1. Introduction

Estimates of growth and regularity of solutions to the $\bar\partial$-equation in domains in $\mathbb{C}^n$ were obtained by L^2-methods in the 1960s by Kohn, Hörmander et al. In the 1970s Henkin, Skoda, and others introduced formulas for representation of solutions that gave further information such as L^p-estimates, Hölder estimates, and so on.

In the unit disk, the solution to $\bar\partial u = f$ that has minimal norm in $L^2_\alpha = L^2((1-|\zeta|^2)^{\alpha-1})$, $\alpha > 0$, is given by by the formula

$$K_\alpha f(z) = \frac{i}{2\pi}\int_D \left(\frac{1-|\zeta|^2}{1-\bar\zeta z}\right)^\alpha \frac{f(\zeta)\wedge d\zeta}{\zeta - z},$$

and

$$P_\alpha u(z) = \frac{1}{\pi}\int_D \frac{(1-|\zeta|^2)^{\alpha-1}u(\zeta)d\lambda(\zeta)}{(1-\bar\zeta z)^{\alpha+1}}$$

is the orthogonal projection of $L^2(D)$ onto the subspace of holomorphic functions. Both of these claims follow from the easily checked formula

$$K_\alpha \bar\partial u = u - P_\alpha u. \tag{1.1}$$

In higher dimensions a relation like (1.1), where P_α is some holomorphic projection, only determines the action of K_α on $\bar\partial$-closed forms f. The canonical operator due to Kohn (corresponding to $P = P_1$), K^{Kohn}, is the operator that vanishes on forms that are orthogonal to the $\bar\partial$-closed forms, with respect to the Euclidean metric

$$(f, g)_E = \int_D \sum f_j \bar g_j.$$

However, the explicitly given solution operators for $\bar\partial$ found in the 1970s (which indeed seem to be natural as several different approaches give operators with at least the same boundary values) do not coincide with K^{Kohn} even in the ball . Much later Harvey and Polking, [10], actually found an explicit expression for K^{Kohn} in the ball, essentially expressed by rational functions, but anyway not as simple as the previously known solution formulas.

It is known since long ago that if D is strictly pseudoconvex, then the $\bar\partial$-operator behaves differently in different directions; roughly speaking it acts as half a derivative in the complex tangential directions. This is reflected in the standard estimates for $\bar\partial$. For instance, the well-known Henkin-Skoda estimate, [14] and [15], states that $\bar\partial u = f$ has a solution such that

$$\int_{\partial D} |\bar\partial\rho \wedge u|_E \le C\int_D (-\rho)^{-1/2}\left[\sqrt{-\rho}|f|_E + |\bar\partial\rho\wedge f|_E\right]. \tag{1.2}$$

Here, $|\ |_E$ denotes the Euclidean norm of a form and ρ is a defining function for D, so that $-\rho$ is approximately the distance to the boundary, and $\bar\partial\rho \wedge f$ determines the complex tangential part of f near the boundary. This estimate was the first important success for weighted integral formulas, and once they were constructed the estimate follows nicely, as the very feature of the formulas reflects this difference in normal and complex tangential part. This suggests that these operators better should be understood in terms of a metric that takes this difference into account.

The geometrical meaning of these operators in the ball was studied in [2]. They can be described as the canonical operators with respect to the metric form $(1-|z|^2)i\partial\bar\partial\log(1/(1-|z|^2))$, and certain weight factors $(1-|z|^2)^\alpha$ on the fiber (of the trivial line bundle over D). The objective of this paper is to generalize to the strictly pseudoconvex case. For simplicity we restrict to smoothly bounded sets and throughout this paper $D=\{\rho<0\}$ denotes a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, and ρ is a strictly plurisubharmonic defining function. There is no hope even in one variable to find precise explicit expressions for the canonical operators. However it was shown by Kerzman-Stein [11] and Ligocka [12] that one can construct a projection operator P which is approximately the orthogonal projection $L^2(D)\to L^2(D)\cap\mathcal{O}(D)$, in the sense that the difference between P and the orthogonal projection is compact. Analogously we show that certain (essentially) well-known homotopy operators for $\bar\partial$ are approximately canonical with respect to the metric $\Omega=(-\rho)i\partial\bar\partial\log(-1/\rho)$ and weights $(-\rho)^\alpha$ on the fiber. For the precise statements, see Section 5.

If the defining function ρ is real-analytic and $v(\zeta,z)$ is the unique function near the diagonal that is holomorphic in z, anti-holomorphic in ζ, and such that $v(\zeta,\zeta)=-\rho(\zeta)$, then the principal term of the canonical solution operator acting on $(0,q)$-forms has the simple kernel

$$c_q\frac{\partial_\zeta\bar v\wedge(\bar\partial_z\partial_\zeta\bar v)^q}{v^{\alpha-1+n-q}\bar v^{q+1}}.$$

We also discuss the boundary complex and provide formulas for approximate canonical homotopy operators for $\bar\partial_b$. These formulas have an independent interest but they are furthermore intimately connected to the previously discussed operators.

For weights corresponding to $\alpha\geq1$, we prove that the orthogonal projection $L^2_\alpha\to L^2_\alpha\cap\operatorname{Ker}\bar\partial$ preserves regularity. Moreover, we show that the formal adjoint, considered as a densely defined operator on L^2_α, coincides with the von Neumann adjoint. When $\alpha=1$, the corresponding holomorphic projection is the (unweighted) Bergman projection. In this case we get, contrary to the Euclidean case, a nice Hodge decomposition, as in the case with a complete metric; see Section 8.

It will be clear from the construction that most results, with appropriate modifications of the formulations, still hold if the boundary is just C^4, but in order to avoid some technicalities we restrict to the smooth case.

The plan of the paper is as follows. In Section 2 we introduce the weighted Bergman type norms and define the corresponding canonical homotopy operators. We also discuss the canonical homotopy operator for the boundary complex. In the next section we compute a formula for the formal adjoint $\bar\partial^*_\alpha$ and discuss its relation to the von Neumann adjoint of $\bar\partial$. It turns out that $\bar\partial^*_\alpha$ is a first order differential operator with coefficients that are smooth up to the boundary, that all forms smooth up to the boundary is in its domain and that, for $\alpha\geq1$, it is equal to the von Neumann adjoint.

In Section 4 we show that the canonical homotopy operator in our domain D can be represented as the complex tangential boundary values of a canonical operator in a certain domain $\tilde{D}$ in $\mathbb{C}^{n+1}$ with respect to a weight one unit less. In Section 5 we introduce the boundary values of some well-known explicit homotopy operators for $\bar{\partial}$ (and $\bar{\partial}_b$). Expressed in our special metric one can write them in a simple way that immediately suggests that they provide the boundary values of some approximate canonical operators. The interplay between canonical operators in D and $\tilde{D}$ then suggests how to define our candidates for approximate homotopy operators in the interior. This continuation to the interior is not previously found in the literature.

The main results in Section 5 are proved in Section 6. In Section 7 we use the results in Section 5 to express the canonical operators with approximate integral formulas, which leads to regularity results in Section 8.

Acknoledgement: We are grateful to blablablablabla

2. Canonical homotopy operators

Recall that $D = \{\rho < 0\}$ is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, and that ρ is a strictly plurisubharmonic C^∞ defining function. For forms f and g, we let $\langle f, g \rangle$ be the inner product generated by the metric form

$$\Omega = (-\rho) i\partial\bar{\partial} \log(-1/\rho),$$

and for positive α we let

$$(f, g)_\alpha = \frac{\Gamma(n+\alpha)}{2^n \pi^n \Gamma(\alpha)} \int_D (-\rho)^\alpha \langle f, g \rangle \, dV,$$

where $dV = \Omega_n$ ($\Omega_k = \Omega^k/k!$ and similarily for other metric forms). The corresponding norms are denoted $|f|^2 = \langle f, f \rangle$ and $\|f\|_\alpha^2 = (f, f)_\alpha$. The inner product $\langle f, g \rangle$ degenerates on the boundary of D, and in order to understand its asymptotic behaviour, we can express it in terms of $\beta = i\partial\bar{\partial}\rho$, which is equivalent to the Euclidean metric since ρ is strictly plurisubharmonic. If $\langle f, g \rangle_\beta$ denotes the inner product with respect to β, for $(0, q)$-forms f and g we have that, see Section 3,

$$\langle f, g \rangle = \Big((-\rho) \langle f, g \rangle_\beta + \big\langle \bar{\partial}\rho \wedge f, \bar{\partial}\rho \wedge g \big\rangle_\beta \Big) / B \quad \text{and} \quad (-\rho)\Omega_n = B\beta_n, \tag{2.1}$$

where the function $B = -\rho + |\bar{\partial}\rho|_\beta^2$ is smooth up to the boundary and nonvanishing. Hence dV is equivalent to the Lebesgue measure divided by the distance to the boundary.

Recall that a vector at a point $p \in \partial D$ is complex tangential if it is annihilated by both $d\rho|_p$ and $d^c\rho|_p$. If f is any form over ∂D, we denote its restriction to the complex tangential vectors by $f|_b$. This restriction is determined by $d\rho\wedge d^c\rho\wedge f$, and in particular if f is a $(0,q)$-form then $f|_b$ is determined simply by $\bar\partial\rho\wedge f$. On the boundary, $\langle f,g\rangle$ degenerates to an inner product of the complex tangential parts $f|_b$ and $g|_b$ of f and g. When α tends to 0 we get the following inner product for complex tangential $(0,q)$-forms $f|_b$ and $g|_b$:

$$(f,g)_b=\frac{(n-1)!}{2^n\pi^n}\int_{\partial D}\langle f,g\rangle\,d\sigma=\frac{(n-1)!}{2^n\pi^n}\int_{\partial D}\left\langle\bar\partial\rho\wedge f,\bar\partial\rho\wedge g\right\rangle_\beta d\sigma/B,$$

where $d\sigma=dS/|d\rho|_\beta$ and dS is the surface measure induced by β.

With this notation the estimate (1.2) becomes

$$\int_{\partial D}|u|\le C\int_D(-\rho)^{1/2}|f|dV.$$

Thus roughly speaking one gains $\sqrt{-\rho}$ on the solution. There are many variants of this estimate in various norms. The following weighted L^2-estimates are fundamental for this paper.

Theorem 2.1. *Suppose that $\alpha>0$. For any $\bar\partial$-closed $(0,q+1)$-form f in $L^2_{\rm loc}$ there is a form u such that $\bar\partial u=f$, and*

$$\|u\|^2_\alpha\le C\|f\|^2_{\alpha+1}. \tag{2.2}$$

For any $\epsilon>0$ we also have a solution whose boundary values u satisfy

$$\int_{\partial D}|u|^2\le C_\epsilon\int_D(-\rho)^{1-\epsilon}|f|^2dV.$$

Thus one gains $\sqrt{-\rho}$ on the solution in all cases but $\alpha=0$. This theorem is readily proved by integral formulas; see the end of Section 6. It is worth to notice that this theorem, however, does not immediately follow from the standard L^2-technique, but it requires an extra argument due to Donelly and Fefferman, [7].

Remark 1. If $\omega=i\partial\bar\partial\log(-1/\rho)$, thus ω is precisely the Bergman metric in the ball case, and

$$\|f\|^2_{\ell,\omega}=\int_D(-\rho)^\ell|f|^2_\omega\omega_n,$$

then for an (p,q)-form f, see Section 3,

$$\|f\|^2_\alpha=\frac{\Gamma(n+\alpha)}{2^n\pi^n\Gamma(\alpha)}\|f\|^2_{n-p-q+\alpha,\omega}, \tag{2.3}$$

and hence (2.2) states that $\|u\|_{\ell,\omega}\le C_\ell\|f\|_{\ell,\omega}$ for all $\ell>n-q$.

By Theorem 2.1 we can define our canonical homotopy operators. Let $L^2_{\alpha,q}$ be the space of locally square integrable $(0,q)$-forms in D such that $(f,f)_\alpha<\infty$ and let

$$\mathcal{K}_{\alpha,q}=L^2_{\alpha,q}\cap\operatorname{Ker}\bar\partial.$$

Theorem 2.1, in particular, implies that the densely defined operator

$$\bar\partial\colon L^2_{\alpha,q}\to\mathcal{K}_{\alpha,q+1}$$

is surjective for each $q \geq 0$ and $\alpha > 0$. We define the canonical operator

$$K^{\text{can}}_{\alpha,q} \colon L^2_{\alpha,q+1} \to L^2_{\alpha,q},$$

so that $K^{\text{can}}_{\alpha,q} f$ is the minimal solution to $\bar\partial u = f$ if $\bar\partial f = 0$, and $K^{\text{can}}_{\alpha,q} f = 0$ if $f \in \mathcal{K}^{\perp}_{\alpha,q+1}$. Let

$$P^{\text{can}}_{\alpha} \colon L^2_{\alpha,0} \to \mathcal{K}_{\alpha,0}$$

be the orthogonal projection onto the Bergman space $\mathcal{K}_{\alpha,0}$ (the Bergman projection). One easily verifies that

$$\bar\partial K^{\text{can}}_{\alpha,q} + K^{\text{can}}_{\alpha,q+1} \bar\partial = I \tag{2.4}$$

acting on $f \in L^2_{\alpha,q+1}$ in $\operatorname{Dom} \bar\partial$, i.e., such that also $\bar\partial f \in L^2_{\alpha,q+2}$, and that actually (2.4) provides the orthogonal decomposition of $L^2_{\alpha,q+1}$, i.e., $\bar\partial K^{\text{can}}_{\alpha,q} f$ is the orthogonal projection of f onto $\mathcal{K}_{\alpha,q+1}$. We also have

$$K^{\text{can}}_{\alpha,0} \bar\partial = I - P^{\text{can}}_{\alpha}$$

on $\operatorname{Dom} \bar\partial$, and therefore, if we let

$$K^{\text{can}}_{\alpha} = \sum_{q=0}^{n-1} K^{\text{can}}_{\alpha,q},$$

we can simply write

$$K^{\text{can}}_{\alpha} \bar\partial + \bar\partial K^{\text{can}}_{\alpha} = I - P^{\text{can}}_{\alpha} \tag{2.5}$$

on $\operatorname{Dom} \bar\partial$. An operator K, satisfying (2.5) (for some holomorphic projection P), is called a homotopy operator for $\bar\partial$.

