}))}{8d}$ and let m and n be as in Lemma A for this choice of q, p, ϵ and δ. Let us now investigate the probability that $B(N)$ is occupied. The probability that $B(m)$ is a q,p-open m-pad is small, but positive. It serves only as a starting point for the procedure, so this does not impose any problems.

Let $C_1^v = B(m)^v$ and $C_1^e = \{\text{the open edges of } B(m)\}$ (p-open for edges of length one). Create four functions, two defined on the edges of length one of the original lattice $\mathbb{Z}^d$ and two defined on the vertices of the original lattice $\mathbb{Z}^d$. Let $\boldsymbol{\gamma}_1$ take the value p for edges of length one in C_1^e and 1 for all the edges of length one in $\mathbb{Z}^d \setminus C_1^e$. Let $\boldsymbol{\beta}_1$ equal 0 for all edges of length one. Let $\tilde{\boldsymbol{\gamma}}_1$ be q for vertices in C_1^v and 1 for all other vertices, and finally let $\tilde{\boldsymbol{\beta}}_1$ equal 0 for all vertices in $\mathbb{Z}^d$. Having only checked if $B(m)$ is a q,p-open m-pad, we have no information about vertices and edges outside $B(m)$. Now every vertex v of $\mathbb{Z}^d$ is $\tilde{\gamma}_1(v)$-open and $\tilde{\beta}_1(v)$-closed and all edges e of $\mathbb{Z}^d$ of length one are $\gamma_1(e)$-open and $\beta_1(e)$-closed. Apply Lemma A with $S = C_1$, β as β_1 restricted to $\Delta^e C_1$ and $\tilde{\beta}$ as $\tilde{\beta}_1$ restricted to $\Delta^v C_1$. This lemma tells us that the probability for the existence of a path from $B(m)$ to $K(m, n)$ within $B(n)$ is at least $1 - \epsilon$. By symmetry it follows that the event in (*2*) occurs under q and p with probability at least $1 - 2d\epsilon$.

Let $B_1 = B(n) \cup \cup_{j=1}^{2d} T_j(m, n)$ and $C_2 = C_1 \cup E_1 \cup F_1$ where

$$
\begin{aligned}
E_1^v &= \{x \in \Delta^v C_1 : x \text{ is } (\tilde{\beta}_1(x) + \delta)\text{-open}\} \\
E_1^e &= \{e \in \Delta^e C_1 : e \text{ is } (\beta_1(e) + \delta)\text{-open}\} \\
F_1^v &= \{x \in B_1^v : x \ q\text{-open}, x \text{ can be joined by a path in } B_1 \setminus B(m) \text{ to} \\
&\qquad \Delta C_1. \text{ The path being } q,p\text{-open except for the unique element} \\
&\qquad y = y(x) \text{ of } \Delta C_1 \text{ which is } (\beta_1(y) + \delta)\text{-open if } y \in \Delta^e C_1 \text{ and} \\
&\qquad (\tilde{\beta}_1(y) + \delta)\text{-open if } y \text{ in } \Delta^v C_1.\} \\
F_1^e &= \{e \in B_1^e : e \ p\text{-open}, e \text{ can be joined by a path in } B_1 \setminus B(m) \text{ to} \\
&\qquad \Delta C_1. \text{ The path being } q,p\text{-open except for the unique element} \\
&\qquad y = y(e) \text{ of } \Delta C_1 \text{ which is } (\beta_1(y) + \delta)\text{-open if } y \in \Delta^e C_1 \text{ and} \\
&\qquad (\tilde{\beta}_1(y) + \delta)\text{-open if } y \text{ in } \Delta^v C_1.\}
\end{aligned}
$$

