Higher-Order Riesz Operators for the Ornstein-Uhlenbeck Semigroup *

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Abstract. We prove that the second-order Riesz transforms associated to the Ornstein-Uhlenbeck semigroup are weak type $(1,1)$ with respect to the Gaussian measure in finite dimension. We also show that they are given by a principal value integral plus a constant multiple of the identity. For the Riesz transforms of order three or higher, we present a counterexample showing that the weak type $(1,1)$ estimate fails.

Mathematics Subject Classifications (1991). 42B20, 42C10.

Key words: Calderón-Zygmund operators, Gaussian measure, Hermite polynomials, Mehler kernel, Ornstein-Uhlenbeck semigroup, Riesz transforms.

*The authors were partially supported by the European Union under contract ERBCHRXCT930083

1 Introduction and main result

Let $d\gamma(x) = e^{-|x|^2}dx$ be the Gaussian measure in $\mathbb{R}^d$, $d \geq 1$. Then the operator $L = -\frac{1}{2}\Delta + x \cdot \nabla$, defined on the space of test functions (i.e., the space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$), has a self-adjoint extension to $L^2(\gamma)$, also denoted L. The spectral properties of L are well known: L is positive semi-definite with discrete spectrum $\{0, 1, ...\}$. For $d = 1$, the eigenfunctions are the Hermite polynomials, given by $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$, $n = 0, 1, ...$ The eigenfunctions for arbitrary d are tensor products $H_\alpha = \otimes_{i=1}^d H_{\alpha_i}$, where α is a multiindex.

The Ornstein-Uhlenbeck semi-group $(e^{-tL})_{t\geq 0}$ is thus well defined on $L^2(\gamma)$. It is given by the Mehler kernel

$$M_t(x,y) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \exp\left(-\frac{|e^{-t}x - y|^2}{1-e^{-2t}}\right),$$

in the sense that $e^{-tL}f(x) = \int M_t(x,y)f(y)dy$ for $f \in L^2(\gamma)$. This makes it possible to define powers of L and compute their kernels. A negative power L^{-b} will be defined on the subspace $H_0^\perp$ of $L^2(\gamma)$ consisting of functions having vanishing integral against γ. We denote by $\Pi_0 : L^2(\gamma) \to H_0^\perp$ the orthogonal projection.

Letting $D = (\partial_1, ..., \partial_d)$ be the differentiation operator in $\mathbb{R}^d$, one can form products $D^\alpha L^{-b}\Pi_0$ with α a multiindex and $b > 0$. When $|\alpha| = 2b$, this is a Riesz operator. These Riesz operators have been studied by several authors. They are bounded on $L^p(\gamma)$, $1 < p < \infty$. This was first proved by P. A. Meyer [8] and by Gundy [4], who used probabilistic methods. See also [2] for a simpler proof of P. A. Meyer's theorem. Analytic proofs were then found by Urbina [15], Pisier [11], Gutierrez [5] and Gutierrez, Segovia and Torrea [6]. We refer to Fabes, Gutierrez and Scotto [1] for further bibliographical information. These boundedness properties of course imply a priori inequalities of the type $\| D^\alpha u \|_p \leq \| L^b u \|_p$.

To prove the weak type $(1,1)$ inequality when it holds, the methods of these papers do not seem adequate. In dimension $d = 1$, Muckenhoupt [9] obtained this inequality for first-order Riesz operators, i.e. $|\alpha| = 1, b = 1/2$. The extension to arbitrary dimension was done by Fabes, Gutiérrez and Scotto [1], still for first order operators. But for third-order operators and $d = 1$, a counterexample to the weak type $(1,1)$ boundedness is given in Forzani and Scotto [3]. In the present paper, we prove the weak type $(1,1)$ estimate for second-order Riesz operators and arbitrary finite d. Here is the precise statement of our main result

Theorem 1.1 *For any multiindex α with $|\alpha| = 2$, the operator $D^\alpha L^{-1}\Pi_0$ is of weak type $(1,1)$ with respect to γ.*

This result has been obtained independently also by Menárguez, Pérez, and Soria [7], see also [10].

We also determine the distributional kernel of a Riesz operator of any order (see (3.20)).

The proof of Theorem 1.1 consists of two parts, corresponding to the local and the global parts of the operator and to be found in Sections 3 and 4, respectively. We define the local part in Section 2, before Theorem 2.7, by restricting the kernel to pairs of points (x, y) whose mutual distance is no larger than $\frac{1}{1+|x|+|y|}$. The reason for this is that x and y are then contained in a small ball where γ is essentially proportional to Lebesgue measure. Locally, our Riesz operators behave like Euclidean Riesz operators; they are given by principal value singular integrals plus, in some cases, a constant multiple of the identity. Our proof for this part, given in Section 3, is based on comparison with a Calderón-Zygmund convolution kernel. The ordinary L^p or L^1 $-$ weak L^1 estimates for the operator given by this latter kernel can be transferred from Lebesgue measure to Gaussian measure γ, via a summation over balls of the type mentioned above. The details of this transference are given in Section 2 (see Lemma 2.4 and Theorem 2.7). This local argument holds for Riesz operators of any order.

The proof for the remaining, global part of the operator, to be found in Section 4, is based on a technical lemma giving estimates of the absolute value of the kernel (Lemma 4.3). These estimates make it possible to apply the so called "method of forbidden regions", which was previously used in Sjögren [13] to get the weak type $(1, 1)$ inequality for the maximal operator of the Ornstein-Uhlenbeck semigroup. No cancellation is involved here.

Finally, in Section 5, we present a counterexample, valid in arbitrary dimension, to show that the Riesz transforms of order at least three are not of weak type $(1, 1)$ with respect to the Gaussian measure.

2 Some auxiliary results

In the following, we shall always be working in the euclidean space $\mathbb{R}^d$ for $d \geq 1$. Instead of using the heat semigroup e^{-tL}, we shall find it more convenient to work with the operators r^L, $0 \leq r < 1$, whose integral kernel $\mathcal{M}_r$ can be obtained from the Mehler kernel by the change of variables $t = -\log r$. Thus

$$\mathcal{M}_r(x,y) = \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} \exp\left(-\frac{|rx-y|^2}{1-r^2}\right).$$

Lemma 2.1 *Let ρ be a function in $L^1((0,1), \frac{dr}{r})$. If*

$$m(\lambda) = \int_0^1 r^\lambda \; \rho(r) \, \frac{dr}{r},$$

then

$$m(L)f(x) = \int_{\mathbb{R}^d} K(x,y)\, f(y)\, dy, \qquad f \in L^2(\gamma),$$

where

$$K(x,y) = \int_0^1 \mathcal{M}_r(x,y)\, \rho(r) \frac{dr}{r}.$$

The proof follows immediately from the spectral analysis in terms of the semigroup.

For every nonnegative integer n we shall denote by P_n the orthogonal projection onto the space spanned by the Hermite polynomials of degree n.

Remark. Since $m(L)\Pi_0 = m(L) - m(0)P_0$, the kernel of the operator $m(L)\Pi_0$ can be obtained from that of $m(L)$ by subtracting the kernel of $m(0)P_0$, which is

$$m(0)\; \pi^{-d/2} e^{-|y|^2} = \int_0^1 \rho(r)\, \frac{dr}{r}\; \pi^{-d/2} e^{-|y|^2}.$$

For each $b > 0$, let

$$K_b(x,y) = \frac{1}{\Gamma(b)} \int_0^1 \left(\mathcal{M}_r(x,y) - \pi^{-d/2} e^{-|y|^2}\right) (-\log r)^{b-1}\, \frac{dr}{r}.$$

Lemma 2.2 *For each $b > 0$, the kernel of the operator $L^{-b}\Pi_0$ is K_b in the sense that for all test functions ϕ and ψ on $\mathbb{R}^d$, the following identity holds*

$$\left\langle L^{-b}\Pi_0 \phi, \psi \right\rangle = \iint K_b(x,y)\, \phi(y)\, \psi(x)\, d\gamma(x)\, dy.$$

Proof. We begin by proving that the kernel K_b defines a distribution on $\mathbb{R}^d \times \mathbb{R}^d$. Let ϕ and ψ be two test functions on $\mathbb{R}^d$. We claim that the integral

$$\iint \int_0^1 \left(\mathcal{M}_r(x,y) - \pi^{-d/2} e^{-|y|^2}\right) (-\log r)^{b-1}\, \phi(y)\, \psi(x)\, \frac{dr}{r}\, d\gamma(x)\, dy \tag{2.3}$$

is absolutely convergent and that its absolute value is bounded by $C \max|\phi| \max|\psi|$, where C is a constant that depends on the supports of ϕ and ψ. Indeed, we write

$$\iint \int_0^1 \left| \left(\mathcal{M}_r(x,y) - \pi^{-d/2} e^{-|y|^2}\right) (-\log r)^{b-1}\, \phi(y)\, \psi(x) \right| \frac{dr}{r}\, d\gamma(x)\, dy$$
$$= \iint \int_0^{\frac{1}{2}} \cdots + \iint \int_{\frac{1}{2}}^1 \cdots,$$

and we estimate the two integrals separately. Since

$$|\mathcal{M}_r(x,y) - \pi^{-d/2} e^{-|y|^2}| \le r\, \theta(r,x,y)$$

where θ is a function locally bounded in (x,y) uniformly for $0 \le r \le 1/2$, the first integral is convergent and is bounded by $C \max|\phi| \max|\psi|$. To obtain the desired estimate of the second integral, we integrate first with respect to y and then we use the fact that the function $(-\log r)^{b-1} r^{-1}$ is integrable on $[1/2, 1]$.