We have an analogue for the boundary complex. Let $L^2_{b,q}$ denote the space of complex tangential $(0,q)$-forms in L^2 over ∂D, and let $\mathcal{K}_{b,q}$ be the kernel of $\bar\partial_b$. If $f \in L^2_{b,q}$ and $\bar\partial_b f = 0$, then there is a solution $u \in L^2_{b,q-1}$ if $q < n-1$. If $q = n-1$, then a necessary and sufficient condition for solvability is that f is orthogonal to $\operatorname{Ker} \bar\partial^*_b$. This is well known but follows also from Theorem 5.4. We thus have the homotopy relation

$$\bar\partial_b K^{\text{can}}_{b} + K^{\text{can}}_{b} \bar\partial_b = I - P^{\text{can}}_{b} - S^{\text{can}}_{b}, \tag{2.6}$$

where S^{can}_b is the orthogonal projection $L^2_{b,n-1} \to \operatorname{Ker} \bar\partial^*_b \cap L^2_{b,n-1}$.

blablabla Define P^{can}_b. blablabla

3. The adjoint operator $\bar\partial^*_\alpha$

Let $\bar\partial^*_\alpha$ be the formal adjoint of $\bar\partial$ with respect to $(\ ,\ )_\alpha$, i.e., $(\bar\partial f, g)_\alpha = (f, \bar\partial^*_\alpha g)_\alpha$ for all compactly supported smooth f and g. Our first objective is to find a formula for $\bar\partial^*_\alpha$ that reveals its behaviour near the boundary. If θ is a form, we let $\theta\neg$ denote interior multiplication by θ, with respect to the metric β, i.e., $\langle \theta\neg f, g\rangle_\beta = \langle f, \bar\theta\wedge g\rangle_\beta$ for all g. Let $\gamma = i\partial\rho\wedge\bar\partial\rho$ and, as before, $B = -\rho + |\bar\partial\rho|^2_\beta$.

Proposition 3.1. *With the notation above, the formal adjoint is*

$$\bar\partial^*_\alpha = i\,[\partial, (\beta - (1/B)\gamma)\neg] + \frac{\alpha+n-p-q}{B}\partial\rho\neg$$

when acting on a (p,q)*-form.*

Since β is non-degenerate on $\overline{D}$, the operators involved have coefficients that are smooth up to the boundary and hence $\bar\partial^*_\alpha$ is a first order differential operator with smooth coefficients.

Proof. ¿From (2.3) it follows that

$$(-\rho)\bar\partial^*_\alpha = \bar\partial^*_{n-p-q+\alpha,\omega}, \tag{3.1}$$

and since ω is a Kähler metric, see [4] or [9],

$$\bar\partial^*_{\alpha+n-p-q,\omega} = i[\partial, \omega\neg_\omega] + (\alpha+n-p-q)\frac{\partial\rho}{-\rho}\neg_\omega, \tag{3.2}$$

if $\neg_\omega$ denotes interior multiplication with respect to ω. Thus we just have to express $\omega\neg_\omega$ and $\partial\rho\neg_\omega$ in terms of $\neg$ (i.e. $\neg_\beta$). To this end, we choose an orthonormal frame $e_1,\dots,e_n$ with respect to β, for the space of $(1,0)$-forms, such that $e_1 = \partial\rho/|\partial\rho|_\beta$. Then $\beta = i\sum_1^n e_j\wedge\bar e_j$, and

$$\omega \;=\; \frac{\beta}{(-\rho)} + \frac{\gamma}{(-\rho)^2} \;=\; i\frac{\sum_{j=1}^n e_j\wedge\bar e_j}{(-\rho)} + i\frac{\partial\rho\wedge\bar\partial\rho}{(-\rho)^2} \;=\; aie_1\wedge\bar e_1 + bi\sum_{j=2}^n e_j\wedge\bar e_j,$$

where $a = B/(-\rho)^2$ and $b = 1/(-\rho)$. Thus $\bar e_1\neg_\omega e_j = \langle e_1, e_j\rangle_\omega = (1/a)\delta_{1j}$ and hence $\partial\rho\neg_\omega = ((-\rho)^2/B)\partial\rho\neg$. Moreover, it is readily verified that

$$\omega\neg_\omega = (1/a)ie_1\wedge\bar e_1\neg + (1/b)i\sum_{j=2}^n(-\rho)\,(e_j\wedge\bar e_j)\neg = (-\rho)\,(\beta-(1/B)\gamma)\neg.$$

The desired formula now follows from the last two equalities, (3.1) and (3.2). ☐

Proof of (2.1) *and* (2.3). Using the notation in the preceding proof, we have that

$$(-\rho)dV = (-\rho)\Omega^n/n! = (-\rho)^{n+1}\omega^n/n! = (-\rho)^{n+1}ab^{n-1}\beta^n/n! = B\beta^n/n!.$$

The first equality in (2.1) is easily checked for $f = g = \bar e_{I_1}\wedge\ldots\wedge\bar e_{I_q}$, and from this the general case follows. In the same way it follows that if f is a (p,q)-form, then

$$|f|^2 dV = (-\rho)^{n-p-q}|f|^2_\omega\omega^n/n!,$$

which implies (2.3). □

An $f \in L^2_\alpha$ is in $\operatorname{Dom} \bar\partial^*_\alpha$ (the domain of the von Neumann adjoint) if there is a $g \in L^2_\alpha$ such that $(g, u)_\alpha = (f, \bar\partial u)$ for all $u \in \operatorname{Dom} \bar\partial$. If this holds, then clearly $\bar\partial^*_\alpha f = g$ in the distribution sense, but in general the converse is not true, i.e., there are $f \in L^2_\alpha$ with $\bar\partial^*_\alpha f \in L^2_\alpha$ such that yet f does not belong to $\operatorname{Dom} \bar\partial^*_\alpha$. In particular, this is the case in the Euclidean metric and $\alpha = 1$. Our situation is much nicer.

Let $\mathcal{E}_q$ denote the space of $(0, q)$-forms in D that are smooth up to the boundary. In the Euclidean case (and $\alpha = 1$) an $f \in \mathcal{E}_*$ is in the domain of $\bar\partial^*$ if and only if $\partial\rho \neg f = 0$ on the boundary. However, we have

Proposition 3.2. *If $\alpha > 0$ and $f, g \in \mathcal{E}_*$ then $(\bar\partial^*_\alpha f, g)_\alpha = (f, \bar\partial g)_\alpha$.*

Proof. Since $\bar\partial^*_\alpha$ has smooth coefficients, the boundary integral that occurs when integrating by parts must vanish if $\alpha > 1$. Since the expression is analytic in α, the general case follows by analytic continuation. □

Theorem 3.3. *Suppose that $\alpha \ge 1$. If $f, g \in L^2_\alpha$ and $\bar\partial^*_\alpha f$ and $\bar\partial g$ are in L^2_α as well, then*

$$(\bar\partial^*_\alpha f, g)_\alpha = (f, \bar\partial g)_\alpha. \tag{3.3}$$

That is, $f \in L^2_\alpha$ is in $\operatorname{Dom} \bar\partial^*_\alpha$ if and only if $\bar\partial^*_\alpha f$ is in L^2_α.

Since the the image of $\bar\partial\colon L^2_{\alpha,q-1} \to L^2_{\alpha,q}$ is equal to $\mathcal{K}_\alpha$ for $q \ge 1$, we have in particular that $f \in L^2_{\alpha,q}$ is in $\mathcal{K}^\perp_{\alpha,q}$ if and only if $\bar\partial^*_\alpha f = 0$. Theorem 3.3 is an immediate consequence of Proposition 3.2 and the following approximation lemma.

Lemma 3.4. *Suppose that $\mathcal{P}$ is a first order linear differential operator with coefficients that are smooth up to the boundary, and suppose that $\alpha \ge 1$. If f and $\mathcal{P}f$ are in L^2_α, then there are $f_j \in \mathcal{E}_*$ such that $f_j \to f$ and $\mathcal{P}f_j \to \mathcal{P}f$ in L^2_α.*

To prove this lemma one first approximates f by a form defined in a neighborhood of $\overline{D}$ and then makes a standard regularization of this form. We omit the details. In general the lemma fails if $\alpha < 1$; cf. the remark below.

For further reference we need

Lemma 3.5. *For any $\alpha > 0$, we have that $(\bar\partial^*_\alpha \phi, g)_\alpha = 0$ if $\phi \in \mathcal{E}_*$ and $g \in \mathcal{K}_\alpha$.*

Proof. If $g \in L^2_\alpha$ then $g \in L^2_{\alpha'}$ if $\alpha' > \alpha$. Since $\bar\partial^*_{\alpha'}\phi$ is in $\mathcal{E}_*$, it follows from Theorem 3.3 that $(\bar\partial^*_{\alpha'}\phi, g)_{\alpha'} = 0$ for $\alpha' \ge 1$. The desired conclusion then follows by analytic continuation. □

The argument above breaks down if one only assumes that $\phi, \bar\partial^*_\alpha \phi \in L^2_\alpha$, since this does not imply that $\bar\partial^*_{\alpha'}\phi \in L^2_{\alpha'}$ for $\alpha' > \alpha$.

Now let $\alpha > 0$ be arbitrary. As usual we can consider $\overline{\Box}_\alpha = \bar\partial\bar\partial^*_\alpha + \bar\partial^*_\alpha\bar\partial$. Moreover, let $K^{\text{can},*}_\alpha\colon L^2_\alpha \to L^2_\alpha$ be the L^2_α-adjoint of K^{can}_α and $E^{\text{can}}_\alpha = K^{\text{can}}_\alpha K^{\text{can},*}_\alpha + K^{\text{can},*}_\alpha K^{\text{can}}_\alpha$. By simple arguments it follows that

$$\bar\partial K^{\text{can}}_\alpha f + \bar\partial^*_\alpha K^{\text{can},*}_\alpha f = f - P^{\text{can}}_\alpha f \tag{3.4}$$

and

$$\overline{\Box}_\alpha E^{\rm can}_\alpha f = f - P^{\rm can}_\alpha f \tag{3.5}$$

for any $f \in L^2_\alpha$. Moreover, we have

$$E^{\rm can}_\alpha \overline{\Box}_\alpha f = f - P^{\rm can}_\alpha f \tag{3.6}$$

provided $f \in \mathrm{Dom}\,\overline{\Box}_\alpha$, where $f \in \mathrm{Dom}\,\overline{\Box}_\alpha$ means that $f \in \mathrm{Dom}\,\bar\partial \cap \mathrm{Dom}\,\bar\partial^*_\alpha$, $\bar\partial f \in \mathrm{Dom}\,\bar\partial^*_\alpha$ and $\bar\partial^*_\alpha f \in \mathrm{Dom}\,\bar\partial$. For instance, let us verify (3.4). For any $f \in L^2_\alpha$ and $g \in \mathrm{Dom}\,\bar\partial$ we have (using (2.5) and for simplicity assuming that $q > 0$) that

$$(K^{\rm can,*}_\alpha f, \bar\partial g)_\alpha = (f, K^{\rm can}_\alpha \bar\partial g)_\alpha = (f, g - \bar\partial K^{\rm can}_\alpha g)_\alpha = (f - \bar\partial K^{\rm can}_\alpha f, g)_\alpha,$$

since $\bar\partial K^{\rm can}_\alpha$ is self-adjoint. This shows that $K^{\rm can,*}_\alpha f \in \mathrm{Dom}\,\bar\partial^*_\alpha$ and $\bar\partial^*_\alpha K^{\rm can,*}_\alpha f = f - \bar\partial K^{\rm can}_\alpha f$. Thus (3.4) follows.

