ON EXTREMAL THEORY FOR SELF-SIMILAR PROCESSES

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We derive upper and lower asymptotic bounds for the distribution of the supremum for a self-similar stochastic process. As an intermediate step, most proofs relate suprema to sojourns before proceeding to an appropriate discrete approximation.

Our results rely on one or more of three assumptions, which in turn essentially require weak convergence, existence of a first moment, and tightness, respectively. When all three assumptions hold the upper and lower bounds coincide (Corollary 1).

For $\mathbf{P}$-smooth processes weak convergence can be replaced with the use of a certain upcrossing intensity that works even for (a.s.) discontinuous processes (Theorem 7).

Results on extremes for a self-similar process do not on their own imply results for Lamperti's associated stationary process or vice versa. But we show that if the associated process satisfies analogues of our three assumptions, then the assumptions hold for the self-similar process itself. Through this connection new results on extremes for self-similar processes can be derived by invoking the stationary literature.

Examples of application include Gaussian processes in $\mathbb{R}^n$, totally skewed α-stable processes, Kesten-Spitzer processes, and Rosenblatt processes.

Introduction. Methods to study the asymptotic behaviour of $\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}$ for large u, for a stationary stochastic process $\{\xi(t)\}_{t\geq 0}$, have been developed by e.g., Berman (1982) and Albin (1990). These methods require that a few conditions are verified. Although it can be hard to verify the conditions, this often constitute the most convenient (if not the only) way to study extremes.

There do not exist systematic approaches to non-stationary extremes comparable with stationary theories in terms of efficiency: Many arguments that work on a general level for stationary processes do not extend to non-stationary settings. At best one can find methods specific for the particular non-stationary process under consideration making it possible to carry out the corresponding computations also in the non-stationary case: Usually this requires much effort. At worst the non-stationary process cannot be studied along schemes of stationary origin, and one has to start from 'scratch' (often meaning that sharp results cannot be found).

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In Sections 2-8 we shall see that for self-similar non-stationary processes, a theory of extremes can be developed that performs (at least) as well as stationary counterparts. As in the stationary case, given a specific self-similar process one must check a couple of conditions before infering results on extremes: The contribution of the theory is that it usually is much easier to verify these conditions than to start from zero, and often alternative approaches are not available.

In Section 2 we find the asymptotic behaviour of $\int_0^1 h_u(s)\,\mathbf{P}\{\xi(s)>u\}\,ds$ for large u, when $\xi(t)$ is a self-similar process and $\{h_u(\cdot)\}_{u\in\mathbb{R}}$ a uniformly bounded family of functions. This technical result is needed in most proofs of latter sections.

In Section 3 we determine the asymptotic distributional behaviour (for large u) of the sojourn time spent above the level u

$$L(u)\equiv L(1;u) \qquad \text{where} \qquad L(t;u)\equiv\int_0^t I_{(u,\hat{u})}(\xi(s))\;ds \quad \text{for} \quad t>0.$$

To that end we require weak convergence of the conditional finite dimensional distributions $\big\{\big(w^{-1}[\xi(1-qt)-u]\,\big|\,\xi(1)>u\big)\big\}_{t>0}$ to some limit $\{\zeta(t)\}_{t>0}$ as u becomes large, when $w=w(u)$ and $q=q(u)$ are suitably choosen. Further we need a requirement which interprets to $\mathbf{E}\{\int_0^\infty I_{(0,\infty)}(\zeta(s))\,ds\}<\infty$.

In Section 4-6 we study the asymptotic behaviour of $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$ through establishing relationships between the events $\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$ and $\{L(u)>0\}$. To get a sharp relationship we (not suprisingly) have to require that the convergence $\big(w^{-1}[\xi(1-qt)-u]\,\big|\,\xi(1)>u\big)\rightarrow_{\mathcal{L}}\zeta(t)$ is 'tight'.

In Section 7 we investigate how results can be simplified and/or sharpened when the limit $\zeta(t)$ takes the simple form $\zeta(t)=\xi'\cdot t$ for some random variable ξ'.

In Section 8 we develop several sufficient criteria for verifying tightness.

In Section 9 we derive a connection between our findings in Sections 2-8 and extremal theory for the associated stationary process obtained via the transformation of Lamperti [see e.g., Proposition 7.1.4 and the notes to Section 7.1 in Samorodnitsky & Taqqu (1994) ([**S&T**])]: We show that if the associated process fits into the framework of the stationary theory of Albin (1990, Section 2), then the original self-similar process satisfies the hypothesis of our result in Sections 2-8.

Connections to stationary theory are automatic for some global problems like e.g., the law of iterated logarithm. But this is not the case for local extremes, and the relation we establish is non-trivial and new. It is first now that results on extremes for e.g., fractional Brownian motion (fBm.) are implied by the work of Pickands (1969) on stationary Gaussian processes: The reader knowledgeable in the stationary literature will be able to derive many results on self-similar extremes with little effort by invoking the 'stationary connection' established in Section 9.

Our theory is general in the sense that it does not impose additional structural assumptions on the self-similar process (like e.g., Markovianess). Instead it requires that one can carry out a few basic estimates related to the tail behaviour of the one- and two-dimensional distributions of the process. This can be an important or even crucial advantage. But of course there exist processes which are better studied via methods specific for the process under consideration than via our approach.

Our results demonstrate what properties of a self-similar process that affect local extremes and the probabilistic principles involved when proving this. But the main motivation for our work were a wish to provide a systematic method useful to study extremes for particular examples of self-similar processes for which other methods are not available. It is not uninteresting to see how our approach applies to reprove results for processes whose 'extreme behaviour' is already known (and it works very swiftly in most such cases). But the true value of a new method must be judged by it's ability to generate new results for important examples of self-similar processes:

In Section 10 we give an application to $\mathbb{R}^n$-valued self-similar Gaussian processes whose component processes are independent with covariance functions possessing a polynomial modulus of continuity. This class of processes include virtually all processes arising in applications as well as most encountered in theory.

In Section 11 we give an application to the $\mathbb{L}^2$-norm of Brownian motion. Representing 'the action of a Brownian path', this process is theoretically important [see e.g., Yor (1992) for more information]. It is also of applied interest in e.g., physics [e.g., Duplantier (1989) and Chan et al. (1994)].

In Section 12 we study log-fractional α-stable motion that is totally skewed to the left. This process was discovered by Kasahara et al. (1988) as the first example of a self-similar α-stable process with index $\kappa = 1/\alpha$ that is not α-stable motion.

In Section 13 we study linear fractional α-stable motions that are totally skewed to the left. These processes were introduced by Maejima (1983) and Taqqu & Wolpert (1983) as natural stable generalization of fBm., and they constitute the most important class of stable processes.

The most important class of stationary stable processes are moving averages of α-stable motion. In Section 14 we give an application to the family of self-similar processes whose associated stationary processes via the Lamperti transformation are α-stable moving averages that are totally skewed to the left.

Of course, our theory apply also to α-stable processes that are not totally skewed. But we do not dwell on this since the extreme behaviour of such processes is already well-understood through the works of de Acosta (1977) and Samorodnitsky (1988).

In Section 15 we study Kesten-Spitzer processes. These processes appear as functional limit of random walks in random sceneries when the walk and the scenery both belong to domains of attraction of stable laws [Kesten & Spitzer (1979)].

In Sections 16 we give an application to Rosenblatt processes. These processes are important because their role in non-central limit theorems parallells that of fBm. in central limits: See Taqqu (1975) and Dobrushin & Major (1979) [or surveys like Taqqu & Czado (1985) and Taqqu (1986)] for precise statements.

All results derived in Sections 10-16 are new, and we do not know any other way to prove them than via our approach.

1. Preliminaries. In this paper all stochastic variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Further $\{\xi(t)\}_{t\geq 0}$ denotes an $\mathbb{R}$-valued stochastic process which is self-similar with index $\kappa > 0$. Thus the finite dimensional distributions of $\xi(\lambda t)$ coincide with those of $\lambda^{\kappa}\xi(t)$ for $\lambda > 0$.

For each stochastic process under consideration we assume that a separable and measurable version have been choosen. Such a version exists under weak conditions like e.g., $\mathbf{P}$-continuity almost everywhere [Doob (1953, Theorem II.2.6); see Veervaat (1985, Theorem 1.1) for a converse], and it is to that version our results apply. In particular, if $\xi(t)$ has stationary increments, then $\xi(t)$ is $\mathbf{P}$-continuous [e.g., Vervaat (1985, Eq. 1.1)] and thus has a separable and measurable version.

Write G for the distribution function of $\xi(1)$ and $\hat{u} \equiv \sup\{x \in \mathbb{R} : G(x) < 1\}$. We shall assume that G belongs to a domain of attraction of extremes with $\hat{u} > 0$. Thus there exist a constant $\hat{x} \in (0, \infty]$ and functions $w : (-\infty, \hat{u}) \to (0, \infty)$ and $F : (-\hat{x}, \infty) \to (-\infty, 1)$ such that F is a distribution function on $[0, \infty)$ and

$$(1.1) \qquad \lim_{u\uparrow\hat{u}} \big(1 - G(u + xw(u))\big) \big/ (1 - G(u)) = 1 - F(x) \quad \text{for } x \in (-\hat{x}, \infty).$$

Here G can be Type II-attracted $[G \in \mathcal{D}(\mathrm{II})]$, and then we can take $\hat{x} = -1$, $F(x) = 1-(1+x)^{-\gamma}$ for some $\gamma > 0$, and $w(u) = u$ so that $W \equiv \lim_{u\uparrow\hat{u}} w(u)/u = 1$.

Otherwise G is Type I- or Type III-attracted $[G \in \mathcal{D}(\mathrm{I})$ or $G \in \mathcal{D}(\mathrm{III})]$ and then we can assume that $\hat{x} = \infty$ and $F(x) = 1 - \mathrm{e}^{-x}$ for some continuous w satisfying $W \equiv \lim_{u \uparrow \hat{u}} w(u)/u = 0$. Here $\hat{u} = \infty$ for $G \in \mathcal{D}(\mathrm{I})$ while $\hat{u} \in (0, \infty)$ when $G \in \mathcal{D}(\mathrm{III})$. Further w is self-neglecting, i.e.,

(1.2) $\qquad w(u + xw(u))/w(u) \to 1$ locally uniformly for $x \in \mathbb{R}$ as $u \uparrow \hat{u}$.

For future use we define

$$\mathfrak{P}_0 \equiv \int_0^\infty \frac{(1 - F(\hat{s}))\, d\hat{s}}{(1 + W\hat{s})^{1+1/\kappa}} = \begin{cases} 1 & \text{for } G \in \mathcal{D}(\mathrm{I}) \cup \mathcal{D}(\mathrm{III}) \\ (\gamma + \kappa^{-1})^{-1} & \text{for } G \in \mathcal{D}(\mathrm{II}) \end{cases}.$$

The fact that $G \in \mathcal{D}$ is needed in the crucial Proposition 1 below. Most marginal distributions occuring in the study of stochastic processes belong to $\mathcal{D}$, and we are not aware of a specific self-similar process for which $G \notin \mathcal{D}$. See e.g., Resnick (1987, Chapter 1) to learn more about the domains of attraction $\mathcal{D} \equiv \mathcal{D}(\mathrm{I}) \cup \mathcal{D}(\mathrm{II}) \cup \mathcal{D}(\mathrm{III})$.

In the sequel it is assumed that an interval $J \subseteq (-1, \infty)$ with $0 \in J$, and a function $q : (-\infty, \hat{u}) \to (0, \infty)$ such that $Q \equiv \lim_{u \uparrow \hat{u}} q(u)^{-1}$ exists and $\hat{a} \equiv 1 \big/ \big(2 \sup_{u < \hat{u}} q(u)\big) > 0$ have been specified. The function q is featured in all assumptions and theorems, and the first step when applying our results is to choose a suitable q. Inferences then depend on which assumptions hold for this q.

Most results require that the variation of $p(u) \equiv u^{-1/\kappa} q(u)$ is restricted by

(1.3) p is almost decreasing, i.e., $\mathfrak{P}_1 \equiv \overline{\lim}_{v \uparrow \hat{u}} \sup_{u \in [v, \hat{u})} p(u)/p(v) < \infty$, and

(1.4) the limit $\hat{p}(x) \equiv \lim_{u \uparrow \hat{u}} p(u + xw(u))/p(u)$ exists and is continuous for $x > 0$.

In applications q tends to be non-increasing so that (1.3) holds with $\mathfrak{P}_1 = 1$. When $G \in \mathcal{D}(\mathrm{I}) \cup \mathcal{D}(\mathrm{II})$ (1.4) holds if e.g., q is regularly varying at ∞.

Upper bounds on extremes rely on the additional requirement that

(1.5) there is a $\rho \in \mathbb{R}$ such that $\int_0^\infty (1 + Ws)^{\rho - 1 - 1/\kappa} (1 - F(s))\, ds < \infty$ and

(1.6) $u^\rho p(u)$ is almost increasing, i.e., $\mathfrak{P}_2 \equiv \underline{\lim}_{v \uparrow \hat{u}} \inf_{u \in [v, \hat{u})} (u/v)^\rho p(u)/p(v) > 0$.

When $G \in \mathcal{D}(\mathrm{I}) \cup \mathcal{D}(\mathrm{III})$ (1.5) is void and (1.6) means that q have bounded decrease. For $G \in \mathcal{D}(\mathrm{I})$ (1.3)-(1.6) thus hold if e.g., q is non-increasing and regularly varying (at ∞). For $G \in \mathcal{D}(\mathrm{II})$ (1.3)-(1.6) hold if e.g., q is non-increasing and regularly varying with index greater than $-\gamma$.

For $G \in \mathcal{D}(\mathrm{I}) \cup \mathcal{D}(\mathrm{III})$ the fact that $w(u)/u \to 0$ makes it natural to require

(1.7) $u^{-1}w(u)$ is almost decreasing, i.e., $\mathfrak{P}_3 \equiv \overline{\lim}_{v\uparrow\hat{u}} \sup_{u\in[v,\hat{u})} (v/u)w(u)/w(v) < \infty.$

Of course, if $G \in \mathcal{D}(\mathrm{II})$ then (1.7) holds trivially.

The behaviour of extremes will depend on the finiteness of the limits

$$\mathfrak{P}_4 \equiv \underline{\lim}_{u\uparrow\hat{u}}\, u\, q(u)/w(u) \qquad \text{and} \qquad \mathfrak{P}_5 \equiv \overline{\lim}_{u\uparrow\hat{u}}\, u\, q(u)/w(u).$$

Given functions h_1 and h_2, we write $h_1(u) \sim h_2(u)$ if $\lim_{u\uparrow\hat{u}} h_1(u)/h_2(u) = 1$ and $h_1(u) \simeq h_2(u)$ if $\lim_{u\uparrow\hat{u}}(h_1(u) - h_2(u)) = 0$. Further $h_1(u) \preceq h_2(u)$ means that $\overline{\lim}_{u\uparrow\hat{u}}(h_1(u) - h_2(u)) \le 0$, and $h_1(u) \succeq h_2(u)$ that $\underline{\lim}_{u\uparrow\hat{u}}(h_1(u) - h_2(u)) \ge 0$.

2. The mean sojourn time. In order to study the asymptotic behaviour of $L(u)$ in Section 3, we must first understand the behaviour of $\mathbf{P}\{\xi(s) > u\}$ for $s \in (0,1]$: Proposition 1 gives a quantitative statement of the obvious fact that $\mathbf{P}\{\xi(s) > u\}$ increases with s. The idea of the proof is to use (1.1) to obtain (formally)

$$\mathbf{P}\{\xi(s) > u\} = \mathbf{P}\{\xi(1) > s^{-\kappa}u\} \sim \big[1 - F\big((s^{-\kappa} - 1)u/w\big)\big]\, \mathbf{P}\{\xi(1) > u\}.$$

Proposition 1. *Assume that* $G \in \mathcal{D}$ *[so that* (1.1) *holds]. Then we have*

$$\mathbf{E}\{L(u)\} \sim \mathfrak{P}_0\, w(u)\, \mathbf{P}\{\xi(1) > u\} / (\kappa\, u) \qquad \text{as } u \uparrow \hat{u}. \tag{2.1}$$

Writing $s_u \equiv (1 + sw(u)/u)^{-1/\kappa}$ *we further have*

$$\int_0^1 h_u(s)\, \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{E}\{L(u)\}}\, ds \simeq \frac{1}{\mathfrak{P}_0} \int_0^\infty h_u(s_u)\, \frac{(1 - F(s))\, ds}{(1 + Ws)^{1+1/\kappa}} \tag{2.2}$$

for each family $\{h_u\}_{u<\hat{u}}$ *of functions satisfying* $\overline{\lim}_{u\uparrow\hat{u}} \sup_{s\in(0,1)} |h_u(s)| < \infty$.

Lemma 1. *Assume that* $G \in \mathcal{D}$. *For each* $y \in [0,\infty)$ *and* $z \in \mathbb{R}$ *we then have*

$$\lim_{u\uparrow\hat{u}} \frac{\kappa\, u}{w(u)} \int_0^{[1+yw(u)/u]^{-1/\kappa}} s^{-\kappa z - 1}\, \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{P}\{\xi(1) > u\}}\, ds = \int_y^\infty \frac{(1 - F(\hat{s}))\, d\hat{s}}{(1 + W\hat{s})^{1-z}}. \tag{2.3}$$

Proof of Lemma 1. When the right hand side of (2.3) is infinite, the fact that

$$\frac{\kappa\, u}{w} \int_0^{(1+yw/u)^{-1/\kappa}} s^{-\kappa z - 1}\, \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{P}\{\xi(1) > u\}}\, ds = \int_y^\infty \frac{\mathbf{P}\{\xi(1) > u + \hat{s}w\}}{(1 + \hat{s}w/u)^{1-z}\, \mathbf{P}\{\xi(1) > u\}}\, d\hat{s} \tag{2.4}$$

combines with (1.1) and Fatou's Lemma to prove (2.3). When the right hand side of (2.3) is finite, (2.4) readily combines with (1.1) and the fact that $w(u)=O(u)$ to show that it is sufficient to prove

$$\text{(2.5)} \qquad \lim_{y\to\infty}\overline{\lim_{u\uparrow\hat{u}}}\int_y^\infty \hat{s}^{\hat{z}-1}\,\frac{\mathbf{P}\{\xi(1)>u+\hat{s}w\}}{\mathbf{P}\{\xi(1)>u\}}\,d\hat{s}=0 \quad \text{where} \quad \hat{z}\equiv z\vee 1.$$

Now let $\tilde{G}$ be the distribution function of $\xi(1)^+$, choose $\{u_n\}_{n=1}^\infty\subseteq\mathbb{R}$ such that $u_n\uparrow\hat{u}$ as $n\to\infty$, and write N_n for the integer part of $(1-\tilde{G}(u_n))^{-1}$. Defining $Z_n\equiv\max_{1\le i\le N_n}X_i$ where $X_1,X_2,\dots$ are independent random variables with common distribution $\tilde{G}$, we then have

$$\ln\mathbf{P}\big\{(Z_n-u_n)/w(u_n)\le x\big\}=N_n\ln\tilde{G}\big(u_n+xw(u_n)\big)\sim-N_n\big[1-\tilde{G}\big(u_n+xw(u_n)\big)\big]$$

as $n\to\infty$. Since $N_n\sim(1-\tilde{G}(u_n))^{-1}$, (1.1) thus implies that

$$\text{(2.6)} \quad \lim_{n\to\infty}\mathbf{P}\big\{(Z_n-u_n)/w(u_n)\le x\big\}=\lim_{n\to\infty}\tilde{G}^{N_n}\big(u_n+xw(u_n)\big)=\exp\big\{-[1-F(x)]\big\}.$$

Assume that there exist a non-degenerate random variable Z and sequences $\{a_n\}_{n=1}^\infty,\{b_n\}_{n=1}^\infty\subseteq\mathbb{R}$ such that $(\max\{X_1,\dots,X_n\}-b_n)/a_n\to_{\mathcal{L}}Z$ as $n\to\infty$. Then Theorem 2.1 of Pickands (1968) states that

$$\mathbf{E}\Big\{\big[\big((\max_{1\le i\le n}X_i-b_n)/a_n\big)^+\big]^m\Big\}\to\mathbf{E}\big\{(Z^+)^m\big\} \quad \text{as } n\to\infty$$

for each $m\in\mathbb{R}^+$ satisfying $\mathbf{E}\big\{[(X_1)^-]^m\big\}<\infty$ and $\mathbf{E}\{(Z^+)^m\}<\infty$.

Applying Pickands' result to the convergence established in (2.6) we deduce that

$$\text{(2.7)} \qquad \mathbf{E}\bigg\{\bigg[\bigg(\frac{Z_n-u_n}{w(u_n)}\bigg)^+\bigg]^{\hat{z}}\bigg\}\to\int_0^\infty\frac{x^{\hat{z}}\,F'(x)\,dx}{\exp\{1-F(x)\}}\le\int_0^\infty x^{\hat{z}}\,F'(x)\,dx<\infty:$$

Here finiteness follows from finiteness of the right hand side of (2.3) when $\hat{z}>1$ and $G\in\mathcal{D}(\mathrm{II})$. Otherwise it is a consequence of the possible forms of F in (1.1).