While examining step (*2*) we have gained new information about some of the vertices and edges in $B_1 \setminus C_1$. The vertices that appear in C_2 we know to be q-open. As $\Delta^v B(m)$ is empty, even E_1^v must be empty, so there can be no δ-open vertices. (The reason for still including E_1^v is to have this first step look the same as the following ones.) For the edges of length one in C_2 we can say that they are either p-open, those in $C_2 \setminus \Delta C_1$, or δ-open, those in $C_2 \cap \Delta C_1$. We can summarize this knowledge in saying that all vertices v in $\mathbb{Z}^d$ are $\tilde{\gamma}_2(v)$-open and $\tilde{\beta}_2(v)$-closed and all edges e of length one in $\mathbb{Z}^d$ are $\gamma_2(e)$-open and $\beta_2(e)$-closed, where the four functions are defined in the following way;

$$\tilde{\gamma}_2(v) = \begin{cases} \tilde{\beta}_1(v) + \delta & v \in \Delta^v C_1 \cap C_2^v \\ q & v \in C_2^v \setminus (C_1^v \cup \Delta^v C_1) \\ \tilde{\gamma}_1(v) & \text{otherwise} \end{cases}$$

$$\tilde{\beta}_2(v) = \begin{cases} \tilde{\beta}_1(v) + \delta & v \in (\Delta^v C_1 \setminus C_2^v) \cap B_1^v \\ q & v \in (\Delta^v C_2 \setminus \Delta^v C_1) \cap B_1^v \\ \tilde{\beta}_1(v) & \text{otherwise} \end{cases}$$

$$\gamma_2(e) = \begin{cases} \beta_1(e) + \delta & e \in \Delta^e C_1 \cap C_2^e \\ p & e \in C_2^e \setminus (C_1^e \cup \Delta^e C_1) \\ \gamma_1(e) & \text{otherwise} \end{cases}$$

$$\beta_2(e) = \begin{cases} \beta_1(e) + \delta & e \in (\Delta^e C_1 \setminus C_2^e) \cap B_1^e \\ p & e \in (\Delta^e C_2 \setminus \Delta^e C_1) \cap B_1^e \\ \beta_1(e) & \text{otherwise} \end{cases}$$

It is now that the need for the vertex boundary of a set on another set arises. It takes care of all the negative information we have about the elements in the boxes $b_i + B(n)$ for $i = 1, ..., 2d$. For example, it may well be that there is an edge in C_2 that has one endpoint in C_2 and the other one, x, on the border of $b_1 + B(n)$ and that this endpoint does not belong to C_2 but to $B(n)$. We then have negative information about x as we know that it does not belong to C_2. It is clear that x does not belong to $\Delta^v(C_2 \cap (b_1 + B(n)))$, but to $\Delta^v_{C_2, b_1+B(n)}(C_2 \cap (b_1 + B(n)))$, so if we use the ordinary vertex boundary in Lemma A, the event H will not contain all the information known to us.

Let us now investigate the probability of step (*3*). This we do by applying Lemma A $2d$ times — each time centred at b_j for $j = 1, ..., 2d$. The first time centred at b_1 and using the set $S = C_2$ and the above described β_2 and $\tilde{\beta}_2$ restricted to $\Delta^e(C_2 \cap (b_1 + B(n)))$ and to $\Delta^v_{C_2,(b_1+B(n))}(C_2 \cap (b_1 + B(n)))$ respectively. The lemma gives that the (conditional) probability under q and p of the first of the $2d$ events in (*3*) is at least $1 - \epsilon$. For the discussion of the second application of the lemma, we introduce yet some notation. Let $B_2 = b_1 + (B(n) \cup T_1^*(m, n))$ and $C_3 = C_2 \cup E_2 \cup F_2$ where

$$
\begin{array}{rcl}
E_2^v & = & \{x \in \Delta^v C_2 \cap (b_1 + B(n)) : x \text{ is } (\tilde{\beta}_2(x) + \delta)\text{-open}\} \\
E_2^e & = & \{e \in \Delta^e C_2 \cap (b_1 + B(n)) : e \text{ is } (\beta_2(e) + \delta)\text{-open}\} \\
F_2^v & = & \{x \in B_2^v : x \ q\text{-open, } x \text{ can be joined by a path in } B_2 \setminus C_2 \text{ to} \\
& & \Delta C_2. \text{ The path being } q,p\text{-open except for the unique element} \\
& & y = y(x) \text{ of } \Delta C_2 \text{ which is } (\beta_2(y) + \delta)\text{-open if } y \in \Delta^e C_2 \text{ and} \\
& & (\tilde{\beta}_2(y) + \delta)\text{-open if } y \text{ in } \Delta^v C_2.\} \\
F_2^e & = & \{e \in B_2^e : e \ p\text{-open, } e \text{ can be joined by a path in } B_2 \setminus C_2 \text{ to} \\
& & \Delta C_2. \text{ The path being } q,p\text{-open except for the unique element} \\
& & y = y(e) \text{ of } \Delta C_2 \text{ which is } (\beta_2(y) + \delta)\text{-open if } y \in \Delta^e C_2 \text{ and} \\
& & (\tilde{\beta}_2(y) + \delta)\text{-open if } y \text{ in } \Delta^v C_2.\}
\end{array}
$$