Finally, let us take a look at the boundary complex. Since $(\ ,\ )_b$ is the limit of $(\ ,\ )_\alpha$ when $\alpha \to 0$, Proposition 3.1 provides a formula for the adjoint $\bar\partial^*_b$. Moreover, the analogue of Proposition 3.3 holds, since it holds for smooth forms which are dense in the graph norms. In the obvious way $K^{\rm can,*}_b$, $\overline{\Box}_b$ and $E^{\rm can,*}_b$ are defined, and we have that

$$\bar\partial K^{\rm can}_b f + \bar\partial^*_b K^{\rm can,*}_b f = f - P^{\rm can}_b f - S^{\rm can}_b f.$$

Furthermore,

$$\overline{\Box}_b E^{\rm can}_b f = f - P^{\rm can}_b f - S^{\rm can}_b f \tag{3.7}$$

for $f \in L^2_b$ and

$$E^{\rm can}_b \overline{\Box}_b f = f - P^{\rm can}_b f - S^{\rm can}_b f \tag{3.8}$$

for $f \in \mathrm{Dom}\,\overline{\Box}_b$.

Remark 2. Theorem 3.3 is not true for $0 < \alpha < 1$; at least not for $(0, n)$-forms. To see this, let D be the unit disk. Since $\bar\partial\colon L^2_{\alpha,0} \to L^2_{\alpha,1}$ is surjective, the statement would imply that $f \in L^2_{\alpha,1}$ vanishes if $\bar\partial^*_\alpha f = 0$. However, the latter equation means that $\partial(1-|z|^2)^\alpha f = 0$ and hence the kernel of $\bar\partial^*_\alpha\colon L^2_{\alpha,1} \to L^2_{\alpha,0}$ consists of all forms $f = (1-|z|^2)^{-\alpha}\bar h$, where h is holomorphic and $\int(1-|z|^2)^{-\alpha}|h|^2 < \infty$. For $q < n$ the corresponding result is true in the "limit case" when $\alpha \to 0$; therefore, one could guess that it is true even in the intermediate cases $0 < \alpha < 1$. In particular we would then have that

$$f \in L^2_\alpha \text{ and } \bar\partial^*_\alpha f = 0 \quad \text{implies} \quad f \in \mathcal{K}^\perp_\alpha \tag{3.9}$$

for $(0, q)$-forms, $1 \le q \le n-1$. Let us relate this statement to the norms $\|\ \|_{\ell,\omega}$. In view of (2.3) we have that $\bar\partial^*_\alpha f = 0$ if and only if $\bar\partial^*_{\ell,\omega} f = 0$, where $\ell = \alpha + n - q$. Since $|\partial \log(-1/\rho)|_\omega$ is bounded, ω is a complete metric, see [4], and therefore the compactly supported forms are dense in the graph norms with respect to the norms $\|\ \|_{\ell,\omega}$. In particular, this means that the formal adjoint $\bar\partial^*_{\ell,\omega}$ coincides with the corresponding von Neumann adjoint. Hence (3.9) holds if and only if the image of $\bar\partial\colon L^2_{\ell,\omega} \to L^2_{\ell,\omega}$ is dense in $\mathcal{K}_\alpha$ (as it follows that f is orthogonal to this image if $\bar\partial^*_{\ell,\omega} f = 0$). In view of Theorem 3.3, the image is dense if $\alpha \ge 1$. For $\alpha > 1$, it is in fact equal to $\mathcal{K}_\alpha$; this is the content of Theorem 2.1. However,

$\bar\partial\colon L^2_{\ell,\omega}\to L^2_{\ell,\omega}$ is *not* surjective if $0<\alpha\le 1$. To see this, let D be the ball and let $f=\bar\partial\bar h$, where h is some holomorphic function with $h(0)=0$ that is C^1 up to the boundary. Then certainly $f\in\mathcal{K}_{\alpha,1}$, but it is not in the image of $\bar\partial$ unless $h=0$. In fact, if $\bar\partial u=f$ and $\int(1-|\zeta|^2)^{\alpha-2}|u|^2<\infty$, then since $g=u-\bar h$ is holomorphic, (and g and $\bar h$ are orthogonal with respect to radial measures) we would have that $\int(1-|\zeta|^2)^{\alpha-2}(|h|^2+|g|^2)<\infty$ which implies that $h=0$.

4. Up and down in dimension

Given our domain D and defining function ρ in $\mathbb{C}^n$, let $\tilde\rho(z,w)=\rho(z)+|w|^2$ for $(z,w)\in\mathbb{C}^{n+1}$. Then $\tilde\rho$ is a strictly plurisubharmonic defining function for the strictly pseudoconvex domain $\tilde D=\{\tilde\rho<0\}$ in $\mathbb{C}^{n+1}$. Hence, anything done so far applies equally well to $\tilde D$. For a $(0,q)$-form f in D, we put $\tilde f(z,w)=f(z)$.

Proposition 4.1. *Let D and $\tilde D$ be as above. Then for $(0,q)$-forms we have*
o) $f\equiv 0$ in D if and only if $\tilde f|_b\equiv 0$ on $\partial\tilde D$.
i) $(f,g)_\alpha=(\tilde f,\tilde g)_{\alpha-1}$ for $\alpha\ge 1$. In particular, $f\in L^2_\alpha(D)$ if and only if $\tilde f\in L^2_{\alpha-1}(\tilde D)$ for $\alpha>1$ and $f\in L^2_1(D)$ if and only if $\tilde f|_b\in L^2_b(\partial\tilde D)$.
ii) $\bar\partial f=g$ in D if and only if $\bar\partial\tilde f=\tilde g$ in $\tilde D$ if and only if $\bar\partial_b\tilde f|_b=\tilde g|_b$.
iii) $(\bar\partial^*_\alpha f)^\sim=\bar\partial^*_{\alpha-1}\tilde f$ for $f\in\mathcal{E}_*(\overline{D})$ for $\alpha>1$, and $(\bar\partial^*_1 f)^\sim|_b=\bar\partial^*_b\tilde f|_b$.
iv) $f\in\mathcal{K}^\perp_\alpha$ if and only if $\tilde f\in\mathcal{K}^\perp_{\alpha-1}$, and $f\in\mathcal{K}^\perp_1$ if and only if $\tilde f|_b\in\mathcal{K}^\perp_b$.
v) If $f\in L^2_\alpha$, $\alpha>1$, then $f=f_1+f_2$ is the decomposition in $\mathcal{K}_\alpha$ and $\mathcal{K}^\perp_\alpha$ if and only if $\tilde f=\tilde f_1+\tilde f_2$ is the decomposition in $\mathcal{K}_{\alpha-1}$ and $\mathcal{K}^\perp_{\alpha-1}$, and if $\alpha=1$, then $f=f_1+f_2$ is the decomposition in $\mathcal{K}_1$ and $\mathcal{K}^\perp_1$ if and only if $\tilde f|_b=\tilde f_1|_b+\tilde f_2|_b$ is the decomposition in $\mathcal{K}_b$ and $\mathcal{K}^\perp_b$.

Let ϕ be a complex tangential form on $\partial\tilde D$. We say that ϕ is invariant if $\tau^*\phi=\phi$ for all $\tau(z,w)=(z,e^{i\theta}w)$. Notice that $\tau^*\phi$ is well defined since $\tau^*d\tilde\rho=d\tilde\rho$ and $\tau^*d^c\tilde\rho=d^c\tilde\rho$. ¿From [2] we recall the following result.

Proposition 4.2. *There is a one-to-one correspondence between forms $f\in L^2(D)$ and invariant complex tangential forms $\phi\in L^2_b(\partial\tilde D)$. Moreover, f is smooth on $\overline{D}$ if and only if ϕ is smooth.*

The hard direction is to show that any invariant complex tangential form ϕ on $\partial\tilde D$ is $\tilde f$ for some f in D. The idea is the following. If ϕ also denotes a $(0,q)$-form over $\partial\tilde D$ that represents ϕ, then it can be written uniquely as

$$\phi=a+wd\bar w\wedge b,$$

where a and b contain no occurrences of $d\bar{w}$. The invariance condition implies that the coefficients in a and b are invariant in the last variable, and since $wd\bar{w} = -\bar{\partial}\rho$ as complex tangential forms, one get f as

$$f = a - \bar{\partial}\rho \wedge b.$$

For the complete proof of Proposition 4.2, see [2], which also contains a generalization where $\tilde{D}$ is a domain m dimensions higher.

Notice that any invariant $(0,n)$-form $\phi \in L^2_b(\partial\tilde{D})$ is orthogonal to $\operatorname{Ker}\bar{\partial}^*_b$; in fact, this is equivalent to the solvability of $\bar{\partial}_b u = \phi$ in $L^2_b(\partial\tilde{D})$, which in turn follows from Propositions 4.1 and 4.2.

Let $\tilde{K}^{\text{can}}_\alpha$ denote the canonical homotopy operator in $\tilde{D}$ with respect to $\tilde{\rho}$, and let $\tilde{K}^{\text{can}}_b$ be the canonical operator on $\partial\tilde{D}$ (corresponding to $\alpha = 0$), and similarily with $\tilde{P}^{\text{can}}_\alpha$ and $\tilde{P}^{\text{can}}_b$. ¿From Propositions 4.1 and 4.2 we get the following basic result.

Theorem 4.3. *With the notation above we have that*

$$(K^{\text{can}}_\alpha f)\tilde{} = \tilde{K}^{\text{can}}_{\alpha-1}\tilde{f}, \qquad (P^{\text{can}}_\alpha f)\tilde{} = \tilde{P}^{\text{can}}_{\alpha-1}\tilde{f}$$

if $\alpha > 1$ *and*

$$(K^{\text{can}}_1 f)^\sim|_b = \tilde{K}^{\text{can}}_b \tilde{f}|_b, \qquad (P^{\text{can}}_1 f)^\sim|_b = \tilde{P}^{\text{can}}_b \tilde{f}|_b.$$

Thus $K^{\text{can}}_\alpha f$ *(and* $P^{\text{can}}_\alpha f$*) can be reconstructed from the complex tangential boundary values* $\tilde{K}^{\text{can}}_{\alpha-1}\tilde{f}|_b$ *(and* $\tilde{P}^{\text{can}}_{\alpha-1}\tilde{f}|_b$*). In particular,* $K^{\text{can}}_\alpha f$ *is smooth if (and only if)* $\tilde{K}^{\text{can}}_{\alpha-1}\tilde{f}|_b$ *is.*

Proof. First suppose that $f \in \mathcal{K}_\alpha$. Then $u = K^{\text{can}}_\alpha f$ solves $\bar{\partial}u = f$ and $u \in \mathcal{K}^\perp_\alpha$. By Proposition 4.1, therefore, $\bar{\partial}\tilde{u} = \tilde{f}$ and $\tilde{u} \in \mathcal{K}^\perp_{\alpha-1}$, so that $\tilde{u} = \tilde{K}^{\text{can}}_{\alpha-1}\tilde{f}$. On the other hand, if $f \in \mathcal{K}^\perp_\alpha$, then $\tilde{f} \in \mathcal{K}^\perp_{\alpha-1}$ and therefore, $K^{\text{can}}_\alpha f$ as well as $K^{\text{can}}_{\alpha-1}\tilde{f}$ vanish. The other statements follow in the same way. □

Proof of Proposition 4.1. Part o) follows since $0 = \tilde{f}|_b$ if and only if $0 = (\bar{\partial}\rho + \bar{\partial}|w|^2) \wedge f$ on $\partial\tilde{D}$, which holds if and only of $f \equiv 0$ in D.

Since

$$\tilde{\beta} = i\partial\bar{\partial}\rho + idw \wedge d\bar{w} \tag{4.1}$$

we find that, at each point, dw is orthogonal to all dz_j with respect to $\tilde{\beta}$, and moreover dw has norm one. In other words, if a, a', b, b' contain no differentials of w, then

$$\left\langle a + b \wedge \bar{\partial}|w|^2, a' + b' \wedge \bar{\partial}|w|^2 \right\rangle_{\tilde{\beta}} = \langle a, a'\rangle_\beta + |w|^2 \langle b, b'\rangle_\beta .$$

It follows that $|\bar{\partial}\tilde{\rho}|^2_{\tilde{\beta}} = |\bar{\partial}\rho|^2_\beta + |w|^2$ and therefore $\tilde{B} = B$. In view of (2.1) we also get

$$\langle \tilde{f}, \tilde{g}\rangle\tilde{} = \langle f, g\rangle . \tag{4.2}$$

Moreover, cf. (2.1),

$$(-\tilde{\rho})d\tilde{V} = \tilde{B}\tilde{\beta}_{n+1} = B\beta_n \wedge idw \wedge d\bar{w} = (-\rho)dV \wedge idw \wedge d\bar{w},$$

and therefore,

$$\int_{\tilde D}(-\tilde\rho)^{\alpha-1}\left\langle \tilde f,\tilde g\right\rangle^{\sim} d\tilde V=\int_{\tilde D}(-\tilde\rho)^{\alpha-2}\left\langle \tilde f,\tilde g\right\rangle^{\sim}(-\tilde\rho)d\tilde V=$$

$$=\int_D\int_{|w|^2\le-\rho(z)}(-\rho-|w|^2)^{\alpha-2}idw\wedge d\bar w\,\langle f,g\rangle\,(-\rho)dV=\frac{2\pi}{\alpha-1}\int_D(-\rho)^{\alpha}\langle f,g\rangle\,dV.$$

This proves part $i)$ for $\alpha>1$. The case $\alpha=1$ follows by continuity. Part $ii)$ is obvious in view of $o)$, just noting that $\bar\partial_b\tilde f|_b=(\bar\partial\tilde f)|_b=(\bar\partial f)^{\sim}|_b$.