By applying Lemma 2.1 and the remark following it to $\rho(r) = r^{\epsilon}(-\log r)^{b-1}$, $\epsilon > 0$, $b > 0$, we obtain that the integral kernel of the operator $(\epsilon I + L)^{-b}\Pi_0$ is

$$J_{\epsilon,b}(x,y) = \frac{1}{\Gamma(b)} \int_0^1 \left(\mathcal{M}_r(x,y) - \pi^{-d/2} e^{-|y|^2}\right) (-\log r)^{b-1} r^{\epsilon-1}\, dr.$$

Thus

$$\langle (\epsilon I + L)^{-b}\Pi_0 \phi, \psi \rangle = \iint J_{\epsilon,b}(x,y)\, \phi(y)\, \psi(x)\, dy\, d\gamma(x).$$

As ϵ tends to 0, the left hand side tends to $\langle L^{-b}\phi, \psi\rangle$ by the spectral theorem. Thus we only need to show that

$$\lim_{\epsilon\to 0} \iint J_{\epsilon,b}(x,y)\, \phi(y)\, \psi(x)\, d\gamma(x)\, dy = \iint K_b(x,y)\, \phi(y)\, \psi(x)\, d\gamma(x)\, dy$$

for all test functions ϕ and ψ. This is immediate in view of the absolute convergence of (2.3). ∎

We present now a simple covering lemma, which will be basic in passing from estimates with respect to Lebesgue measure for the local part to estimates with respect to the Gaussian measure. The action will take place in the local region, which we define here, once and for all, as

$$N = \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \;:\; |x-y| < \frac{1}{1+|x|+|y|} \right\}.$$

Also, for $x \in \mathbb{R}^d$, we shall denote

$$N_x = \{y \in \mathbb{R}^d \ : \ (x,y) \in N\}.$$

We shall use the notation E^c for the complement of a set E in $\mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{R}^d$, as the case might be. Besides, $|E|$ will stand for the Lebesgue measure of E. We shall freely use the letters $C < \infty$ or $c > 0$ to denote constants, not necessarily equal at different occurrences.

Lemma 2.4 *Let $B(x_j, \frac{\kappa}{4(1+|x_j|)})$ be a maximal family of disjoint balls, where $\kappa = 1/20$. Denote*

$$B_j = B(x_j, \frac{\kappa}{1+|x_j|}).$$

Then

1. *The collection $\{B_j\}_{j\in\mathbb{N}}$ covers $\mathbb{R}^d$ and if $4B_j$ stands for a ball centered at x_j with radius 4 times that of B_j, the balls $\{4B_j\}_{j\in\mathbb{N}}$ have bounded overlap, i.e., there is a constant C_0 such that no point can belong to more than C_0 balls of the family $\{4B_j\}_{j\in\mathbb{N}}$*
2. *$x \in B_j$ and $y \in 4B_j \Rightarrow (x,y) \in N$.*
3. *$x \in B_j \Rightarrow B(x, \frac{\kappa}{1+|x|}) \subset 4B_j$.*

Proof. We start with the simple observation that

$$|x-y| \le 1 \ \Rightarrow \ \frac{1}{2} \le \frac{1+|x|}{1+|y|} \le 2. \tag{2.5}$$

To prove 3., we notice that $z \in B(x, \frac{\kappa}{1+|x|})$ implies

$$|z-x_j| < |z-x| + |x-x_j| \le \frac{\kappa}{1+|x|} + \frac{\kappa}{1+|x_j|} < \frac{4\kappa}{1+|x_j|},$$

where we used (2.5). Similarly 2. follows from

$$|x-y| \le |x-x_j| + |x_j-y| < \frac{5\kappa}{1+|x_j|} \le \frac{1}{1+|x|+|y|},$$

which holds because

$$1+|x|+|y| \le 1+|x|+1+|y| \le 4(1+|x_j|) = \frac{1+|x_j|}{5\kappa}.$$

Finally we come to the proof of 1. Assume $z \notin B_j$. We want to prove that $B(z, \frac{\kappa}{4(1+|z|)})$ and $B(x_j, \frac{\kappa}{4(1+|x_j|)})$ are disjoint. This will be guaranteed once we prove that

$$|z - x_j| \geq \frac{\kappa}{4(1+|z|)} + \frac{\kappa}{4(1+|x_j|)}, \tag{2.6}$$

assuming $|z - x_j| > \frac{\kappa}{1+|x_j|}$. If $|z| > |x_j|/2$, we have $1/(1+|z|) < 2/(1+|x_j|)$ and so

$$|z - x_j| > \frac{\kappa}{1+|x_j|} > \frac{\kappa}{2(1+|x_j|)} + \frac{\kappa}{4(1+|z|)},$$

which implies (2.6). If instead $|z| \leq |x_j|/2$, then $|z - x_j| \geq |x_j|/2$. It follows that

$$|z - x_j| > \frac{1}{2}\left(\frac{\kappa}{1+|x_j|} + \frac{|x_j|}{2}\right) \geq \frac{\kappa}{2}\frac{1+|x_j|+|x_j|^2}{1+|x_j|} \geq \frac{\kappa}{2},$$

which again implies (2.6)

It only remains to prove the bounded overlap of the balls $4B_j$. Suppose $z \in 4B_j$ for J values of j. Then $B(z, \frac{16\kappa}{1+|z|})$ contains $B(x_j, \frac{\kappa}{4(1+|x_j|)})$ for these J values of j because of (2.5). Since the latter balls are pairwise disjoint and the Lebesgue measures of all these balls are comparable, this gives clearly an upper bound for J.

■

We shall use this lemma to get a way of passing from an estimate for an operator with respect to Lebesgue measure to an estimate for the local part of the operator with respect to Gaussian measure.

Let T be a linear operator mapping $\mathcal{C}_0^\infty(\mathbb{R}^d)$ into the space of measurable functions in $\mathbb{R}^d$. Suppose that for $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $x \notin$ supp (f), one has

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy$$

for a kernel $K(x,y)$, which we for simplicity assume continuous off the diagonal. Then we can define the global part of T by

$$T_{glob} f(x) = \int_{\mathbb{R}^d} K(x,y)(1 - \chi_N(x,y)) f(y) dy,$$

and the local part of T by $T_{loc} f = Tf - T_{glob} f$.

Theorem 2.7 *Let T and K be as just described, and assume that T is of weak type $(1,1)$ with respect to Lebesgue measure and that the kernel satisfies the inequality $|K(x,y)| \leq C|x-y|^{-d}$ for $x \neq y$. Then T_{loc} is of weak type $(1,1)$ with respect to γ.*

Proof. Assume that x is in the ball B_j from the covering in Lemma 2.4. Then

$$
\begin{aligned}
(2.8)\qquad & T_{loc}f(x) \\
= {} & Tf(x) - \int K(x,y)\chi_{(4B_j)^c}(y)f(y)dy + \int K(x,y)(\chi_{(4B_j)^c}(y) - \chi_{N^c}(x,y))f(y)dy \\
= {} & T(f\chi_{4B_j})(x) + \int K(x,y)\chi_{N_x\setminus 4B_j}(y)f(y)dy.
\end{aligned}
$$

For the last integral here, we observe that $c/(1+|y|) < |y-x| < C/(1+|y|)$ when the integrand does not vanish, and then $|K(x,y)| \le C(1+|y|)^d$. Thus

$$
\begin{aligned}
T_{loc}f(x) &\le \sum_j \chi_{B_j}(x)|T(f\chi_{4B_j})(x)| + \int_{|y-x|<C/(1+|y|)} (1+|y|)^d |f(y)|dy \\
&= T^1 f(x) + T^2 f(x).
\end{aligned}
$$

The weak type property of T implies that for $\lambda > 0$,

$$
(2.9)\qquad |\{x \in B_j \,:\, |T(f\chi_{4B_j})(x)| > \lambda\}| \le C\lambda^{-1}\int_{4B_j} |f(x)|dx.
$$

Notice that the Gaussian density $e^{-|x|^2}$ is of constant order of magnitude in each $4B_j$. Hence we can replace Lebesgue measure by γ in both sides of (2.9). By summing in j and using the bounded overlap of the balls $4B_j$ from Lemma 2.4, we conclude that T^1 is of weak type $(1,1)$ for γ. Further, since the Gaussian density is also essentially constant in the ball $\{x \,:\, |y-x| < C/(1+|y|)\}$, uniformly in y, we have

$$
\int_{\mathbb{R}^d} T^2 f(x)d\gamma(x) \le \int_{\mathbb{R}^d} |f(y)| \int_{|y-x|<C/(1+|y|)} d\gamma(x)(1+|y|)^d dy \le C\int_{\mathbb{R}^d} |f(y)|e^{-|y|^2}dy.
$$

Thus T^2 is of strong type $(1,1)$ with respect to the Gaussian measure, and the theorem follows.

■

3 The local part

The aim of this section is to estimate the local part of the Riesz operators. After some estimates for the kernel of such an operator, we introduce a convolution kernel which turns out to be homogeneous and of Calderón-Zygmund type. The corresponding operator approximates the Riesz operator near the diagonal. This allows us to see that the Riesz operator is given by a principal value integral plus a constant multiple of the identity.