If the last step was successful, namely we have reached $b_1+T_1^*(m,n)$ from C_2 in B_2, our current knowledge about the edges and vertices of $\mathbb{Z}^d$ is summarized by saying that the vertices are all $\tilde{\gamma}_3$-open and $\tilde{\beta}_3$-closed and the edges of length one are all γ_3-open and β_3-closed, where the four functions are defined in the following way;

$$
\tilde{\gamma}_3(v) = \begin{cases} \tilde{\beta}_2(v) + \delta & v \in \Delta^v C_2 \cap C_3^v \\ q & v \in C_3^v \setminus (C_2^v \cup \Delta^v C_2) \\ \tilde{\gamma}_2(v) & \text{otherwise} \end{cases}
$$

$$
\tilde{\beta}_3(v) = \begin{cases} \tilde{\beta}_2(v) + \delta & v \in (\Delta^v C_2 \setminus C_3^v) \cap B_2^v \\ q & v \in (\Delta^v C_3 \setminus \Delta^v C_2) \cap B_2^v \\ \tilde{\beta}_2(v) & \text{otherwise} \end{cases}
$$

$$
\gamma_3(e) = \begin{cases} \beta_2(e) + \delta & e \in \Delta^e C_2 \cap C_3^e \\ p & e \in C_3^e \setminus (C_2^e \cup \Delta^e C_2) \\ \gamma_2(e) & \text{otherwise} \end{cases}
$$

$$
\beta_3(e) = \begin{cases} \beta_2(e) + \delta & e \in (\Delta^e C_2 \setminus C_3^e) \cap B_2^e \\ p & e \in (\Delta^e C_3 \setminus \Delta^e C_2) \cap B_2^e \\ \beta_2(e) & \text{otherwise} \end{cases}
$$

Applying Lemma A "centred" at b_2 with $S = C_3$, β as β_3 restricted to $\Delta^e(C_3 \cap (b_2 + B(n)))$ and $\tilde{\beta}$ as $\tilde{\beta}_3$ restricted to $\Delta^v_{C_3, b_2+B(n)}(C_3 \cap (b_2 + B(n)))$, we get that the conditional probability of success in this second step of (*3*) also is at least $1-\epsilon$. Continuing in this way we can show that all $2d$ events in (*3*) occur with conditional probability at least $1-\epsilon$.

We note that any particular region can be "updated" at most $2d+1$ times — for vertices this means by an amount of δ at most $2d$ times and by q at most once, for edges of length one it means by δ at most $2d$ times and by an amount of p at most once — implying that the open cluster we find is open under $q+2d\delta$ and $p+2d\delta$ (= under $q+\eta$ and $p+\eta$). The probability, given that $B(m)$ is a q,p-open m-pad, that the site box of the origin is occupied under $q+\eta$ and $p+\eta$ is greater than $(1-2d\epsilon)(1-\epsilon)^{2d} > (1-4d\epsilon)$. This quantity is at least $Q = \frac{1}{2}(1+p_c(site))$, which is strictly larger than $p_c(site)$.

One can in a similar manner show that success of the events described in (*1'*) and (*2'*) have sufficiently high probability under $q+\eta$ and $p+\eta$.