To see part $iii)$, first notice that if ϕ is a form that only contains differentials of w, then $\phi\neg\tilde f=0$. Moreover, $(\partial f)\tilde{}=\partial\tilde f$. Therefore, $iii)$ follows from Proposition 3.1.

Let us now consider part $iv)$. The nontrivial direction is that $\tilde f\in\mathcal{K}_{\alpha-1}^{\perp}$ if $f\in\mathcal{K}_{\alpha}^{\perp}$ (we assume $\alpha>1$, the case $\alpha=1$ is similar). It follows from Theorem 4.1 that the operator $\tilde{}\colon L^2_\alpha(D)\to L^2_{\alpha-1}(\tilde D)$ is bounded. Let $M\colon L^2_{\alpha-1}(\tilde D)\to L^2_\alpha(D)$ be its adjoint. We claim that $\bar\partial Mg=0$ if $\bar\partial g=0$. Clearly, $\bar\partial Mg=0$ in the distribution sense means that $(\bar\partial^*_\alpha\phi,Mg)_\alpha=0$ for all compactly supported smooth forms ϕ. However, for any such ϕ we have $(\bar\partial^*_\alpha\phi,Mg)_\alpha=((\bar\partial^*_\alpha\phi)\tilde{}\,,g)_{\alpha-1}=(\bar\partial^*_{\alpha-1}\tilde\phi,g)_{\alpha-1}$, and the last term vanishes by Lemma 3.5. Now take $f\in\mathcal{K}_\alpha^{\perp}$ and $g\in\mathcal{K}_{\alpha-1}$. Then $(\tilde f,g)_{\alpha-1}=(f,Mg)_\alpha=0$ by the assumption on f since $\bar\partial Mg=0$. Thus part $iv)$ is proved. The last statement is an immediate consequence of $i)$, $ii)$ and $iv)$. □

With the same argument it follows that $\bar\partial Mg=M\bar\partial g$ if $\bar\partial g\in L^2_{\alpha-1}(\tilde D)$. It is possible to compute Mg and verify this directly; see Proposition 6.5.

5. Explicit homotopy operators

In [11] Kerzman and Stein proved that there is an explicitly given projection operator $P_b\colon L^2_{b,0}\to\mathcal{K}_{b,0}$, i.e. a projection onto the holomorphic functions with boundary values in L^2, which is approximatively equal to the Szegö projection $P_b^{\rm can}$ in the sense that $P_b^{\rm can}-P_b$ is compact on L^2_b. There are two main steps in their construction. The first difficulty is to prove that the projection operator, obtained from the Cauchy-Fantappie-Leray formula with a holomorphic support function, which a priori is just defined on say smooth functions, actually extends to a bounded operator on L^2_b. A nice proof of this fact can be found in [13]. The next step is to show that, roughly speaking, the support function can be chosen in such a way that the resulting operator has the extra property that $P_b-P_b^*$ is compact. (This means that $P_{b,0}$ is close to be self-adjoint, i.e. the orthogonal projection.) It is then quite easy to conclude that $P_b^{\rm can}-P_b$ is compact, see Section 7. Following the same lines, Ligocka, [12], obtained the analogous result for the Bergman projection, i.e. she constructed an operator P_1 such that $P_1^{\rm can} - P_1$ is compact on $L^2_{1,0}$. However, this case is simpler since the L^2-boundedness follows from a brutal estimate of the kernel (this works for any P_α to be defined below ablabla) if $\alpha > 1/2$). In the following sections we present generalizations of these results to higher order forms (for $\alpha = b$ and $\alpha \geq 1$); i.e. we show that there are explicit operators K_α which are close to the canonical operators $K_\alpha^{\rm can}$ in a certain sense. For the precise statements, see Section 7. For instance, we then show in Section 8 that one can derive regularity properties for $K_\alpha^{\rm can}$ from the corresponding properties for the explicit operators K_α. The boundary values of our operators K_α coincide with the boundary values of some wellknown homotopy operators for $\bar\partial$, and the interior values will be connected to the boundary values of the corresponding operators in $\tilde D$ (using the notation from the previous section).

Our basic result for K_α is that

$$K_\alpha = H_\alpha \bar\partial^*_\alpha + R_\alpha, \tag{5.1}$$

where H_α is self-adjoint and R_α and $\bar\partial R_\alpha$ are compact on L^2_α. In particular, this implies that K_α approximately vanishes on $\mathcal{K}_\alpha^\perp$. ¿From (5.1) it is quite easy to see that $\bar\partial K_\alpha$ is L^2_α-bounded, and that $\bar\partial K_\alpha - (\bar\partial K_\alpha)^*$ is compact. Following [11] one then gets that $\bar\partial K_\alpha^{\rm can} - \bar\partial K_\alpha$ is compact; recall that $\bar\partial K_\alpha^{\rm can}$ is the orthogonal projection $L^2_\alpha \to \mathcal{K}_\alpha$. It should be noted that the kernel for $\bar\partial K_\alpha$ is genuinely singular, so the L^2_α boundedness of $\bar\partial K_\alpha$ cannot be obtained by a brutal estimate.

We have an analogous result for the boundary complex. However, the explicit operator S_b corresponding to $S_b^{\rm can}$ will approximate the latter in the sense that $I - S_b$ is a projection onto $\mathcal{K}_b^\perp$ and $S_b^{\rm can} - S_b$ is compact. That is, the image of S_b will not be exactly $\mathcal{K}_{b,n-1}$.

Recall that $D = \{\rho < 0\}$ and ρ is a smooth strictly plurisubharmonic defining function. Define the function $v(\zeta, z)$ near the diagonal $\Delta = \{\zeta = z\}$ by

$$-v(\zeta, z) = \rho + \sum \rho_j (z_j - \zeta_j) + \frac{1}{2} \sum \rho_{jk}(z_j - \zeta_j)(z_k - \zeta_k) \tag{5.2}$$

($\rho = \rho(\zeta), \rho_j = \partial\rho/\partial\zeta_j$ and so on). Then certainly $v(\zeta, z)$ is holomorphic in z near Δ and we have

$$v(z,\zeta) = \overline{v(\zeta, z)} + \mathcal{O}(|\zeta - z|^3) \quad \text{and} \quad \bar\partial_\zeta v(\zeta, z) = \mathcal{O}(|\zeta - z|^2). \tag{5.3}$$

Since we assume that ρ is smooth in this paper we can add higher order terms in (5.2) and get (5.3) with any integer k instead of 3 (and 2). In particular, if ρ is real analytic we can take

$$-v(\zeta, z) = \sum_{|\alpha|=0}^{\infty} \rho_\alpha (z - \zeta)^\alpha \tag{5.4}$$

near Δ. Then $v(\zeta, z)$ is the unique function that is holomorphic in z, satisfies $v(\zeta, \zeta) = -\rho(\zeta)$ and $v(z, \zeta) = \overline{v(\zeta, z)}$, i.e., the polarization of ρ. If ρ is real analytic and v is defined by (5.4) then (5.3) is easily verified. If ρ is just smooth one can obtain (5.3) by approximation. In what follows it is convenient to think of ρ as real analytic and $v(\zeta, z)$

being its polarization, even though the property (5.3) is enough. Moreover, since ρ is strictly plurisubharmonic it follows that

$$2\operatorname{Re} v(\zeta,z) \geq -\rho(\zeta) - \rho(z) + \delta|\zeta - z|^2 \tag{5.5}$$

near the diagonal. We then define $v(\zeta,z)$ globally by patching essentially with $|\zeta - z|^2$ (to be precise, with $|\zeta - z|^2 - \rho(\zeta)$, see Section 6) so that (5.5) (essentially) holds globally.

Let $\sigma(\zeta,z)$ be defined on $\overline{D\times D}$ by

$$\sigma(\zeta,z)^2 = |v(\zeta,z)|^2 - (-\rho(\zeta))(-\rho(z)).$$

Since ($\rho = \rho(\zeta)$)

$$\sigma^2 = v\bar{v} - \rho\rho(z) = |v+\rho|^2 + (-\rho)(v + \rho + \bar{v} + \rho(z)), \tag{5.6}$$

it follows from (5.5) that σ is nonnegative (?como ?)

For a fixed $\alpha > 0$ we let an integral operator H_α and its kernel $h_\alpha(\zeta,z)$ be connected by the relation

$$H_\alpha f(z) = (f, \overline{h(\cdot,z)})_\alpha.$$

Notice that H_α is self-adjoint with respect to $(\ ,\ )_\alpha$ if and only if h_α is hermitean, i.e. $h_{\alpha,q}(\zeta,z) = (-1)^q\,\overline{h_{\alpha,q}(z,\zeta)}$ for the $(0,q)$ part $h_{\alpha,q}$ of h_α).

Let $\mathcal{E}_q$ denote the space of $(0,q)$-forms that are smooth up to the boundary and let $\mathcal{E}_q^b$ denote the space of smooth complex tangential $(0,q)$-forms. We also let $\mathcal{H}_q = \mathcal{E}_q \cap \operatorname{Ker}\bar\partial$ and $\mathcal{H}_q^b = \mathcal{E}_q^b \cap \operatorname{Ker}\bar\partial_b$. Thus $\mathcal{H}_0$ is the space of holomorphic functions that are smooth up to the boundary.

Theorem 5.1. *Let $\alpha \geq 1$. There are explicit operators $K_\alpha\colon \mathcal{E}_{*+1} \to \mathcal{E}_*$ and $P_\alpha\colon \mathcal{E}_0 \to \mathcal{H}_0$ such that*

$$\bar\partial K_\alpha f + K_\alpha \bar\partial f = f - P_\alpha f, \tag{5.7}$$

with the following additional properties. The kernels satisfy the estimates

$$|k_\alpha(\zeta,z)| \lesssim \frac{1}{|v|^{n+\alpha-1/2}}\left(\frac{|v|}{\sigma}\right)^{2n-1}, \tag{5.8}$$

and

$$|p_\alpha(\zeta,z)| \lesssim 1/|v|^{n+\alpha}. \tag{5.9}$$

Moreover,

$$p_\alpha(\zeta,z) = \frac{1}{v(\zeta,z)^{n+\alpha}} + \varrho_\alpha(\zeta,z), \tag{5.10}$$

where

$$|\varrho_\alpha(\zeta,z)| \leq C\frac{1}{|v|^{n+\alpha-1/2}}, \quad \zeta,z \in \overline{D},$$

and if $z \in \partial D$, *then*

(5.11)

$$k_\alpha(\zeta, z) = \sum_{q=0}^{n-1} c_{\alpha,n,q} \frac{\partial_\zeta \bar{v}(\zeta, z) \wedge (\bar{\partial}_z \partial_\zeta \bar{v}(\zeta, z))^q}{v(\zeta, z)^{n+\alpha-q-1} \bar{v}(\zeta, z)^{q+1}} + r_\alpha(\zeta, z), \quad c_{\alpha,n,q} = i^{1-q} \frac{\Gamma(\alpha + n - q - 1)}{\Gamma(n + \alpha)},$$

where

$$|r_\alpha(\zeta, z)| \le C \frac{1}{|v|^{n+\alpha-1}}, \quad |\bar{\partial}_z r_\alpha(\zeta, z)| \le C \frac{1}{|v|^{n+\alpha-1/2}}. \tag{5.12}$$

Notice that $|v|/\sigma$ is equal to 1 if z (or ζ) is on ∂D.

??Notice that the

Thus the leading terms of k_α (for z on the boundary) and p_α have quite simple expressions.

Remark 3. If the domain is $\tilde{D}$ for some D and if any of these operators is applied to a form that is independent of the last variable then the resulting form is (rotation) invariant in the last variable. By construction it will be the case that $K_\alpha f$ is connected to the boundary vakues of $\tilde{K}_{\alpha-1} \tilde{f}$ in the same way as for the canonical operators, cf. Theorem 4.3.

Although the kernels for the interior values of these operators can be computed explicitly, for many purposes it is anyway most convenient to use this representation as the boundary values of the corrseponding operators in $\tilde{D}$.

Using analytic continuation it is possible to define the interior values of the operators K_α et cetera even for $0 < \alpha < 1$ (actually for $\alpha > -n$).