As we showed in Lemma 2.2, the kernel of $L^{-b}\Pi_0$ is for $b > 0$

$$K_b(x,y) = c_b \int_0^1 \left(\log \frac{1}{r}\right)^{b-1} \left((1-r^2)^{-\frac{d}{2}} e^{-\frac{|rx-y|^2}{1-r^2}} - e^{-|y|^2}\right) \frac{dr}{r}.$$

Fix a multiindex α with $0 < a = |\alpha|$. We shall need the formula

$$D_x^\alpha \left(e^{-\frac{|sx-y|^2}{1-r^2}}\right) = \frac{(-s)^a}{(1-r^2)^{a/2}} H_\alpha \left(\frac{sx-y}{\sqrt{1-r^2}}\right) e^{-\frac{|sx-y|^2}{1-r^2}}, \tag{3.1}$$

which is a consequence of the definition of the Hermite polynomials. Then one verifies

$$D_x^\alpha K_b(x,y) = c_b \int_0^1 \left(\log \frac{1}{r}\right)^{b-1} (1-r^2)^{-\frac{d}{2}} \frac{(-r)^a}{(1-r^2)^{a/2}} H_\alpha \left(\frac{rx-y}{\sqrt{1-r^2}}\right) e^{-\frac{|rx-y|^2}{1-r^2}} \frac{dr}{r}. \tag{3.2}$$

We shall consider this kernel near the diagonal. Since

$$|H_\alpha(x)| e^{-|x|^2} \le C e^{-\frac{|x|^2}{2}}, \tag{3.3}$$

where C may depend on α, we get, for $(x,y) \in N$ and $0 < r < 1$,

$$\left| H_\alpha \left(\frac{rx-y}{\sqrt{1-r^2}}\right)\right| e^{-\frac{|rx-y|^2}{1-r^2}} \le C e^{-\frac{|rx-y|^2}{4(1-r)}} \le C e^{-\frac{|x-y|^2}{4(1-r)}}, \tag{3.4}$$

where the last inequality is a consequence of the following estimate, valid in N :

$$\begin{aligned} |rx-y|^2 = |(x-y)-(1-r)x|^2 &= |x-y|^2 + (1-r)^2|x|^2 - 2(1-r)\langle x, x-y\rangle \\ &\ge |x-y|^2 + (1-r)^2|x|^2 - 2(1-r). \end{aligned} \tag{3.5}$$

Lemma 3.6 *For* $|\alpha| = a > 0,\ b > 0$ *and* $(x,y) \in N,$ *we have*

$$|D_x^\alpha K_b(x,y)| \le \begin{cases} C|x-y|^{2b-a-d}, & 2b-a-d < 0, \\ C \log \frac{C}{|x-y|}, & 2b-a-d = 0, \\ C, & 2b-a-d > 0. \end{cases}$$

Proof. We split the integral in (3.2) into two parts. It is easy to see from (3.4) that

$$\left|\int_0^{1/2} \cdots\right| \le C.$$

Similarly

$$\left|\int_{1/2}^1 \cdots\right| \le C\int_{1/2}^1 (1-r)^{b-a/2-d/2-1} e^{-\frac{|x-y|^2}{4(1-r)}}dr,$$

and, after the change of variables $u = (1-r)^{-1}|x-y|^2/4$, the last quantity becomes

$$C|x-y|^{2b-a-d}\int_{|x-y|^2/2}^{\infty} u^{-(b-a/2-d/2)-1}e^{-u}du.$$

From here, the estimates of the lemma readily follow .

■

For $a < 2b$, this lemma implies that the singularity of $D_x^\alpha K_b$ on the diagonal is integrable in y. One can then differentiate under the integral sign when K_b is integrated against a test function $f \in C_0^\infty$, to obtain

$$D_x^\alpha \int K_b(x,y)f(y)dy = \int D_x^\alpha K_b(x,y)f(y)dy, \quad a < 2b.$$

A derivative $D_x^\alpha K_b$ of order $|\alpha| = 2b$ has a nonintegrable singularity on the diagonal, and we shall compare it with a convolution kernel. Let

$$\tilde{K}_b(x) = c_b\int_0^1 \left(\log\frac{1}{r}\right)^{b-1}(1-r^2)^{-\frac{d}{2}}r^{2b}e^{-\frac{|x|^2}{1-r^2}}\frac{dr}{r}.$$

Then, for any α,

$$D^\alpha\tilde{K}_b(x) = c_b\int_0^1 \left(\log\frac{1}{r}\right)^{b-1}(1-r^2)^{-\frac{d}{2}}\frac{(-1)^a r^{2b}}{(1-r^2)^{a/2}}H_\alpha\left(\frac{x}{\sqrt{1-r^2}}\right)e^{-\frac{|x|^2}{1-r^2}}\frac{dr}{r} \tag{3.7}$$

Assume now that $a = 2b$. Write for a test function f

$$\begin{aligned} & D_x^\alpha \int K_b(x,y)f(y)dy \\ = \; & D_x^\alpha\int \tilde{K}_b(x-y)f(y)dy + D_x^\alpha\int (K_b(x,y)-\tilde{K}_b(x-y))f(y)dy \\ & = I + II. \end{aligned} \tag{3.8}$$

We shall see that one can differentiate under the integral sign in II. This will be a consequence of the following lemma, which says that the differentiated kernel has an integrable singularity on the diagonal.

Lemma 3.9 *For $|\alpha| = 2b > 0$ and $(x, y) \in N$, we have*

$$|D_x^\alpha K_b(x, y) - D^\alpha \tilde{K}_b(x - y)| \le \begin{cases} C\frac{1+|x|}{|x-y|^{d-1}}, & d > 1, \\ C(1 + |x|)\log \frac{C}{|x||x-y|}, & d = 1. \end{cases}$$

Proof. We shall estimate the difference between the integrals in (3.2) and (3.7), with x replaced by $x - y$ in the integrand of (3.7). For either of these integrands, we easily see that

$$\left| \int_0^{1/2} \cdots \right| \le C,$$

exactly as in the proof of Lemma 3.6. Let us define $r(x) = \max(\frac{1}{2}, 1 - |x|^{-2})$.

Consider now the integrals between $1/2$ and $r(x)$ of the same integrands. By using (3.4) and a simpler version of it in which rx is replaced by x, we obtain, in both cases

$$\left| \int_{1/2}^{r(x)} \cdots \right| \le C \int_{1/2}^{r(x)} (1 - r)^{-d/2-1} e^{-\frac{|x-y|^2}{4(1-r)}} dr.$$

The same change of variables as in Lemma 3.6 transforms the last integral into

$$C|x - y|^{-d} \int_{|x-y|^2/2}^{|x|^2|x-y|^2/4} u^{d/2-1} e^{-u} du.$$

Observing that the upper limit of integration here is small, we conclude that

$$\left| \int_{1/2}^{r(x)} \cdots \right| \le C|x|^d.$$

We finally consider the difference between the integrals from $r(x)$ to 1. This difference can be estimated by

$$\text{(3.10)} \quad C \int_{r(x)}^1 (1 - r^2)^{-d/2-1} \left| H_\alpha \left(\frac{rx - y}{\sqrt{1 - r^2}} \right) e^{-\frac{|rx-y|^2}{1-r^2}} - H_\alpha \left(\frac{x - y}{\sqrt{1 - r^2}} \right) e^{-\frac{|x-y|^2}{1-r^2}} \right| dr.$$

For fixed r, let us define

$$\varphi(s) = e^{-\frac{|sx-y|^2}{1-r^2}} H_\alpha \left(\frac{sx - y}{\sqrt{1 - r^2}} \right).$$

The derivative will be

$$\begin{aligned} \varphi'(s) = & -\frac{2\langle x, sx - y \rangle}{1 - r^2} e^{-\frac{|sx-y|^2}{1-r^2}} H_\alpha \left(\frac{sx - y}{\sqrt{1 - r^2}} \right) \\ & + e^{-\frac{|sx-y|^2}{1-r^2}} \sum_{j=1}^d \partial_j H_\alpha \left(\frac{sx - y}{\sqrt{1 - r^2}} \right) \frac{x_j}{\sqrt{1 - r^2}} \\ = & \; e^{-\frac{|sx-y|^2}{1-r^2}} \left\{ -\frac{2\langle x, sx - y \rangle}{1 - r^2} H_\alpha \left(\frac{sx - y}{\sqrt{1 - r^2}} \right) + 2 \sum_{j=1}^d \alpha_j H_{\alpha - e_j} \left(\frac{sx - y}{\sqrt{1 - r^2}} \right) \frac{x_j}{\sqrt{1 - r^2}} \right\}, \end{aligned}$$

so that, by using (3.3), we see that

$$|\varphi'(s)| \le C\frac{|x|}{\sqrt{1-r^2}}e^{-\frac{|sx-y|^2}{4(1-r)}}. \tag{3.11}$$

Applying the mean value theorem together with (3.11), we can estimate (3.10) by

$$C\int_{r(x)}^{1}(1-r^2)^{-d/2-1}\frac{|x|}{\sqrt{1-r^2}}\left\{\sup_{r\le s\le 1} e^{-\frac{|sx-y|^2}{4(1-r^2)}}\right\}(1-r)dr. \tag{3.12}$$

From (3.5) we see that

$$\sup_{r\le s\le 1} e^{-\frac{|sx-y|^2}{4(1-r^2)}} \le Ce^{-\frac{|x-y|^2}{8(1-r)}}. \tag{3.13}$$

By applying (3.13) to (3.12), we finally see that (3.10) is bounded by

$$C|x|\int_{1-|x|^{-2}}^{1}(1-r)^{-d/2-1/2}e^{-\frac{|x-y|^2}{8(1-r)}}dr, \tag{3.14}$$

which, after the change of variables $u=(1-r)^{-1}|x-y|^2/8$, becomes

$$C|x||x-y|^{-d+1}\int_{\frac{|x|^2|x-y|^2}{8}}^{\infty} u^{d/2-3/2}e^{-u}du.$$

When $d>1$, this is at most $C|x||x-y|^{-d+1}$. For the case $d=1$, we split the last integral at the point 1 and get the bound

$$C|x|\log\frac{C}{|x||x-y|}.$$

The estimates we have obtained for the different parts lead us immediately to the conclusion of the lemma.