It is an easy exercise in integration theory to see that (2.6) and (2.7) imply

$$\text{(2.8)} \qquad \lim_{n\to\infty}\mathbf{E}\bigg\{\bigg(\frac{Z_n-u_n}{w(u_n)}\bigg)^{\hat{z}}I_{\{(Z_n-u_n)/w(u_n)>y\}}\bigg\}=\int_y^\infty\frac{x^{\hat{z}}\,F'(x)\,dx}{\exp\{1-F(x)\}}<\infty.$$

Now the fact that $1-(1-\varepsilon)^N\ge N(1-\varepsilon)^{N-1}\varepsilon$ for $N\ge1$ and $\varepsilon\in[0,1]$ yields

$$\text{(2.9)} \qquad \mathbf{E}\bigg\{\bigg(\frac{Z_n-u_n}{w(u_n)}\bigg)^{\hat{z}}I_{\{(Z_n-u_n)/w(u_n)>y\}}\bigg\}$$

$$= \hat{z} \int_y^{\infty} x^{\hat{z}-1} \left[1-\tilde{G}^{N_n}\big(u_n+xw(u_n)\big)\right] dx$$

$$\geq \hat{z} \int_y^{\infty} x^{\hat{z}-1} N_n\, \tilde{G}^{N_n-1}(u_n) \left[1-\tilde{G}\big(u_n+xw(u_n)\big)\right] dx$$

$$\sim \frac{\hat{z}}{\mathrm{e}} \int_y^{\infty} x^{\hat{z}-1} \frac{\mathbf{P}\{\xi(1)>u_n+xw(u_n)\}}{\mathbf{P}\{\xi(1)>u_n\}}\, dx.$$

But combining (2.8)-(2.9) it follows that (2.5) holds. □

Proof of Proposition 1. Clearly (2.3) implies (2.1), while (2.1) in turn implies that

$$\begin{aligned}
(2.10)\qquad & \int_0^1 h_u(s)\, \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{E}\{L(u)\}}\, ds \\
&\simeq \frac{\kappa\, u}{\mathfrak{P}_0 w} \int_0^1 h_u(s)\, \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{P}\{\xi(1) > u\}}\, ds \\
&= \frac{1}{\mathfrak{P}_0} \int_0^y h_u\big((1+\hat{s}w/u)^{-1/\kappa}\big)\, \frac{\mathbf{P}\{\xi(1)>u+\hat{s}w\}}{(1+\hat{s}w/u)^{1+1/\kappa}\, \mathbf{P}\{\xi(1)>u\}}\, d\hat{s} \\
&\quad + \frac{\kappa\, u}{\mathfrak{P}_0 w} \int_0^{(1+yw/u)^{-1/\kappa}} h_u(\hat{s})\, \frac{\mathbf{P}\{\xi(\hat{s})>u\}}{\mathbf{P}\{\xi(1)>u\}}\, d\hat{s}.
\end{aligned}$$

Since the convergence in (1.1) is locally uniform, an application of (2.3) followed by sending $y\to\infty$ in (2.10) proves (2.2). □

3. Asymptotic distributions for sojourns. First we need two assumptions:

Assumption 1. *There is an* $(\mathbb{R}\cup\{-\infty,\infty\})$*-valued process* $\{\zeta(t)\}_{t\geq 0}$ *such that*

$$\lim_{u\uparrow\hat{u}} \mathbf{P}\bigg\{\bigcap_{i=1}^{n}\Big\{\frac{\xi(1-q(u)t_i)-u}{w(u)}>x_i\Big\}\,\bigg|\,\xi(1)>u\bigg\} = \mathbf{P}\bigg\{\bigcap_{i=1}^{n}\{\zeta(t_i)>x_i\}\bigg\}$$

for $n\in\mathbb{Z}^+(=\{1,2,\ldots\})$, $t_1,\ldots,t_n\in[0,Q)$ *and continuity points* $x_1,\ldots,x_n\in J$ *for the functions* $\mathbf{P}\{\zeta(t_1)>\cdot\},\ldots,\mathbf{P}\{\zeta(t_n)>\cdot\}$.

In view of the fact that (1.1) implies that $\mathbf{P}\{w^{-1}[\xi(1)-u] > x \mid \xi(1) > u\}$ converges [to $1-F(x)$], Assumption 1 is a quite natural requirement.

Assumption 2. *We have*

$$\lim_{d\to\infty} \overline{\lim_{u\uparrow\hat{u}}} \int_{d\wedge(1/q(u))}^{1/q(u)} \mathbf{P}\big\{\xi(1-q(u)t)>u\,\big|\,\xi(1)>u\big\}\, dt = 0.$$

Assumption 2 is void when $Q<\infty$, and more generally Proposition 2 below shows that it holds when $\mathfrak{P}_4>0$. Assumption 2 requires that if $\xi(1)>u$, then $\xi(t)$

have not spent too much time above the level u before time $t=1$. Assumption 2 can be interpreted as $\mathbf{E}\{\int_0^Q I_{(0,\infty)}(\zeta(s))\,ds\}<\infty$ when Assumption 1 holds.

In Theorem 1 we find the asymptotic distribution of $L(u)=L(1;u)$ as $u\uparrow\hat{u}$. The idea of the proof is that if $\big(w^{-1}[\xi(1-qt)-u]\,\big|\,\xi(1)>u\big)\to_{\mathcal{L}}\zeta(t)$, then also

$$\big(L(1;u)/q\,\big|\,\xi(1)>u\big)=_{\mathcal{L}}\left(\int_0^{1/q} I_{(0,\infty)}\big(w^{-1}[\xi(1-qt)-u]\big)\,dt\,\middle|\,\xi(1)>u\right)\quad\text{converges.}$$

By self-similarity this transfers to $\big(L(s;u)/q\,\big|\,\xi(s)>u\big)$, and combining the relation

$$\int_x^\infty \mathbf{P}\{L(u)/q>y\}\,dy=\frac{1}{q}\int_0^1 \mathbf{P}\big\{L(s;u)/q>x\,\big|\,\xi(s)>u\big\}\,\mathbf{P}\{\xi(s)>u\}\,ds \tag{3.0}$$

with Proposition 1, the asymptotic behaviour of $L(u)$ follows: Eq. (3.0) is discussed below. It's significance has long been understood and utilized by Berman.

Theorem 1. *Assume that Assumption 1 holds with $G\in\mathcal{D}$, and that (1.4) holds. Defining*

$$\Lambda(x)\equiv\frac{1}{\mathfrak{P}_0}\int_0^\infty \mathbf{P}\bigg\{\int_0^Q I_{(0,\infty)}(\zeta(t))\,dt>\frac{x}{\hat{p}(s)}\bigg\}\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa}}\qquad\textit{for } x\geq 0,$$

we then have

$$\varliminf_{u\uparrow\hat{u}}\int_x^\infty\frac{\mathbf{P}\{L(u)/q(u)>y\}}{\mathbf{E}\{L(u)/q(u)\}}\,dy\geq\Lambda(x)\qquad\textit{for each } x>0.$$

If in addition Assumption 2 and (1.3) hold, then we have

$$\varlimsup_{u\uparrow\hat{u}}\int_x^\infty\frac{\mathbf{P}\{L(u)/q(u)>y\}}{\mathbf{E}\{L(u)/q(u)\}}\,dy\leq\Lambda(x^-)\qquad\textit{for each } x>0.$$

Of course, the asymptotic behaviour of $\mathbf{E}\{L(u)\}$ is described by (2.1).

Lemma 2. *Assume that Assumption 1 holds with $G\in\mathcal{D}$, and that (1.4) holds. Then $\mathbf{P}\{\zeta(t)>x\}$ is continuous at $x=0$ for each $t\in(0,Q)$, and*

$$\textit{the conditional law of}\quad\left(\int_0^{T\wedge(s_u/q)} I_{(u,\hat{u})}\big(\xi(s_u-qt)\big)\,dt\,\middle|\,\xi(s_u)>u\right)$$

$$\textit{converge weakly to that of}\quad \hat{p}(s)\int_0^{(T/\hat{p}(s))\wedge Q} I_{(0,\infty)}(\zeta(t))\,dt.$$

Proof of Lemma 2. Take a $u_0<\hat{u}$ such that $w(u)/u\leq 1$ for $u\geq u_0$, and note that

$$(1-qt)^{-\kappa}(u-\varepsilon w)>u+\big(\varepsilon^{-1/2}-\varepsilon^{1/2}-\varepsilon\big)w\quad\text{if}\quad(1-qt)^{-\kappa}>1+\varepsilon^{-1/2}(w/u)$$

$$\tilde{u} \equiv (1-qt)^{-\kappa}(u+\varepsilon w) \leq u+\big(\varepsilon^{-1/2}+\varepsilon^{1/2}+\varepsilon\big)w \quad \text{if} \quad (1-qt)^{-\kappa} \leq 1+\varepsilon^{-1/2}(w/u)$$

for $u \geq u_0$ and $\varepsilon \in (0,1)$. Since $(1-qt)^{-\kappa}w \leq 2(1+\varepsilon^{-1/2})w(\tilde{u})$ when $(1-qt)^{-\kappa} \leq 1+\varepsilon^{-1/2}(w/u)$ and $u \geq u_1$, for some $u_1 = u_1(\varepsilon) \geq u_0$ [recall (1.2)], it follows that

$$\begin{aligned}
&\mathbf{P}\big\{\zeta(t) > -\tfrac{1}{2}\varepsilon\big\} - \mathbf{P}\big\{\zeta(t) > \tfrac{1}{2}\varepsilon\big\} \\
&\leq \varlimsup_{u\uparrow\hat{u}} \frac{\mathbf{P}\big\{u-\varepsilon w < \xi(1-qt) \leq u+\varepsilon w\big\}}{\mathbf{P}\{\xi(1) > u\}} \\
&\leq \varlimsup_{u\uparrow\hat{u}} \frac{\mathbf{P}\big\{\xi(1) > u+(\varepsilon^{-1/2}-\varepsilon^{1/2}-\varepsilon)w\big\}}{\mathbf{P}\{\xi(1) > u\}} + \varlimsup_{u\uparrow\hat{u}} \frac{\mathbf{P}\big\{\tilde{u}-4\varepsilon(1+\varepsilon^{-1/2})w(\tilde{u}) < \xi(1) \leq \tilde{u}\big\}}{\mathbf{P}\{\xi(1) > \tilde{u}\}} \\
&= \big(1 - F(\varepsilon^{-1/2}-\varepsilon^{1/2}-\varepsilon)\big) + \big(-F(-4(\varepsilon+\varepsilon^{1/2}))\big) \\
&\to 0 \quad \text{as } \varepsilon \downarrow 0.
\end{aligned}$$

Now put $\tilde{u} \equiv s_u^{-\kappa}u$ and $\tilde{q} \equiv q(\tilde{u})$. Since (1.4) implies that $\tilde{q}s_u/q \to \hat{p}(s)$, Assumption 1 then combine with the above established continuity to give

$$\begin{aligned}
&\mathbf{E}\bigg\{\bigg(\int_0^{T\wedge(s_u/q)} I_{(u,\hat{u})}\big(\xi(s_u-qt)\big)\,dt\bigg)^m \bigg|\, \xi(s_u) > u\bigg\} \\
&= \int\limits_{0<\tilde{t}_1,\ldots,\tilde{t}_m<(qT/(\tilde{q}s_u))\wedge(1/\tilde{q})} \mathbf{P}\bigg\{\bigcap_{i=1}^m \bigg\{\frac{\xi\big(1-\tilde{q}\tilde{t}_i\big)-\tilde{u}}{w(\tilde{u})} > 0\bigg\}\,\bigg|\, \xi(1) > \tilde{u}\bigg\} \frac{d\tilde{t}_m \ldots d\tilde{t}_1}{(q/(\tilde{q}s_u))^m} \\
&\to \hat{p}(s)^m \int\limits_{0<\tilde{t}_1,\ldots,\tilde{t}_m<(T/\hat{p}(s))\wedge Q} \mathbf{P}\bigg\{\bigcap_{i=1}^m \{\zeta(\tilde{t}_i) > 0\}\bigg\}\, d\tilde{t}_m \ldots d\tilde{t}_1.
\end{aligned}$$

The lemma now follows from recalling the elementary fact that convergence of moments for bounded random variables implies weak convergence. $\square$

Proof of Theorem 1. In view of the elementary fact that

$$I_{(x,\infty)}(L(u)) \int_0^1 I_{(-\infty,x]}\big(L(s;u)\big)\, I_{(u,\hat{u})}(\xi(s))\, ds = I_{(x,\infty)}(L(u))\, x \quad \text{for } x > 0,$$

we readily obtain

$$\begin{aligned}
(3.1) \qquad &\int_x^\infty \frac{\mathbf{P}\{L(u)/q > y\}}{\mathbf{E}\{L(u)/q\}}\, dy \\
&= \frac{1}{\mathbf{E}\{L(u)\}} \int_0^\infty \mathbf{P}\big\{(L(u)-qx) > \hat{y}\big\}\, d\hat{y} \\
&= \frac{1}{\mathbf{E}\{L(u)\}}\, \mathbf{E}\big\{(L(u)-qx)\, I_{(qx,\infty)}(L(u))\big\}
\end{aligned}$$

$$= \frac{1}{\mathbf{E}\{L(u)\}}\,\mathbf{E}\bigg\{ I_{(qx,\infty)}(L(u)) \int_0^1 \big(1 - I_{(-\infty,qx]}(L(s;u))\big)\, I_{(u,\hat{u})}(\xi(s))\, ds \bigg\}$$

$$= \frac{1}{\mathbf{E}\{L(u)\}}\,\mathbf{E}\bigg\{ \int_0^1 I_{(qx,\infty)}(L(s;u)) I_{(u,\hat{u})}(\xi(s))\, ds \bigg\}$$

$$= \frac{1}{\mathbf{E}\{L(u)\}} \int_0^1 \mathbf{P}\big\{ L(s;u)/q > x,\, \xi(s) > u \big\}\, ds.$$

Taking $\varepsilon \in (0,1)$, (3.1) combines with Lemma 2 and (2.2) to show that

$$\text{(3.2)} \qquad \varlimsup_{u \uparrow \hat{u}} \int_x^\infty \frac{\mathbf{P}\{L(u)/q > y\}}{\mathbf{E}\{L(u)/q\}}\, dy$$

$$\leq \varlimsup_{u \uparrow \hat{u}} \int_0^1 \mathbf{P}\bigg\{ \int_0^{d \wedge (s/q)} I_{(u,\hat{u})}\big(\xi(s-qt)\big)\, dt > x - \tfrac{1}{2}\varepsilon x \,\bigg|\, \xi(s) > u \bigg\} \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{E}\{L(u)\}}\, ds$$

$$+ \varlimsup_{u \uparrow \hat{u}} \int_0^1 \mathbf{P}\bigg\{ \int_{d \wedge (s/q)}^{s/q} I_{(u,\hat{u})}\big(\xi(s-qt)\big)\, dt > \tfrac{1}{2}\varepsilon x \,\bigg|\, \xi(s) > u \bigg\} \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{E}\{L(u)\}}\, ds$$

$$\leq \frac{1}{\mathfrak{P}_0} \int_0^\infty \mathbf{P}\bigg\{ \int_0^{(d/\hat{p}(s)) \wedge Q} I_{(0,\infty)}(\zeta(t))\, dt > \frac{x - \varepsilon x}{\hat{p}(s)} \bigg\} \frac{(1 - F(s))\, ds}{(1 + Ws)^{1+1/\kappa}}$$

$$+ \frac{2}{\varepsilon x} \varlimsup_{u \uparrow \hat{u}} \int_{s=0}^{s=1} \int_{t=d \wedge (s/q)}^{t=s/q} \mathbf{P}\big\{ \xi(s-qt) > u \,\big|\, \xi(s) > u \big\}\, dt\, \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{E}\{L(u)\}}\, ds.$$

But writing $\tilde{u} \equiv s^{-\kappa} u$ and $\tilde{q} \equiv q(\tilde{u})$, (1.3) yields that $s\tilde{q}/q = p(s^{-\kappa}u)/p(u) \leq 2\mathfrak{P}_1$ for u large. In view of Assumption 2 we thus have

$$\text{(3.3)} \qquad \sup_{u \geq u_2} \int_{d \wedge (s/q)}^{s/q} \mathbf{P}\big\{ \xi(s-qt) > u \,\big|\, \xi(s) > u \big\}\, dt$$

$$= \sup_{u \geq u_2} \int_{(dq/s\tilde{q}) \wedge (1/\tilde{q})}^{1/\tilde{q}} \mathbf{P}\big\{ \xi(1 - \tilde{q}\tilde{t}) > \tilde{u} \,\big|\, \xi(1) > \tilde{u} \big\} \frac{s\,\tilde{q}}{q}\, d\tilde{t}$$

$$\leq 2\mathfrak{P}_1 \sup_{u \geq u_2} \int_{(d/2\mathfrak{P}_1) \wedge (1/q)}^{1/q} \mathbf{P}\big\{ \xi(1 - q\tilde{t}) > u \,\big|\, \xi(1) > u \big\}\, d\tilde{t}$$

$$\leq \varepsilon^2 \quad \text{for } d \geq d_0 \text{ and } s \in (0,1], \text{ for some } d_0 \geq 1 \text{ and } u_2 < \hat{u}.$$

Inserting in (3.2), the upper bound now follows from sending $d \to \infty$ and $\varepsilon \downarrow 0$.

The proof of the lower bound is analogous (but easier): Using (3.1), Fatou's Lemma, and Lemma 2, [but neither (1.3) nor Assumption 2], we obtain

$$\varliminf_{u \uparrow \hat{u}} \int_x^\infty \frac{\mathbf{P}\{L(u)/q > y\}}{\mathbf{E}\{L(u)/q\}}\, dy$$

$$\geq \varliminf_{u \uparrow \hat{u}} \int_0^1 \mathbf{P}\bigg\{ \int_0^{d \wedge (s/q)} I_{(u,\hat{u})}\big(\xi(s-qt)\big)\, dt > x \,\bigg|\, \xi(s) > u \bigg\} \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{E}\{L(u)\}}\, ds$$

$$\geq \frac{1}{\mathfrak{P}_0}\int_0^\infty \mathbf{P}\bigg\{\int_0^{(d/\hat{p}(s))\wedge Q} I_{(0,\infty)}(\zeta(t))\,dt > \frac{x}{\hat{p}(s)}\bigg\}\,\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa}}$$
$$\to \varLambda(x) \quad \text{as } d\to\infty. \quad \square$$

Berman (1982) used versions of Assumptions 1-2, Lemma 2 and (3.1) [=(3.0)] to study sojourns, and also worked on relations to extremes. See also Berman (1992).

For $q(u)$ large enough to make $\mathfrak{P}_4>0$, the trivial estimate

$$\mathbf{P}\big\{\xi(1-qt)>u\,\big|\,\xi(1)>u\big\} \;\leq\; \mathbf{P}\{\xi(1-qt)>u\}\;/\;\mathbf{P}\{\xi(1)>u\}$$

combines with Proposition 1 to show that Assumptions 2 holds:

Proposition 2. *If $G\in\mathcal{D}$ and $\mathfrak{P}_4>0$, then Assumption 2 holds.*

Proof. Write $y(u)=\kappa\,d\,u\,q/w$, so that $\underline{\lim}_{u\uparrow\hat{u}}\, y=\kappa\, d\,\mathfrak{P}_4$ and $(1+yw/u)^{-1/\kappa}\geq(1-qd)^+$. Invoking (2.2) we then obtain

$$\begin{aligned}
\overline{\lim_{u\uparrow\hat{u}}}\int_{d\wedge(1/q)}^{1/q}\frac{\mathbf{P}\{\xi(1-qt)>u\}}{\mathbf{P}\{\xi(1)>u\}}\,dt &= \overline{\lim_{u\uparrow\hat{u}}}\,\frac{1}{q}\int_0^{(1-qd)^+}\frac{\mathbf{P}\{\xi(s)>u\}}{\mathbf{P}\{\xi(1)>u\}}\,ds\\
&\leq \overline{\lim_{u\uparrow\hat{u}}}\,\frac{\mathfrak{P}_0}{\mathfrak{P}_4\kappa}\int_0^1 I_{(0,(1+yw/u)^{-1/\kappa}]}(s)\,\frac{\mathbf{P}\{\xi(s)>u\}}{\mathbf{E}\{L(u)\}}\,ds\\
&= \frac{1}{\mathfrak{P}_4\kappa}\int_{\kappa d\mathfrak{P}_4}^\infty\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa}}. \quad \square
\end{aligned}$$

4. First bounds on extremes. By Theorem 1, Asumptions 1 and 2 imply

$$\begin{aligned}
\text{(4.0)} \qquad \mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\} &\geq \max\bigg\{\mathbf{P}\{\xi(1)>u\},\,\frac{1}{x}\int_0^x\mathbf{P}\{L(u)/q>y\}\,dy\bigg\}\\
&\geq \max\bigg\{\mathbf{P}\{\xi(1)>u\},\,\frac{1-\varLambda(x^-)}{x}\,\mathbf{E}\{L(u)/q\}\bigg\},
\end{aligned}$$

and assuming 'tightness' this inequality can be reversed. Whether $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$ behaves like $\mathbf{P}\{\xi(1)>u\}$ or $\mathbf{E}\{L(u)/q\}$ thus depends on the ratio $\mathbf{P}\{\xi(1)>u\}\,/\,\mathbf{E}\{L(u)/q\}$, which in turn by (2.1) behaves like $u\,q(u)\,/\,(\mathfrak{P}_0 w(u))$.

In Section 4 we derive bounds for $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$ without assuming knowledge of the size of $u\,q(u)/w(u)$. In Sections 5 and 6 we give more precise results requiring that $\overline{\lim}_{u\uparrow\hat{u}}\,u\,q(u)/w(u)$ is infinite and finite, respectively.

The ideas behind all theorems in Sections 4-6 are versions of the estimate (4.0). As indicated above, it is easy to derive a lower bound for $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$:

Theorem 2. *If Assumption 2 holds with $G\in\mathcal{D}$, and if* (1.3) *holds, then we have*

$$\varliminf_{u\uparrow\hat{u}} \frac{1}{\mathbf{E}\{L(u)\}/q(u)+\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\} > 0.$$

Proof. Clearly we have [cf. (4.0)]

$$\varliminf_{u\uparrow\hat{u}} \frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\} \geq \frac{1}{x}\bigg[1-\varlimsup_{u\uparrow\hat{u}}\int_x^\infty \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\bigg] \tag{4.1}$$

for each $x>0$. Given an $\varepsilon\in(0,1)$, (3.2) and (3.3) further show that

$$\begin{aligned}&\varlimsup_{u\uparrow\hat{u}}\int_x^\infty \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\\ &\leq \varlimsup_{u\uparrow\hat{u}}\int_0^1 \mathbf{P}\bigg\{\int_0^{d_0\wedge(s/q)} I_{(u,\hat{u})}(\xi(s-qt))\,dt>(1-\tfrac{1}{2}\varepsilon)x\,\bigg|\,\xi(s)>u\bigg\}\frac{\mathbf{P}\{\xi(s)>u\}}{\mathbf{E}\{L(u)\}}\,ds+\frac{2\varepsilon}{x}.\end{aligned}$$

Since the first term on the right hand side vanishes for $x=d_0/(1-\frac{1}{2}\varepsilon)$, (4.1) yields

$$\varliminf_{u\uparrow\hat{u}} \frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\} \geq \frac{1-\frac{1}{2}\varepsilon}{d_0}\bigg(1-\frac{2\varepsilon(1-\frac{1}{2}\varepsilon)}{d_0}\bigg) > 0.$$

The (virtually stronger) statement of the theorem now follows easily. $\square$

Our upper bounds use one of two tightness assumptions: Define $t_a^u(0)\equiv 1$ and

$$t_a^u(k+1)\equiv t_a^u(k)\,\big(1-a\,q(t_a^u(k)^{-\kappa}u)\big)\quad\text{for}\quad k\leq K(a,u)\equiv\sup\big\{k\in\mathbb{N}:t_a^u(k)^{-\kappa}u<\hat{u}\big\}$$

when $u\in(-\infty,\hat{u})$ and $a\in(0,\hat{a}]$. Note that $t_a^u(k)\geq\frac{1}{2}t_a^u(k-1)\geq\ldots\geq 2^{-k}>0$.