Furthermore we have

Theorem 5.2. *There is a hermitean kernel* $h_\alpha(\zeta, z)$ *and a kernel* $r_\alpha(\zeta, z)$ *such that*

$$k_\alpha(\zeta, z) = \partial_{\zeta,\alpha} h_\alpha(\zeta, z) + r_\alpha(\zeta, z), \tag{5.13}$$

and such that both h_α *and* r_α *are*

$$\lesssim \frac{1 + |\log|v|| + \log(\sigma/|v|)}{|v|^{n+\alpha-1}} \left(\frac{|v|}{\sigma}\right)^{2n-2} \tag{5.14}$$

whereas $\bar{\partial}_z r_\alpha(\zeta, z)$ *satisfies* (5.8). *The adjoint operators* K_α^* *and* P_α^* *map smooth forms onto smooth forms, as well as corresponding operators* H_α *and* R_α *and their adjoints.*

The proofs of Theorems 5.1 and 5.2 are postponed to the next section. ¿From these two theorems we obtain

Theorem 5.3.
i) The operators P_α *and* K_α *extend to bounded operators on* L^2_α, K_α *is even compact, and the homotopy relation* (5.7) *holds for* f *such that* $f, \bar{\partial} f \in L^2_\alpha$
ii) The operators H_α, R_α *and* $\bar{\partial} R_\alpha$ *are compact on* L^2_α *and*

$$K_\alpha f = H_\alpha \bar{\partial}_\alpha^* f + R_\alpha f \quad \text{if} \quad f, \bar{\partial}_\alpha^* f \in L^2_\alpha. \tag{5.15}$$

iii) The operator H_α *is self-adjoint.*

iv) The operator $\bar\partial K_\alpha$ *is bounded on* L^2_α.
v) The operators $P_\alpha - P^*_\alpha$ *and* $\bar\partial K_\alpha - (\bar\partial K_\alpha)^*$ *are compact on* L^2_α.

Proof. ¿From (5.6) we have that $v+\rho = \partial\rho\cdot(\zeta - z) + \mathcal{O}(|\zeta - z|^2)$, and therefore $|v+\rho|$ is approximately the length of the projection of $\zeta - z$ onto the normal space to the complex hyperplane $\{z;\ \partial\rho\cdot(z-\zeta)=0\}$. Moreover, since

$$2v = -\rho(\zeta) - \rho(z) + \delta(\zeta,z)|\zeta - z|^2 + ia(\zeta,z) + \mathcal{O}(|\zeta - z|^3)$$

(which in fact proves (5.5)), where a is real and δ strictly positive, we have that $v + \bar v + \rho + \rho(z) = 2\delta(\zeta,z)|\zeta - z|^2 + \mathcal{O}(|\zeta - z|^3)$. For fixed $z\in D$ therefore $\{\sigma < t\}$ has extension $\sim t/\sqrt{-\rho(z)}$ in $2n-2$ directions and $\sim t$ in the last two ones (for $t \le c|v|$), and therefore we have

$$|\{\zeta;\ \sigma(\zeta,z) < t\}| \le C\frac{t^{2n}}{(-\rho(z))^{n-1}}$$

If $\sigma(\zeta,z)\le 1/\sqrt{2}|v(\zeta,z)|$ we get (by (5.5)

$$\frac{1}{2}|v|^2 \le (-\rho)(-\rho(z)) \le (-\rho)2|v|$$

implying that $|v| \lesssim (-\rho)$ and hence we have

$$-\rho(z)\sim |v(\zeta,z)| \sim -\rho(\zeta).$$

¿From (5.5) and the fact that

$$d_\zeta v|_{\zeta=z} = d_z\bar v|_{z=\zeta} = -\bar\partial\rho|_\zeta, \quad \zeta = z\in\partial D,$$

we get the usual estimates

$$\int_{\partial D}\frac{d\sigma(\zeta)}{|v(\zeta,z)|^{n+\alpha}} \lesssim \Big(\frac{1}{-\rho(z)}\Big)^\alpha, \quad \alpha>0,$$

and

$$\int_D\frac{(-\rho(\zeta))^\beta d\lambda(\zeta)}{|v(\zeta,z)|^{n+1+\alpha+\beta}} \lesssim \Big(\frac{1}{-\rho(z)}\Big)^\alpha, \quad \alpha>0, \quad \beta>-1.$$

By standard techniques it now follows from (5.8) and (5.9) that that

$$\|K_\alpha f\|_\alpha \le C\|f\|_\alpha \text{ and } \|P_\alpha f\|_\alpha \le C\|f\|_\alpha \tag{5.16}$$

for all $\alpha\ge 1$. If χ_N is 1 where $|k_\alpha|\le N$ and 0 elsewhere, then $\chi_N k_\alpha$ is a Hilbert-Schmidt kernel, and hence its corresponding operator $K_{\alpha,N}$ is compact. It is readily verified that $K_{\alpha,N}\to K_\alpha$ in operator norm, and hence K_α is compact. Since (5.7) holds for smooth f, the general case now follows from Proposition 3.4. Thus i) is proved. Part ii) follows in the same way, and part iii) is obvious.

If $f\in\mathcal{K}_\alpha$, then it follows from (5.7) that $\bar\partial K_\alpha f = f$. On the other hand, if $f\in\mathcal{K}_\alpha^\perp$, then $\bar\partial^*_\alpha f = 0$ and, by (5.15), then $\bar\partial K_\alpha f = \bar\partial R_\alpha f \in L^2_\alpha$. Hence $\bar\partial K_\alpha$ is bounded on L^2_α.

It remains to prove part v). ¿From (5.10), (5.9), (5.3) and (5.5) it follows that the kernel for $P_\alpha - P^*_\alpha$ is $\lesssim |v|^{-(n-1/2)}$, and as before the compactness then follows. Since H_α is self-adjoint, at least for smooth f it follows that

$$(\bar\partial K_\alpha - (\bar\partial K_\alpha)^*)f = (\bar\partial R_\alpha - (\bar\partial R_\alpha)^*)f.$$

It is then true for general f by the approximation lemma, and hence $\bar\partial K_\alpha - (\bar\partial K_\alpha)^*$ is compact since $\bar\partial R_\alpha$ and its adjoint are, because of the estimates of the kernels r_α and $\bar\partial_z r_\alpha$. □

In the same way it follows that

$$\|K_\ell f\|_\alpha \le C\|f\|_{\alpha+1}$$

if $2\ell > \alpha + 1$. Since $K_\alpha f$ is a solution to $\bar\partial u = f$ if $\bar\partial f = 0$, we obtain Theorem 2.1.

For the boundary complex we have analogous results.

Theorem 5.4. *There are operators* $K_b\colon \mathcal{E}^b_{*+1} \to \mathcal{E}^b_*$, $P_b\colon \mathcal{E}^b_0 \to \mathcal{H}^b_0$ *and* $I - S_b\colon \mathcal{E}^b_{n-1} \to \mathcal{H}^{b,\perp}_{n-1}$ *with the following properties. To begin with,*

$$\bar\partial_b K_b + K_b \bar\partial_b = I - P_b - S_b, \tag{5.17}$$

where the kernels satisfy

$$|k_b(\zeta,z)| \le C\frac{1}{|v|^{n-1/2}}, \quad |p_b(\zeta,z)| \le C\frac{1}{|v|^n}, \quad |s_b(\zeta,z)| \le C\frac{1}{|v|^n}.$$

Moreover,

$$k_b(\zeta,z) = \sum_{q=0}^{n-1} i^{1-q}\frac{(n-q-2)!}{(n-1)!}\frac{\partial_\zeta \bar v(\zeta,z)\wedge(\bar\partial_z\partial_\zeta \bar v(\zeta,z))^q}{v(\zeta,z)^{n-q-1}\bar v(\zeta,z)^{q+1}} + r_b(\zeta,z),$$

where

$$|r_b(\zeta,z)| \le C\frac{1}{|v|^{n-1}}, \quad |\bar\partial_{z,b} r_b(\zeta,z)| \le C\frac{1}{|v|^{n-1/2}} \tag{5.18}$$

and

$$p_b(\zeta,z) = \frac{1}{v(\zeta,z)^n} + \varrho_b(\zeta,z), \quad s_b(\zeta,z) = \frac{(\partial_\zeta\bar\partial_z\bar v)^n}{\bar v(\zeta,z)^{n-1}} + \varrho_b',$$

where

$$|\varrho_b'(\zeta,z)| \le C\frac{1}{|v|^{n-1/2}}, \quad |\varrho_b(\zeta,z)| \le C\frac{1}{|v|^{n-1/2}}, \quad \zeta\in\partial D, z\in D.$$

Both $P_b f$ and $S_b f$ are defined by first evaluating for $z \in D$ and then taking the boundary values.

¿From the estimates of the kernels it follows that K_b extends to a bounded (in fact compact) operator on L^2_b. It is also true but nontrivial that the singular integral operators S_b and P_b are L^2_b-bounded, see [13] for a nice proof. Since $\mathcal{E}^b_*$ is dense in the graph norms it follows that the homotopy relation (5.17) holds for all $f \in \operatorname{Dom}\bar\partial_b$.

We claim that

$$k_b(\zeta,z) = \partial_{\zeta,b} h_b(\zeta,z) + r_b'(\zeta,z), \tag{5.19}$$

where $h_b(\zeta,z)$ is hermitean (i.e., $h_{b,q}(\zeta,z) = (-1)^q\overline{h_{b,q}(z,\zeta)}$ holds for the $(0,q)$ part of h_b), which means that the corresponding operator H_b is self-adjoint, and where $r_b'(\zeta,z)$ fulfills the same estimates, (5.18), as $r_b(\zeta,z)$ but with an extra logarithm. When ρ is real analytic the leading term in the expression for $k_b(\zeta,z)$ is (modulo a smooth kernel arising from the fact that v only is z-holomorphic near the diagonal) $\partial_{\zeta,b}h(\zeta,z)$, where

$$h_b(\zeta,z)=-\sum_{q=1}^{n-1} i^{1-q}\frac{(n-q-2)!}{q(n-1)!}\frac{(\bar\partial_z\partial_\zeta\bar v(\zeta,z))^q}{v(\zeta,z)^{n-q-1}\bar v(\zeta,z)^q}+\frac{i}{n-1}\frac{\log\bar v(\zeta,z)}{v(\zeta,z)^{n-1}}, \tag{5.20}$$

which gives (5.19). In the general case, the right-hand expression ϕ of (5.20) is not hermitean, since $\overline{v(\zeta,z)}\neq v(z,\zeta)$. However, if we choose (??? ϕ_q denotes the ??part of ϕ ???)

$$h_{b,q}=\left(\phi_q(\zeta,z)+(-1)^q\overline{\phi_q(z,\zeta)}\right)/2,$$

then anyway (5.19) will hold with the estimate

$$|r_b'(\zeta,z)|\le C\frac{1+|\log|v||}{|v|^{n-1}} \tag{5.21}$$

of the remainder term. This follows from (5.3) and (5.5). Moreover, $\left(\phi_q(\zeta,z)-(-1)^q\overline{\phi_q(z,\zeta)}\right)/2$ fulfills (5.21).

It is also true that all these operators as well as their adjoints map smooth forms onto smooth forms.

Summing up we have

Corollary 5.5. *The operators S_b and P_b are bounded on L^2_b. The operators K_b, H_b, R_b and $\bar\partial_b H_b$ are even compact. Furthermore,* (5.17) *holds on $f\in\operatorname{Dom}\bar\partial_b$, and*

$$K_bf=H_b\bar\partial_b^*f+R_bf$$

for $f\in\operatorname{Dom}\bar\partial_b^*$, where H_b is self-adjoint. Finally $\bar\partial_bK_b$ is bounded on L^2_b and

$$\bar\partial_bK_b-(\bar\partial_bK_b)^*$$

is compact.

6. Construction of the kernels.

In order to prove Theorems 5.1 and 5.2 our first step is the following result.

Proposition 6.1. *For each $\alpha>0$ we have an operator $K^b_\alpha\colon\mathcal{E}_{*+1}\to\mathcal{E}^b_*$ and a projection $P_\alpha\colon\mathcal{E}_0\to\mathcal{H}_0$ such that*

$$\bar\partial_bK^b_\alpha+K^b_\alpha\bar\partial=I-P_\alpha|_b, \tag{6.1}$$

given by

$$K^b_\alpha f(z)=\left(f,\overline{k^b_\alpha(\cdot,z)}\right)_\alpha,\ z\in\partial D\quad\text{and}\quad P_\alpha f(z)=\left(f,\overline{p_\alpha(\cdot,z)}\right)_\alpha,\ z\in D,$$

where
(6.2)

$$k^b_\alpha(\zeta,z)=\sum_{q=0}^{n-1}c_{\alpha,n,q}\frac{\partial_\zeta\bar v(\zeta,z)\wedge(\bar\partial_z\partial_\zeta\bar v(\zeta,z))^q}{v(\zeta,z)^{n+\alpha-q-1}\bar v(\zeta,z)^{q+1}}+r^b_\alpha(\zeta,z),\quad c_{\alpha,n,q}=i^{1-q}\frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)},$$

(6.3)
$$|r^b_\alpha(\zeta,z)|\le C\frac{1}{|v|^{n+\alpha-1}},\quad |\bar\partial_z r^b_\alpha(\zeta,z)|\le C\frac{1}{|v|^{n+\alpha-1/2}}$$

and

$$p_\alpha(\zeta,z)=\frac{1}{v(\zeta,z)^{n+\alpha}}+\varrho_\alpha(\zeta,z),\ \text{where } |\varrho_\alpha(\zeta,z)|\le C\frac{1}{|v|^{n+\alpha-1/2}},\quad \zeta,z\in\overline{D}.$$

Moreover, if the domain is $\tilde D$ for some D and if any of these operators is applied to a form that is independent of the last variable then the resulting form is (rotation) invariant in the last variable.