■

To deal with I in (3.8), we study the singularities of some derivatives of $\tilde{K}_b$.

Lemma 3.15 *Let α be a multiindex of length $a=2b-1$ or $2b$, $b>0$. Then there exists a function h, homogeneous of degree $2b-a-d$ and smooth in $\mathbb{R}^d\setminus\{0\}$, such that*

$$D^\alpha\tilde{K}_b(x)=h(x)+O(|x|^{2b-a-d+1}) \quad \textit{as } x\to 0,$$

except when $d=1$ and $a=2b-1$. In this exceptional case, we have instead

$$D^\alpha\tilde{K}_b(x)=c_1\log|x|+c_2\mathrm{sign}(x)+c_3+o(1) \quad \textit{as } x\to 0.$$

Proof. Write the integral in (3.7) as $\int_0^{1/2} + \int_{1/2}^1$. Clearly

$$\int_0^{1/2} = c_0 + o(1) \quad \text{as } x \to 0, \tag{3.16}$$

so that this part of $D^\alpha \tilde{K}_b$ verifies the conclusion of the lemma.

In $\int_{1/2}^1$ we observe that

$$(\log 1/r)^{b-1} r^{2b-1} = c(1-r^2)^{b-1} r(1 + (1-r)g(r)),$$

with g a bounded function in $[1/2, 1]$, and expand the Hermite polynomial. Then $\int_{1/2}^1$ is seen to be a sum of terms

$$c_\delta \int_{1/2}^1 (1-r^2)^{b-1-d/2-a/2-|\delta|/2} x^\delta e^{-\frac{|x|^2}{1-r^2}} r(1 + (1-r)g(r))dr, \tag{3.17}$$

taken over multiindices δ between 0 and α. The factor r here is introduced to facilitate the change of variables $s = (1-r^2)/|x|^2$. If we neglect the term $(1-r)g(r)$for a moment, the expression (3.17) is transformed to

$$cx^\delta |x|^{2b-a-d-|\delta|} \int_0^{3/(4|x|^2)} s^{b-1-d/2-a/2-|\delta|/2} e^{-1/s} ds. \tag{3.18}$$

In the main case $2b - a - d - |\delta| < 0$, the integral in (3.18) converges even if extended to $+\infty$, and for small $|x|$ its value is then $c + O(|x|)$, for some $c > 0$. Then the expression in (3.18) is clearly

$$cx^\delta |x|^{2b-a-d-|\delta|} + O(|x|^{2b-a-d+1}), \quad \text{as } x \to 0.$$

The remaining, exceptional case $2b - a - d - |\delta| = 0$ occurs precisely when $a = 2b-1,\ d = 1,\ \delta = 0$. Then (3.18) is instead

$$c_1 \log |x| + c_2 + o(1).$$

The contribution from the term $(1-r)g(r)$ can be transformed similarly, and is seen to be no larger than

$$Q = C|x|^{2b-a-d+2} \int_0^{3/(4|x|^2)} s^{b-d/2-a/2-|\delta|/2} e^{-1/s} ds.$$

If $2b - a < d + |\delta| - 2$, the integral here converges at ∞, and $Q = O(|x|^{2b-a-d+2}),\ x \to 0$. If $2b - a = d + |\delta| - 2$, we similarly find that $Q = O(|x|^{2b-a-d+2} \log 1/|x|)$. In these two cases, we thus have $Q = O(|x|^{2b-a-d+1})$. If finally $2b - a > d + |\delta| - 2$, we get $Q = O(|x|^{|\delta|})$. Then $Q = O(|x|^{2b-a-d+1})$ again, provided $|\delta| \geq 2b - a - d + 1$. This last inequality is easily seen to be satisfied as soon as we are not in the exceptional case defined above. With this sole exception, we thus have $Q = O(|x|^{2b-a-d+1})$. In the exceptional case, we observe instead directly that the contribution to (3.17) coming from the term $(1-r)g(r)$ is

$$\int_{1/2}^{1} e^{-\frac{|x|^2}{1-r^2}} h(r) r dr,$$

for some bounded function h which does not depend on x. This expression is $c + o(1)$ as $x \to 0$. From this the lemma follows.

■

Consider now term I in (3.8), with $|\alpha| = 2b$ and $f \in C_0^\infty$. Let $D^\alpha = \partial_j D^{\alpha'}$, for some j and some α' with $|\alpha'| = 2b - 1$, and write $P = D^{\alpha'} \tilde{K}_b$ for short. Then

$$\begin{aligned} D^\alpha(\tilde{K}_b * f)(x) &= \frac{\partial}{\partial x_j} \int P(y) f(x-y) dy \\ = \int P(y) f'_j(x-y) dy &= -\lim_{\rho \to 0} \int_{|y|>\rho} P(y) \frac{\partial}{\partial y_j} f(x-y) dy. \end{aligned}$$

We integrate by parts with respect to y_j in the last integral. Let p_j denote the projection $\mathbb{R}^d \to \mathbb{R}^{d-1}$ obtained by deleting the jth coordinate. If $y' \in \mathbb{R}^{d-1}$ satisfies $|y'| < \rho$, the line $p_j^{-1}(y')$ has two intersections with the sphere $|y| = \rho$ denoted $y_+(y')$ and $y_-(y')$, where the subscript indicates the sign of the jth coordinate. We get

$$\begin{aligned} (3.19) \quad & D^\alpha(\tilde{K}_b * f)(x) \\ = \ & \lim_{\rho \to 0} \left(\int_{|y'|<\rho} (P(y_+(y')) f(x - y_+(y')) - P(y_-(y')) f(x - y_-(y'))) dy' \right. \\ + \ & \left. \int_{|y|>\rho} \partial_j P(y) f(x-y) dy \right). \end{aligned}$$

From Lemma 3.15 it follows that the integral in y' in the last expression tends to $a_\alpha f(x)$ as $\rho \to 0$. Here a_α is a constant which is nonzero for instance when $D^\alpha = \partial_j^2$. Thus the last integral converges, and its limit is a principal value.

Combining I and II, we finally get

$$(3.20) \qquad D_x^\alpha \left(\int_{\mathbb{R}^d} K_b(x,y) f(y) dy \right) = a_\alpha f(x) + \text{p.v.} \int_{\mathbb{R}^d} D_x^\alpha K_b(x,y) f(y) dy$$

for $|\alpha| = 2b > 0$.

Off the diagonal, the operator $R_\alpha = D^\alpha L^{-b} \Pi_0$ is thus given by the smooth kernel $D^\alpha K_b$. In particular, we have a decomposition $R_\alpha = R_{\alpha,loc} + R_{\alpha,glob}$, as defined in Section 2.

Theorem 3.21 *Let α be a multiindex with $|\alpha| = 2b > 0$. Then $R_{\alpha,loc}$ is of weak type $(1,1)$ with respect to the Gaussian measure γ.*

Proof. Let h be the homogeneous function obtained in Lemma 3.15. It then follows from Lemmas 3.9 and 3.15 that

$$\Psi(x,y) = D^\alpha K_b(x,y) - h(x-y)$$

satisfies

$$\int_{(x,y)\in N} |\Psi(x,y)|dx \le C,$$

where C is independent of y. Thus the kernel $\Psi(x,y)\chi_N(x,y)$ defines a strong type $(1,1)$ operator both for Lebesgue measure and for γ.

The function h must have mean value 0 on the unit sphere, since the last integral in (3.19) has a limit for every test function f. Thus h is a Calderón-Zygmund kernel and convolution by p.v. h defines an operator T of weak type $(1,1)$ for Lebesgue measure in $\mathbb{R}^d$. Hence, Theorem 2.7 implies that T_{loc} is of weak type $(1,1)$ for γ. It now follows that the operator $R_{\alpha,\,loc}$, defined by the kernel

$$\text{p.v. } D^\alpha K_b(x,y)\chi_N(x,y)$$

is of weak type $(1,1)$ for γ.

■

4 The global part of the second-order operators

The major part of this section consists of the technical Lemma 4.3, giving size estimates for Riesz kernels of order 2. Then this lemma is applied, via the "forbidden regions" method, to obtain the weak type $(1,1)$ estimate of the corresponding Riesz operators.