Assumption 3′. *For some choice of $\sigma>0$ and $a\in(0,\hat{a}]$ we have*

$$\nu(a,\sigma)\equiv\varlimsup_{u\uparrow\hat{u}}\frac{\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w(u),\ \max_{0\leq k\leq K(a,u)}\xi(t_a^u(k))\leq u\big\}}{\mathbf{E}\{L(u)\}/q(u)+\mathbf{P}\{\xi(1)>u\}}<\infty.$$

Assumption 3. *Assumption 3′ holds with $\lim_{a\downarrow 0}\nu(a,\sigma)=0$ for each $\sigma>0$.*

Assumptions 3 and 3′ are often verified via Propositions 3-5 in Section 8.

Theorem 3. *Assume that Assumption 3′ holds with $G\in\mathcal{D}$ and that* (1.5) *and* (1.6) *hold. If in addition either* (1.3) *or* (1.4) *holds, then we have*

$$\varlimsup_{u\uparrow\hat{u}} \frac{1}{\mathbf{E}\{L(u)\}/q(u)+\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\} < \infty.$$

Proof. Since w is continuous, and since [recall (1.2)] $u-2\sigma w+\sigma w(u-2\sigma w) \leq u \leq u+\sigma w$ for $u \in [u_3, \hat{u})$, for some $u_3 < \hat{u}$, we can to each sequence $u_n \uparrow \hat{u}$ find a sequence $u_n' \uparrow \hat{u}$ such that $u_n = u_n' + \sigma w(u_n')$ for n large. Consequently

$$\begin{aligned}
&\overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)/q\}+\mathbf{P}\{\xi(1)>u\}} \mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\} \\
&= \overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\big\{L(u+\sigma w)\,/\,q(u+\sigma w)\big\}+\mathbf{P}\big\{\xi(1)>u+\sigma w\big\}} \mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w\big\}.
\end{aligned}$$

Further note that (1.1)-(1.2) and (2.1) yield $\mathbf{E}\{L(u+\sigma w)\}/\mathbf{E}\{L(u)\} \to 1-F(\sigma)$, while $\overline{\lim}_{u\uparrow\hat{u}}\, q(u+\sigma w)/q \leq C_\sigma(1+\sigma W)^{1/\kappa}$, where $C_\sigma = \mathfrak{P}_1$ if (1.3) holds, and $C_\sigma = \hat{p}(\sigma)$ if (1.4) holds. Hence we have [using (1.1) again]

$$\begin{aligned}
(4.2)\quad &\overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)/q\}+\mathbf{P}\{\xi(1)>u\}} \mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\} \\
&\leq \frac{C_\sigma(1+\sigma W)^{1/\kappa}}{1-F(\sigma)} \overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)/q\}+\mathbf{P}\{\xi(1)>u\}} \mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w\big\}.
\end{aligned}$$

In view of Assumption 3′ we now readily conclude that it is sufficient to prove

$$\overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)/q\}} \mathbf{P}\Big\{\max_{1\leq k\leq K}\xi(t_a^u(k))>u\Big\} < \infty \quad \text{for } a\in(0,\hat{a}].$$

To that end we note that (1.6) combines with (2.1) and (2.3) to give

$$\begin{aligned}
(4.3)\quad q\,\mathbf{P}\Big\{\max_{1\leq k\leq K}\xi(t_a^u(k))>u\Big\} &\leq q\sum_{k=1}^{K}\mathbf{P}\{\xi(t_a^u(k))>u\} \\
&\leq \sum_{k=1}^{K}\int_{t_a^u(k)}^{t_a^u(k-1)} \frac{q\,\mathbf{P}\{\xi(t)>u\}}{t_a^u(k-1)-t_a^u(k)}\,dt \\
&= \sum_{k=1}^{K}\int_{t_a^u(k)}^{t_a^u(k-1)} \frac{p(u)\,\mathbf{P}\{\xi(t)>u\}}{a\,p(t_a^u(k-1)^{-\kappa}u)}\,dt \\
&\leq \sum_{k=1}^{K}\int_{t_a^u(k)}^{t_a^u(k-1)} \frac{\mathbf{P}\{\xi(t)>u\}}{a\,\frac{1}{2}\mathfrak{P}_2\,t_a^u(k-1)^{\rho\kappa}}\,dt \\
&\leq \frac{2}{\mathfrak{P}_2 a}\int_0^1 t^{-\rho\kappa}\,\mathbf{P}\{\xi(t)>u\}\,dt \\
&\sim \frac{2\,\mathbf{E}\{L(u)\}}{\mathfrak{P}_0\mathfrak{P}_2 a}\int_0^\infty \frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa-\rho}}. \qquad \square
\end{aligned}$$

5. Sharp (results on) extremes when $\mathfrak{P}_5 = \infty$.

Theorem 4. *Assume that Assumption 3 holds with* $G\in\mathcal{D}$ *and that* (1.5) *and* (1.6) *hold. Then the following implications hold*

$$\text{(5.1)}\qquad \mathfrak{P}_5=\infty \implies \lim_{u\uparrow\hat u}\frac{1}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\big\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\big\}=1,$$

$$\text{(5.2)}\qquad \mathfrak{P}_4=\infty \implies \lim_{u\uparrow\hat u}\frac{1}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\big\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\big\}=1.$$

Proof of (5.1). Clearly we have

$$\text{(5.3)}\quad \mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w\Big\}=\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w,\ \max_{1\le k\le K}\xi(t_a^u(k))\le u,\ \xi(1)>u\Big\}$$
$$+\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w,\ \max_{0\le k\le K}\xi(t_a^u(k))\le u\Big\}$$
$$+\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w,\ \max_{1\le k\le K}\xi(t_a^u(k))>u\Big\}.$$

Taking $\{u_n\}_{n=1}^{\infty}$ such that $u_n\uparrow\hat u$ and $u_nq(u_n)/w(u_n)\to\infty$, (2.1) and (4.3) imply

$$\text{(5.4)}\qquad \frac{1}{\mathbf{P}\{\xi(1)>u_n\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u_n+\sigma w(u_n),\ \max_{1\le k\le K(a,u_n)}\xi(t_a^{u_n}(k))>u_n\Big\}$$
$$\le\frac{2\,w(u_n)}{\kappa\,u_nq(u_n)}\,\frac{1}{\mathbf{E}\{L(u_n)/q(u_n)\}}\,\mathbf{P}\Big\{\max_{1\le k\le K(a,u_n)}\xi(t_a^{u_n}(k))>u_n\Big\}$$
$$\le\frac{2\,w(u_n)}{\kappa\,u_nq(u_n)}\,\frac{2}{\mathfrak{P}_0\mathfrak{P}_2a}\int_0^\infty\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa-\rho}}$$
$$\to 0\quad\text{as}\quad n\to\infty.$$

Further (2.1) combines with Assumption 3 to show that

$$\text{(5.5)}\qquad \frac{\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u_n+\sigma w(u_n),\ \max_{0\le k\le K(a,u_n)}\xi(t_a^{u_n}(k))\le u_n\big\}}{\mathbf{P}\{\xi(1)>u_n\}}$$
$$\sim\bigg(\frac{\mathfrak{P}_0w(u_n)}{\kappa\,u_nq(u_n)}+1\bigg)\frac{\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u_n+\sigma w(u_n),\ \max_{0\le k\le K(a,u_n)}\xi(t_a^{u_n}(k))\le u_n\big\}}{\mathbf{E}\{L(u_n)/q(u_n)\}+\mathbf{P}\{\xi(1)>u_n\}}$$
$$\to(0+1)\,f(a)\quad\text{as}\quad n\to\infty,$$

where $f(a)\to 0$ as $a\downarrow 0$. Combining (5.3)-(5.5) and sending $a\downarrow 0$ we conclude that

$$\text{(5.6)}\qquad \varlimsup_{n\to\infty}\frac{1}{\mathbf{P}\{\xi(1)>u_n\}}\,\mathbf{P}\big\{\sup\nolimits_{t\in[0,1]}\xi(t)>u_n+\sigma w(u_n)\big\}\le 1.$$

But as in the proof of Theorem 3, a change of variable in the limit shows that

$$\lim_{u\uparrow\hat u}\frac{1}{\mathbf{P}\{\xi(1)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}=\lim_{u\uparrow\hat u}\frac{1}{\mathbf{P}\{\xi(1)>u+\sigma w\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w\Big\}$$

$$= \frac{1}{1-F(\sigma)} \lim_{u\uparrow\hat{u}} \frac{1}{\mathbf{P}\{\xi(1)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,1]} \xi(t)>u+\sigma w\Big\}.$$

In view of (5.6), (5.1) now follows from sending $\sigma\downarrow 0$. □

Proof of (5.2). Now (5.4) and (5.5) hold for any sequence $u_n\uparrow\hat{u}$, and [in view of (5.3)] so does (5.6). By a change of variable in the limit we thus get

$$\overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{P}\{\xi(1)>u\}} \mathbf{P}\{\sup\nolimits_{t\in[0,1]} \xi(t)>u\}$$

$$= \frac{1}{1-F(\sigma)} \overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{P}\{\xi(1)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,1]} \xi(t)>u+\sigma w\Big\} \leq \frac{1}{1-F(\sigma)} \to 1 \quad \text{as } \sigma\downarrow 0. \quad \square$$

6. Sharp extremes when $\mathfrak{P}_5<\infty$.

Theorem 5. *Assume that Assumptions 1 and 2 hold with* $G\in\mathcal{D}$ *and that* (1.3) *and* (1.4) *hold. Then we have*

$$\underline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)\}/q(u)} \mathbf{P}\{\sup\nolimits_{t\in[0,1]} \xi(t)>u\} \geq \overline{\lim}_{x\downarrow 0}\,(1-\Lambda(x))/x.$$

Proof. In view of (4.1) an application of Theorem 1 shows that

$$\underline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)\}/q(u)} \mathbf{P}\{\sup\nolimits_{t\in[0,1]} \xi(t)>u\} \geq (1-\Lambda(x^-))/x \quad \text{for each } x>0. \quad \square$$

Theorem 6. *Assume that Assumptions 1 and 3 hold with* $G\in\mathcal{D}$ *and* $\mathfrak{P}_5<\infty$, *and that* (1.3)-(1.7) *hold. Then we have*

$$\overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)\}/q(u)} \mathbf{P}\{\sup\nolimits_{t\in[0,1]} \xi(t)>u\} \leq \underline{\lim}_{x\downarrow 0}\,(1-\Lambda(x))/x.$$

Lemma 3. *Assume that Assumption 1 holds with* $G\in\mathcal{D}$, $\Lambda(0)=1$ *and* $\mathfrak{P}_5<\infty$. *If in addition* (1.4)-(1.7) *hold, then we have*

$$\lim_{x\downarrow 0} \overline{\lim_{u\uparrow\hat{u}}} \frac{1}{\mathbf{E}\{L(u)\}/q(u)} \mathbf{P}\Big\{L(u-\sigma w)/q(u-\sigma w)\leq x,\, \max\nolimits_{1\leq k\leq K}\xi(t_a^u(k))>u\Big\}=0$$

for $\sigma\in(0,1)$ *and for* $a\in(0,\hat{a}]$ *sufficiently small.*

Proof of Lemma 3. Using (1.6) and (1.7) we obtain

$$q(u) \leq 2\mathfrak{P}_5 w(u)/u \leq 4\mathfrak{P}_3\mathfrak{P}_5 w(v)/v \leq 4\mathfrak{P}_3\mathfrak{P}_5 \quad \text{for } u_4\leq v\leq u<\hat{u}, \text{ for some } u_4<\hat{u}.$$

Choosing a $c>0$ such that $(1-x)^{\kappa} \geq 1-cx$ for $x\in[0,\frac{1}{2}]$, we therefore get

$$\big(t_a^u(k+1)/t_a^u(k)\big)^{\kappa}u \geq \big(1-aq(t_a^u(k)^{-\kappa}u)\big)^{\kappa}u \geq u-ca4\mathfrak{P}_3\mathfrak{P}_5 w \geq u-\sigma w \equiv \tilde{u}$$

for $u\geq u_4$ and for $a\in(0,\hat{a}]$ sufficiently small. Since self-similarity yields

$$\Big(L(u)\,,\,\xi(t_a^u(k))\Big) =_{\mathcal{L}} \Big((t_a^u(k)/t)\,L\big(t/t_a^u(k),\,(t/t_a^u(k))^{\kappa}u\big)\,,\,(t_a^u(k)/t)^{\kappa}\xi(t)\Big),$$

we now conclude, adding things up, and using (1.6) as in (4.3),

$$
\begin{aligned}
&q\,\mathbf{P}\Big\{L(\tilde{u})/q(\tilde{u})\leq x,\ \max\nolimits_{1\leq k\leq K}\xi(t_a^u(k))>u\Big\}\\
&\leq q\sum_{k=1}^{K}\mathbf{P}\Big\{L(\tilde{u})/q(\tilde{u})\leq x,\ \xi(t_a^u(k))>u\Big\}\\
&=\sum_{k=1}^{K}\int_{t_a^u(k+1)}^{t_a^u(k)}\frac{q\,\mathbf{P}\big\{L\big(t/t_a^u(k),\,(t/t_a^u(k))^{\kappa}\tilde{u}\big)/q(\tilde{u})\leq(t/t_a^u(k))x,\ \xi(t)>(t/t_a^u(k))^{\kappa}u\big\}}{t_a^u(k)-t_a^u(k+1)}\,dt\\
&\leq\sum_{k=0}^{K}\int_{t_a^u(k+1)}^{t_a^u(k)}\frac{\mathbf{P}\big\{L(t;\tilde{u})/q(\tilde{u})\leq x,\ \xi(t)>\tilde{u}\big\}}{a\,\frac{1}{2}\mathfrak{P}_2\,t_a^u(k)^{\rho\kappa}}\,dt\\
&\leq\frac{2}{\mathfrak{P}_2 a\sqrt{1-\Lambda(x)}}\int_{(1-\Lambda(x))^{1/(2\rho\kappa)}}^{1}\Big(\mathbf{P}\{\xi(t)>\tilde{u}\}-\mathbf{P}\big\{L(t;\tilde{u})/q(\tilde{u})>x,\ \xi(t)>\tilde{u}\big\}\Big)\,dt\\
&\qquad\qquad+\frac{2}{\mathfrak{P}_2 a}\int_0^{(1-\Lambda(x))^{1/(2\rho\kappa)}}t^{-\rho\kappa}\,\mathbf{P}\{\xi(t)>\tilde{u}\}\,dt\qquad\text{for } u \text{ sufficiently large.}
\end{aligned}
$$

Here (2.1), (3.1), Theorem 1, and the fact that $\Lambda(x)\to 1$, show that

$$
\begin{aligned}
&\overline{\lim_{u\uparrow\hat{u}}}\,\frac{1}{\mathbf{E}\{L(u)\}\sqrt{1-\Lambda(x)}}\bigg[\int_0^1\mathbf{P}\{\xi(t)>\tilde{u}\}\,dt-\int_0^1\mathbf{P}\big\{L(t;\tilde{u})/q(\tilde{u})>x,\ \xi(t)>\tilde{u}\big\}\,dt\bigg]\\
&=\frac{1-F(-\sigma)}{\sqrt{1-\Lambda(x)}}\bigg[1-\lim_{u\uparrow\hat{u}}\int_x^{\infty}\frac{\mathbf{P}\{L(\tilde{u})/q(\tilde{u})>y\}}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,dy\bigg]\leq(1-F(-\sigma))\sqrt{1-\Lambda(x)}\to 0
\end{aligned}
$$

as $x\downarrow 0$. Moreover (1.5), (2.1) and (2.3) easily give

$$\overline{\lim_{u\uparrow\hat{u}}}\,\frac{1}{\mathbf{E}\{L(u)\}}\int_0^{(1-\Lambda(x))^{1/(2\rho\kappa)}}t^{-\rho\kappa}\,\mathbf{P}\{\xi(t)>\tilde{u}\}\,dt\to 0\qquad\text{as } x\downarrow 0.\qquad\square$$

Proof of Theorem 6. We can without loss assume that $\Lambda(0)=1$ (since the statement of the theorem is void otherwise). Writing $\tilde{u}\equiv u-\sigma w$ and $\tilde{q}\equiv q(\tilde{u})$ we then have

(6.1) $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w\big\}$

$$\le \frac{1}{x}\int_0^x \mathbf{P}\Big\{\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w\Big\}\cup\{L(\tilde u)/\tilde q>y\}\cup\Big\{\max_{0\le k\le K}\xi(t_a^u(k))>u\Big\}\Big\}\,dy$$

$$\le \frac{1}{x}\bigg[\mathbf{E}\{L(\tilde u)/\tilde q\}-\int_x^\infty \mathbf{P}\{L(\tilde u)/\tilde q>y\}\,dy\bigg]$$

$$+\mathbf{P}\{\xi(1)>\tilde u\}\,\frac{1}{x}\int_0^x \mathbf{P}\big\{L(\tilde u)/\tilde q\le y\,\big|\,\xi(1)>\tilde u\big\}\,dy$$

$$+\mathbf{P}\Big\{L(\tilde u)/\tilde q\le x,\ \max_{1\le k\le K}\xi(t_a^u(k))>u\Big\}$$

$$+\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w,\ \max_{0\le k\le K}\xi(t_a^u(k))\le u\Big\}.$$

Since $\mathfrak{P}_5<\infty$, Lemma 3 and Assumption 3 show that the last two terms are asymptotically neglible. Further Lemma 2 and the fact that $\Lambda(0)=1$ imply

$$\overline{\lim_{u\uparrow\hat u}}\,\frac{1}{x}\int_0^x \mathbf{P}\big\{L(\tilde u)/\tilde q\le y\,\big|\,\xi(1)>\tilde u\big\}\,dy$$

$$\le \overline{\lim_{d\to\infty}}\ \overline{\lim_{u\uparrow\hat u}}\,\frac{1}{x}\int_0^x \mathbf{P}\bigg\{\int_0^{d\wedge(1/q)} I_{(u,\hat u)}(\xi(1-qt))\,dt\le y\,\bigg|\,\xi(1)>u\bigg\}\,dy$$

$$\le \mathbf{P}\bigg\{\int_0^Q I_{(0,\infty)}(\zeta(t))\,dt\le x\bigg\}$$

$$\to 0 \quad \text{as } x\downarrow 0.$$

Adding things up and invoking (1.1), (1.3) and Theorem 1, we now conclude

$$\overline{\lim_{u\uparrow\hat u}}\,\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w\big\}$$

$$\le \underline{\lim_{x\downarrow 0}}\,\overline{\lim_{u\uparrow\hat u}}\,\frac{\mathbf{E}\{L(\tilde u)/\tilde q\}}{\mathbf{E}\{L(u)/q\}}\,\frac{1-\Lambda(x)}{x}+\overline{\lim_{x\downarrow 0}}\,\overline{\lim_{u\uparrow\hat u}}\,\frac{\mathbf{P}\{\xi(1)>\tilde u\}}{\mathbf{E}\{L(u)/q\}}\times\mathbf{P}\bigg\{\int_0^Q I_{(0,\infty)}(\zeta(t))\,dt\le x\bigg\}$$

$$\le \frac{\mathfrak{P}_1(1-F(-\sigma))}{(1-\sigma W)^{1/\kappa}}\,\underline{\lim_{x\downarrow 0}}\,\frac{1-\Lambda(x)}{x}+\frac{(1-F(-\sigma))\,\mathfrak{P}_5\kappa}{\mathfrak{P}_0}\times 0.$$

The argument used to establish (4.2) therefore carries over to show that

$$\overline{\lim_{u\uparrow\hat u}}\,\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\le\frac{\mathfrak{P}_1(1-F(-\sigma))\,C_\sigma(1+\sigma W)^{1/\kappa}}{(1-\sigma W)^{1/\kappa}\,(1-F(\sigma))}\,\underline{\lim_{x\downarrow 0}}\,\frac{1-\Lambda(x)}{x}.\ \square$$

Corollary 1. *Assume that Assumptions 1-3 hold with $G\in\mathcal{D}$ and $\mathfrak{P}_5<\infty$, and that (1.3)-(1.7) hold. Then the limits*

$$\lim_{u\uparrow\hat u}\frac{1}{\mathbf{E}\{L(u)\}/q(u)}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}\;=\;\lim_{x\downarrow 0}(1-\Lambda(x))/x\;\equiv\;-\Lambda'(0)$$

exist with common value $-\Lambda'(0)\in(0,\infty)$.

Proof. The facts that the limits exist and are equal follows from Theorems 5 and 6. Further Theorems 2 and 3 show that the limit is strictly positive and finite. $\square$

7. Sharp extremes for P-smooth processes with $G\in\mathcal{D}(\mathrm{I})\cup\mathcal{D}(\mathrm{III})$**.** One often encounters processes $\xi(t)$ which are asymptotically smooth in the sense that

(7.1)
$$\lim_{u\uparrow\hat{u}}\mathbf{P}\bigg\{\bigg|\frac{\xi(1-qt)-u}{w}-\frac{\xi(1)-u}{w}+\frac{qt\,\xi'}{w}\bigg|>\varepsilon\,\bigg|\,\xi(1)>u\bigg\}=0\quad\text{for }\varepsilon>0\text{ and }t\in[0,Q),$$

for some variable ξ' [usually a derivative of $\xi(t)$ at $t=1$]. Also assuming that

$$\overline{\lim_{u\uparrow\hat{u}}}\,\mathbf{E}\bigg\{\bigg(\frac{q(u)\,(\xi')^+}{w(u)}\bigg)^{\varrho}\,\bigg|\,\xi(1)>u\bigg\}<\infty\qquad\text{for some}\quad\varrho>1,\tag{7.2}$$

and that $G\in\mathcal{D}(\mathrm{I})\cup\mathcal{D}(\mathrm{III})$ possesses a density g for which

$$\lim\nolimits_{u\uparrow\hat{u}}\,w(u)\,g(u+xw(u))/(1-G(u))=\mathrm{e}^{-x}\qquad\text{for}\quad x\geq 0,\tag{7.3}$$

we shall prove a version of Corollary 1 where Assumption 1 is not needed.