Having come this far, we have proved the following theorem;

Theorem 7.1. *Consider a long range mixed percolation model with exponentially decaying connections on $\mathbb{Z}^d$, $d \geq 2$. Let q be the probability for a vertex to be on, p be the probability for edges of length one to be on and let p_i for $i \geq 2$ be the probability for edges of length i to be on. Suppose that $q, p \in (0,1)$ and $\theta(q,p,p_2,...) > 0$. For any $\eta > 0$ there exist m and n such that the renormalized site percolation model with underlying long range mixed percolation model $q+\eta$, $p+\eta$, p_2, p_3, ... described above percolates with positive probability as a nearest neighbour model.*

Now we are ready to give a proof of the main theorem.

Proof of Theorem 5.1. We have chosen p and q supercritical, so we can find some small $\epsilon > 0$ so that the long range percolation model with parameters $q-\epsilon$, $p-\epsilon$, p_2, p_3, ... percolates. By Theorem 7.1 we know that for any choice of $\epsilon > 0$ there exist m and n such that the renormalized site percolation model with underlying long range mixed percolation model $(q-\epsilon)+\epsilon$, $(p-\epsilon)+\epsilon$, p_2, p_3, ... $(= (q,p,p_2,p_3,...))$ percolates. We observe that the event that a certain site box in this renormalization is occupied depends on the state of a uniformly bounded number of edges which are all of L_1-length at most $4(n+m+1) = 4N$. Taking away all edges of L_1-length greater than $4N$ does not affect the percolation of the renormalized site model. As the construction of the renormalization was made, we know that if the renormalized site model percolates, so does the underlying long range mixed percolation model. We can now draw the conclusion that also the long range mixed model without its longest edges percolates. ∎

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References

Aizenman, M., Kesten, H. and Newman, C.M. (1987). Uniqueness of the Infinite Cluster and Continuity of Connectivity Functions for Short and Long Range Percolation. *Comm. Math. Phys.* **111**, (505-531).

Aizenman, M. and Newman, C.M. (1986). Discontinuity of the Percolation Density in One Dimensional $1/|x-y|^2$ Percolation Models. *Comm. Math. Phys.* **107**, (611-647).

Broadbent, S.R. (1954). In discussion of symposium on Monte Carlo methods. *J. Roy. Statist. Soc. Ser. B.* **16**, (68).

Broadbent, S.R. and Hammersley, J.M. (1957). Percolation processes. I. Crystals and mazes. *Proc. Cambridge Phil. Soc.* **53**, (629-641).

BURTON, R. AND KEANE, M. (1989). Density and Uniqueness in Percolation. *Comm. Math. Phys.* **121**, (501-505).

DURRETT, R. (1991). *Probability: Theory and Examples.* Wadsworth, Inc., Belmont, California.

GRIMMETT, G. (1989). *Percolation.* Springer-Verlag, New York Inc.

GRIMMETT, G. AND MARSTRAND, J. (1990). The supercritical phase of percolation is well behaved. *Proc. R. Soc. Lond. A.* **430**, (439-457).

HARA, T. AND SLADE, G. (1994). Mean-field behaviour and the lace expansion. *Probability Theory and Phase Transitions*, Proceedings of the NATO ASI meeting in Cambridge 1993, Kluwer.

HARRIS, T.E. (1960). A lower bound on the critical probability in a certain percolation process. *Proc. Cambridge Phil. Soc.* **56**, (13-20).

KESTEN, H. (1980). The Critical Probability of Bond Percolation on the Square Lattice Equals $\frac{1}{2}$. *Comm. Math. Phys.* **74**, (41-59).

KESTEN, H. (1987). Percolation theory and first-passage percolation. *Ann. Prob.* **15**, (1231-1271).

MEESTER, R. AND ROY, R. (1995). *Continuum percolation.* Cambridge University Press.

MEESTER, R. AND STEIF, J.E. (1996). On the continuity of the critical value for long range percolation in the exponential case. *Comm. Math. Phys.* **180**, (483-504).

NEWMAN, C.M. AND SCHULMAN, L.S. (1986). One Dimensinal $1/|j-i|^s$ Percolation Models: The Existence of a Transition for $s \leq 2$. *Comm. Math. Phys.* **104**, (547-571).

SCHULMAN, L.S. (1983). Long range percolation in one dimension. *J. Phys. A: Mathematical and General.* **16**, (L639-L641).