Off the diagonal, the kernel of the second order Riesz operator $R_{jk} = \partial_j \partial_k L^{-1} \Pi_0$ is

$$4c_1 \int_0^1 (1-r^2)^{-d/2-2} r^2 (rx_j - y_j)(rx_k - y_k) \exp\left(-\frac{|rx-y|^2}{1-r^2}\right) \frac{dr}{r}$$
$$-2\delta_{jk}\, c_1 \int_0^1 (1-r^2)^{-d/2-1} r^2 \exp\left(-\frac{|rx-y|^2}{1-r^2}\right) \frac{dr}{r}.$$

The absolute value of this kernel can be estimated by the positive kernel

$$K(x,y) = \int_0^1 \psi(r;x,y)\, e^{-\phi(r;x,y)}\, dr \tag{4.1}$$

where

$$\psi(r;x,y) = r\,(1-r^2)^{-d/2-1}\left(1+\frac{|rx-y|^2}{1-r^2}\right) \quad \text{and} \quad \phi(r;x,y) = \frac{|rx-y|^2}{1-r^2}. \tag{4.2}$$

In this section we shall prove that the global part T_{glob} of the integral operator with kernel K, i.e.

$$T_{glob} f(x) = \int K(x,y)\, \chi_{N^c}(x,y)\, f(y)\, dy,$$

is of weak type $(1,1)$. This clearly implies that the global part of any second order Riesz operator is of weak type $(1,1)$.

We cover the set $\{x : |x| > 1\}$ with nonoverlapping cubes $Q_i, i = 1, 2, \ldots$, centred at points x_i, $|x_i| \geq 1$ and of diameters d_i such that $c/|x_i| \leq d_i < 1/(10|x_i|)$, for some $c > 0$. This can clearly be done. We number the cubes in such a way that $|x_i|$ is nondecreasing. Let

$$K^*(x,y) = \sup\{K(x',y) : x' \text{ in the same } Q_i \text{ as } x, \text{ or } x' = x\}.$$

For $|y| > 1$ we let $\eta = |y|$ and write $x = \xi y/\eta + v$, where $\xi \in \mathbb{R}$ and $v \perp y$. Define for such a y regions $D_i = D_i(y)$ by

$$\begin{aligned}
D_0 &= \{x : \xi < 0\} \\
D_1 &= \{x : 0 < \xi < \eta,\ |x-y| > \beta\eta\} \\
D_2 &= \{x : (x,y) \in N^c,\ 0 < \xi < \eta,\ |x-y| < \beta\eta\} \\
D_3 &= \{x : (x,y) \in N^c,\ \xi > \eta,\ |x-y| < \beta\eta\} \\
D_4 &= \{x : \xi > \eta,\ |x-y| > \beta\eta\},
\end{aligned}$$

where $\beta > 0$ is sufficiently small.

Lemma 4.3 *Let $(x,y) \in N^c$. For every $\beta > 0$ sufficiently small there exists a constant C depending only on β and d such that the following estimates hold for $|y| > 1$:*

(a) if $x \in D_0$, then $K(x,y) \leq Ce^{-\eta^2}$;

(b) if $x \in D_1$, then $K(x,y) \leq C(1+\xi)e^{\xi^2-\eta^2}$;

(c) if $x \in D_2$, then $K(x,y) \leq C(\eta^{-1}+|v|)^{-d}e^{\xi^2-\eta^2} + C\sqrt{\eta(\eta-\xi)}(\sqrt{\frac{\eta-\xi}{\eta}}+|v|)^{-d}e^{\xi^2-\eta^2}$;

(d) if $x \in D_3$, then $K(x,y) \leq C(\eta^{-1}+|v|)^{-d}$;

(e) if $x \in D_4$, then $K(x,y) \leq C$.

Moreover if $|y| \leq 1$ or $|x| \leq 1$, then

(f) $K(x,y) \leq Ce^{-\eta^2}$.

These estimates remain valid with K^* instead of K.

Proof. Since the kernel K is defined by the integral (4.1), to prove the lemma we shall need to estimate the function ψ from above and the function ϕ from below over the various regions D_i. Clearly this requires that we estimate the function $r \mapsto |rx-y|$ both from above and from below in each region. In the following we shall often use the observation that, if $|y| > 1$ and $(x,y) \in N^c$ then $|x-y|\,|y| \geq 1/4$.

Case (a). Assume that $|y| \geq 1$ and $x \in D_0$. Since $\xi < 0$,

$$\frac{|rx-y|^2}{1-r^2} \geq \frac{r^2|x|^2+|y|^2}{1-r^2} = |y|^2 + \frac{r^2}{1-r^2}(|x|^2+|y|^2).$$

Therefore

$$e^{-\frac{|rx-y|^2}{1-r^2}} \leq e^{-|y|^2}\, e^{-\frac{r^2}{1-r^2}(|x|^2+|y|^2)}. \tag{4.4}$$

Thus to estimate $K(x,y)$ we write

$$K(x,y) = \int_0^{1/2} \cdots + \int_{1/2}^1 \cdots = K_1(x,y) + K_2(x,y). \tag{4.5}$$

Then by (4.4)

$$K_1(x,y) \leq Ce^{-|y|^2} \int_0^{1/2} r(1+|x|^2+|y|^2)\, e^{-r^2(|x|^2+|y|^2)}\, dr \leq Ce^{-\eta^2}.$$

On the other hand

$$K_2(x,y) \leq e^{-|y|^2} \int_{1/2}^1 (1-r^2)^{-d/2-1} \left(1+\frac{|x|^2+|y|^2}{1-r^2}\right) e^{-\frac{1}{4}\frac{|x|^2+|y|^2}{1-r^2}}\, r\, dr$$

Here the exponential can be estimated by

$$e^{-\frac{1}{8(1-r^2)}}e^{-\frac{1}{8}\frac{|x|^2+|y|^2}{1-r^2}}$$

and these factors take care of the first two factors in the integrand. Thus $K_2(x,y) \le Ce^{-\eta^2}$, which completes the proof of case *(a)*.

Case (b). We shall use the identity

$$\frac{|rx-y|^2}{1-r^2} = \frac{|x-ry|^2}{1-r^2} - |x|^2 + |y|^2 = \frac{(\xi - r\eta)^2 + r^2|v|^2}{1-r^2} - \xi^2 + \eta^2 \tag{4.6}$$

to factor out $e^{\xi^2-\eta^2}$ in the estimate. Since $0 < \xi < \eta$, we also have the estimates

$$|rx-y|^2 = (r\xi - \eta)^2 + r^2|v|^2 \le \eta^2 + r^2|v|^2 \tag{4.7}$$

$$\le \eta^2 + |v|^2. \tag{4.8}$$

Next we write

$$K(x,y) = \left(\int_0^{1-\beta/2} + \int_{1-\beta/2}^1\right) = K_3(x,y) + K_4(x,y).$$

In estimating K_3 we may bound the negative powers of $1-r^2$ by a constant. Thus by (4.6) and (4.7) we get

$$\begin{aligned} K_3(x,y) &\le C\, e^{\xi^2-\eta^2}\int_0^1 r(1+\eta^2+r^2|v|^2)\; e^{-c\left[(\xi-r\eta)^2+r^2|v|^2\right]}\,dr \\ &\le Ce^{\xi^2-\eta^2}\left\{1+\int_0^1 r\,\eta^2\; e^{-c(\xi-r\eta)^2}\,dr + \int_0^1 r^3|v|^2\; e^{-cr^2|v|^2}\,dr\right\} \end{aligned}$$

Performing the change of variables $\xi - r\eta = t$ in the first integral and $r|v| = t$ in the second, we get

$$\begin{aligned} K_3(x,y) &\le e^{\xi^2-\eta^2}\left\{1+\int_{-\infty}^{+\infty}(|t|+\xi)\; e^{-ct^2}\,dt + \frac{1}{|v|^2}\int_0^{|v|} t^3\; e^{-ct^2}\,dt\right\} \\ &\le C\, e^{\xi^2-\eta^2}(1+\xi). \end{aligned}$$

To estimate $K_4(x,y)$ we remark that $|x-ry| > \frac{1}{2}|x-y|$ for every $r > 1-\beta/2$. Indeed, if $r > 1-\beta/2$ and $x \in D_1$, one has that $(1-r)|y| < \frac{\beta}{2}|y| < \frac{1}{2}|x-y|$. Thus $|x-ry| \ge |x-y| - (1-r)|y| > \frac{1}{2}|x-y|$. Therefore

$$\begin{aligned} (\xi - r\eta)^2 + r^2|v|^2 \ge r^2|x-ry|^2 > \frac{r^2}{4}|x-y|^2 &\ge \frac{r^2}{4}\max\left(\beta^2\eta^2, |v|^2\right) \\ &> C(\eta^2+|v|^2). \end{aligned} \tag{4.9}$$

Thus, by (4.6), (4.8) and (4.9)

$$K_4(x,y) \le C e^{\xi^2-\eta^2} \int_0^1 r(1-r^2)^{-d/2-1} \left(1 + \frac{\eta^2+|v|^2}{1-r^2}\right) e^{-C\frac{\eta^2+|v|^2}{1-r^2}}\, dr.$$

Performing the change of variables $\frac{\eta^2+|v|^2}{1-r^2} = t$, we get

$$\begin{aligned} K_4(x,y) &\le C e^{\xi^2-\eta^2} (\eta^2+|v|^2)^{-d/2} \int_0^{+\infty} t^{d/2-1}(1+t)\, e^{-t}\, dt \\ &\le C\, e^{\xi^2-\eta^2}, \end{aligned}$$

since $\eta > 1$. This completes the proof of *(b)*.