Every infinitely divisible process $\xi(t)$ can be written $\xi(t)=_{\mathcal{L}}\int f_t(x)\,dM(x)$ where $f_t(\cdot)$ is a deterministic function and M an independently scattered random measured. When $f_{(\cdot)}(x)$ is smooth this suggests that $\xi'=\xi'(1)=\int f_1'(x)\,dM(x)$, and so it can be quite easy to prove (7.1) (cf. Sections 11-12 and 14).

Also the verification of (7.2) can be suprisingly easy: See the proof of Theorem 10 for a swift strategy for verifying (7.2) that works for 'light-tailed' processes.

It is well-known that (7.3) holds if e.g., g is ultimately decreasing [e.g., Resnick (1987, Propositions 1.16 and 1.17)]. In view of (1.1) and (1.2), it is also obvious that if (7.3) holds for $x=0$, then (7.3) holds for all $x\in\mathbb{R}$.

Theorem 7. *Assume that Assumptions 2 and 3 hold with* $G\in\mathcal{D}(\mathrm{I})\cup\mathcal{D}(\mathrm{III})$ *and* $\mathfrak{P}_5<\infty$*. If in addition* (1.3)-(1.7) *and* (7.1)-(7.3) *hold, then we have*

$$0<\underline{\lim_{u\uparrow\hat{u}}}\,\frac{q(u)\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>u\}}{w(u)}\leq\overline{\lim_{u\uparrow\hat{u}}}\,\frac{q(u)\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>u\}}{w(u)}<\infty,\tag{7.4}$$

and moreover

$$\mathbf{P}\big\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\big\}\sim(\kappa\,u)^{-1}\,\mathbf{E}\big\{(\xi')^+\,\big|\,\xi(1)>u\big\}\,\mathbf{P}\{\xi(1)>u\}\quad\text{as }u\uparrow\hat{u}.\tag{7.5}$$

Proof. Given an $s\in\mathbb{R}$, (1.1) and (7.3) imply that $w\,g(u+(s+z)w)/(1-G(u+sw))\to\mathrm{e}^{-z}$ for $z\in[0,\infty)$. Here the functions on both sides are densities on $[0,\infty)$, and the convergence theorem of Scheffé (1947) thus shows that

$$\int_0^\infty h_u(z)\,\frac{w\,g(u+(s+z)w)}{1-G(u+sw)}\,dz\simeq\int_0^\infty h_u(z)\,\mathrm{e}^{-z}\,dz\quad\text{when}\quad\overline{\lim_{u\uparrow\hat{u}}}\sup_{z\geq 0}|h_u(z)|<\infty.\tag{7.6}$$

Writing $\tilde{u}\equiv u+sw$, $\tilde{w}\equiv w(\tilde{u})$ and $\tilde{q}\equiv q(\tilde{u})$, (2.2), (3.1), and (7.6) yield that

$$\frac{1}{x}\left[1-\int_x^\infty\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\right] \tag{7.7}$$

$$
\begin{aligned}
&=\frac{1}{x}\int_0^1\mathbf{P}\big\{L(s;u)/q\leq x\;\big|\;\xi(s)>u\big\}\,\frac{\mathbf{P}\{\xi(s)>u\}}{\mathbf{E}\{L(u)\}}\,ds\\
&\simeq\int_0^\infty\mathbf{P}\bigg\{\int_0^{1/\tilde{q}}I_{(0,\infty)}\bigg(\frac{\xi(1-\tilde{q}t)-\tilde{u}}{w}\bigg)\,dt\leq\frac{qx}{s_u\tilde{q}}\;\bigg|\;\xi(1)>\tilde{u}\bigg\}\,\frac{\mathrm{e}^{-s}ds}{x}\\
&\leq\int_0^\infty\mathbf{P}\bigg\{\int_0^{(dq/\tilde{q})\wedge(1/\tilde{q})}I_{(x^2,\infty)}\bigg(\frac{\xi(1)-\tilde{u}}{w}-\frac{\tilde{q}t\,\xi'}{w}\bigg)\,dt\leq\frac{q(1+\varepsilon)x}{s_u\tilde{q}}\;\bigg|\;\xi(1)>\tilde{u}\bigg\}\,\frac{\mathrm{e}^{-s}ds}{x}\\
&\quad+\int_0^\infty\mathbf{P}\bigg\{\int_0^{dq/\tilde{q}}I_{(x^2,\infty)}\bigg(\frac{\xi(1)-\tilde{u}}{w}-\frac{\tilde{q}t\,\xi'}{w}-\frac{\xi(1-\tilde{q}t)-\tilde{u}}{w}\bigg)\,dt\geq\frac{q\varepsilon x}{\tilde{q}}\;\bigg|\;\xi(1)>\tilde{u}\bigg\}\,\frac{\mathrm{e}^{-s}ds}{x}\\
&\preceq\int_0^\infty\mathbf{P}\bigg\{\int_0^{(dq/\tilde{q})\wedge(1/\tilde{q})}I_{(0,\infty)}\bigg(\frac{\xi(1)-\tilde{u}}{w}-\frac{\tilde{q}t\,\xi'}{w}\bigg)\,dt\leq\frac{q(1+\varepsilon)^2x}{\tilde{q}}\;\bigg|\;\xi(1)>\tilde{u}+x^2w\bigg\}\\
&\qquad\qquad\times\frac{(1+\varepsilon)\,\tilde{w}\,g(\tilde{u})\,ds}{x\,(1-G(u))}\\
&\quad+\int_{\{s\geq 0\,:\,s_u^{-1}>(1+\varepsilon)\}\cup\{s\geq 0\,:\,w>(1+\varepsilon)\tilde{w}\}}\frac{\mathrm{e}^{-s}ds}{x}\\
&\quad+\int_0^\infty\frac{\mathbf{P}\big\{\tilde{u}<\xi(1)\leq\tilde{u}+x^2w\big\}}{\mathbf{P}\{\xi(1)>\tilde{u}\}}\,\frac{\mathrm{e}^{-s}ds}{x}\\
&\quad+\int_0^\infty\frac{\tilde{q}}{q\varepsilon x}\int_0^{dq/\tilde{q}}\mathbf{P}\bigg\{\frac{\xi(1)-\tilde{u}}{\tilde{w}}-\frac{\tilde{q}t\,\xi'}{\tilde{w}}-\frac{\xi(1-\tilde{q}t)-\tilde{u}}{\tilde{w}}\geq\frac{w\,x^2}{\tilde{w}}\;\bigg|\;\xi(1)>\tilde{u}\bigg\}\,dt\,\frac{\mathrm{e}^{-s}ds}{x},
\end{aligned}
$$

where $d,\varepsilon>0$ are constants. Here the second integral on the right hand side tends to zero (as $u\uparrow\hat{u}$) by (1.2), while the third integral tends to $(1-\mathrm{e}^{-x^2})/x$ by (1.1). Using (1.4) and (7.1) we further obtain

$$\frac{\tilde{q}}{q\varepsilon x}\int_0^{dq/\tilde{q}}\mathbf{P}\bigg\{\frac{\xi(1)-\tilde{u}}{\tilde{w}}-\frac{\tilde{q}t\,\xi'}{\tilde{w}}-\frac{\xi(1-\tilde{q}t)-\tilde{u}}{\tilde{w}}\geq\frac{w\,x^2}{\tilde{w}}\;\bigg|\;\xi(1)>\tilde{u}\bigg\}\;\begin{cases}\to 0\\ \leq d/(\varepsilon x)\end{cases},$$

and so the fourth integral tends to zero. Upon conditioning on the value of $\xi(1)$, the first integral on the right hand side of (7.7) finally becomes

$$
\begin{aligned}
&\int_0^\infty\int_x^\infty\mathbf{P}\bigg\{\int_0^{(dq/\tilde{q})\wedge(1/\tilde{q})}I_{(x^2,\infty)}\bigg(yx-\frac{\tilde{q}t\,\xi'}{w}\bigg)\,dt\leq\frac{q(1+\varepsilon)^2x}{\tilde{q}}\;\bigg|\;\frac{\xi(1)-u}{w}=s+yx\bigg\}\\
&\qquad\qquad\times\frac{w\,g(\tilde{u}+yxw)\,dy}{1-G(\tilde{u})}\,\frac{(1+\varepsilon)\,\tilde{w}\,g(\tilde{u})\,ds}{1-G(u)}\\
&\quad\simeq\int_x^\infty\int_0^\infty\mathbf{P}\bigg\{\frac{q(1+\varepsilon)^2\xi'}{w}\geq y-x\;\bigg|\;\frac{\xi(1)-u}{w}=s+yx\bigg\}\,\frac{(1+\varepsilon)\,w\,g(\tilde{u}+yxw)\,dsdy}{1-G(u)}
\end{aligned}
$$

$$= \int_x^\infty \mathbf{P}\biggl\{\frac{q(1+\varepsilon)^2\xi'}{w} \geq y-x \,\biggm|\, \frac{\xi(1)-u}{w} > yx\biggr\}\,\frac{(1+\varepsilon)\,(1-G(u+yxw))\,dy}{1-G(u)}$$

$$\leq (1+\varepsilon)\int_x^\infty \mathbf{P}\biggl\{\frac{q(1+\varepsilon)^2\xi'}{w} \geq y-x \,\biggm|\, \xi(1)>u\biggr\}\,dy$$

provided that $x,\varepsilon>0$ are sufficiently small compared with $d>0$ and Q. Evaluating the integral on the right hand side and inserting in (7.7) we therefore conclude

$$\text{(7.8)}\qquad \frac{1}{x}\biggl[1-\int_x^\infty \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\biggr] \preceq \frac{(1+\varepsilon)^3 q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w} + \frac{1-\mathrm{e}^{-x^2}}{x}.$$

In a similar (but less complicated) manner we get [using (1.1) and (7.1)]

$$\text{(7.9)}\qquad \frac{1}{x}\int_0^x \mathbf{P}\bigl\{L(u)/q\leq y \mid \xi(1)>u\bigr\}\,dy$$

$$\leq \mathbf{P}\biggl\{\int_0^{d\wedge(1/q)} I_{(0,\infty)}\biggl(\frac{\xi(1-qt)-u}{w}\biggr)\,dt \leq x \,\biggm|\, \xi(1)>u\biggr\}$$

$$\leq \mathbf{P}\biggl\{\int_0^{d\wedge(1/q)} I_{(x^2,\infty)}\biggl(\frac{\xi(1)-u}{w} - \frac{qt\,\xi'}{w}\biggr)\,dt \leq (1+\varepsilon)x \,\biggm|\, \xi(1)>u\biggr\}$$

$$+\,\mathbf{P}\biggl\{\int_0^{d\wedge(1/q)} I_{(x^2,\infty)}\biggl(\frac{\xi(1)-u}{w} - \frac{qt\,\xi'}{w} - \frac{\xi(1-qt)-u}{w}\biggr)\,dt \geq \varepsilon x \,\biggm|\, \xi(1)>u\biggr\}$$

$$\leq \mathbf{P}\biggl\{\frac{q(1+\varepsilon)x\,\xi'}{w} \geq \frac{\xi(1)-u}{w} - x^2 \,\biggm|\, \xi(1)>u+\sqrt{x}w\biggr\}$$

$$+\,\frac{\mathbf{P}\bigl\{u<\xi(1)\leq u+\sqrt{x}w\bigr\}}{\mathbf{P}\{\xi(1)>u\}}$$

$$+\,\frac{1}{\varepsilon x}\int_0^d \mathbf{P}\biggl\{\frac{\xi(1)-u}{w} - \frac{qt\,\xi'}{w} - \frac{\xi(1-qt)-u}{w} \geq x^2 \,\biggm|\, \xi(1)>u\biggr\}\,dt$$

$$\preceq \mathbf{P}\biggl\{\frac{q(1+\varepsilon)x\,\xi'}{w} \geq \sqrt{x}-x^2 \,\biggm|\, \xi(1)>u\biggr\}$$

$$+\,\frac{G(u+\sqrt{x}\,w)-G(u)}{1-G(u)}$$

$$\preceq \frac{(1+\varepsilon)x}{\sqrt{x}-x^2}\;\frac{q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w}$$

$$+\,(1-\mathrm{e}^{-\sqrt{x}})$$

provided that $x,\varepsilon>0$ are sufficiently small compared with $d>0$ and Q.

On the other hand (3.3) and (7.2) combine with the arguments above to yield

$$\text{(7.10)}\qquad \frac{1}{x}\biggl[1-\int_x^\infty \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\biggr]$$

$$
\begin{aligned}
&\succeq \int_0^\infty \mathbf{P}\bigg\{\int_0^{(dq/s_u\tilde q)\wedge(1/\tilde q)} I_{(0,\infty)}\Big(\frac{\xi(1-\tilde q t)-\tilde u}{w}\Big)\,dt \le \frac{q(1-\varepsilon)x}{s_u\tilde q}\ \Big|\ \xi(1)>\tilde u\bigg\}\ \frac{\mathrm{e}^{-s}ds}{x}\\
&\quad -\int_0^1 \mathbf{P}\bigg\{\int_{d\wedge(s/q)}^{s/q} I_{(u,\hat u)}(\xi(s-qt))\,dt \ge \varepsilon x\ \Big|\ \xi(s)>u\bigg\}\ \frac{\mathbf{P}\{\xi(s)>u\}}{\mathbf{E}\{L(u)\}}\,ds\\
&\succeq \int_0^\infty \mathbf{P}\bigg\{\int_0^{dq/s_u\tilde q} I_{(0,\infty)}\Big(\frac{\xi(1-\tilde q t)-\tilde u}{w}\Big)\,dt \le \frac{q(1-\varepsilon)x}{s_u\tilde q}\ \Big|\ \xi(1)>\tilde u\bigg\}\ \frac{\mathrm{e}^{-s}ds}{x}\\
&\quad -\int_{\{s\ge 0\,:\,w<(1-\varepsilon)\tilde w\}} \frac{\mathrm{e}^{-s}ds}{x}\\
&\quad -\frac{1}{\varepsilon x}\int_0^1\int_{d\wedge(s/q)}^{s/q} \mathbf{P}\{\xi(s-qt)>u\ |\ \xi(s)>u\}\,dt\ \frac{\mathbf{P}\{\xi(s)>u\}}{\mathbf{E}\{L(u)\}}\,ds\\
&\succeq \int_0^\infty \mathbf{P}\bigg\{\int_0^{dq/s_u\tilde q} I_{(-x^2,\infty)}\Big(\frac{\xi(1)-\tilde u}{w}-\frac{\tilde q t\,\xi'}{w}\Big) \le \frac{q(1-2\varepsilon)x}{s_u\tilde q}\ \Big|\ \xi(1)>\tilde u\bigg\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \frac{(1-\varepsilon)\,\tilde w\,g(\tilde u)\,ds}{x\,(1-G(u))}\\
&\quad -\int_0^\infty \mathbf{P}\bigg\{\int_0^{dq/s_u\tilde q} I_{(x^2,\infty)}\Big(\frac{\xi(1-\tilde q t)-\tilde u}{w}-\frac{\xi(1)-\tilde u}{w}+\frac{\tilde q t\,\xi'}{w}\Big)\,dt \ge \frac{q\varepsilon x}{s_u\tilde q}\ \Big|\ \xi(1)>\tilde u\bigg\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \frac{\mathrm{e}^{-s}ds}{x}\\
&\quad -\varepsilon/x\\
&\succeq \int_0^\infty\int_0^\infty \mathbf{P}\bigg\{\frac{q(1-2\varepsilon)\xi'}{w}\ge y+x\ \Big|\ \frac{\xi(1)-u}{w}=s+yx\bigg\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \frac{w\,g(\tilde u+yxw)\,dy}{1-G(\tilde u)}\ \frac{(1-\varepsilon)\,\tilde w\,g(\tilde u)\,ds}{1-G(u)}\\
&\quad -\int_0^\infty \frac{s_u\tilde q}{q\varepsilon x}\int_0^{dq/s_u\tilde q} \mathbf{P}\bigg\{\frac{\xi(1-\tilde q t)-\tilde u}{\tilde w}-\frac{\xi(1)-\tilde u}{\tilde w}+\frac{\tilde q t\,\xi'}{\tilde w}\ge \frac{w\,x^2}{\tilde w}\ \Big|\ \xi(1)>\tilde u\bigg\}\,dt\ \frac{\mathrm{e}^{-s}ds}{x}\\
&\quad -\varepsilon/x\\
&\simeq \int_0^\infty \mathbf{P}\bigg\{\frac{q(1-2\varepsilon)\xi'}{w}\ge y+x\ \Big|\ \frac{\xi(1)-u}{w}>yx\bigg\}\ \frac{(1-\varepsilon)\,(1-G(u+yxw))\,dy}{1-G(u)}\\
&\quad -\varepsilon/x\\
&\succeq (1-\varepsilon)\int_0^\infty \mathbf{P}\bigg\{\frac{q(1-2\varepsilon)\xi'}{w}-x\ge y\ \Big|\ \xi(1)>u\bigg\}\,dy\\
&\quad -x^{-1/3}\,(1-\varepsilon)\,\frac{G(u+x^{-1/3}xw)-G(u)}{1-G(u)}\\
&\quad -(1-\varepsilon)\,\mathbf{E}\bigg\{\Big(\frac{q(1-2\varepsilon)(\xi')^+}{w}\Big)^{\varrho}\ \Big|\ \xi(1)>u\bigg\}\int_{x^{-1/3}}^\infty y^{-\varrho}\,dy
\end{aligned}
$$

$$- \varepsilon/x$$

$$\succeq (1-\varepsilon)\,(1-2\varepsilon)\,\frac{q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w} \;-\; x$$

$$- x^{-1/3}\,(1-\mathrm{e}^{-x^{2/3}})$$

$$- \mathbf{E}\biggl\{\biggl(\frac{q\,(\xi')^+}{w}\biggr)^{\varrho}\,\biggm|\,\xi(1)>u\biggr\}\,\frac{x^{(\varrho-1)/3}}{\varrho-1}$$

$$- \varepsilon/x \qquad \text{for } x \text{ sufficiently small.}$$

Since $\succeq$ is transitive, (7.10) shows that

$$\begin{aligned}
&\varliminf_{u\uparrow\hat{u}}\biggl(\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\bigl\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\bigr\} - \frac{q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w}\biggr)\\
&\qquad\geq \varliminf_{u\uparrow\hat{u}}\biggl(\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\bigl\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\bigr\} - \frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\biggr)\\
&\qquad\quad + \varliminf_{u\uparrow\hat{u}}\biggl(\frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy - \frac{q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w}\biggr)\\
&\qquad\geq -x \;-\; \frac{1-\mathrm{e}^{-x^{2/3}}}{x^{1/3}} \;-\; \varlimsup_{u\downarrow\hat{u}}\mathbf{E}\biggl\{\biggl(\frac{q\,(\xi')^+}{w}\biggr)^{\varrho}\,\biggm|\,\xi(1)>u\biggr\}\,\frac{x^{(\varrho-1)/3}}{\varrho-1}\\
&\qquad\to 0 \qquad \text{as } x\downarrow 0.
\end{aligned}$$

Hence we conclude that

$$\varliminf_{u\uparrow\hat{u}}\biggl(\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\bigl\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\bigr\} - \frac{q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w}\biggr) \geq 0. \tag{7.11}$$

On the other, defining $\Gamma:\mathbb{R}^+\to[0,1]$ by

$$\Gamma(x) = \varliminf_{u\uparrow\hat{u}}\int_x^\infty \frac{\mathbf{P}\{L(u)/q(u)>y\}}{\mathbf{E}\{L(u)/q(u)\}}\,dy,$$

we obviously have

$$\frac{1-\Gamma(x)}{x} = \varlimsup_{u\uparrow\hat{u}}\frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy \leq \varlimsup_{u\uparrow\hat{u}}\frac{1}{\mathbf{E}\{L(u)/q\}}\mathbf{P}\bigl\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\bigr\}.$$

Here the right hand side is finite by Theorem 3. In particular $\lim_{x\downarrow 0}\Gamma(x)=1$, and replacing Λ with Γ in the proof of Lemma 3 we therefore easily deduce

$$\lim_{x\downarrow 0}\varlimsup_{u\uparrow\hat{u}}\frac{1}{\mathbf{E}\{L(u)\}/q}\,\mathbf{P}\Bigl\{L(\tilde{u})/q(\tilde{u})\leq x,\ \max\nolimits_{1\leq k\leq K}\xi(t_a^u(k))>u\Bigr\}=0 \tag{7.12}$$

[without using Theorem 1 or Assumption 1], where $\tilde{u} \equiv u - \sigma w(u)$ and $\sigma > 0$. Writing $\overline{u} \equiv u + \sigma w$ for $\sigma > 0$ we further have

$$(7.13)\quad \varlimsup_{u\uparrow\hat{u}}\bigg(\frac{1}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\big\} - \frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>\overline{u}\}}{w(\tilde{u})}\bigg)$$

$$\leq \varlimsup_{u\uparrow\hat{u}}\bigg(\frac{1}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\big\} - \frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(\tilde{u})/q(\tilde{u})>y\}}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,dy\bigg)$$

$$+\varlimsup_{u\uparrow\hat{u}}\bigg(\frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(\tilde{u})/q(\tilde{u})>y\}}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,dy - \frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>\tilde{u}\}}{w(\tilde{u})}\bigg)$$

$$+\varlimsup_{u\uparrow\hat{u}}\bigg(\frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>\tilde{u}\}}{w(\tilde{u})} - \frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>\overline{u}\}}{w(\tilde{u})}\bigg).$$