The kernels here are essentially wellknown, the very point of the proposition is formula (6.2), and that the leading term is essentially ∂_ζ of a hermitean kernel; cf. the discussion preceding Corollary 5.5.

Proof. Let $\alpha>0$ be fixed. To begin with we let $\eta_j=z_j-\zeta_j$ and let $\chi=\chi(|\eta|)$ be a smooth function supported and identically 1 near Δ and set

$$q_j(\zeta,z)=\chi\left(\rho_j+\frac12\sum_k\rho_{jk}\eta_k\right)-(1-\chi)\bar\eta_j$$

(or possibly with some more terms if ρ is real analytic, cf. Section 5). Then we define v globally by

$$-v(\zeta,z)=q(\zeta,z)\cdot\eta+\rho(\zeta),$$

and if we let $s(\zeta,z)=-q(z,\zeta)$ we also get, cf. (5.3),

(6.4)
$$-s(\zeta,z)\cdot\eta-\rho(z)=v(z,\zeta)=\overline{v(\zeta,z)}+\mathcal{O}(|\eta|^3).$$

Using the notation $s\sim\sum s_jd\zeta_j$ and $q\sim\sum q_jd\zeta_j$, we define the operators

(6.5) $\hat K f(z)=$
$$=\int_D\sum_{k=0}^{n-1}c'_{\alpha,n,k}\frac{(-\rho)^{\alpha-1}f\wedge s\wedge(\bar\partial s)^k(-\rho(\bar\partial q)^{n-k-1}-(n-k-1)q\wedge\bar\partial\rho\wedge(\bar\partial q)^{n-k-2})}{(-q\cdot\eta-\rho)^{\alpha+n-k-1}(-s\cdot\eta)^{k+1}}$$

for $z\in\partial D$, where

$$c'_{\alpha,n,k}=\left(\frac{i}{2\pi}\right)^n\frac{\Gamma(\alpha+n-k-1)}{(n-k-1)!\Gamma(\alpha)}$$

and

$$\hat P f(z)=c_{\alpha,n,-1}\int_D\frac{(-\rho)^{\alpha-1}f(-\rho(\bar\partial q)^n-n\bar\partial\rho\wedge q\wedge(\bar\partial q)^{n-1})}{(-q\cdot\eta-\rho)^{n+\alpha}},\quad z\in D.$$

If f is smooth, then $\bar\partial_b \hat K f$ is continuous, $\hat P f$ has continuous boundary values and, see [5], the relation

$$\bar\partial_b \hat K f + \hat K \bar\partial f = f|_b - \hat P f|_b. \tag{6.6}$$

holds. It is well-known that in fact $\hat K f$ is smooth and $\hat P f$ has smooth boundary values if f is smooth.

Let $\hat K_q$ and $\hat P_q$ denote the components that are $(0,q)$ in dz. Since $q(\zeta, z)$ is holomorphic in z near the diagonal, all components $\hat P_q$ but $\hat P_0$ are smooth since the singularities are cancelled. Notice that the leading term in $\hat K_q$ is the one corresponding to $k = q$ in the sum, since in all the other ones the singularity is cancelled; the leading term is in fact the desired one, but since $\hat P_q$ does not vanish identically for $q > 0$, the operator $\hat K_q f$ is just (the boundary values of) an approximate solution if $\bar\partial f = 0$. To get rid of this flaw, let $\mathcal{L}$ be a C^∞ homotopy for $\bar\partial$ and $\mathcal{Q}$ the corresponding holomorphic projection, i.e. both of them map $\mathcal{E}_*$ into itself ($\mathcal{L}$ decreasing the degree one unit) and

$$\bar\partial \mathcal{L} + \mathcal{L}\bar\partial = I - \mathcal{Q}. \tag{6.7}$$

Such a homotopy can e.g. be obtained by the formula above, choosing s and q in an appropriate way in $\overline{D \times D}$ so that q is holomorphic in z, see e.g. [1] for details.

It is unknown, at least for us, if there is a $v(\zeta, z)$ that is globally holomorphic in z and that furthermore satisfies the first equality in (5.3).

So far we have only used $s(\zeta, z)$ when z is on the boundary. It can be extended inwards so that the relation (6.6) holds in D; see [5]. Applying $\bar\partial$ both from the left and from the right of the equality (6.6) we get the additional relation

$$\bar\partial \hat P = \hat P \bar\partial. \tag{6.8}$$

Let us now define

$$K^b = \hat K + \mathcal{L}\hat P, \quad P = \mathcal{Q}\hat P.$$

It is readily verified that $(\mathcal{L}\hat P)_q$ takes smooth forms to smooth forms for all $q \geq 0$, and therefore

$$K^b = \hat K + \text{smooth operator.} \tag{6.9}$$

Moreover,

$$\mathcal{Q}\hat P = \hat P_0 - \mathcal{L}\bar\partial \hat P_0,$$

where $\hat P_0$ defined as above just is the component acting on $(0,0)$-forms. Since the kernel for $\hat P_0$ is holomorphic near the diagonal, $\bar\partial \hat P_0$ has no singularity, and hence

$$P = \hat P + \text{smooth operator.}$$

¿From (6.6) and (6.8) we get

$$\bar\partial_b K^b + K^b \bar\partial = I - P|_b, \tag{6.10}$$

where P is holomorphic and only acts on $(0,0)$-forms, and so K^b is an exact homotopy operator for $\bar\partial$ whose leading term is $\hat K$, and $\hat P_0$ is the leading term of the corresponding projection.

We are now going to rewrite the leading terms and to this end we need

Proposition 6.2. *If ρ is C^3 and v, s, q are defined as above, then*

$$\partial_\zeta \bar v = -s + \mathcal{O}(|\eta|) = q + \mathcal{O}(|\eta|) = -\partial\rho + \mathcal{O}(|\eta|) \tag{6.11}$$

$$s \wedge q = \mathcal{O}(|\eta|), \quad \partial\rho \wedge \partial_\zeta \bar v = \mathcal{O}(|\eta|) \tag{6.12}$$

$$\bar\partial_\zeta q = \partial\bar\partial\rho + \mathcal{O}(|\eta|) \tag{6.13}$$

$$\partial\rho \wedge \partial_\zeta \bar v = s \wedge q + \mathcal{O}(|\eta|^2) \tag{6.14}$$

and

$$\bar\partial_z s - \bar\partial_z \partial_\zeta \bar v = \mathcal{O}(|\eta|). \tag{6.15}$$

The estimates in the lemma is with respect to the Euclidean metric. The crucial part is (6.14) which first occurred in [3]. The other ones are more or less direct consequences of the definitions. All of them but (6.15) can be found in [1] so let us restrict ourselves to this one.

Proof of (6.15). ¿From (6.4) we get

$$\partial_\zeta \overline{v(\zeta,z)} + \mathcal{O}(|\eta|^2) = \sum s_j d\zeta_j - \sum (\partial_\zeta s_j)(z_j - \zeta_j) = s - \sum (\partial_\zeta s_j)(z_j - \zeta_j),$$

and so

$$\bar\partial_z \partial_\zeta \overline{v(\zeta,z)} + \mathcal{O}(|\eta|) = \bar\partial_z s - \sum (\bar\partial_z \partial_\zeta s_j)(z_j - \zeta_j) + \mathcal{O}(|\eta|) = \bar\partial_z s + \mathcal{O}(|\eta|).$$

□

By repeated use of Proposition 6.2 we can rewrite $\hat K_q f$ and (by (6.9)) obtain

(6.16)
$$K_q^b f = c_{\alpha,n,q} \int_D \frac{(-\rho)^{\alpha-1} f \wedge \partial_\zeta \bar v \wedge (\bar\partial_z \partial_\zeta \bar v)^q}{v^{\alpha+n-1-q}\bar v^{q+1}} \wedge (-\rho\beta + (n-q-1)\gamma) \wedge \beta^{n-q-2} + R_q^b f$$

where

$$R_q^b f = \int_D (-\rho)^{\alpha-1} \mathcal{O}\left(\frac{1}{|v|^{n+\alpha}}\right) f \wedge (\mathcal{O}(-\rho)\mathcal{O}(|\eta|) + \mathcal{O}(|\eta|^2) \wedge \bar\partial\rho). \tag{6.17}$$

Now we need the following simple observations.

Lemma 6.3. *Suppose that the kernels $K(\zeta,z)$ and $k(\zeta,z)$ are connected by the relation*

$$\int_D f\wedge K=\int_D \langle f,\bar k\rangle\,dV \tag{6.18}$$

(f being a $(0,q)$ form and thus K being $(n,n-q)$ in ζ). Then

$$K=c_q\bar k\wedge\Omega_{n-q}=c_q\bar k\wedge(-\rho\beta_{n-q}+\gamma\wedge\beta_{n-q-1})/(-\rho),\quad k=\pm*K \text{ and } |K|=|k|,$$

where $*$ is the Hodge star with respect to Ω, and $c_q=1$ if q is even and $c_q=-i$ if q is odd.

Proof. For any forms, f and g,

$$\langle f,g\rangle\,dV=f\wedge *g.$$

Moreover, if they are $(0,q)$ forms, then

$$\langle f,g\rangle\,\Omega_n=c_qf\wedge\bar g\wedge\Omega_{n-q}.$$

Now, the lemma follows since $**=\pm1$ and $*$ is an isometry. □

Notice that (2.1) and (6.11) imply that $|\partial\rho|\sim\sqrt{-\rho}$ and $|\partial_\zeta\overline{v(\zeta,z)}|\sim\sqrt{-\rho}+|\zeta-z|$. In view of the lemma, (6.16) and (6.17) therefore imply (6.2) and the first estimate in (6.3). The second estimate in (6.3) follows in the same way, just noting that the operator $\bar\partial$ at most increases the singularity half a unit. In the same way one can rewrite the expression for $\hat Pf$ and obtain the desired properties of Pf.

Finally, if the domain is $\tilde D$ for some D then $\tilde v=v(\zeta,z)-\bar\zeta_{n+1}z_{n+1}$ and so on. If $\tau_\theta(z,w)=(z,e^{i\theta}w)$, then by an easy calculation one can check that $\tau_\theta^*\hat K\tilde f=\hat K\tilde f$ and similarily for the other operators. If we then use $\mathcal{L}$ and $\mathcal{Q}$ with the same invariance property, then K and P will have this property as well. This concludes the proof of Proposition 6.1. □

To have K_α defined even for $\alpha=1$ we need the the corresponding result for the boundary complex, but this is nothing but Theorem 5.4, with the additional observation that the corresponding operators on $\tilde D$ preserve rotational invariance in the last variable.

Proof of Theorem 5.4. The proof is performed along the same lines as the previous one. Let $\hat K_bf$ be the limit when $\alpha\to0$ of the $n-2$ first terms in the expression for $\hat Kf$ above. Then

$$\hat K_bf=\int_{\partial D}\sum_{k=0}^{n-2}\left(\frac{i}{2\pi}\right)^n\frac{f\wedge s\wedge(\bar\partial s)^k\wedge q\wedge(\bar\partial q)^{n-k-2}}{v^{n-k-1}u^{k+1}}$$

if $u(\zeta,z)=v(z,\zeta)$. In the same way we let $\alpha\to0$ in the expression for $\hat P$ and get

$$\hat P_b=\left(\frac{i}{2\pi}\right)^n\int_{\partial D}\frac{f\wedge q\wedge(\bar\partial q)^n}{v^n}.$$

Let $F=Tf$ be some reasonable (linear) extension of the tangential form f into D, and let VF be the limit of the term corresponding to $k=n-1$ (with F instead of f) when

$\alpha \to 0$. Then

$$VF = \left(\frac{i}{2\pi}\right)^n \int_D \frac{F \wedge s \wedge (\bar\partial s)^{n-1}}{u^n}.$$

If we let $\hat S_b f = V\bar\partial F + \bar\partial_b VF$, then the relation

$$\bar\partial_b \hat K_b + \hat K_b \bar\partial_b = I - \hat P_b - \hat S_b$$

holds for tangential forms. If F is $(0,q)$ and $q \le n-1$, then the expression for VF has no singularity and hence it can be considered as a smoothing operator VT acting on f. Now,

$$V\bar\partial F = \left(\frac{i}{2\pi}\right)^n \int_{\partial D} \frac{f \wedge s \wedge (\bar\partial s)^{n-1}}{u^n} - RF \tag{6.19}$$

where R has no singularity since u is holomorphic in ζ near the diagonal. Thus $\hat S_b f$ has the integral in (6.19) as its leading term and is smoothing on all but $(0,n-1)$-forms. Notice that if we had used a v that is globally holomorphic in z then we would have obtained an exact homotopy formula. Let $\mathcal{L}$, $\mathcal{Q}$, $\mathcal{U}$ be the operators in such a formula, cf. the proof of Proposition 6.1, where $\mathcal{U}$ is an operator acting on $(0,n-1)$-forms. If we define

$$K_b = \hat K_b - \mathcal{L}(\hat P_b + \hat S_b), \quad P_b = \mathcal{Q}(\hat P_b + \hat S_b), \quad S_b = \mathcal{U}(\hat P_b + \hat S_b),$$

it follows that K_b, P_b and S_b satisfy the exact homotopy formula and that their leading terms are the desired ones, after rewriting. □

We now want to extend the definition of K^b_α to the interior values, so that the homotopy relation holds. In view of Section 4, Proposition 6.1 and Theorem 5.4, the following definition is natural.