Case (c). For $|y| > 1$ and $x \in D_2$ denote by $\epsilon_0 = \epsilon_0(x,y)$ and $\epsilon_1 = \epsilon_1(x,y)$ the two positive numbers defined by

$$\epsilon_0 = 2\left(1 - \frac{\xi}{\eta}\right) + \frac{|v|}{\eta}, \qquad \epsilon_1 = \frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) + 2\frac{|v|}{\eta},$$

and let $r_0 = 1 - \epsilon_0$, $r_1 = 1 - \epsilon_1$. Since $1 - \xi/\eta < \beta$ and $|v| < \beta\eta$, one can choose β so small that $r_0,\ r_1 \in [1/2, 1]$. Thus

$$\begin{aligned} K(x,y) &\le \left(\int_0^{r_0} + \int_{r_0}^{r_1} + \int_{r_1}^{1}\right) \cdots\, dr \\ &= K_5(x,y) + \int_{r_0}^{r_1} \cdots\, dr + K_6(x,y), \end{aligned}$$

where the middle integral is taken as 0 when $r_0 > r_1$. To estimate $K_5(x,y)$ we let $r \in [0, r_0]$. Since $(1-r)\eta \ge \epsilon_0\eta = 2(\eta - \xi) + |v|$, one has that

$$|\xi - r\eta| \ge (1-r)\eta - (\eta - \xi) \ge \frac{1}{2}(1-r)\eta \tag{4.10}$$

and

$$\begin{aligned} |rx - y| &\le |r\xi - \eta| + r|v| \\ &\le (1-r)\eta + r(\eta - \xi) + r|v| \\ &\le (1-r)\eta + 2(\eta - \xi) + |v| \\ &\le 2(1-r)\eta. \end{aligned} \tag{4.11}$$

Thus by (4.6), (4.10) and (4.11)

$$\begin{aligned} K_5(x,y) &\le C\, e^{\xi^2-\eta^2} \int_0^{r_0} r(1-r^2)^{-d/2-1} \left(1 + \frac{(1-r)^2\eta^2}{1-r^2}\right) e^{-\frac{1}{4}\frac{(1-r)^2}{1-r^2}\eta^2}\, dr \\ &\le C\, e^{\xi^2-\eta^2} \int_0^{1-\epsilon_0} (1-r)^{-d/2-1}(1 + (1-r)\eta^2) e^{-\frac{1}{8}(1-r)\eta^2}\, dr. \end{aligned}$$

Performing the change of variables $(1-r)\eta^2 = t$, we get

$$K_5(x,y) \le C\ e^{\xi^2-\eta^2}\eta^d \int_{\epsilon_0\eta^2}^{+\infty} t^{-d/2-1}(1+t)\,e^{-t/8}\,dt \tag{4.12}$$

But

$$\epsilon_0\eta^2 = (2(\eta-\xi)+|v|)\eta \ge \frac{1}{2}(|x-y|+|v|)\eta \ge \frac{1}{8}\left(\frac{1}{\eta}+|v|\right)\eta, \tag{4.13}$$

because $(x,y)\notin N$; and, in particular $\epsilon\eta^2 \ge 1/8$. Since the integral in (4.12) decays exponentially in $\epsilon\eta^2$, we get

$$K_5(x,y) \le Ce^{\xi^2-\eta^2}\eta^d(\epsilon\eta^2)^{-d} \le Ce^{\xi^2-\eta^2}\left(\frac{1}{\eta}+|v|\right)^{-d}$$

by virtue of (4.13). This is the desired estimate for K_5. Next we estimate K_6. We claim that there exist two constants $0 < C_0 < C_1$ such that

$$|x-ry|^2 \ge C_0\,|x-y|^2 \quad \text{and} \quad |rx-y|^2 \le C_1\,|x-y|^2. \tag{4.14}$$

Indeed, since $r > r_1$, one has $(1-r)\eta < \frac{1}{2}(\eta-\xi)+2|v|$. Thus

$$\begin{aligned} |x-ry| &\ge |\xi - r\,\eta| \\ &\ge |\xi-\eta| - (1-r)\eta \\ &\ge (\eta-\xi)/2 - 2|v| \end{aligned}$$

and clearly $|x-ry| \ge |v|$. We conclude that

$$\begin{aligned} |x-ry| &\ge \frac{1}{4}\left(\frac{\eta-\xi}{2} - 2|v|\right) + \frac{3}{4}|v| \\ &\ge \frac{1}{8}(\eta-\xi+|v|) \\ &\ge \frac{1}{8}|x-y|. \end{aligned}$$

This yields the first estimate in (4.14). The second estimate follows from

$$\begin{aligned} |rx-y| \le (1-r)\xi + \eta - \xi &\le (1-r)\eta + \eta - \xi \\ &\le C(\eta-\xi+|v|) \\ &\le C|x-y|. \end{aligned}$$

Therefore, by (4.14), one has

$$\begin{aligned} K_6(x,y) &\le C\ e^{\xi^2-\eta^2}\int_{r_1}^{1}(1-r)^{-d/2-1}\left(1+\frac{|x-y|^2}{1-r}\right)e^{-\frac{C_0}{2}\frac{|x-y|^2}{1-r}}\,dr \\ &\le C\ e^{\xi^2-\eta^2}|x-y|^{-d}\int_0^{+\infty} t^{d/2-1}(1+t)\,e^{-C_0 t}\,dt \\ &\le C\ e^{\xi^2-\eta^2}|x-y|^{-d}. \end{aligned}$$

Since $|x-y| \geq 1/2(|x-y|+|v|) \geq C(\eta^{-1}+|v|)$ for $(x,y) \in N^c$ and $|y|>1$, the last estimate implies that

$$K_6(x,y) \leq C\ e^{\xi^2-\eta^2}\left(\frac{1}{\eta}+|v|\right)^{-d}.$$

Finally, if $r_0 < r_1$, we have to estimate also the integral over the interval $[r_0, r_1]$. Notice that $r_0 < r < r_1$ implies that $\epsilon_1 < 1-r < \epsilon_0$. Thus

$$|v| < \frac{3}{2}(\eta-\xi) \quad \text{and} \quad \frac{1}{2}\frac{\eta-\xi}{\eta} < 1-r < 4\frac{\eta-\xi}{\eta}, \tag{4.15}$$

whence we get the estimates

$$|rx-y| \leq \eta - r\xi + |v| = (1-r)\eta + r(\eta-\xi) + |v| \leq C(\eta-\xi) \tag{4.16}$$

and

$$\frac{|x-ry|^2}{1-r^2} = \frac{(\xi-r\eta)^2}{1-r^2} + \frac{|v|^2}{1-r^2} \geq \frac{1}{8}\frac{\eta(\xi-r\eta)^2}{\eta-\xi} + \frac{1}{8}\frac{\eta|v|^2}{\eta-\xi}. \tag{4.17}$$

Define the new variable s by $\xi - r\eta = \eta s$. Then by (4.15), (4.16) and (4.17)

$$\begin{aligned}\int_{r_0}^{r_1} \psi\, e^{-\phi}\, dr &\leq C\ e^{\xi^2-\eta^2}\left(\frac{\eta-\xi}{\eta}\right)^{-d/2-1}(1+\frac{\eta-\xi}{\eta}\eta^2)\ e^{-\frac{1}{8}\frac{\eta}{\eta-\xi}|v|^2}\int_{\mathbb{R}} e^{-\frac{1}{8}\frac{\eta^3}{\eta-\xi}s^2}\, ds \\ &\leq C_n\ e^{\xi^2-\eta^2}\left(\frac{\eta-\xi}{\eta}\right)^{-d/2+1/2}\left(\frac{1}{\eta-\xi}+\eta\right)\left(1+\frac{\eta}{\eta-\xi}|v|^2\right)^{-n}\end{aligned}$$

for each $n>0$. By choosing $n = d/2$ we get

$$\int_{r_0}^{r_1} \psi\, e^{-\phi}\, dr \leq C\ e^{\xi^2-\eta^2}\left(\frac{1}{\eta-\xi}+\eta\right)\left(\frac{\eta-\xi}{\eta}\right)^{1/2}\left(\frac{\eta-\xi}{\eta}+|v|^2\right)^{-d/2}.$$

To obtain the desired estimate we only need to observe that by the first estimate in (4.15) $\eta-\xi > (\eta-\xi)/2 + |v|/3 > |x-y|/3 > C\eta^{-1}$ for some $C>0$. This completes the proof of case *(c)*.