Here (7.9) and (7.12) plus (1.6), (2.1), (6.1) and Assumption 3 give that

$$\varlimsup_{u\uparrow\hat{u}}\bigg(\frac{1}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\big\} - \frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(\tilde{u})/q(\tilde{u})>y\}}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,dy\bigg)$$

$$\leq \varlimsup_{u\uparrow\hat{u}}\frac{\mathbf{P}\{\xi(1)>\tilde{u}\}}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,\frac{1}{x}\int_0^x \mathbf{P}\big\{L(\tilde{u})/\tilde{q}\leq y\,\big|\,\xi(1)>\tilde{u}\big\}\,dy$$

$$+\varlimsup_{u\uparrow\hat{u}}\frac{1}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,\mathbf{P}\Big\{L(\tilde{u})/q(\tilde{u})\leq x,\ \max_{1\leq k\leq K}\xi(t_a^u(k))>u\Big\}$$

$$+\varlimsup_{u\uparrow\hat{u}}\frac{1}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>\overline{u},\ \max_{0\leq k\leq K}\xi(t_a^u(k))\leq u\Big\}$$

$$\leq \mathfrak{P}_5\kappa\,\varlimsup_{u\uparrow\hat{u}}\bigg(\frac{x}{\sqrt{x}-x^2}\,\frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>\tilde{u}\}}{w(\tilde{u})} + (1-\mathrm{e}^{-\sqrt{x}})\bigg)$$

$$+\frac{1}{\mathfrak{P}_2}\varlimsup_{u\uparrow\hat{u}}\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\Big\{L(\tilde{u})/q(\tilde{u})\leq x,\ \max_{1\leq k\leq K}\xi(t_a^u(k))>u\Big\}$$

$$+\frac{1}{\mathfrak{P}_2}\varlimsup_{u\uparrow\hat{u}}\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>\overline{u},\ \max_{0\leq k\leq K}\xi(t_a^u(k))\leq u\Big\}$$

$$\to 0 \quad \text{as } x\downarrow 0 \text{ and } a\downarrow 0 \text{ (in that order)},$$

where we also used the obvious fact that (7.2) implies

$$(7.14)\quad \varlimsup_{u\uparrow\hat{u}}\mathbf{E}\bigg\{\frac{q(u)\,(\xi')^+}{w(u)}\,\bigg|\,\xi(1)>u\bigg\} \leq \varlimsup_{u\uparrow\hat{u}}\bigg[\mathbf{E}\bigg\{\bigg(\frac{q(u)\,(\xi')^+}{w(u)}\bigg)^{\varrho}\,\bigg|\,\xi(1)>u\bigg\}\bigg]^{1/\varrho} < \infty.$$

Moreover (7.8) yields

$$\varlimsup_{u\uparrow\hat{u}}\bigg(\frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(\tilde{u})/q(\tilde{u})>y\}}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,dy - \frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+\,|\,\xi(1)>\tilde{u}\}}{w(\tilde{u})}\bigg) \leq \frac{1-\mathrm{e}^{-x^2}}{x} \to 0$$

as $x\downarrow 0$. Finally (7.2) and Hölder's inequality combine with (2.1) to show that

$$\begin{aligned}
&\overline{\lim_{u\uparrow\hat{u}}}\bigg(\frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+|\,\xi(1)>\tilde{u}\}}{w(\tilde{u})}-\frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+|\,\xi(1)>\overline{u}\}}{w(\tilde{u})}\bigg)\\
&\qquad\le \overline{\lim_{u\uparrow\hat{u}}}\,\mathbf{E}\bigg\{\frac{q(\tilde{u})\,(\xi')^+}{w(\tilde{u})}\,I_{\{\tilde{u}<\xi(1)\le\overline{u}\}}\,\bigg|\,\xi(1)>\tilde{u}\bigg\}\\
&\qquad\le \overline{\lim_{u\uparrow\hat{u}}}\bigg[\mathbf{E}\bigg\{\bigg(\frac{q(\tilde{u})\,(\xi')^+}{w(\tilde{u})}\bigg)^{\varrho}\,\bigg|\,\xi(1)>\tilde{u}\bigg\}\bigg]^{1/\varrho}\bigg[\frac{\mathbf{P}\{\tilde{u}<\xi(1)\le\overline{u}\}}{\mathbf{P}\{\xi(1)>\tilde{u}\}}\bigg]^{(\varrho-1)/\varrho}\\
&\qquad\to 0\qquad\text{as }\ \sigma\downarrow 0.
\end{aligned}$$

Inserting all these facts in (7.13) we obtain

$$\overline{\lim_{u\uparrow\hat{u}}}\bigg(\frac{1}{\mathbf{E}\{L(\tilde{u})/q(\tilde{u})\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\}-\frac{q(\tilde{u})\,\mathbf{E}\{(\xi')^+|\,\xi(1)>\overline{u}\}}{w(\tilde{u})}\bigg)\le 0,$$

which in turn [by (1.2) and (1.3)] implies that

$$\overline{\lim_{u\uparrow\hat{u}}}\bigg(\frac{w(\tilde{u})/w(\overline{u})}{\mathbf{E}\{L(\tilde{u})/q(\overline{u})\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\}-\frac{q(\overline{u})\,\mathbf{E}\{(\xi')^+|\,\xi(1)>\overline{u}\}}{w(\overline{u})}\bigg)\le 0.$$

But here (1.1) and (2.1) combine with Theorem 3 to show that

$$\begin{aligned}
&\frac{w(\tilde{u})/w(\overline{u})}{\mathbf{E}\{L(\tilde{u})/q(\overline{u})\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\}-\frac{1}{\mathbf{E}\{L(\overline{u})/q(\overline{u})\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\}\\
&\qquad=\bigg(\frac{\mathbf{E}\{L(\overline{u})\}\,w(\tilde{u})}{\mathbf{E}\{L(\tilde{u})\}\,w(\overline{u})}-1\bigg)\frac{1}{\mathbf{E}\{L(\overline{u})/q(\overline{u})\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\}\\
&\qquad\sim\bigg(\frac{\mathbf{P}\{\xi(1)>\overline{u}\}\,\tilde{u}}{\mathbf{P}\{\xi(1)>\tilde{u}\}\,\overline{u}}-1\bigg)\frac{1}{\mathbf{E}\{L(\overline{u})/q(\overline{u})\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>\overline{u}\}\\
&\qquad\to 0\qquad\text{as }\ u\uparrow\hat{u}\ \text{ and }\ \sigma\downarrow 0\ \text{ (in that order).}
\end{aligned}$$

Consequently we have

$$\overline{\lim_{u\uparrow\hat{u}}}\bigg(\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}-\frac{q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w}\bigg)\le 0. \tag{7.15}$$

Combining (7.11) with (7.15), observing that by (7.15) and Theorem 2,

$$\begin{aligned}
&\underline{\lim_{u\uparrow\hat{u}}}\,\mathbf{E}\bigg\{\frac{q(u)\,(\xi')^+}{w(u)}\,\bigg|\,\xi(1)>u\bigg\}\\
&\qquad\ge-\overline{\lim_{u\uparrow\hat{u}}}\bigg(\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}-\frac{q\,\mathbf{E}\{(\xi')^+|\,\xi(1)>u\}}{w}\bigg)\\
&\qquad\quad+\underline{\lim_{u\uparrow\hat{u}}}\,\frac{1}{\mathbf{E}\{L(u)/q\}}\,\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}\\
&\qquad>0,
\end{aligned}$$

and recalling (7.14), we finally conclude that (7.4) and (7.5) holds. □

8. Sufficient criteria for tightness. It is an old idea to derive tightness for a process from requirements on it's increments. Generally speaking, most derivations of this type have many steps in common. Thus they should not be regarded as inaccessible for readers despite their often both long and technical proofs.

Proposition 3. *Assume that* (1.5)-(1.7) *hold with* $G\in\mathcal{D}$ *and* $\xi(t)$ $\mathbf{P}$*-continuous.*
(i) *If there exist* $\lambda_0,\, c,\, e,\, C>0,\ u_5<\hat{u}$ *and* $d>1$ *such that*

$$\mathbf{P}\Big\{\xi(1)>u+(\lambda+\nu)w(u),\,\xi(1-q(u)t)\le u+\nu w(u)\Big\}\le C\,t^d\,\lambda^{-e}\,\mathbf{P}\{\xi(1)>u\}\tag{8.1}$$

for $0\le t^c\le\lambda\le\lambda_0,\ \nu\ge 0$ *and* $u\in[u_5,\hat{u})$, *then Assumption 3 holds and* $\mathfrak{P}_5<\infty$.
(ii) *If there exist* $\lambda_0,\, c,\, e,\, C>0,\ u_5<\hat{u}$ *and* $d>1$ *such that*

$$\mathbf{P}\Big\{\xi(1-q(u)t)>u+(\lambda+\nu)w(u),\,\xi(1)\le u+\nu w(u)\Big\}\le C\,t^d\,\lambda^{-e}\,\mathbf{P}\{\xi(1)>u\}\tag{8.2}$$

for $0<t^c\le\lambda\le\lambda_0,\ \nu\ge 0$ *and* $u\in[u_5,\hat{u})$, *then Assumption 3 holds.*

Proposition 5 below describes one method potentially useful to verify (8.1). Other such methods include estimates related to Tjebysjev's inequality like e.g.,

$$\frac{\mathbf{P}\big\{\xi(1)>u+(\lambda+\nu)w,\,\xi(1-qt)\le u+\nu w\big\}}{\mathbf{P}\{\xi(1)>u\}}\le(\lambda w)^{-2}\,\mathbf{E}\big\{[\xi(1)-\xi(1-qt)]^2\,\big|\,\xi(1)>u\big\}.$$

If $\xi(t)$ has a (super-exponetially) 'light-tailed' distribution the estimate

$$\mathbf{P}\big\{\xi(1)>u+(\lambda+\nu)w,\,\xi(1-qt)\le u+\nu w\big\}\le\mathbf{P}\big\{\xi(1)+\varDelta[\xi(1)-\xi(1-qt)]>u+\varDelta\lambda w\big\}$$

may also work provided that $\varDelta$ is suitably choosen.

When Assumption 1 holds (8.1) is interpreted as $\mathbf{P}\{\zeta(t)\le-\lambda\}\le C\,t^d\,\lambda^{-e}$.

Although often useful, (8.1) is a stronger condition than (8.2). In particular (8.1) cannot hold when $\mathfrak{P}_5=\infty$ while (8.2) still may work well (cf. Section 13).

Proof of Proposition 3.(i). Take $a\in(0,\hat{a}]$ and choose $j_k^n=j_{a,k}^{u,n}\in\mathbb{N}$ such that $t_{a2^{-n}}^u(j_k^n)\ge t_{a2^{-(n+1)}}^u(k)>t_{a2^{-n}}^u(j_k^n+1)$ for $n\in\mathbb{N}$. Further let $T_u\equiv\inf\big\{t\in[0,1]:t^{-\kappa}u<\hat{u}\big\}$ and $\mathcal{C}_u\equiv\bigcup_{n=0}^{\infty}\mathcal{C}_{u,n}$ where $\mathcal{C}_{u,n}\equiv\big\{t_{a2^{-n}}^u(1),t_{a2^{-n}}^u(2),\ldots\big\}$. Since $t_{a2^{-n}}^u(k)-t_{a2^{-n}}^u(k+1)\le a2^{-n}\sup_{v<\hat{u}}q(v)$, the fact that $\mathcal{C}_u$ is dense in $[T_u,1]$ will follow if we can prove that $\lim_{k\to\infty}t_{a2^{-n}}^u(k)=T_u$ whenever $K(a2^{-n},u)=\infty$. But if the limit were greater that T_u, then we would have

$$1\;=\;\lim_{k\to\infty}t_{a2^{-n}}^u(k+1)/t_{a2^{-n}}^u(k)\;=\;1-\lim_{k\to\infty}a2^{-n}q\big(t_{a2^{-n}}^u(k)^{-\kappa}u\big).$$

In view of (1.6) this implies $\lim_{k\to\infty} t^u_{a2^{-n}}(k)^{-\kappa}u = \hat{u}$, so that $t^u_{a2^{-n}}(k)\to T_u$!

Taking $\lambda_n \equiv (1-2^{-\varrho})\sum_{k=0}^n 2^{-\varrho k}$ where $\varrho\in\big(0, c\wedge((d-1)/e)\big)$, (1.7) yields

$$t^u_{a2^{-(n+1)}}(k)^{-\kappa}\sigma(\lambda_n-\lambda_{n-1})w \geq (2\mathfrak{P}_3)^{-1}\sigma(1-2^{-\varrho})2^{-\varrho n}\, w\big(t^u_{a2^{-(n+1)}}(k)^{-\kappa}u\big) \tag{8.3}$$

for u sufficiently large. Further (1.6) shows that

$$\begin{aligned}&\frac{1-t^u_{a2^{-n}}(j^n_k+1)/t^u_{a2^{-(n+1)}}(k)}{q\big(t^u_{a2^{-(n+1)}}(k)^{-\kappa}u\big)} \qquad (8.4)\\ &\leq \frac{a2^{-n}q\big(t^u_{a2^{-n}}(j^n_k)^{-\kappa}u\big)}{q\big(t^u_{a2^{-(n+1)}}(k)^{-\kappa}u\big)} \leq \frac{a2^{-n}}{\frac{1}{2}\mathfrak{P}_2}\bigg(\frac{t^u_{a2^{-(n+1)}}(k)^{-\kappa}}{t^u_{a2^{-n}}(j^n_k)^{-\kappa}}\bigg)^{\rho-1/\kappa} \leq \mathfrak{P}_2^{-1}a2^{1-n+(\kappa\rho-1)^+}.\end{aligned}$$

Since $\mathcal{C}_u$ separates $\{\xi(t)\}_{t\in[T_u,1]}$ (being dense in $[T_u,1]$), (4.3) and (8.1) now give

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w,\ \max_{1\leq k\leq K}\xi(t^u_a(k))\leq u\Big\} \qquad (8.5)\\
&\leq \mathbf{P}\bigg\{\bigcup_{n=0}^{\infty}\Big\{\sup_{t\in\mathcal{C}_{u,n+1}}\xi(t)>u+\sigma\lambda_n w,\ \sup_{t\in\mathcal{C}_{u,n}}\xi(t)\leq u+\sigma\lambda_{n-1}w\Big\}\bigg\}\\
&\leq \sum_{n=0}^{\infty}\sum_{k=1}^{K(a2^{-(n+1)},u)}\mathbf{P}\Big\{\xi\big(t^u_{a2^{-(n+1)}}(k)\big)>u+\sigma\lambda_n w,\ \xi\big(t^u_{a2^{-n}}(j^n_k+1)\big)\leq u+\sigma\lambda_{n-1}w\Big\}\\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{K(a2^{-(n+1)},u)}\mathbf{P}\bigg\{\xi(1)>\frac{u+\sigma\lambda_n w}{t^u_{a2^{-(n+1)}}(k)^{\kappa}},\ \xi\bigg(\frac{t^u_{a2^{-n}}(j^n_k+1)}{t^u_{a2^{-(n+1)}}(k)}\bigg)\leq\frac{u+\sigma\lambda_{n-1}w}{t^u_{a2^{-(n+1)}}(k)^{\kappa}}\bigg\}\\
&\leq \sum_{n=0}^{\infty}\sum_{k=1}^{K(a2^{-(n+1)},u)}\frac{C\,\big(\mathfrak{P}_2^{-1}a2^{1-n+(\kappa\rho-1)^+}\big)^d}{\big((2\mathfrak{P}_3)^{-1}\sigma(1-2^{-\varrho})2^{-\varrho n}\big)^e}\mathbf{P}\big\{\xi(1)>t^u_{a2^{-(n+1)}}(k)^{-\kappa}u\big\}\\
&\leq \sum_{n=0}^{\infty}\frac{C\,\big(\mathfrak{P}_2^{-1}a2^{1-n+(\kappa\rho-1)^+}\big)^d}{\big((2\mathfrak{P}_3)^{-1}\sigma(1-2^{-\varrho})2^{-\varrho n}\big)^e}\,\frac{2\,\mathbf{E}\{L(u)\}}{q\,\mathfrak{P}_0\mathfrak{P}_2 a2^{-(n+1)}}\int_0^\infty\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa-\rho}}.\end{aligned}$$

Hence the following strong version of Assumption 3 holds;

$$\varlimsup_{u\uparrow\hat{u}}\frac{1}{\mathbf{E}\{L(u)/q(u)\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w(u),\ \max_{1\leq k\leq K}\xi(t^u_a(k))\leq u\Big\}\to 0 \tag{8.6}$$

as $a\downarrow 0$. Moreover an inspection of the proof of Theorem 3 reveals that the fact that the left-hand side of (8.6) is finite for a small implies

$$\varlimsup_{u\uparrow\hat{u}}\frac{\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}}{\mathbf{E}\{L(u)/q(u)\}}<\infty \quad\text{so that}\quad \mathfrak{P}_5=\varlimsup_{u\uparrow\hat{u}}\frac{q(u)\,\mathbf{P}\{\xi(1)>u\}}{\kappa\,\mathbf{E}\{L(u)\}}<\infty. \ \square$$

Proof of Proposition 3.(ii). Now (8.3) and (8.4) change to

$$t^u_{a2^{-n}}(j^n_k)^{-\kappa}\sigma(\lambda_n-\lambda_{n-1})w \geq (2\mathfrak{P}_3)^{-1}\sigma(1-2^{-\varrho})2^{-\varrho n}\, w\big(t^u_{a2^{-n}}(j^n_k)^{-\kappa}u\big), \quad\text{and}$$

$$\frac{1 - t^u_{a2^{-(n+1)}}(k)/t^u_{a2^{-n}}(j^n_k)}{q\big(t^u_{a2^{-n}}(j^n_k)^{-\kappa}u\big)} \;\le\; \frac{1 - t^u_{a2^{-n}}(j^n_k{+}1)/t^u_{a2^{-n}}(j^n_k)}{q\big(t^u_{a2^{-n}}(j^n_k)^{-\kappa}u\big)} \;=\; a2^{-n}.$$

Consequently (8.5) modifies to

$$\textbf{P}\Big\{\sup\nolimits_{t\in[0,1]}\xi(t)>u+\sigma w,\ \max\nolimits_{0\le k\le K}\xi(t^u_a(k))\le u\Big\} \tag{8.7}$$

$$\le \sum_{n=0}^{\infty}\sum_{k=1}^{K(a2^{-(n+1)},u)} \textbf{P}\Big\{\xi\big(t^u_{a2^{-(n+1)}}(k)\big)>u+\sigma\lambda_n w,\ \xi\big(t^u_{a2^{-n}}(j^n_k)\big)\le u+\sigma\lambda_{n-1}w\Big\}$$

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{K(a2^{-(n+1)},u)} \textbf{P}\bigg\{\xi\bigg(\frac{t^u_{a2^{-(n+1)}}(k)}{t^u_{a2^{-n}}(j^n_k)}\bigg)>\frac{u+\sigma\lambda_n w}{t^u_{a2^{-n}}(j^n_k)^{\kappa}},\ \xi(1)\le\frac{u+\sigma\lambda_{n-1}w}{t^u_{a2^{-n}}(j^n_k)^{\kappa}}\bigg\}$$

$$\le \sum_{n=0}^{\infty}\sum_{k=1}^{K(a2^{-(n+1)},u)} \frac{C\,(a2^{-n})^d}{\big((2\mathfrak{P}_3)^{-1}\sigma(1-2^{-\varrho})2^{-\varrho n}\big)^e}\,\textbf{P}\big\{\xi(1)>t^u_{a2^{-n}}(j^n_k)^{-\kappa}u\big\}.$$

But here (1.3) and (1.6) show that

$$\begin{aligned} &t^u_{a2^{-(n+1)}}(k-\ell) - t^u_{a2^{-(n+1)}}(k)\\ &\quad= \sum_{i=1}^{\ell} t^u_{a2^{-(n+1)}}(k-i)\,a2^{-(n+1)}\,q\big(t^u_{a2^{-(n+1)}}(k-i)^{-\kappa}u\big)\\ &\quad= a2^{-(n+1)}\,t^u_{a2^{-n}}(j^n_k{+}1)\,q\big(t^u_{a2^{-n}}(j^n_k{+}1)^{-\kappa}u\big)\sum_{i=1}^{\ell}\frac{p\big(t^u_{a2^{-(n+1)}}(k-j)^{-\kappa}u\big)}{p\big(t^u_{a2^{-n}}(j^n_k{+}1)^{-\kappa}u\big)}\\ &\quad\ge a2^{-(n+1)}\,\tfrac12 t^u_{a2^{-n}}(j^n_k)\,\tfrac12\mathfrak{P}_2\,2^{(\kappa\rho-1)^+}q\big(t^u_{a2^{-n}}(j^n_k)^{-\kappa}u\big)\,\ell\,(2\mathfrak{P}_1)^{-1} \end{aligned}$$

for $a\in(0,\hat a]$ and u large. Hence there is an $\ell\in\mathbb{Z}^+$ such that $t^u_{a2^{-n}}(j^n_k)\le t^u_{a2^{-(n+1)}}(k-\ell)$ for $k\ge\ell+1$, and inserting in (8.7) and using (4.3) we thus conclude

$$\begin{aligned} &\textbf{P}\Big\{\sup\nolimits_{t\in[0,1]}\xi(t)>u+\sigma w,\ \max\nolimits_{0\le k\le K}\xi(t^u_a(k))\le u\Big\}\\ &\quad\le \sum_{n=0}^{\infty}\frac{\ell\,C\,(a2^{-n})^d}{\big((2\mathfrak{P}_3)^{-1}\sigma(1-2^{-\varrho})2^{-\varrho n}\big)^e}\,\textbf{P}\{\xi(1)>u\}\\ &\qquad+\sum_{n=0}^{\infty}\frac{C\,(a2^{-n})^d}{\big((2\mathfrak{P}_3)^{-1}\sigma(1-2^{-\varrho})2^{-\varrho n}\big)^e}\,\frac{2\,\textbf{E}\{L(u)\}}{q\,\mathfrak{P}_0\mathfrak{P}_2 a2^{-(n+1)}}\int_0^\infty\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa-\rho}}. \quad\square \end{aligned}$$