Definition 1. With the notation from §3 we define the operators K_α and P_α by

$$(K_\alpha f)\tilde{}|_b = \tilde K^b_{\alpha-1}\tilde f,$$

and analogously for P_α. If $\alpha = 1$ we use instead the operators $\tilde K_b$ and $\tilde P_b$. □

The definition of our operators in D depends on some choices, e.g., there is a cut-off function involved, but in the construction of P_α and K^b_α there is a further choice involved when going from $\hat K$ to K^b_α, cf. the proof of Theorem 6.1. Let us assume that we define $\tilde v = v - \bar\zeta_{n+1} z_{n+1}$, i.e. without making any unnecessary cutoff in the last variable.

Proof of Theorem 5.1. Let us assume that $\alpha > 1$; the case $\alpha = 1$ is similar. To begin with it follows from Proposition 6.1 and Section 4 that K_α and P_α map smooth forms onto smooth forms and that (5.7) holds.

Let now $\tilde{\hat K}$ denote the $\hat K$ in the proof of Proposition 6.1 but on $\tilde D$ and for $\alpha - 1$ instead of α. Since it differs from $\tilde K^b_{\alpha-1}$ only with an operator with smooth kernel, it follows that K_α is obtained from $\tilde{\hat K}$ on $\tilde D$ (modulo operator with smooth kernel). Let $(\zeta,\xi;z,w)$ be coordinates on $\tilde D \times \tilde D$ and define the kernel $\Psi(\zeta,\xi;z,w)$ in $\tilde D$ by

$$\tilde{\hat K}\tilde f(z,w) = \int_{\tilde D} \left(-\rho - |\xi|^2\right)^{\alpha-2} \tilde f(\zeta,\xi) \wedge \Psi(\zeta,\xi;z,w) \wedge d\xi \wedge d\bar\xi. \tag{6.20}$$

Modulo an operator with smooth kernel, following Section 4, then $K_\alpha f(z)$ is obtained from $\tilde{\hat{K}}\tilde{f}(z,w)$ by replacing w by $\sqrt{-\rho(z)}$ and $wd\bar{w}$ by $-\bar{\partial}\rho(z)$. By a straight-forward computation one verifies that this latter object is equal to $\hat{K}f(z)$ in D when $z \in \partial D$. This proves that (modulo operator with smooth kernel), the boundary values of $K_\alpha f$ are equal to K_α^b from Proposition 6.1, and hence (5.11) follows.

In the same way one can check that (the boundary values of) P_α coincides with P_α^b from Proposition 6.1, modulo a smoothing operator. Hence (5.10) and the estimate (5.9) follow from Proposition 6.1.

Let us now verify the proposed estimate (5.8) of the kernel $k_\alpha(\zeta,z)$. Rather than first computing a formula for the kernel in the interior (which in quite possible, see [2] where the corresponding computation in the ball is made) we make a direct estimate. Here $d\lambda$ stands for the Lebesgue measure. Notice that

$$\tilde{v}(\zeta,\xi;z,w) = v(\zeta,z) - \sqrt{-\rho(z)}\bar{\xi}$$

since w is to be replaced by $\sqrt{-\rho(z)}$, cf. Section 4. By (4.2),

$$\begin{aligned}
|K_\alpha f(z)| &= |(K_\alpha f(z))^\sim| = \left|\tilde{K}_{\alpha-1}^b\tilde{f}\left(z,\sqrt{-\rho(z)}\right)\right| = \left|\left(\tilde{f},\overline{\tilde{k}_{\alpha-1}^b\left(\cdot;z,\sqrt{-\rho(z)}\right)}\right)^{\sim}_{\alpha-1}\right| \leq \\
&\leq \int_{\tilde{D}}\left(-\rho-|\xi|^2\right)^{\alpha-2}\left|\tilde{f}\right|\left|\tilde{k}_{\alpha-1}^b\left(\zeta,\xi;z,\sqrt{-\rho(z)}\right)\right| d\lambda(\zeta,\xi) \lesssim \\
&\lesssim \int_{\tilde{D}}\left(-\rho-|\xi|^2\right)^{\alpha-2}|f|\frac{d\lambda(\zeta,\xi)}{\left|v(\zeta,z)-\bar{\xi}\sqrt{-\rho(z)}\right|^{n+\alpha-1/2}} = \\
&= \int_D |f| \int_{|\xi|^2<-\rho}\frac{\left(-\rho-|\xi|^2\right)^{\alpha-2}d\lambda(\xi)}{\left|1-\bar{\xi}\sqrt{-\rho(z)}/v\right|^{n+\alpha-1/2}|v|^{n+\alpha-1/2}}.
\end{aligned}$$

By the substitution $\xi = \sqrt{-\rho(\zeta)}\tau$ in the inner integral we get

$$\lesssim \int_D \frac{(-\rho)^{\alpha-1}|f|}{|v|^{n+\alpha-1/2}}\int_{|\tau|<1}\frac{\left(1-|\tau|^2\right)^{\alpha-2}d\lambda(\tau)}{|1-a\bar{\tau}|^{n+\alpha-1/2}} \lesssim \int_D (-\rho)^{\alpha-1}|f|\frac{1}{\left(1-|a|^2\right)^{n-1/2}|v|^{n+\alpha-1/2}},$$

where $a = \sqrt{-\rho(z)}\sqrt{-\rho(\zeta)}/v(\zeta,z)$, and therefore

$$|k_\alpha(\zeta,z)| \leq C\left(\frac{|v|}{\sigma}\right)^{2n-1}\frac{1}{|v|^{n+\alpha-1/2}}. \tag{6.21}$$

This completes the proof of Theorem 5.1. □

It remains to prove Theorem 5.2. To do this we need an auxiliary result. Let $(\zeta,\xi;z,w)$ be coordinates on $\tilde{D}\times\tilde{D}$. Suppose that $\phi = \phi(\zeta,\xi;z,w)$ is a form on $\tilde{D}\times\tilde{D}$ that is $(0,q)$ in (z,w), $(p,0)$ in (ζ,ξ) and that only depends on ζ, z and $\bar{\xi}w$. In particular this implies that the corresponding operator, defined via $(\ ,\ )_{\alpha-1}$ on $\tilde{D}$, maps invariant forms to invariant forms. By going up and down in dimension, as in Section 4, we then get an induced operator in D. The following proposition relates the adjoints of their kernels.

Proposition 6.4. *Let $(\zeta,\xi;z,w)$ be coordinates on $\tilde D\times\tilde D$. Suppose that $\phi=\phi(\zeta,\xi;z,w)$ is a form on $\tilde D\times\tilde D$ that is $(0,q)$ in (z,w), $(p,0)$ in (ζ,ξ) and that only depends on ζ, z and $\bar\xi w$. If $\Phi(\zeta,z)$ denotes the induced kernel on D (with respect to $(\ ,\ )_\alpha$), then $\Phi^*(\zeta,z)=\overline{\Phi(z,\zeta)}$ is induced by ϕ^*.*

For the proof of Proposition 6.4 we need the following result.

Proposition 6.5. *Let $M\colon L^2_{\alpha-1}(\tilde D)\to L^2_\alpha(D)$ be the adjoint of $\tilde{}\colon L^2_\alpha(D)\to L^2_{\alpha-1}(\tilde D)$. Suppose that $g(\zeta,\xi)=a(\zeta,\xi)+d\bar\xi\wedge b(\zeta,\xi)$. Then*

$$Mg(\zeta)=\frac{\alpha-1}{\pi}\int_{|\tau|<1}(1-|\tau|^2)^{\alpha-2}\left(a(\zeta,\sqrt{-\rho}\tau)-\frac{\bar\tau}{\sqrt{-\rho}}\bar\partial\rho\wedge b(\zeta,\sqrt{-\rho}\tau)\right)d\lambda(\tau).$$

¿From this formula for M one can check directly that $M\bar\partial=\bar\partial M$, cf. the proof of Proposition 4.1.

Proof. We need the following slightly more general version of (4.2),

$$\left\langle\tilde f,a+d\bar\xi\wedge b\right\rangle^{\sim}=\langle f,a\rangle-\frac{\xi}{-\rho}\langle f,\bar\partial\rho\wedge b\rangle, \tag{6.22}$$

which is obtained in the same way. If $g=a+d\bar\xi\wedge b$,

$$Mg(\zeta)=\frac{\alpha-1}{\pi}\int_{|\xi|<\sqrt{-\rho}}(-\rho)^{-(\alpha-1)}(-\rho-|\xi|^2)^{\alpha-2}\left(a(\zeta,\xi)-\frac{\bar\xi}{-\rho}\bar\partial\rho\wedge b(\zeta,\xi)\right)d\lambda(\xi),$$

and after the change of variables $\xi=\sqrt{-\rho}\tau$ we get the desired expression. □

Notice, by the way, that since the right-hand side of (6.22) equals $\left\langle\tilde f,a-(-\rho)^{-1}\bar\xi\bar\partial\rho\wedge b\right\rangle^{\sim}$ this means that the mapping $a+d\bar\xi\wedge b\mapsto a-(-\rho)^{-1}\bar\xi\bar\partial\rho\wedge b$ is the orthogonal projection onto the image of $\tilde{}$.

Proof of Proposition 6.4. Either $\phi(\zeta,\xi;z,w)$ has the form $\psi(\zeta,z,\bar\xi w)$, $\psi(\zeta,z,\bar\xi w)\wedge\xi d\bar w$, $\psi(\zeta,z,\bar\xi w)\wedge d\xi\wedge d\bar w$ or $\psi(\zeta,z,\bar\xi w)\wedge\bar w d\xi$, where ψ has no occurrences of $d\xi$ or $d\bar w$.

Now, $(\Phi f)(z)^\sim=(\tilde f,\bar\phi)^\sim_{\alpha-1}=(f,M\bar\phi)_\alpha$ and hence $\overline{\Phi(\zeta,z)}=M\bar\phi$, where each occurrence of w is replaced by $\sqrt{-\rho(z)}$ and $\bar w dw$ by $-\partial\rho\,(z)$. It is now readily checked by Proposition 6.5 that $\left(M\overline{\phi}\right)^*=M\overline{\phi^*}$. □

Proof of Theorem 5.2. Let us now define H_α and R_α. The idea is to copy the argument for the boundary operators preceding Corollary 5.5.

The leading term in the expression for $\hat k^b_{\alpha-1}(\zeta,\xi;z,w)$ in $\tilde D$ from Proposition 6.1 though a priori only defined for $(z,w)\in\partial\tilde D$, has a natural meaning for any $(z,w)\in\overline{\tilde D}$, since $\tilde v(\zeta,\xi;z,w)=v(\zeta,z)+\bar\xi w$ has. (In fact, if $a(\zeta,\xi;z,w)$ is any kernel that is defined for $(\zeta,\xi;z,w)\in\partial\tilde D\times\overline{\tilde D}$ and only depends on (z,ζ) and $\bar\xi w$, then one easily checks that there is a unique extension to $\overline{\tilde{D}} \times \overline{\tilde{D}}$, that only depends on (z, ζ) and $\bar{\xi}w$.) Moreover, it is of the type in Proposition 6.4. We find that this leading term is approximately $\partial_{\zeta,\xi}\phi^b_\alpha$, where

$$\tilde{\phi}^b = -\sum_{q=1}^{n} i^{1-q}\frac{\Gamma(n+\alpha-q-1)}{q\Gamma(n+\alpha)}\frac{(\bar{\partial}_z\partial_\zeta\bar{v} + d\xi\wedge d\bar{w})^q}{(v-\bar{\xi}w)^{n+\alpha+q-1}(\bar{v}-\xi\bar{w})^q} + \frac{i}{n}\frac{\log(\bar{v}-\xi\bar{w})}{(v-\bar{\xi}w)^{n+\alpha-1}}.$$

Also this kernel is defined for all $(z, w) \in \tilde{D}$, and it is of the type in Proposition 6.4.

Therefore, we can consider its adjoint $\phi^{b,*} = \overline{\phi^b_{\alpha-1}(z, w, \zeta, \xi)}$, $(z, w) \in \partial\tilde{D}$ as well. Analogously to the boundary case (cf. the discussion preceding Corollary 5.5) we define H_α as the operator in D induced by the kernel $(\phi^b + (-1)^q\phi^{b,*})/2$. Since this kernel is $\lesssim |\tilde{v}|^{-(n+\alpha-1)}(1 + |\log|\tilde{v}||)$, the estimate (5.14) for $h_\alpha(\zeta, z)$ follows in the same way as the estimate of $k_\alpha(\zeta, z)$ in the proof of Theorem 5.1. As before, H_α preserves regularity. We then define R_α so that (5.15) holds for smooth forms. The proposed estimates follow as before.