Case (d). Since

$$\left(1+\frac{|rx-y|^2}{1-r^2}\right)\ e^{-\frac{|rx-y|^2}{1-r^2}} \leq C\ e^{-\frac{1}{2}\frac{|rx-y|^2}{1-r^2}}, \tag{4.18}$$

we have that

$$\begin{aligned}K(x,y) &\leq \int_0^1 C\,(1-r^2)^{-d/2-1}\ e^{-\frac{1}{2}\frac{|rx-y|^2}{1-r^2}}\, dr \\ &= \int_0^{1/2} + \int_E + \int_F \\ &= K_7(x,y) + K_8(x,y) + K_9(x,y),\end{aligned}$$

where $E = \{r \;:\; r \geq 1/2,\ |r\xi - \eta| < (\xi - \eta)/2\}$ and $F = [1/2, 1] \setminus E$. The integral over $[0, 1/2]$ is clearly bounded by a constant. Now we claim that there exists a positive constant C such that for x in D_3

$$\frac{\xi - \eta}{\xi} + |v|^2 \geq C\,(\eta^{-2} + |v|^2). \tag{4.19}$$

If $|v| > C\eta^{-1}$ this is obvious. Otherwise $\xi - \eta + |v| \geq |x - y| > (4\eta)^{-1}$ implies that $\xi - \eta > (8\eta)^{-1}$. Since in this region $\xi \sim \eta$ we have that $(\xi - \eta)/\xi > C(\xi - \eta)/\eta > C/\eta^2$ and (4.19) follows again. To estimate K_8 we observe that if $r \in E$ then

$$\frac{1}{2}\,\frac{\xi - \eta}{\xi} < 1 - r < \frac{3}{2}\,\frac{\xi - \eta}{\xi}.$$

Therefore there exist two positive constants C and C_0 such that $K_8(x, y)$ is bounded by

$$C\left(\frac{\xi - \eta}{\xi}\right)^{-d/2-1} e^{-C_0\frac{\xi}{\xi-\eta}|v|^2} \int_{\mathbb{R}} e^{-C_0\frac{\xi}{\xi-\eta}|r\xi-\eta|^2}\,dr. \tag{4.20}$$

By making the change of variables $\left(\frac{\xi}{\xi-\eta}\right)^{1/2}(r\xi - \eta) = t$, we see that (4.20) equals

$$\begin{aligned}
& C\left(\frac{\xi - \eta}{\xi}\right)^{-d/2-1} e^{-C_0\frac{\xi}{\xi-\eta}|v|^2} \left(\frac{\xi}{\xi - \eta}\right)^{-1/2} \xi^{-1} \\
\leq\ & C_n\left(\frac{\xi - \eta}{\xi}\right)^{-d/2-1/2} (1 + \frac{\xi}{\xi - \eta}|v|^2)^{-n}\ \xi^{-1}
\end{aligned}$$

for every positive n. By choosing $n = (d+1)/2$ and using (4.19) and the fact that $\xi \sim \eta$, we get

$$\begin{aligned}
K_8(x, y) &\leq C\left(\frac{\xi - \eta}{\xi} + |v|^2\right)^{-(d+1)/2} \xi^{-1} \\
&\leq C(\eta^{-2} + |v|^2)^{-d/2}(\eta^{-2} + |v|^2)^{-1/2}\ \eta^{-1} \\
&\leq C(\eta^{-1} + |v|)^{-d}.
\end{aligned}$$

Thus K_8 satisfies the desired estimate. To estimate K_9 we remark that for r in F

$$|rx - y|^2 = (r\xi - \eta)^2 + r^2|v|^2 \geq \frac{1}{4}\left((\xi - \eta)^2 + |v|^2\right).$$

Thus

$$\begin{aligned}
K_9(x, y) &\leq C\int_{1/2}^{1} (1 - r^2)^{-d/2-1} e^{-\frac{1}{8}\frac{(\xi-\eta)^2+|v|^2}{1-r^2}}\,dr \\
&\leq C\left(|\xi - \eta| + |v|\right)^{-d} \\
&\leq C(\eta^{-1} + |v|)^{-d}.
\end{aligned}$$

The last inequality follows from the estimates

$$\eta^{-1}+|v|\leq C|x-y|+|v|\leq C(\xi-\eta+|v|).$$

This completes the proof of case *(d)*.

Case (e). We shall actually prove that the desired estimate holds for $|x-y|>\beta|y|$. By (4.18) we have that

$$\begin{aligned} K(x,y) &\leq C \int_0^1 (1-r^2)^{-d/2-1}\, e^{-\frac{1}{2}\frac{|rx-y|^2}{1-r^2}}\, dr \\ &= \int_0^{1-\beta/2} + \int_{1-\beta/2}^1 \\ &= K_{10}(x,y)+K_{11}(x,y). \end{aligned}$$

The kernel K_{10} is clearly bounded by a constant. To estimate K_{11} we remark that, since $|x-y|>\beta\eta$, if $r\geq 1-\beta/2$ then

$$|rx-y|\geq r|x-y|-(1-r)|y|\geq C_0\eta,$$

where $C_0=\frac{1}{2}\beta(1-\beta)>0$. Thus

$$\begin{aligned} K_{11}(x,y) &\leq C\int_{1-\beta/2}^1 (1-r^2)^{-d/2-1} e^{-\frac{1}{2}\frac{C_0^2\eta^2}{1-r^2}}\, dr \\ &\leq C\eta^{-d} \\ &\leq C. \end{aligned}$$

This completes the proof of case *(e)*.

Case (f). Assume first that $|y|\leq 1$. Since now $|x-y|>1/4$ there exists an $\alpha<1$ such that $|rx-y|\geq C>0$ for $\alpha\leq r\leq 1$. From (4.18) we get

$$K(x,y)\leq C\int_0^\alpha dr + C\int_\alpha^1 (1-r^2)^{-d/2-1} e^{-\frac{C_0}{1-r^2}}\, dr.$$

The desired estimate follows, since the last two integrals are clearly bounded. The remaining case $|y|>1$, $|x|\leq 1$, is contained in *(a)* and *(b)*.

It remains to prove that the kernel K^* satisfies the same estimates. If we replace the point x by a point x' which is in the same cube Q_i and in the same region D_i as x, the order of magnitude of the right hand sides in the estimates *(a)*-*(e)* does not change. Thus we must examine only the cases when x and x' are in the same cube Q_i but in different regions D_i. Notice that in the proof of case *(e)* we have actually proved that the estimate holds for all x outside the ball $\{x:\ |x-y|>\beta|y|\}$. This remark takes care of the case when x and x' lie across the boundary between D_1 and D_4. When x and x' lie on different sides of the sphere $\{z : |z-y| = \beta\eta\}$, we observe that the value of β can be modified slightly in the above arguments. Thus we can make x and x' belong to the same region D_i, which takes care of this case. It remains to consider x and x' on different sides of the common boundary of D_0 and D_1 or that of D_2 and D_3. In the first case (D_0 and D_1), we have $|\xi| < 1/\eta$. But this inequality implies that the right-hand sides of the estimates in (a) and (b) are of the same order of magnitude, and so this case is settled. The other case (D_2 and D_3) is similar and goes via the inequality $|\xi - \eta| < 1/\eta$. This concludes the proof of estimates *(a)-(e)* for the kernel K^*. The proof of the estimate *(f)* is easy. ∎

We shall now use Lemma 4.3 to prove the main result of this section, namely

Theorem 4.21 *For every $j,k \in \{1,2,\cdots,d\}$, $R_{jk,glob}$ is of weak type $(1,1)$ with respect to the Gaussian measure γ.*

Proof. Let $0 < f \in L^1(\gamma)$.

For $|x| < 1$, case *(f)* of the lemma implies that

$$\int K(x,y)\,\chi_{N^c}(x,y)\,f(y)dy \leq C \parallel f \parallel_{L^1(\gamma)},$$

which gives a strong type (1,1) estimate for this part of the operator. The same argument applies to arbitrary x but with the integral taken only over $|y| < 1$.

We can thus restrict ourselves to that part of the kernel with $|x|$ and $|y| > 1$, and start with $K\chi_{\{|y|>1,\ x\in D_0(y)\}}$. Observe that Lemma 4.3 implies that

$$\int_{D_0(y)} K(x,y)d\gamma(x) \leq Ce^{-|y|^2}. \tag{4.22}$$

Write $D^0(x) = \{y : |y| > 1,\ x \in D_0(y)\}$. We conclude by changing the order of integration that

$$\int d\gamma(x) \int_{D^0(x)} K(x,y)f(y)dy \leq C \int e^{-|y|^2} f(y)dy,$$

so that this part of the operator is of strong type (1,1).

We split D_1 into two disjoint parts D_1' and D_1'' with

$$D_1' = \{x \in D_1 : |v| > \xi/2\}.$$

For D_1' we have

$$\begin{aligned}\int_{D_1'(y)} K(x,y)d\gamma(x) &\leq C \int\int_{D_1'(y)} (1+\xi)e^{\xi^2-\eta^2}e^{-\xi^2-|v|^2}d\xi dv \\ &\leq C\int_0^{\eta}(1+\xi)d\xi \int_{|v|>\xi/2} e^{-|v|^2}dv\ e^{-\eta^2} \leq Ce^{-\eta^2}.\end{aligned}$$

The estimate thus obtained is similar to (4.22), and we can argue as above to take care of D_1'.

When $x \in D_4$, we observe that $|y|/|x| < \lambda$ for some $\lambda < 1$, so that $|x|^2 > |y|^2 + \epsilon|x|^2$ where $\epsilon > 0$ depends only on β. Together with Lemma 4.3, this implies that

$$\int_{D_4(y)} K(x,y)d\gamma(x) \le Ce^{-|y|^2} \int e^{-\epsilon|x|^2}dx \le Ce^{-|y|^2}.$$

Thus we also control the D_4 part of the operator.