Proposition 4. **(i)** *If* (1.5)-(1.7) *hold with* $G\in\mathcal{D}$, *and if*

$$\nu_1(a,\sigma)\equiv\varlimsup_{u\uparrow\hat u}\frac{\textbf{P}\big\{\sup_{t\in[1,1+aq(u)]}\xi(t)>u+\sigma w(u),\ \xi(1)\le u\big\}}{\textbf{P}\{\xi(1)>u\}}<\infty \tag{8.8}$$

for some $\sigma>0$ *and* $a\in(0,\hat a]$, *then Assumption 3′ holds and* $\mathfrak{P}_5<\infty$. *If in addition* $\nu_1(a,\sigma)/a\to0$ *as* $a\downarrow0$ *for each* $\sigma>0$, *then Assumption 3 holds.*

(ii) *If* (1.5)-(1.7) *hold with* $G\in\mathcal{D}$, *and if*

$$\nu_2(a,\sigma)\equiv\overline{\lim_{u\uparrow\hat{u}}}\,\frac{\mathbf{P}\big\{\sup_{t\in[1-aq(u),1]}\xi(t)>u+\sigma w(u),\,\xi(1)\le u\big\}}{\mathbf{P}\{\xi(1)>u\}}<\infty \tag{8.9}$$

for some $\sigma>0$ *and* $a\in(0,\hat{a}]$, *then Assumption 3′ holds. If in addition* $\nu_2(a,\sigma)/a\to 0$ *as* $a\downarrow 0$ *for each* $\sigma>0$, *then Assumption 3 holds.*

Proof of (i). Since $(1-aq)^{-1}\le 1+2aq$ for $aq\le\frac{1}{2}$, (1.6) shows that

$$\big(1-aq(t_a^u(k-1)^{-\kappa}u)\big)^{-1}\le 2a\,q(t_a^u(k-1)^{-\kappa}u)\le 2a\,(\tfrac{1}{2}\mathfrak{P}_2)^{-1}\,2^{(\kappa\rho-1)^+}q(t_a^u(k)^{-\kappa}u)$$

for $a\in(0,\hat{a}]$ and u sufficiently large. Using (1.7) and (8.8) we thus get

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{t\in[t_a^u(k),t_a^u(k-1)]}\xi(t)>u+\sigma w,\,\xi(t_a^u(k))\le u\Big\} \qquad (8.10)\\
&=\mathbf{P}\Bigg\{\sup_{t\in[1,\,(1-aq(t_a^u(k-1)^{-\kappa}u))^{-1}]}\xi(t)>\frac{u+\sigma w}{t_a^u(k)^{\kappa}},\,\xi(1)\le\frac{u}{t_a^u(k)^{\kappa}}\Bigg\}\\
&\le\mathbf{P}\Bigg\{\sup_{t\in[1,\,1+\mathfrak{P}_2^{-1}a2^{2+(\kappa\rho-1)^+}q(t_a^u(k)^{-\kappa}u)]}\xi(t)>\frac{u}{t_a^u(k)^{\kappa}}+\frac{\sigma w(t_a^u(k)^{-\kappa}u)}{2\mathfrak{P}_3},\,\xi(1)\le\frac{u}{t_a^u(k)^{\kappa}}\Bigg\}\\
&\le 2\nu_1(a,\sigma)\,\mathbf{P}\big\{\xi(1)>t_a^u(k)^{-\kappa}u\big\}\qquad\text{for } u \text{ sufficiently large.}\end{aligned}$$

In a by now familiar manner (4.3) therefore yields that

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w,\,\max_{1\le k\le K}\xi(t_a^u(k))\le u\Big\}\\
&\qquad\le\frac{4\,\nu_1(a,\sigma)\,\mathbf{E}\{L(u)\}}{q\,\mathfrak{P}_0\mathfrak{P}_2\,a}\int_0^\infty\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa-\rho}}.\end{aligned}$$

Hence the left-hand side of (8.6) is finite so that $\mathfrak{P}_5<\infty$ (cf. the proof of Proposition 3). Further Assumption 3′ holds, and Assumption 3 holds if $\nu_1(a,\sigma)/a\to 0$. $\Box$

Proof of (ii). Using (8.9) instead of (8.8), (8.10) changes to

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{t\in[t_a^u(k),t_a^u(k-1)]}\xi(t)>u+\sigma w,\,\xi(t_a^u(k-1))\le u\Big\}\\
&=\mathbf{P}\Bigg\{\sup_{t\in[1-aq(t_a^u(k-1)^{-\kappa}u),\,1]}\xi(t)>\frac{u}{t_a^u(k-1)^{\kappa}}+\frac{\sigma w(t_a^u(k-1)^{-\kappa}u)}{2\mathfrak{P}_3},\,\xi(1)\le\frac{u}{t_a^u(k-1)^{\kappa}}\Bigg\}\\
&\le 2\nu_2(a,\sigma)\,\mathbf{P}\big\{\xi(1)>t_a^u(k-1)^{-\kappa}u\big\}\qquad\text{for } u \text{ sufficiently large.}\end{aligned}$$

Thus we readily conclude [again invoking (4.3)]

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u+\sigma w,\,\max_{0\le k\le K}\xi(t_a^u(k))\le u\Big\}\\
&\qquad\le 2\nu_2(a,\sigma)\Bigg[\mathbf{P}\{\xi(1)>u\}+\frac{2\,\mathbf{E}\{L(u)\}}{q\,\mathfrak{P}_0\mathfrak{P}_2\,a}\int_0^\infty\frac{(1-F(s))\,ds}{(1+Ws)^{1+1/\kappa-\rho}}\Bigg].\qquad\Box\end{aligned}$$

For some processes there exist constants $C,c>0$ and $u_6<\hat u$ such that

$$\mathbf{P}\{\xi(t)>u,\ \xi(s)\le v\}\le C\int_{-\infty}^{v}\mathbf{P}\{\xi(c(t-s))>u-x\}\,dF_{\xi(s)}(x) \tag{8.11}$$

for $0<s<t$ and $u_6\le v<u<\hat u$. Obviously (8.11) holds when $\xi(t)$ has stationary independent increments. But (8.11) is a much weaker requirement than that.

Proposition 5. *Assume that* (1.5)-(1.7) *hold, and that* (8.11) *holds with* $\xi(t)$ **P**-*continuous. If in addition* $G\in\mathcal{D}(\mathrm{I})$ *with*

$$L_1\equiv\overline{\lim}_{u\to\infty}\ln(1-G(u))/u<0 \quad \text{and} \quad L_2\equiv\underline{\lim}_{u\to\infty}q(u)^{-\kappa}w(u)>0, \tag{8.12}$$

or if $G\in\mathcal{D}(\mathrm{II})$ *with* $\gamma\kappa>1$, *then Assumption 3 holds.*

Proof. First assume that $G\in\mathcal{D}(\mathrm{I})$. Then de Haan's Theorem (e.g., Resnick, 1987, Proposition 1.4) states that there exist a constant $u_7\in\mathbb{R}$, and functions ϕ,Φ: $[u_7,\infty)\to(0,\infty)$ with ϕ self-neglecting, such that

$$\lim_{u\to\infty}\Phi(u)\ \text{ exists and }\quad 1-G(y)=\Phi(y)\,\exp\left\{-\int_{u_7}^{y}\frac{dx}{\phi(x)}\right\}\quad\text{for }\ y\ge u_7. \tag{8.13}$$

Since $\phi(u)/\phi(u+x\phi(u))\to 1$ locally uniformly it follows that

$$\frac{1-G(u+t\phi(u))}{1-G(u)}=\frac{\Phi(u+t\phi(u))}{\Phi(u)}\exp\left\{-\int_{u}^{u+t\phi(u)}\frac{dx}{\phi(x)}\right\}\to \mathrm{e}^{-t}\quad\text{as }\ u\uparrow\hat u.$$

Consequently $w(u)\sim\phi(u)$ [e.g., Resnick (1987, p. 26)]. Since (1.7) yields $w/w(u-yw)\le 2\mathfrak{P}_3u/(u-yw)\le 4\mathfrak{P}_3$ for $0\le y\le u/(2w)$ and u large, we further obtain

$$\frac{1-G(u-yw)}{1-G(u)}\le 2\exp\left\{\int_{u-yw}^{u}\frac{dx}{\phi(x)}\right\}\le 2\exp\left\{\int_{u-yw}^{u}\frac{2\,dx}{w(x)}\right\}\le 2\,\mathrm{e}^{8\mathfrak{P}_3y}$$

for $0\le y\le u/(2w)$ and u large. Hence (8.11)-(8.12) show that

$$\begin{aligned}(8.14)\quad &\mathbf{P}\{\xi(1)>u+(\lambda+\nu)w,\ \xi(1-qt)\le u+\nu w\}\\ &\le C\sum_{\ell=0}^{[u/(2w)]-1}\mathbf{P}\{\xi(cqt)>(\lambda+\ell)w\}\,\mathbf{P}\{u+(\nu-\ell-1)w<\xi(1-qt)\le u+(\nu-\ell)w\}\\ &\qquad+C\,\mathbf{P}\{\xi(cqt)>\big(\lambda+[u/(2w)]\big)w\}\\ &\le C\sum_{\ell=0}^{[u/(2w)]-1}\Big(1-G\big((cqt)^{-\kappa}(\lambda+\ell)w\big)\Big)\Big(1-G\big(u+(\nu-\ell-1)w\big)\Big)\end{aligned}$$

$$
\begin{aligned}
&+C\left(1-G(\tfrac{1}{3}(cqt)^{-\kappa}u)\right)\\
&\leq 2C\,(1-G(u))\sum_{\ell=0}^{\infty}\exp\bigl\{-\tfrac{1}{2}(L_1\wedge 1)(L_2\wedge 1)\,(ct)^{-\kappa}(\lambda+\ell)\bigr\}\,\exp\bigl\{8\mathfrak{P}_3(\ell+1-\nu)\bigr\}\\
&\quad+\frac{C\,(1-G(u))\,\Phi(\frac{1}{3}(cqt)^{-\kappa}u)}{\Phi(u)}\exp\biggl\{-\int_u^{\frac{1}{3}(cqt)^{-\kappa}u}\frac{dx}{\phi(x)}\biggr\}\quad\text{for } u \text{ large.}
\end{aligned}
$$

Since $\phi(xu)\leq 2w(xu)\leq 4\mathfrak{P}_3 xw$ for $x\geq 1$ and u large, we here have

$$
\text{(8.15)}\quad \exp\biggl\{-\int_u^{\frac{1}{3}(cqt)^{-\kappa}u}\frac{dx}{\phi(x)}\biggr\}\leq\exp\biggl\{-\int_1^{\frac{1}{3}(cqt)^{-\kappa}}\frac{u\,dx}{4\mathfrak{P}_3 x\,w}\biggr\}=(3(cqt)^{\kappa})^{u/(4\mathfrak{P}_3 w)}.
$$

Inserting in (8.14) and recalling that $\Phi(u)$ converges it follows that (8.1) holds.

Now assume that $G\in\mathcal{D}(\mathrm{II})$. Given an $\varepsilon\in(0,1)$ Potter's Theorem (e.g., Resnick, 1987, Proposition 0.8) then claims that there is a $u_8=u_8(G,\varepsilon)>0$ such that

$$
x^{-\gamma(1+\varepsilon)}/(1+\varepsilon)\;\leq\;(1-G(ux))/(1-G(u))\;\leq\;x^{-\gamma(1-\varepsilon)}/(1-\varepsilon)\quad\text{for } x\geq 1 \text{ and } u\geq u_8.
$$

Invoking (8.11) we therefore deduce (8.1) through the estimates

$$
\begin{aligned}
&\mathbf{P}\bigl\{\xi(1)>u(1+\lambda+\nu),\,\xi(1-qt)\leq u(1+\nu)\bigr\}\\
&\qquad\leq C\left(1-G((cqt)^{-\kappa}\lambda u)\right)\leq C\,(1-\varepsilon)^{-1}\,((cqt)^{-\kappa}\lambda)^{-(1-\varepsilon)\gamma}\,(1-G(u)).\;\Box
\end{aligned}
$$

9. Lamperti's associated stationary process. In essence, Assumptions 1-3 consist of a set of asymptotic distributional requirements on 'events' of the type

$$
\bigl(\xi(1+qt)>\tilde{u}\,\big|\,\xi(1)>u\bigr)\quad\text{where}\quad\tilde{u}=u+\delta w(u)\ \text{ and }\ \delta,t\in\mathbb{R}\ \text{ are constants.}
$$

Expressed in terms of the stationary process $X(t)\equiv\mathrm{e}^{-\kappa t}\xi(\mathrm{e}^t)$, this event becomes

$$
\bigl((1+qt)^{\kappa}X(\ln(1+qt))>\tilde{u}\,\big|\,X(0)>u\bigr)\approx\bigl(X(qt)>\hat{u}\,\big|\,X(0)>u\bigr)\quad\text{where}\quad\hat{u}=(1-\kappa qt)\tilde{u}
$$

and the right hand side is a Taylor expansion. Thus it is not surprising that Assumptions 1-3 can be expressed in terms of similar assumptions on $X(t)$ or $Y(t)\equiv X(-t)$, which is done in Propositions 6-9. These assumptions in turn essentially coincide with those used by Albin (1990) to study stationary extremes.

Since stationary processes often allow neat and 'balanced' calculations, it can be rewarding to first analyze $X(t)$ and then transfer results to $\xi(t)$ via Propositions 6-9. Further these propositions yield 'gratis' results for self-similar processes obtainable by invoking estimates in the literature for the associated stationary process.

Proposition 6. *If for each* $K\in[1,\infty)$ *we have*

$$\mathbf{P}\big\{X(-q(u)t)>e^{\kappa q(u)t}u\,\big|\,X(0)>u\big\}\leq f_1(t)+f_2(u)\quad\textit{for}\ \ 0\leq q(u)t\leq K-\ln(q(u)),$$

for some $f_1\in\mathbb{L}^1(\mathbb{R}^+)$ *and* $f_2(u)=o\big(q(u)/\ln(q(u))\big)$, *then Assumption 2 holds.*

Proof. Since Assumption 2 holds when $Q<\infty$, we can assume $Q=\infty$ so that $q(u)\to 0$. Using stationarity and that $-\ln(1-x)\geq x$ for $x\in[0,1)$, we then obtain

$$(9.1)\qquad \begin{aligned}&\int_d^{1/q}\mathbf{P}\big\{\xi(1-qt)>u\,\big|\,\xi(1)>u\big\}\,dt\\
&\quad=\int_{(-\ln(1-dq))/q}^{\infty}\mathbf{P}\big\{X(-q\hat{t})>\mathrm{e}^{\kappa q\hat{t}}u\,\big|\,X(0)>u\big\}\,\mathrm{e}^{-q\hat{t}}\,d\hat{t}\\
&\quad\leq\int_d^{(K-\ln(q))/q}f_1(t)\,dt+\frac{(K-\ln(q))\,f_2(u)}{q}+\int_{(K-\ln(q))/q}^{\infty}\mathrm{e}^{-q\hat{t}}\,d\hat{t}\\
&\quad\to\int_d^{\infty}f_1(t)\,dt+0+\mathrm{e}^{-K}\qquad\text{as}\ \ u\uparrow\hat{u}.\ \ \square\end{aligned}$$

Proposition 7. *Assume that there are* $f_3\in\mathbb{L}^1(\mathbb{R}^+)$ *and* $f_4(u)=o(q(u))$ *such that*

$$(9.2)\qquad \mathbf{P}\big\{X(q(u)t)>u\,\big|\,X(0)>u\big\}\leq f_3(t)+f_2(u)\quad\textit{for}\ \ 0\leq q(u)t\leq h,$$

for some $h>0$. *If in addition* (1.7) *holds with* $\lim_{u\to\infty}w(u)\ln(q(u))/u=0$ *and* $G\in\mathcal{D}(\mathrm{I})$, *then Assumption 2 holds.*

Proof. Inspecting (9.1), and using de Haan's result (8.13) as in (8.15), we obtain

$$\begin{aligned}&\int_d^{1/q}\mathbf{P}\big\{\xi(1-qt)>u\,\big|\,\xi(1)>u\big\}\,dt\\
&\leq\int_{-\ln(1-dq)/q}^{h/q}\mathbf{P}\big\{X(-qt)>\mathrm{e}^{\kappa qt}u\,\big|\,X(0)>u\big\}\,\mathrm{e}^{-qt}\,dt+\int_{h/q}^{\infty}\frac{1-G(\mathrm{e}^{\kappa qt}u)}{1-G(u)}\,dt\\
&\leq\int_d^{h/q}f_3(t)\,dt+\frac{f_4(u)}{q}+\frac{2}{q}\int_h^{\infty}\exp\bigg\{-\int_u^{\mathrm{e}^{\kappa t}u}\frac{dx}{\phi(x)}\bigg\}\,dt\\
&\leq\int_d^{\infty}f_3(t)\,dt+\frac{f_4(u)}{q}+\frac{2}{q}\int_h^{\infty}(\mathrm{e}^{-\kappa t})^{u/(4\mathfrak{P}_3 w)}\,dt\\
&\to\int_d^{\infty}f_3(t)\,dt+0+0\qquad\text{as}\ \ u\uparrow\hat{u}\end{aligned}$$

[where the second zero follows readily using that $w/u\to 0$ and $w\,\ln(q)/u\to 0$.] $\square$

Proposition 8. *Assume that for each choice of* $\sigma>0$ *we have*

$$\lim_{a\downarrow 0}\overline{\lim_{u\uparrow\hat u}}\,\frac{1}{a\,\mathbf{P}\{X(0)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,aq(u)]}X(t)>u+\sigma w(u),\ X(0)\leq u\Big\}=0,\quad\textit{or}\tag{9.3}$$

$$\lim_{a\downarrow 0}\overline{\lim_{u\uparrow\hat u}}\,\frac{1}{a\,\mathbf{P}\{X(0)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,aq(u)]}X(t)>u+\sigma w(u),\ X(aq(u))\leq u\Big\}=0.\tag{9.4}$$

If in addition (1.5)-(1.7) *hold with* $G\in\mathcal{D}$ *and* $\mathfrak{P}_5<\infty$, *then Assumption 3 holds.*

By Albin (1992a, Proposition 2), the requirement (9.3) holds if

$$\frac{\mathbf{P}\{X(qt)>u+\lambda w,\ X(0)\leq u\}}{\mathbf{P}\{X(0)>u\}}\leq C\,t^d\,\lambda^{-e}\quad\text{for }\ 0<t^c\leq\lambda\leq\lambda_1\ \text{ and }\ u\in[u_9,\hat u),$$

for some constants λ_1, c, e, $C>0$, $u_9<\hat u$ and $d>1$, while (9.4) holds if

$$\frac{\mathbf{P}\{X(0)>u+\lambda w,\ X(qt)\leq u\}}{\mathbf{P}\{X(0)>u\}}\leq C\,t^d\,\lambda^{-e}\quad\text{for }\ 0<t^c\leq\lambda\leq\lambda_1\ \text{ and }\ u\in[u_9,\hat u).$$

Proof of Proposition 8. Since $\ln(1+aq)\leq aq$ and $(1+aq)^{-\kappa}(u+\sigma w)\geq u+\frac{1}{2}\sigma w$ for u sufficiently large and $a>0$ sufficiently small, (9.3) implies that

$$\nu_1(a,\sigma)\leq\overline{\lim_{u\uparrow\hat u}}\,\frac{1}{1-G(u)}\,\mathbf{P}\Big\{\sup\nolimits_{t\in[0,\ln(1+aq)]}X(t)>u+\tfrac{1}{2}\sigma w,\ X(0)\leq u\Big\}=o(a).$$

Similarly (9.4) yields that $\nu_2(a,\sigma)=o(a)$. $\square$

Proposition 9. *Assume that* (1.3) *and* (1.4) *hold with* $G\in\mathcal{D}$, $Q=\infty$ *and* $\mathfrak{P}_4=\mathfrak{P}_5<\infty$, *and that* (9.3) *or* (9.4) *holds for each* $\sigma>0$.

(i) *If for each* $y\in J$ *there is an* $(\mathbb{R}\cup\{-\infty,\infty\})$*-valued process* $\{\eta_y(t)\}_{t>0}$ *such that*

$$\lim_{u\uparrow\hat u}\mathbf{P}\bigg\{\bigcap_{i=1}^{n}\Big\{\frac{X(q(u)t_i)-u}{w(u)}>x_i\Big\}\,\bigg|\,\frac{X(0)-u}{w(u)}>y\bigg\}=\mathbf{P}\bigg\{\bigcap_{i=1}^{n}\{\eta_y(t_i)>x_i\}\bigg\}\tag{9.5}$$

for $t_1,\ldots,t_n>0$ *and continuity points* $x_1,\ldots,x_n\in J$ *for* $\mathbf{P}\{\eta_y(t_1)>\cdot\},\ldots,\mathbf{P}\{\eta_y(t_n)>\cdot\}$, *then Assumption 1 holds.*

(ii) *If there is an* $(\mathbb{R}\cup\{-\infty,\infty\})$*-valued process* $\{\eta(t)\}_{t>0}$ *such that*

$$\lim_{u\uparrow\hat u}\mathbf{P}\bigg\{\bigcap_{i=1}^{n}\Big\{\frac{Y(q(u)t_i)-u}{w(u)}>x_i\Big\}\,\bigg|\,Y(0)>u\bigg\}=\mathbf{P}\bigg\{\bigcap_{i=1}^{n}\{\eta(t_i)>x_i\}\bigg\}\tag{9.6}$$

for $t_1,\dots,t_n>0$ *and continuity points* $x_1,\dots,x_n\in J$ *for* $\mathbf{P}\{\eta(t_1)>\cdot\},\dots,$ $\mathbf{P}\{\eta(t_n)>\cdot\}$, *then Assumption 1 holds with* $\zeta(t)=_{\mathcal{L}}\eta(t)-\mathfrak{P}_5\kappa t$.