Notice that in the real analytic case, $\phi^b = \phi^{b,*}$, hence H_α is simply the operator induced by ϕ^b.

In view of Proposition 6.4 it follows that the kernel $h_\alpha(\zeta, z)$ is hermitean, and that H_α is self-adjoint.

Finally we prove that K^*_α maps smooth forms to smooth forms. Recall from the proof of Theorem 5.1 that, modulo an operator with smooth kernel, $K_\alpha f$ is obtained from $\tilde{\tilde{K}}\tilde{f}(z, w)$ in (6.20). We claim that if $*$ denotes the Hodge star with respect to $\Omega(\zeta)$, i.e. ignoring the variable ξ, then

$$\tilde{\tilde{K}}\tilde{f}(z, w) = \int_{\tilde{D}} \left(-\rho - |\xi|^2\right)^{\alpha-1} \left\langle \tilde{f}(\zeta, \xi), \overline{*\Psi/\sqrt{-\rho(\zeta)}} \right\rangle d\tilde{V}.$$

In fact,

$$f \wedge \Psi d\xi \wedge \bar{\partial}\xi = \left\langle f, \overline{*\Psi} \right\rangle dV \wedge d\xi \wedge \bar{\partial}\xi = (-\rho - |\xi|^2) \left\langle \tilde{f}, \overline{*\Psi/\sqrt{-\rho(\zeta)}} \right\rangle d\tilde{V}$$

and since neither $\tilde{f}$ nor $*\Psi$ have any differentials $d\xi$, the inner product is the same as the one in $\tilde{D}$. Therefore, the kernel $*\Psi/\sqrt{-\rho(\zeta)}$ induces K_α (modulo smooth). Furhermore, it is readily checked that Ψ only depends on ζ, z and $\bar{\xi}$ w; we omit that computation. Therefore $*\Psi/\sqrt{-\rho(\zeta)}$ satisfies the condition in Proposition 6.4 and therefore (modulo smooth), K^*_α is induced by the adjoint of $*\Psi/\sqrt{-\rho(\zeta)}$. But this kernel is of the standard ???? form and therefore it induces an operator that maps smooth forms onto smooth forms. This concludes the proof of Theorem 5.2. □

7. Approximate formulas for the canonical operators

Let F_α denote any of the operators from Theorem 5.3 that satisfies (5.8). Then we know that F_α maps $\mathcal{E}_*$ into itself and is compact on L^2_α. In fact, it is regularizing one half unit in a certain sense, but for our purposes it is enough to notice the following proposition, see [1].

Proposition 7.1. *The kernel of* $F^\ell_\alpha = F_\alpha \circ \ldots \circ F_\alpha$ *is less then*

$$C\frac{1}{|v|^{n+\alpha-\ell/2}}\left(\frac{|v|}{\sigma}\right)^{2n-\ell} \tag{7.1}$$

if $\ell < 2n$. *For any* $r > 0$, *there is a* k *such that* $F^\ell_\alpha \colon L^2_\alpha \to C^r(\overline{D})$ *if* $\ell \geq k$.

We will say that an operator as in Proposition 7.1 is of type F^ℓ_α. Following the ideas in Kerzman-Stein [11] we can use Theorem 5.3 and Proposition 7.1 to derive an asymptotic expansion of the canonical operators K^{can}_α and $\Pi^{\mathrm{can}}_\alpha = \bar\partial K^{\mathrm{can}}_\alpha$ in terms of our explicit operators. Let $\Pi_\alpha = \bar\partial K_\alpha$. Fix $\alpha \geq 1$, and let us suppress the subscripts on the operators for the remainder of the section.

Since Π and Π^{can} are both projections $L^2_\alpha \to \mathcal{K}_\alpha$ (and since Π^{can} is self-adjoint) we get

$$\Pi\Pi^{\mathrm{can}} = \Pi^{\mathrm{can}}, \quad \Pi^*\Pi^{\mathrm{can}} = \Pi^*.$$

By Theorem 5.3, $\Pi - \Pi^* = F$ and therefore we get $\Pi^{\mathrm{can}} - \Pi^* = F\Pi^{\mathrm{can}}$. By iteration we get

$$\Pi^{\mathrm{can}} = \Pi^* + F\Pi^* + F^2\Pi^* + \ldots + F^{k-1}\Pi^* + F^k\Pi^{\mathrm{can}}$$

The same argument works for the orthogonal projection P^{can} as well as for the corresponding projection Π_b on the boundary and the orthogonal projections P^{can}_b and S^{can}_b. For the latter statement, apply the argument to $I - S_b$ and $I - S^{\mathrm{can}}_b$.

We conclude

Theorem 7.2. *Suppose that* $\alpha \geq 1$. *The orthogonal projections* $\Pi^{\mathrm{can}}_\alpha \colon L^2_\alpha \to \mathcal{K}_\alpha$ *and* P^{can}_α *can be written*

$$\Pi^{\mathrm{can}}_\alpha = \Pi_\alpha + \mathcal{L}_\alpha + G_\alpha \quad \text{and} \quad P^{\mathrm{can}}_\alpha = P_\alpha + \mathcal{L}_\alpha + G_\alpha \tag{7.2}$$

where $\mathcal{L}_\alpha$ *is explicit, compact on* L^2_α *and preserves* C^∞ *regularity, and* G_α *maps* L^2_α *into* $C^r(\overline{D})$. *Correspondingly for the boundary complex we have*

$$\Pi^{\mathrm{can}}_b = \Pi_b + \mathcal{L}_b + G_b, \quad P^{\mathrm{can}}_b = P_b + \mathcal{L}_b + G_b$$

and

$$S^{\mathrm{can}}_b = S_b + \mathcal{L}_b + G_b.$$

The principal meaning of this theorem is that the canonical projection Π^{can} is approximately equal to the explicitly given operator Π, and questions about regularity for Π^{can} can be reduced to the corresponding questions for explicitly given operators. In particular, it follows immediately that Π^{can} maps smooth forms on $\overline{D}$ to smooth forms.

So far we have said nothing about in what sense K approximates K^{can}. Since they both are of type F^1 it is reasonable to expect that $K^{\text{can}} - K$ is regularizing one half unit more. More precisely, we have the following theorem.

Theorem 7.3. *Given $r > 0$ and $\alpha \geq 1$ we have*

$$K_\alpha^{\text{can}} = K_\alpha + \mathcal{L}_\alpha^2 + G_\alpha,$$

where G_α maps L_α^2 into $C^r(\overline{D})$ and $\mathcal{L}_\alpha^2$ is an explicitly given operator that preserves regularity, with a kernel satisfying the estimate (7.1) for $\ell = 2$, and such that $\bar\partial\mathcal{L}_\alpha^2$ as well as $\bar\partial_\alpha^*\mathcal{L}_\alpha^2$ are compact. Analogously,

$$K_b^{\text{can}} = K_b + \mathcal{L}_b^2 + G_b. \tag{7.3}$$

Proof. In this proof we will let Π equal $\bar\partial K$ on forms and equal P on functions. It follows from the definition of K^{can} that

$$K^{\text{can}} = (I - \Pi^{\text{can}})K\Pi^{\text{can}}.$$

Using (7.2) we get

$$K^{\text{can}} = (I - \Pi)K\Pi^{\text{can}} - \mathcal{L}K(\Pi + \mathcal{L} + G) - GK\Pi^{\text{can}} = (I - \Pi)K\Pi^{\text{can}} - A$$

where $A = \mathcal{L}K(\Pi + \mathcal{L} + G) + GK\Pi^{\text{can}}$. ¿From the homotopy formula for K it follows that $(I - \Pi)K = K\Pi$ and hence we get

$$K^{\text{can}} = K\Pi\Pi^{\text{can}} - A = K\Pi^{\text{can}} - A.$$

Writing $K\Pi^{\text{can}} = K - K(I - \Pi^{\text{can}})$ and using that $K = H\bar\partial^* + R$ (see Theorem 5.3) we get

$$K^{\text{can}} = K - R(I - \Pi^{\text{can}}) - A = K - R(I - \Pi - \mathcal{L} - G) - A.$$

Now $R(I - \Pi) = RK\bar\partial$ is of type $\mathcal{L}^2$ as well as $R\mathcal{L}$ (which is even of type F^3). Moreover, RG is clearly of type G. In the same way one verifies that the terms in A are either $\mathcal{L}^2$ or G. □

8. Regularity properties

In view of Theorem 7.2, questions about regularity for our canonical operators are reduced to the corresponding questions for (leading terms of) the explicit approximate operators. This works perfectly well for various norms such as L^p, Hölder etc, but here we restrict to the C^∞ regularity.

Recall that $\mathcal{E}_q$ denotes the set of $(0,q)$-forms that are smooth up to the boundary and that $\mathcal{H}_q = \mathcal{E}_q \cap \operatorname{Ker}\bar\partial$. In particular, $\mathcal{H}_0 = \mathcal{E}_0 \cap \mathcal{O}$. It turns out that we have regularity for the orthogonal decomposition for each $\alpha \geq 1$. Let $\mathcal{H}_q^{\perp_\alpha}$ be the orthogonal complement with respect to $\| \ \|_\alpha$ of $\mathcal{H}_q$ in $\mathcal{E}_q$. In the same way, let $\mathcal{E}_q^b$ be the space of smooth complex tangential forms and let $\mathcal{H}_q^b = \operatorname{Ker} \bar\partial_b \cap \mathcal{E}_q^b$.

Theorem 8.1. *Suppose that $\alpha \geq 1$. The canonical homotopy operator $K_\alpha^{\rm can}$ as well as the Bergman projection $P_\alpha^{\rm can}$ preserve regularity. In particular the orthogonal projections $P_\alpha^{\rm can}$ and $\Pi_{\alpha,q}^{\rm can} = \bar\partial K_{\alpha,q}^{\rm can}$ preserve regularity, and therefore we have a smooth orthogonal decomposition*

$$\mathcal{E}_q = \mathcal{H}_q \oplus \mathcal{H}_q^{\perp_\alpha}.$$

The boundary complex operators $K_b^{\rm can}$, $P_b^{\rm can}$ and $S_b^{\rm can}$ also preserve regularity and hence we have the smooth decompositions

$$\mathcal{E}_q^b = \mathcal{H}_q^b \oplus \mathcal{H}^{b,\perp_\alpha}, \quad q < n-1$$

and

$$\mathcal{E}_{n-1}^b = \operatorname{Ker} \bar\partial_b^* \cap \mathcal{E}_{n-1}^b \oplus (\operatorname{Ker} \bar\partial_b^*)^\perp \cap \mathcal{E}_{n-1}^b.$$

Proof of Theorem 8.1. The theorem follows immediately from Theorem 5.3 and Theorem 7.2. □

The regularity result for the boundary operators was proved by Kohn, see [8]. From this result one immediately gets the regularity result for $\Pi_\alpha^{\rm can}$ for all integer values of α by using Theorem 4.3.

Recall the definitions and discussion of the operators $\overline{\Box}_\alpha$ and $E_\alpha^{\rm can}$ in Section 3. ¿From (3.6) (and the definitions) it follows that the $\overline{\Box}_\alpha$-harmonic forms in $\mathcal{E}_*$ are precisely $\mathcal{H}_0$. Furthermore, from (3.5), we obtain regularity for the $\overline{\Box}_\alpha$-problem, i.e. we have smooth solutions to $\overline{\Box}_\alpha u = f$ whenever f is smooth (and, in the case of functions, f is orthogonal to $\mathcal{H}_0$). In the same way, from (3.7) and (3.8) we get that the $\overline{\Box}_b$-harmonic forms in $\mathcal{E}_*^b$ are $\mathcal{H}_0^b \oplus \mathcal{H}_{n-1}^{b,*}$ (definition????????) and that we have regularity for the $\overline{\Box}_b$-problem. We collect these facts in:

Corollary 8.2. *Suppose that $\alpha \geq 1$. The space of smooth harmonic forms with respect to $\overline{\Box}_\alpha$ is $\mathcal{H}_0$ and we have the Hodge decomposition*

$$\mathcal{E}_* = \overline{\Box}_\alpha \mathcal{E}_* \oplus \mathcal{H}_0 = \bar\partial \mathcal{E}_* \oplus \bar\partial_\alpha^* \mathcal{E}_* \oplus \mathcal{H}_0.$$

The space of smooth harmonic forms with respect to $\overline{\Box}_b$ on ∂D is $\mathcal{H}_0^b \oplus \mathcal{H}_{n-1}^{b,*}$ and we have the orthogal decomposition

$$\mathcal{E}_*^b = \overline{\Box}_b \mathcal{E}_*^b \oplus \mathcal{H}_0^b \oplus \mathcal{H}_{n-1}^{b,*} = \bar\partial_b \mathcal{E}_* \oplus \bar\partial_b^* \mathcal{E}_* \oplus \mathcal{H}_0^b \oplus \mathcal{H}_{n-1}^{b,*}.$$

Remark 4. In the ball, Theorem 8.1 is true for all $\alpha > 0$ and therefore it is probably true even in this case but our proof fails for $0 < \alpha < 1$. □

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