For the remaining regions D_1'', D_2, and D_3, we shall deduce the weak-type estimate. Fix y with $|y| > 1$, and let χ_y be the characteristic function of the union of those cubes Q_i which intersect $D_1''(y) \cup D_2(y) \cup D_3(y)$. Define $K_*(x,y) = K^*(x,y)\chi_y(x)$. Notice that $K_*(x,y)$ stays constant as x moves within a Q_i. Our task is to estimate the level set

$$L = \{x : \int K_*(x,y)f(y)dy > \alpha\}$$

for $\alpha > 0$. We first observe that it would be enough to prove the inequality

$$\int_L K_*(x,y)d\gamma(x) \le Ce^{-|y|^2}, \quad y \in \mathbb{R}^d, \tag{4.23}$$

Indeed, (4.23) implies

$$\begin{aligned} \gamma(L) &\le \int_L d\gamma(x)\frac{1}{\alpha}\int K_*(x,y)f(y)dy \\ &= \frac{1}{\alpha}\int f(y)dy\int_L K_*(x,y)d\gamma(x) \le \frac{C}{\alpha}\int f(y)d\gamma(y), \end{aligned} \tag{4.24}$$

which is the desired estimate.

However, (4.23) is false in general. We shall therefore construct a smaller set $E \subset L$ for which (4.23) holds, i.e., such that

$$\int_E K_*(x,y)d\gamma(x) \le Ce^{-|y|^2}, \quad y \in \mathbb{R}^d. \tag{4.25}$$

Still E must be large enough so that

$$\gamma(E) \ge c\gamma(L) \tag{4.26}$$

for some $c > 0$. Given such a set E, we can carry through the argument (4.24) with E instead of L and finish the proof.

We remark that our construction of E follows a method from Sjögren [12], also used in [13].

Notice that L is a union of cubes Q_i. We shall construct E as the union of some of these cubes. Roughly speaking, the reason why (4.23) fails is that L contains too many cubes near each other. If we decide to include a cube in E, we should therefore not include its close neighbours. Still, (4.26) must be respected. To achieve this, we associate with each cube Q_i a forbidden region F_i, which is essentially a cone with vertex in Q_i and directed away from 0. More precisely, let C_i be the cone of vectors forming an angle of at most $\pi/4$ with the centre x_i of Q_i. We define F_i as the union of those cubes Q_j intersecting $Q_i + C_i$. The inequality

$$\gamma(F_i) \leq C\gamma(Q_i) \tag{4.27}$$

is then rather easy to verify. One integrates first in each affine hyperplane orthogonal to x_i, see [13, proof of formula (3.1)].

The procedure to construct E goes as follows. We consider the $Q_i, i = 1, 2, ...$, in this order. For each Q_i, we decide in the following way whether to select it, i.e., include it in E. The cube Q_i is selected if and only if it is contained in L and is not contained in the forbidden region F_j of any Q_j already selected. Then E is defined as the union of the selected Q_i. Since L is covered by the selected cubes and their forbidden regions, (4.26) follows from (4.27).

Now it only remains to verify that the selected cubes are so sparse that (4.25) holds. We fix y with $\eta = |y| > 1$. An elementary geometric argument shows that $K_*(x, y)$ can be nonzero only when x is in the cone C_y of vectors forming an angle of at most $\pi/4$ with y. Consider a line ℓ parallel to y. We claim that the intersection of ℓ and $C_y \cap E$ is contained in at most one Q_i. Indeed, any Q_i intersecting this intersection is contained in E, and its forbidden region contains the half-line of ℓ in the y direction starting at Q_i.

We parametrise the lines ℓ by

$$\ell_v = \{x = \xi y/\eta + v : \xi \in \mathbb{R}\},$$

where $v \perp y$. The observation we just made means that $\ell_v \cap C_y \cap E$ is contained in a segment corresponding to an interval in the parameter ξ given by $\xi_v < \xi < \xi_v + \xi_v^{-1}$, for some $\xi_v > 1/2$. Now we apply Fubini's theorem to the integral in (4.25), getting

$$\int_E K_*(x, y) d\gamma(x) \leq \int e^{-|v|^2} dv \int_{\xi_v}^{\xi_v + \xi_v{}^{-1}} e^{-\xi^2} K_*(\xi y/|y| + v, y) d\xi.$$

The integration in v is with respect to $(d-1)$-dimensional Lebesgue measure in the hyperplane orthogonal to y. To estimate the right-hand side here, we insert the inequalities from Lemma 4.3 in the three regions we are considering.

For D_1'', we get at most

$$C \int e^{-|v|^2} dv \int_{\xi_v}^{\xi_v + \xi_v{}^{-1}} e^{-\xi^2} (1 + |\xi|) e^{\xi^2 - \eta^2} d\xi \leq C \int e^{-|v|^2} \frac{1 + \xi_v}{\xi_v} dv \; e^{-\eta^2} \leq Ce^{-\eta^2}.$$

Using the estimates valid in D_2, we get two terms. The first term is

$$C\int e^{-|v|^2}dv\int_{\xi_v}^{\xi_v+\xi_v{}^{-1}} e^{-\xi^2}(\eta^{-1}+|v|)^{-d}e^{\xi^2-\eta^2}d\xi \leq \frac{C}{\xi_v}\int(\eta^{-1}+|v|)^{-d}dv\; e^{-\eta^2} \leq Ce^{-\eta^2},$$

where we used the fact that $\eta \leq C\xi_v$ here. The second term is

$$C\int e^{-|v|^2}dv\int_{\xi_v}^{\xi_v+\xi_v{}^{-1}} e^{-\xi^2}\sqrt{\eta(\eta-\xi)}\left(\sqrt{\frac{\eta-\xi}{\eta}}+|v|\right)^{-d} e^{\xi^2-\eta^2}d\xi.$$

If $\eta-\xi_v \leq 2/\eta$, this is like the first term. If not, we can estimate this expression by

$$C\int e^{-|v|^2}\frac{1}{\xi_v}\sqrt{\eta(\eta-\xi_v)}\left(\sqrt{\frac{\eta-\xi_v}{\eta}}+|v|\right)^{-d} dv\, e^{-\eta^2} \leq Ce^{-\eta^2},$$

almost as the first term.

Using finally the estimates in D_3, we get at most

$$\begin{aligned} C\int e^{-|v|^2}dv\int_{\xi_v}^{\xi_v+\xi_v{}^{-1}} e^{-\xi^2}(\eta^{-1}+|v|)^{-d}d\xi &\leq C\int\frac{1}{\xi_v}e^{-\xi_v^2}(\eta^{-1}+|v|)^{-d}dv \\ &\leq Ce^{-\eta^2}\frac{1}{\eta}\int(\eta^{-1}+|v|)^{-d}dv \leq Ce^{-\eta^2}. \end{aligned}$$

These estimates together imply (4.25) and thus complete the proof of the theorem. ■

By putting together Theorems 3.21 and 4.21, we have finally proved Theorem 1.1.

5 A counterexample for operators of order at least three

Theorem 5.1 *Let $a = |\alpha| \geq 3$ and $b > 0$. Then the operator $D^\alpha L^{-b}\Pi_0$ is not of weak type $(1,1)$ with respect to the Gaussian measure γ.*

Letting $b = |\alpha|/2$ here, we clearly obtain Riesz operators of order at least 3.

Proof. We estimate the kernel $D_x^\alpha K_b(x,y)$, as given in (3.2), for $\eta = |y|$ large and $y_i \geq c\eta$, $i = 1, \cdots, d$ and $c > 0$. Write $x = \xi y/\eta + v$ as before, with $\xi \in \mathbb{R}$ and $v \perp y$. We let x be in the tube J defined by $\eta/2 < \xi < 3\eta/4$ and $|v| < 1$. Then for each $i = 1, \cdots, d$,

$$-\frac{rx_i - y_i}{\sqrt{1-r^2}} \geq \frac{c\eta}{\sqrt{1-r^2}} \geq c\eta.$$

It follows that

$$(-1)^a H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right) > c\eta^a.$$

In particular, the integrand in (3.2) is positive for $0 < r < 1$. We observe that

$$e^{-\frac{|rx-y|^2}{1-r^2}} = e^{\xi^2-\eta^2} e^{-\frac{|\xi - r\eta|^2 + r^2|v|^2}{1-r^2}}$$

so that for $1/4 < r < 3/4$ and $x \in J$

$$e^{-\frac{|rx-y|^2}{1-r^2}} \geq c e^{\xi^2-\eta^2} e^{-C|\xi-r\eta|^2}.$$

These estimates imply that

$$D^\alpha K_b(x,y) \geq c\eta^a e^{\xi^2-\eta^2} \int_{1/4}^{3/4} e^{-C(\xi-r\eta)^2} dr \geq c\eta^{a-1} e^{\xi^2-\eta^2}.$$

Now let $0 \leq f \in L^1(\gamma)$ be a close approximation of a point mass at y, with norm 1 in $L^1(\gamma)$. Then

$$D^\alpha L^{-b}\Pi_0 f(x) = \int_{\mathbb{R}^d} D^\alpha K_b(x,y) f(y) dy$$

will be close to $e^{\eta^2} D^\alpha K_b(x,y)$ when $x \in J$. We conclude that

$$D^\alpha L^{-b}\Pi_0 f(x) \geq c\eta^{a-1} e^{\xi^2} \geq c\eta^{a-1} e^{(\eta/2)^2}$$

for $x \in J$.

Since $\gamma(J) \geq c\eta^{-1} e^{-(\eta/2)^2}$, as is easily verified, the $L^{1,\infty}(\gamma)$ quasi-norm of $D^\alpha L^{-b}\Pi_0 f$ is at least $c\eta^{a-2} \to \infty$ as $\eta \to \infty$ when $a \geq 3$. This violates the weak type $(1,1)$ property. ∎

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