Proof of (i). Given $x_1,\dots,x_n\in J$ and writing $x_0=t_0\equiv 0$, we have

$$\text{(9.7)}\quad \mathbf{P}\bigg\{\bigcap_{i=1}^{n}\Big\{\frac{\xi(1-qt_i)-u}{w}>x_i\Big\}\,\bigg|\,\xi(1)>u\bigg\}$$
$$=\frac{1}{1-G(u)}\,\mathbf{P}\bigg\{\bigcap_{i=0}^{n}\Big\{\frac{X\big(\ln(1-qt_i)\big)-(1-qt_i)^{-\kappa}u}{w}>(1-qt_i)^{-\kappa}x_i\Big\}\bigg\}.$$

Here (1.1) easily combine with the fact that $q(u)\to 0$ to show that

$$A_i\equiv\Big\{w^{-1}\big(X(\ln(1-qt_i))-(1-qt_i)^{-\kappa}u\big)>(1-qt_i)^{-\kappa}x_i\Big\}$$
$$B_i\equiv\Big\{w^{-1}\big(X(\ln(1-qt_i))-u\big)>x_i+\mathfrak{P}_5\kappa t_i\Big\}$$

satisfy

$$\text{(9.8)}\quad \bigg|\mathbf{P}\Big\{\bigcap_{i=0}^{n}A_i\Big\}-\mathbf{P}\Big\{\bigcap_{i=0}^{n}B_i\Big\}\bigg|\le\sum_{i=0}^{n}\big(\mathbf{P}\{A_i\cap B_i^c\}+\mathbf{P}\{A_i^c\cap B_i\}\big)=o\big(1-G(u)\big)$$

as $u\uparrow\hat{u}$. Inserting in (9.7) and using stationarity it therefore follows that

$$\text{(9.9)}\quad \mathbf{P}\bigg\{\bigcap_{i=1}^{n}\Big\{\frac{\xi(1-qt_i)-u}{w}>x_i\Big\}\,\bigg|\,\xi(1)>u\bigg\}$$
$$\sim\frac{1}{1-G(u)}\,\mathbf{P}\bigg\{\bigcap_{i=0}^{n}\Big\{\frac{X\big(\ln(1-qt_i)-\ln(1-qt_n)\big)-u}{w}>x_i+\mathfrak{P}_5\kappa t_i\Big\}\bigg\}.$$

Now write $\tilde{u}_x=u+xw$ for $x\in J$, and define

$$A_i\equiv\Big\{X\big(\ln(1-qt_i)-\ln(1-qt_n)\big)>\tilde{u}_x\Big\}\quad\text{and}\quad B_i\equiv\Big\{X(q(t_n-t_i))>\tilde{u}_x\Big\}.$$

Given an $\varepsilon>0$ it is easy to see that

$$0\le\ln(1-qt_i)-\ln(1-qt_n)-q(t_n-t_i)\le\varepsilon q(t_n-t_i)\quad\text{for } u \text{ sufficiently large.}$$

Since (1.3) and (1.4) show that $C_x q(\tilde{u}_x)\ge q$ for u large, for some $C_x<\infty$, an application of (9.3) now yields

$$\mathbf{P}\{A_i\cap B_i^c\}=\mathbf{P}\Big\{X\big(\ln(1-qt_i)-\ln(1-qt_n)\big)>\tilde{u}_x,\;X(q(t_n-t_i))\le\tilde{u}_x\Big\}$$

$$\le \mathbf{P}\Big\{\sup_{s\in[0,\,\ln(1-qt_i)-\ln(1-qt_n)-q(t_n-t_i)]} X(s)>\tilde u_x+\sigma w(\tilde u_x),\ X(0)\le\tilde u_x\Big\}$$
$$+\,\mathbf{P}\big\{\tilde u_x < X(0)\le \tilde u_x+\sigma w(\tilde u_x)\big\}$$
$$\le \mathbf{P}\Big\{\sup_{s\in[0,\,\varepsilon C_x q(\tilde u_x)(t_n-t_i)]} X(s)>\tilde u_x+\sigma w(\tilde u_x),\ X(0)\le\tilde u_x\Big\}$$
$$+\,\big(G(\tilde u_x+\sigma w(\tilde u_x))-G(\tilde u_x)\big)$$
$$\sim\Big(o\big(\varepsilon C_x(t_n-t_i)\big)+F(\sigma)\Big)\,\big(1-F(x)\big)\,\big(1-G(u)\big)\quad\text{as}\ \ u\uparrow\hat u.$$

Sending $\varepsilon,\sigma\to 0$ it follows that $\mathbf{P}\{A_i\cap B_i^c\}=o\big(1-G(u)\big)$ and similarly (9.4) implies $\mathbf{P}\{A_i^c\cap B_i\}=o\big(1-G(u)\big)$. Since $\mathbf{P}\{A_i\cap B_i^c\}=\mathbf{P}\{A_i^c\cap B_i\}$ by stationarity, we get (9.8). At continuity points, (9.9) thus combines with (9.5) to show

$$\text{(9.10)}\quad \mathbf{P}\Big\{\bigcap_{i=1}^{n}\Big\{\frac{\xi(1-qt_i)-u}{w}>x_i\Big\}\,\Big|\,\xi(1)>u\Big\}$$
$$\sim\frac{1}{1-G(u)}\,\mathbf{P}\Big\{\bigcap_{i=0}^{n}\Big\{\frac{X(q(t_n-t_i))-u}{w}>x_i+\mathfrak{P}_5\kappa t_i\Big\}\Big\}$$
$$\to\big(1-F(x_n+\mathfrak{P}_5\kappa t_n)\big)\,\mathbf{P}\Big\{\bigcap_{i=0}^{n-1}\Big\{\eta_{x_n+\mathfrak{P}_5\kappa t_n}(t_n-t_i)>x_i+\mathfrak{P}_5\kappa t_i\Big\}\Big\}.\ \square$$

Proof of (ii). At continuity points, the first relation in (9.10) plus (9.6) yield

$$\mathbf{P}\Big\{\bigcap_{i=1}^{n}\Big\{\frac{\xi(1-qt_i)-u}{w}>x_i\Big\}\,\Big|\,\xi(1)>u\Big\}$$
$$\sim\frac{1}{1-G(u)}\mathbf{P}\Big\{\bigcap_{i=0}^{n}\Big\{\frac{Y(q(t_i-t_n))-u}{w}>x_i+\mathfrak{P}_5\kappa t_i\Big\}\Big\}\to\mathbf{P}\Big\{\bigcap_{i=1}^{n}\{\eta(t_i)>x_i+\mathfrak{P}_5\kappa t_i\}\Big\}.\ \square$$

10. Gaussian processes in $\mathbb{R}^n$. Let $\chi_1(t),\ldots,\chi_n(t)$ be independent zero-mean Gaussian processes that are self-similar with index κ and whose covariances satisfy

$$\text{(10.1)}\qquad \mathbf{E}\{\chi_i(1)\chi_i(1+t)\}\;=\;1+\kappa t-C_i|t|^\alpha+o\big(|t|+|t|^\alpha\big)\quad\text{as}\ \ t\to 0,$$

for some constants $\alpha\in(0,2]$ and $C_1,\ldots,C_n>0$. Then $\xi(t)\equiv|(\chi_1(t),\ldots,\chi_n(t))|$ $=\sqrt{\chi_1(t)^2+\ldots+\chi_n(t)^2}$ has associated process $X(t)\equiv|(\mathfrak{X}_1(t),\ldots,\mathfrak{X}_n(t))|$ with standardized Gaussian components $\mathfrak{X}_i(t)=\mathrm{e}^{-\kappa t}\chi_i(\mathrm{e}^t)$ satisfying

$$\text{(10.2)}\qquad \mathbf{E}\{\mathfrak{X}_i(0)\mathfrak{X}_i(t)\}\;=\;1-C_i|t|^\alpha+o\big(|t|+|t|^\alpha\big)\quad\text{as}\ \ t\to 0.$$

The class of processes satisfying (10.2) is very rich, and since an associated process $X(t)=|(\mathfrak{X}_1(t),\ldots,\mathfrak{X}_n(t))|$ generates a self-similar process $\xi(t)=t^{\kappa}X(\ln(t))$ for each $\kappa>0$, the class of self-similar processes satisfying (10.1) is very rich indeed.

In particular (10.1) holds with $C_i=\frac{1}{2}$ and $\kappa=\alpha/2$ when $\chi_i(t)=B_i(t)$ and $B_i(t)$ is fBm. with $\mathbf{E}\{B_i(s)B_i(t)\}=\frac{1}{2}\big(|s|^{\alpha}+|t|^{\alpha}-|t-s|^{\alpha}\big)$.

Pickands (1969) studied stationary Gaussian extremes when (10.2) holds, and the first extension to $\mathbb{R}^n$ is Sharpe's (1978). Our results for $\xi(t)$ below are new for $n\geq 2$. When $n=1$ the behaviour of extremes follows from e.g., Konstant and Pitebarg (1993, Section 2), but the sojourn result still is new.

Now observe that (10.1) implies the existence of an $h>0$ such that

$$\text{(10.3)} \qquad \sup\nolimits_{1\leq i\leq n}\sup\nolimits_{t\in[\varepsilon,h]}\mathrm{e}^{-\kappa t}\,\mathbf{E}\{\chi_i(1)\chi_i(\mathrm{e}^t)\}<1 \quad \text{for } \varepsilon\in(0,h].$$

Further let U be a unit-mean exponentially distributed variable and $\overline{\omega}$ a variable uniformly distributed over the unit sphere $\{\overline{x}\in\mathbb{R}^n:|\overline{x}|=1\}$ such that U, $\overline{\omega}$, $\{B_1(t)\}_{t\geq 0},\ldots,\{B_n(t)\}_{t\geq 0}$ are mutually independent.

Theorem 8. *Assume that* (10.1) *and* (10.3) *hold with* $\alpha\in(0,1]$. *Then* (2.1) *and the conclusions of Theorem 1 and Corollary 1 hold with* $w(u)=(1\vee u)^{-1}$, $q(u)=(1\vee u)^{-2/\alpha}$, $\mathbf{P}\{\xi(1)>u\}\sim(u/\sqrt{2})^{n-2}(\Gamma(\frac{n}{2}))^{-1}e^{-\frac{1}{2}u^2}$ *and*

$$\Lambda(x)=\mathbf{P}\bigg\{\int_0^{\infty}I_{(0,\infty)}\bigg(U-\mathfrak{P}_5\kappa t+\sum_{i=1}^{n}\sqrt{C_i}\,\omega_i\big(B_i(t)-\sqrt{C_i}\,\omega_i|t|^{\alpha}\big)\bigg)\,dt>x\bigg\}.$$

The proof uses Propositions 7-9, and the hypothesis of these propositions follows using results for $X(t)$ in Albin (1990, proof of Theorem 9) ([**A9**]).

Proof of Theorem 8. The asymptotic behaviour of $\mathbf{P}\{\xi(1)>u\}=1-G(u)$ is elementary and shows that $G\in\mathcal{D}(\mathrm{I})$. Further (1.3)-(1.7) hold and $\mathfrak{P}_4=\mathfrak{P}_5=1$ for $\alpha=1$, while $\mathfrak{P}_4=\mathfrak{P}_5=0$ for $\alpha<1$. An inspection of [**A9**] also yields

$$\big(w^{-1}(Y(qt)-u)\,\big|\,Y(0)>u\big)\ \to_{\mathcal{L}}\ \eta(t)\ =\ U+\sum_{i=1}^{n}\sqrt{C_i}\,\omega_i\big(B_i(t)-\sqrt{C_i}\,\omega_i|t|^{\alpha}\big)$$

in the sense of weak convergence of the finite dimensional distributions.

In [**A9**] it is further shown that there are $A,B>0$ and $\varepsilon\in(0,h]$ such that

$$\mathbf{P}\big\{X(qt)>u\,\big|\,X(0)>u\big\}\leq\begin{cases}4n\,\mathbf{P}\big\{\mathcal{N}(0,1)>At^{\alpha/2}\big\} & \text{for } qt\in[0,\varepsilon]\\ 4n\,\mathbf{P}\big\{\mathcal{N}(0,1)>Bu\big\} & \text{for } qt\in(\varepsilon,h]\end{cases},$$

and constants $C,\lambda_2>0$ such that

$$\frac{\mathbf{P}\{X(qt)>u+(\lambda+\delta)w,\, X(0)\le u+\delta w\}}{\mathbf{P}\{X(0)>u\}} \le 2n\,\mathbf{P}\{\mathcal{N}(0,1)>Ct^{-\alpha/2}\} \tag{10.4}$$

for $0<t^{\alpha/2}<\lambda<\lambda_2$. Hence (9.2), (9.4), and (9.6) hold, and Propositions 7, 8 and 9.(ii) apply to prove that Assumptions 1-3 hold with $\zeta(t)=_{\mathcal{L}}\eta(t)-\mathfrak{P}_5\kappa t$. □

Theorem 9. *Assume that* (10.1) *holds with* $\alpha\in(1,2]$. *Then* (2.1), (5.2) *and the conclusion of Theorem 1 hold with* $w(u)=(1\vee u)^{-1}$, $q(u)=(1\vee u)^{-2}$, $\mathbf{P}\{\xi(1)>u\}\sim(u/\sqrt{2})^{n-2}(\Gamma(\frac{n}{2}))^{-1}e^{-\frac{1}{2}u^2}$ *and* $\Lambda(x)=e^{-\kappa x}$.

Proof. Now [**A9**] shows that the finite dimensional distributions of $\big(w^{-1}(Y(qt)-u)\,\big|\,Y(0)>u\big)$ converge weakly to those of the random variable U. Further the fact that $\mathbf{E}\{\mathfrak{X}_i(0)\mathfrak{X}_i(t)\}\ge 1-c|t|$ for t small, for some $c>0$, combines with an inspection of [**A9**] to show that (10.4) holds. Since $\mathfrak{P}_4=\mathfrak{P}_5=1$, Propositions 6 and 9.(ii) show that Assumption 1 and 3 hold with $\zeta(t)=_{\mathcal{L}}U-\kappa t$, while Proposition 2 yields Assumption 2. A trivial calculation finally gives $\Lambda(x)=\mathrm{e}^{-\kappa x}$. □

11. The $\mathbb{L}^2$-norm of Brownian motion. Let $\{W(s)\}_{s\ge0}$ be standard Brownian motion and define $\xi(t)\equiv\int_{\theta t}^{t}W(s)^2\,ds$ where $\theta\in[0,1)$. Then $\xi(t)$ is self-similar with index $\kappa=2$, and so the associated stationary process is given by $X(t)=\mathrm{e}^{-2t}\xi(\mathrm{e}^t)$. Of course, the quantity $\xi(t)$ were first studied by Cameron & Martin (1944).

Theorem 10. *Writing* $\xi'(1)\equiv W(1)^2-\theta\,W(\theta)^2$ *we have*

$$\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}\sim(\kappa\,u)^{-1}\,\mathbf{E}\{\xi'(1)^+|\,\xi(1)>u\}\,\mathbf{P}\{\xi(1)>u\}\quad \textit{as } u\to\infty.$$

Proof. We prove the theorem by application of Theorem 7: Writing

$$\lambda(\theta)\;\equiv\;\sup\{\lambda>0:\cot((1-\theta)\sqrt{\lambda})=\theta/\sqrt{\lambda}\},$$

Lemma 2 in Li (1992) states that there is a constant $K(\theta)>0$ such that

$$\mathbf{P}\{\xi(1)>u\}\;\sim\;K(\theta)\,u^{-1/2}\,\mathrm{e}^{-\frac{1}{2}u/\lambda(\theta)}\qquad\text{as } u\to\infty. \tag{11.1}$$

Thus $G\in\mathcal{D}(\mathrm{I})$ and (1.1) holds with $w(u)=2\lambda(\theta)$. Defining $q(u)\equiv(1\vee u)^{-1}$ we further get $\mathfrak{P}_4=\mathfrak{P}_5=(2\lambda(\theta))^{-1}$, and so Proposition 2 yields Assumption 2. Further, since by Albin (1995, Eq. 2.17) we have

$$\overline{\lim_{a\downarrow0}}\,\overline{\lim_{u\uparrow\hat{u}}}\,\frac{1}{a\,\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,aq]}X(t)>u+\sigma,\,X(0)\le u\Big\}=0\quad\text{for }\sigma>0,$$

eq. (9.3) holds. Consequently Proposition 6 proves Assumption 3.

The fact that (7.1) holds follows using (11.1) in the calculation

$$\begin{aligned}
&\mathbf{P}\bigg\{\frac{1}{2\lambda(\theta)}\bigg|\int_{\theta-\theta qt}^{1-qt}W(s)^2ds-\int_{\theta}^{1}W(s)^2ds+qt[W(1)^2-\theta W(\theta)^2]\bigg|>\varepsilon\;\bigg|\int_{\theta}^{1}W(s)^2ds>u\bigg\}\\
&\leq\frac{1}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\bigg\{\bigg|\int_{\theta-\theta qt}^{\theta}[W(s)^2-W(\theta)^2]\,ds-\int_{1-qt}^{1}[W(s)^2-W(1)^2]\,ds\bigg|>2\varepsilon\lambda(\theta)\bigg\}\\
&\leq\frac{1}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\Big\{\sup_{s\in[0,1]}|W(s)|>\tfrac{1}{2}\sqrt{2\varepsilon\lambda(\theta)}\,u^{3/4}\Big\}\\
&\quad+\frac{1}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\Big\{\sup_{s\in[\theta-\theta qt,\theta]}\theta qt\,|W(s)-W(\theta)|>\tfrac{1}{2}\sqrt{2\varepsilon\lambda(\theta)}\,u^{-3/4}\Big\}\\
&\quad+\frac{1}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\Big\{\sup_{s\in[1-qt,1]}qt\,|W(s)-W(1)|>\tfrac{1}{2}\sqrt{2\varepsilon\lambda(\theta)}\,u^{-3/4}\Big\}\\
&\leq\frac{4}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\Big\{\mathcal{N}(0,1)>\tfrac{1}{2}\sqrt{2\varepsilon\lambda(\theta)}\,u^{3/4}\Big\}\\
&\quad+\frac{4}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\Big\{\mathcal{N}(0,1)>\tfrac{1}{2}\sqrt{2\varepsilon\lambda(\theta)}\,(\theta qt)^{-3/2}\,u^{-3/4}\Big\}\\
&\quad+\frac{4}{\mathbf{P}\{\xi(1)>u\}}\,\mathbf{P}\Big\{\mathcal{N}(0,1)>\tfrac{1}{2}\sqrt{2\varepsilon\lambda(\theta)}\,(qt)^{-3/2}\,u^{-3/4}\Big\}\\
&\to 0\qquad\text{as }\;u\to\infty.
\end{aligned}$$

In order to prove (7.2) we observe that [by (11.1)]

$$\begin{aligned}
\mathbf{P}\big\{[q\,\xi'(1)^+]^2>x\;\big|\;\xi(1)>u\big\}&\leq\mathbf{P}\big\{q\,W(1)^2>\sqrt{x}/2\big\}\;/\;\mathbf{P}\{\xi(1)>u\}\\
&\leq 2\,\mathbf{P}\big\{\mathcal{N}(0,1)>x^{1/4}\sqrt{u/2}\,\big\}\;/\;\mathbf{P}\{\xi(1)>u\}\\
&\leq x^{-2}\qquad\text{for }\;x\geq x_0\;\text{ and }\;u\geq 1,
\end{aligned}$$

for some choice of $x_0>1$. Hence it follows that

$$\mathbf{E}\big\{[q\,\xi'(1)^+]^2>x\;\big|\;\xi(1)>u\big\}=\int_0^\infty\mathbf{P}\big\{[q\,\xi'(1)^+]^2>x\;\big|\;\xi(1)>u\big\}\leq x_0+\frac{1}{x_0}.$$

Now recall that $\xi(1)=_{\mathcal{L}}\sum_{k=1}^{\infty}\lambda_k N_k^2$ where $N_1,N_2,\ldots$ are independent $\mathcal{N}(0,1)$ and $\lambda_1\geq\lambda_2\geq\ldots\geq 0$ [e.g., Li (1992)]. But in the first part of the proof of Albin (1992b, Theorem 4) we show that the density for such a sum satisfies (7.3)! □

12. Totally skewed log-fractional α-stable motion. We write $Z\in S_\alpha(\sigma,\beta)$ when Z is an α-stable random variable with characteristic function

$$\mathbf{E}\big\{\exp[i\theta Z]\big\}=\exp\big\{-|\theta|^\alpha\sigma^\alpha[1-i\beta\tan(\tfrac{\pi\alpha}{2})\mathrm{sign}(\theta)]\big\}\qquad\text{for }\;\theta\in\mathbb{R}.$$

Here $\alpha\in(1,2]$, the scale $\sigma=\sigma_Z\geq 0$ and the skewness $\beta=\beta_Z\in[-1,1]$ are parameters. Also let $\{M(t)\}_{t\in\mathbb{R}}$ be an α-stable motion that is totally skewed to the left, so that $M(t)$ has stationary independent increments and $M(t)\in S_\alpha(|t|^{1/\alpha},-1)$.

Given an $n\in\mathbb{Z}^+$ and functions $h\in\mathbb{L}^\alpha(\mathbb{R})$ and $\hat{h}\in\mathbb{L}^0(\mathbb{R})$ we define

$$\|h\|_\alpha\equiv\big(\textstyle\int_{\mathbb{R}}|h(x)|^\alpha\,dx\big)^{1/\alpha}\quad\text{and}\quad\langle\hat{h},h\rangle_{\alpha,n}\equiv\textstyle\int_{\mathbb{R}}\hat{h}(x)^n\,\mathrm{sign}(h(x))\,|h(x)|^{\alpha-n}\,dx.$$

When $h,\hat{h}\in\mathbb{L}^\alpha(\mathbb{R})$ the random variable $(Z,\hat{Z})\equiv\big(\int_{\mathbb{R}}h\,dM,\int_{\mathbb{R}}\hat{h}\,dM\big)$ satisfies

$$\theta Z+\varphi\hat{Z}\in S_\alpha\Big(\|\theta h+\varphi\hat{h}\|_\alpha,-\big(\textstyle\int_{\mathbb{R}}\mathrm{sign}(\theta h(x)+\varphi\hat{h}(x))\,|\theta h(x)+\varphi\hat{h}(x)|^\alpha\,dx\big)/\|\theta h+\varphi\hat{h}\|_\alpha^\alpha\Big),$$

and each $\mathbb{R}^2$-valued stable random variable $(Z,\hat{Z})$ has the representation $\big(\int_{\mathbb{R}}h\,dM,\int_{\mathbb{R}}\hat{h}\,dM\big)$ in law for some choice of h and $\hat{h}$. When $h\geq 0$ a.s. and $\langle\hat{h},h\rangle_{\alpha,2}<\infty$, Corollary 2.2 of Albin (1997) further states that

$$\begin{aligned}&(12.1)\quad \mathbf{E}\big\{(\hat{Z}-Z)^2\,\big|\,Z>u\big\}\\&=\int_u^\infty\bigg[\bigg(\frac{\langle\hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}-\frac{\langle\hat{h}-h,h\rangle_{\alpha,1}^2}{\|h\|_\alpha^{2\alpha}}\bigg)\int_y^\infty\frac{(\alpha-1)zf_Z(z)}{\mathbf{P}\{Z>u\}}\,dz+\frac{\langle\hat{h}-h,h\rangle_{\alpha,1}^2}{\|h\|_\alpha^{2\alpha}}\,\frac{y^2f_Z(y)}{\mathbf{P}\{Z>u\}}\bigg]dy.\end{aligned}$$

Kasahara et al. (1988) first noted that the process

$$\xi(t)\equiv\int_0^\infty\big(\ln(t+x)-\ln(x)\big)\,dM(x)=\int_0^\infty\ln\big(1+t/x\big)\,dM(x)$$

is self-similar with $\kappa=1/\alpha$. Here $\beta_{\xi(t)}=-1$ so that $\xi(t)$ is totally skewed. Moreover $\xi(t)$ is unbounded a.s. on every interval when $\alpha<2$ (e.g., [**S&T**], Example 10.2.6), but as we shall see below, it is bounded above a.s. with very light tails.

Theorem 11. *Writing* $\xi'(1)\equiv\int_0^\infty(1+x)^{-1}\,dM(x)$ *we have*

$$\mathbf{P}\big\{\sup\nolimits_{t\in[0,1]}\xi(t)>u\big\}\sim(\kappa\,u)^{-1}\,\mathbf{E}\big\{\xi'(1)^+\,\big|\,\xi(1)>u\big\}\,\mathbf{P}\{\xi(1)>u\}\qquad\text{as}\quad u\to\infty.$$

Proof. The proof goes via Theorem 7: According to e.g., [**S&T**], p. 17, we have

$$\mathbf{P}\big\{S_\alpha(\sigma,-1)>u\big\}\sim A(\alpha)\Big(\frac{u}{\sigma}\Big)^{-\alpha/2(\alpha-1)}\exp\bigg\{-B(\alpha)\Big(\frac{u}{\sigma}\Big)^{\alpha/(\alpha-1)}\bigg\}\tag{12.2}$$

as $u\to\infty$, for some constants $A(\alpha),B(\alpha)>0$. Hence $G\in\mathcal{D}(\mathrm{I})$ and (1.1) holds with $w(u)\equiv B(\alpha)^{-1}\sigma_{\xi(1)}^{-\alpha/(\alpha-1)}(1\vee u)^{-1/(\alpha-1)}$. Taking $q(u)\equiv w(u)/u$ we further obtain $\mathfrak{P}_4=\mathfrak{P}_5=1$, so that Assumption 2 follows from Proposition 2.

Since by Hölder's inequality $\langle \hat{h}-h,h\rangle_{\alpha,1}^2 \leq \langle \hat{h}-h,h\rangle_{\alpha,2}\,\|h\|_\alpha^\alpha$, (12.1) combines with (2.3) in a straightforward calculation to show that

$$(12.3)\qquad \mathbf{E}\big\{(\hat{Z}-Z)^2\big|\,Z>u\big\}$$

$$\leq \frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}\bigg(\frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}+1\bigg)\int_u^\infty\bigg[\int_y^\infty \frac{(\alpha-1)\,z\,f_Z(z)\,dz}{\mathbf{P}\{Z>u\}}+\frac{y^2 f_Z(y)}{\mathbf{P}\{Z>u\}}\bigg]dy$$

$$= \frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}\bigg(\frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}+1\bigg)\bigg[u^2+\int_1^\infty \frac{[2\alpha y-(\alpha-1)u]\,\mathbf{P}\{Z>yu\}}{\mathbf{P}\{Z>u\}}\,dy\bigg]$$

$$\leq \frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}\bigg(\frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}+1\bigg)\bigg[u^2+2\alpha\int_0^1 \kappa\,\hat{y}^{-2\kappa-1}\frac{\mathbf{P}\{Z>\hat{y}^{-\kappa}u\}\,d\hat{y}}{\mathbf{P}\{Z>u\}}\bigg]$$

$$\leq \frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}\bigg(\frac{\langle \hat{h}-h,h\rangle_{\alpha,2}}{\|h\|_\alpha^\alpha}+1\bigg)\bigg[u^2+\frac{4\alpha\,w(u)}{u}\bigg]\qquad\text{for } u \text{ sufficiently large.}$$

Noting that $0\leq \ln(1+x)-\ln(1-t+x)\leq 2t(1+x)^{-1}$ for $0\leq t\leq\frac{1}{2}$, we thus get

$$\mathbf{P}\big\{\xi(1)>u+(\lambda+\nu)w,\,\xi(1-qt)\leq u+\nu w\big\}\big/\,\mathbf{P}\{\xi(1)>u\}$$

$$\leq \mathbf{E}\big\{[\xi(1)-\xi(1-qt)]^2\big|\,\xi(1)>u\big\}\;/\;(\lambda w)^2$$

$$\leq \int_0^\infty \frac{4q^2t^2\,[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^2\,\sigma_{\xi(1)}^\alpha}\,dx\bigg(\int_0^\infty \frac{4q^2t^2\,[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^2\,\sigma_{\xi(1)}^\alpha}\,dx+1\bigg)\,\frac{u^2+4\alpha\,w/u}{(\lambda\,w)^2}.$$

Here $w^{-2}q^2u^2=1$ so that (8.1) holds, and so Proposition 3.(i) gives Assumption 3.

In order to prove (7.1) we observe that $0\leq \ln(1+x)-\ln(1-t+x)-t(1+x)^{-1}\leq t^2(1+x)^{-2}$ for $0\leq t\leq\frac{1}{2}$. By another application of (12.3) we therefore obtain

$$\mathbf{E}\bigg\{\bigg(\frac{\xi(1-qt)-u}{w}-\frac{\xi(1)-u}{w}+\frac{qt\,\xi'(1)}{w}\bigg)^2\,\bigg|\,\xi(1)>u\bigg\}$$

$$\int_0^\infty \frac{q^4t^4\,[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^4\,\sigma_{\xi(1)}^\alpha}\,dx\bigg(\int_0^\infty \frac{q^4t^4\,[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^4\,\sigma_{\xi(1)}^\alpha}\,dx+1\bigg)\,\frac{u^2+4\alpha\,w/u}{w^2}$$

$$= O(q^2)\qquad\text{as } u\to\infty.$$

In a by now familiar manner we deduce (7.2) from the [(12.3)-based] esimates

$$\mathbf{E}\big\{[q\,\xi'(1)/w]^2\,\big|\,\xi(1)>u\big\}$$

$$\leq \frac{q^2}{w^2}\int_0^\infty \frac{[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^2\,\sigma_{\xi(1)}^\alpha}\,dx\bigg(\int_0^\infty \frac{[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^2\,\sigma_{\xi(1)}^\alpha}\,dx+1\bigg)\bigg(u^2+\frac{4\alpha\,w(u)}{u}\bigg)$$

$$\to \int_0^\infty \frac{[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^2\,\sigma_{\xi(1)}^\alpha}\,dx\bigg(\int_0^\infty \frac{[\ln(1+x^{-1})]^{\alpha-2}}{(1+x)^2\,\sigma_{\xi(1)}^\alpha}\,dx+1\bigg).$$

Since α-stable distributions are unimodal (e.g., [**S&T**] p. 574), (7.3) holds. □

13. Totally skewed linear fractional α-stable motion. Define $M(t)$ as in Section 12 and choose an $H\in(0,1-1/\alpha)$. Then the process

$$\xi(t)\equiv\int_{\mathbb{R}}|((t+x)^+)^H-(x^+)^H|\,dM(x)=\int_{-t}^{0}(t+x)^H\,dM(x)+\int_{0}^{\infty}((t+x)^H-x^H)\,dM(x)$$

is self-similar with index $H+1/\alpha$, and for $\alpha=2$ it is fBm. (e.g., [**S&T**], Eq. 7.2.7).

Theorem 12. *We have* $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}\sim\mathbf{P}\{\xi(1)>u\}$ *as* $u\to\infty$.

Proof. By (12.2) we have $G\in\mathcal{D}(\mathrm{I})$ and (1.1) holds with $w(u)\equiv B(\alpha)^{-1}\sigma_{\xi(1)}^{-\alpha/(\alpha-1)}$ $(1\vee u)^{-1/(\alpha-1)}$. Taking $q(u)\equiv(1\vee u)^{-\alpha/[(\alpha-1)(H\alpha+1)]}$ we further have $\mathfrak{P}_4=\mathfrak{P}_5=\infty$, and in view of Theorem 4 it only remains to verify Assumption 3.

Now recall that (e.g., according to [**S&T**], Property 1.2.15)

$$\mathbf{P}\big\{S_\alpha(\sigma,-1)<-x\big\}\le C_1(x/\sigma)^{-\alpha}\quad\text{for } x>0,\quad\text{for some constant } C_1>0.$$

Writing $\varepsilon=1-(1+\frac{1}{2}H\alpha)/(1+H\alpha)$ we therefore obtain

$$\mathbf{P}\bigg\{\int_{-1}^{qt^{1-\varepsilon}-1}(1+x)^H dM(x)<-\tfrac{1}{3}\lambda w\bigg\}\le C_1\bigg(\frac{\lambda w\,(1+H\alpha)^{1/\alpha}}{3\,(qt^{1-\varepsilon})^{H+1/\alpha}}\bigg)^{-\alpha}$$
$$\le C_1(C_2\lambda\,t^{-(H/2+1/\alpha)})^{-\alpha}$$

for some $C_2>0$. Defining

$$h(x)\equiv\begin{cases}(1-qt+x)^H-x^H, & 0<x\\ (1-qt+x)^H, & qt^{1-\varepsilon}-1<x<0\\ 0, & x<qt^{1-\varepsilon}-1\end{cases}\qquad \hat{h}(x)\equiv\begin{cases}(1+x)^H-x^H, & 0<x\\ (1+x)^H, & qt^{1-\varepsilon}-1<x<0\\ 0, & x<qt^{1-\varepsilon}-1\end{cases}$$

an analogous calculation shows that

$$\mathbf{P}\bigg\{\xi(1-qt)>u+\frac{\lambda w}{3},\int_{qt-1}^{qt^{1-\varepsilon}-1}(1-qt+x)^H dM(x)>\frac{\lambda w}{3}\bigg\}$$
$$\le\sum_{k=1}^{\infty}\mathbf{P}\bigg\{\int_{\mathbb{R}}h(x)\,dM(x)>u-\frac{k\lambda w}{3}\bigg\}\mathbf{P}\bigg\{\frac{k\lambda w}{3}<\int_{-1}^{qt^{1-\varepsilon}-1}(1+x)^H dM(x)\le\frac{(k+1)\lambda w}{3}\bigg\}$$
$$\le\sum_{k=1}^{\infty}\mathbf{P}\big\{\xi(1)>u-\tfrac{1}{3}k\lambda w\big\}\,\mathbf{P}\big\{S_\alpha(1,-1)>C_2k\,\lambda\,t^{-(H/2+1/\alpha)}\big\}.$$

Moreover it is an elementary matter to prove $h(x)\ge h(x)+(t^{-\varepsilon}-1)\,(h(x)-\hat{h}(x))\ge 0$ for $x\in\mathbb{R}$ and $t\ge1$, and it follows that

$$\mathbf{P}\bigg\{\int_{\mathbb{R}}h(x)\,dM(x)>u+(\tfrac{2}{3}\lambda+\nu)w,\int_{\mathbb{R}}\hat{h}(x)\,dM(x)\le u+(\tfrac{1}{3}\lambda+\nu)w\bigg\}$$

$$\leq \mathbf{P}\Big\{S_\alpha\big(\|h+(t^{-\varepsilon}-1)(h-\hat{h})\|_\alpha,-1\big)>u+(t^{-\varepsilon}-1)\tfrac{1}{3}\lambda w\Big\} \leq \mathbf{P}\big\{\xi(1)>u+\tfrac{1}{6}t^{-\varepsilon}\lambda w\big\}$$

for t small. Adding things up we now readily conclude

$$\begin{aligned}
&\mathbf{P}\big\{\xi(1-qt)>u+(\lambda+\nu)w,\ \xi(1)\leq u+\nu w\big\}\\
&\quad\leq \sum_{k=1}^{\infty}\mathbf{P}\big\{\xi(1)>u-\tfrac{1}{3}k\lambda w\Big\}\,\mathbf{P}\big\{S_\alpha(1,-1)>C_2 k\,\lambda\, t^{-(H/2+1/\alpha)}\big\}\\
&\qquad+\mathbf{P}\bigg\{\int_{\mathbb{R}} h(x)\,dM(x)>u+(\tfrac{2}{3}\lambda+\nu)w\bigg\}\,C_1(C_2\lambda\, t^{-(H/2+1/\alpha)})^{-\alpha}\\
&\qquad+\mathbf{P}\big\{\xi(1)>u+\tfrac{1}{6}t^{-\varepsilon}\lambda w\big\}\\
&\quad\leq C_3\,\lambda^{-e}\,t^{1+H\alpha/2}\,\mathbf{P}\big\{\xi(1)>u\big\}\quad \text{for } t\geq 1,\quad \text{for some constants } C_3, e>0.
\end{aligned}$$

Hence (8.2) holds and Proposition 3.(ii) yields Assumption 3. □

14. Smooth stable and Gaussian moving averages. Clearly the process

$$\xi(t)\equiv t^{\kappa}\int_{\mathbb{R}} f(\ln(t)+x)\,dM(x)\quad \text{is self-similar with index } \kappa$$

for each $f\in\mathbb{L}^\alpha(\mathbb{R})$ [where $M(t)$ is defined as in Section 12]. The proof of the next theorem is a simple adaption of the proof of Theorem 11 and is left to the reader:

Theorem 13. *Take a non-negative absolutely continuous* $f\in\mathbb{L}^\alpha(\mathbb{R})$ *such that*

$$\kappa f+f'\in\mathbb{L}^\alpha(\mathbb{R})\quad \textit{with}\quad \lim_{t\to 0} t^{-2}\Big\langle(1-t)^k f(\ln(1-t)+\cdot)+t(\kappa f(\cdot)+f'(\cdot)),\ f(\cdot)\Big\rangle_{\alpha,2}=0.$$

Writing $\xi'(1)\equiv\int_0^\infty(\kappa f(x)+f'(x))\,dM(x)$ *we then have*

$$\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}\sim(\kappa\, u)^{-1}\,\mathbf{E}\big\{\xi'(1)^+\,\big|\,\xi(1)>u\big\}\,\mathbf{P}\big\{\xi(1)>u\big\}\quad \textit{as}\ \ u\to\infty.$$

15. Kesten-Spitzer processes. Take $\alpha_1,\alpha_2\in(1,2]$ and $\beta_1,\beta_2\in[-1,1]$ and let $\{M(s)\}_{s\in\mathbb{R}}$ be an α_1-stable motion with skewness β_1. Thus $M(s)$ has stationary independent increments and $M(s)\in S_{\alpha_1}(|s|^{1/\alpha_1},\beta_1)$. Further let $\{N(s)\}_{s\geq 0}$ be an α_2-stable motion with skewness β_2 that is independent of $\{M(s)\}_{s\in\mathbb{R}}$, and define

$$\xi(t)\equiv\int_{x\in\mathbb{R}} L_t(x)\,dM(x)\quad \text{where}\quad L_t(x)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t I_{(x-\varepsilon,x+\varepsilon)}(N(s))\,ds$$

is the local time of $N(s)$ at x up to time t: By Boylan (1964), $L_t(x)$ exists and can be choosen as a continuous (random) function $[0,\infty)\times\mathbb{R}\ni(t,x)\to L_t(x)\in[0,\infty)$.

The process $\{\xi(t)\}_{t\geq 0}$ were introduced by Kesten & Spitzer (1979) and is self-similar with index $\kappa=(\alpha_1\alpha_2-\alpha_1+1)/(\alpha_1\alpha_2)>1/\alpha_1$.

Theorem 14. *When $\beta_1 > -1$ the Kesten-Spitzer process $\xi(t)$ satisfies the hypothesis of Theorem 1 and Corollary 1 with $G \in D(\mathrm{II})$ and $q(u) = 1$.*

Proof. According to e.g., [**S&T**], Property 1.2.15, we have

$$\mathbf{P}\big\{S_{\alpha_1}(\sigma,\beta_1) > u\big\} \sim C_{\alpha_1,\beta_1}(u/\sigma)^{-\alpha_1} \quad \text{as } u \to \infty, \text{ for some constant } C_{\alpha_1,\beta_1} > 0.$$

Hence $G \in D(\mathrm{II})$ follows from the easily established fact that $\mathbf{E}\big\{\int_{x\in\mathbb{R}} L_t(x)^\alpha\,dx\big\} = \mathbf{E}\{\|L_t\|_\alpha^\alpha\} < \infty$ for $\alpha \in (1,2]$ [e.g., Kesten & Spitzer (1979, Remark 1)] implies

$$\mathbf{P}\{\xi(1) > u\} = \mathbf{P}\big\{S_{\alpha_1}(\|L_t\|_{\alpha_1},\beta_1) > u\big\} \sim C_{(\alpha_1,\beta_1)}\,\mathbf{E}\{\|L_t\|_{\alpha_1}^{\alpha_1}\}\,u^{-\alpha_1}.$$

Given a $\beta_1 > -1$ and non-negative functions $f_1,\ldots,f_n \in \mathbb{L}^{\alpha_1}(\mathbb{R})$ with $\|f_1\|_{\alpha_1} > 0$, Theorem 4.1 of Samorodnitsky (1988) states that

$$\lim_{u\to\infty} \mathbf{P}\bigg\{\bigcap_{i=2}^{n}\Big\{\int_{x\in\mathbb{R}} f_i(x)\,dM(x) > u\Big\}\bigg|\int_{x\in\mathbb{R}} f_1(x)\,dM(x) > u\bigg\} = \frac{\|f_1\wedge\ldots\wedge f_n\|_{\alpha_1}^{\alpha_1}}{\|f_1\|_{\alpha_1}^{\alpha_1}}.$$

When specializing this result to our specific setting it is readily seen that

$$\begin{aligned}&\lim_{u\to\infty} \mathbf{P}\bigg\{\bigcap_{i=1}^{n}\Big\{\frac{\xi(1-t_i)-u}{u} > x_i\Big\}\bigg|\,\xi(1) > u\bigg\}\\ &\qquad\qquad = \mathbf{E}\Big\{\|L_1 \wedge (1+x_1)^{-1}L_{1-t_1} \wedge \ldots \wedge (1+x_n)^{-1}L_{1-t_n}\|_{\alpha_1}^{\alpha_1} / \|L_1\|_{\alpha_1}^{\alpha_1}\Big\},\end{aligned}$$

and so Assumption 1 holds. Further the fact that $Q = 1$ implies Assumption 2.

Since the finite dimensional distributions of $\{L_1(x) - L_{1-t}(x)\}_{x\in\mathbb{R}}$ coincide with those of $\{\hat{L}_t(x+N(1-t))\}_{x\in\mathbb{R}}$, where $\hat{L}_t(x)$ is the local time of an independent copy $\{\hat{N}(s)\}_{s\geq 0}$ of $\{N(s)\}_{s\geq 0}$ that is also independent of $\{M(s)\}_{s\in\mathbb{R}}$, we have

$$\begin{aligned}\mathbf{P}\big\{\xi(1) > u+(\lambda+\nu)u,\ \xi(1-t) \leq u+\nu u\big\} &\leq \mathbf{P}\big\{\xi(1)-\xi(1-t) > \lambda u\big\}\\ &= \mathbf{P}\bigg\{\int_{x\in\mathbb{R}} \hat{L}_t(x+N(1-t))\,dM(x) > \lambda u\bigg\}\\ &= \mathbf{P}\bigg\{t^\kappa \int_{x\in\mathbb{R}} \hat{L}_1(x)\,dM(x) > \lambda u\bigg\}\\ &\leq 2\,C_{\alpha_1,\beta_1}\mathbf{E}\{\|L_t\|_{\alpha_1}^{\alpha_1}\}\,t^{\kappa\alpha_1}\,(\lambda u)^{-\alpha_1}\end{aligned}$$

for $t^{-\kappa}\lambda u$ large. Hence (8.1) holds and Proposition 3.(i) yields Assumption 3. □

16. Rosenblatt processes. Let $\{B(t)\}_{t\in\mathbb{R}}$ be standard Brownian motion and

$$R(t) \equiv K_\gamma^{-1}\int_{x\in\mathbb{R}^2}\int_0^t [(s-x_1)^+(s-x_2)^+]^{-(1+\gamma)/2} ds\, dB(x_1)dB(x_2) \qquad \text{for } t\geq 0.$$

Here $\gamma\in(0,\frac{1}{2})$ and $K_\gamma>0$ are constants such that $\mathbf{Var}\{R(1)\}=1$. The process $R(t)$ has stationary increments and is self-similar with index $1-\gamma$. Using a different (but equivalent) definition, it was introduced by Rosenblatt (1961) [and named after him by Taqqu (1975)].

For the convenience of the reader we now supply a result from Albin (1998 $^+$):

Theorem A. *There exist constants $C,c>0$ and $j\in\mathbb{Z}$ such that*

$$\mathbf{P}\{R(1)>u\} \sim C\,u^{j/2}\exp\{-c\,u\} \qquad \text{as } u\to\infty.$$

Theorem 15. *For the Rosenblatt process $\{R(t)\}_{t\geq 0}$ we have*

$$\varliminf_{u\to\infty}\frac{\mathbf{P}\big\{\sup_{t\in[0,1]}R(t)>u\big\}}{\mathbf{P}\{R(1)>u\}}>0 \qquad \textit{and} \qquad \varlimsup_{u\to\infty}\frac{\mathbf{P}\big\{\sup_{t\in[0,1]}R(t)>u\big\}}{u^{\gamma/(1-\gamma)}\,\mathbf{P}\{R(1)>u\}}<\infty.$$

Proof. Only the upper bound requires a proof. But by Theorem A the distribution function G of $R(1)$ satisfies (1.1) with $\hat{u}=\infty$, $w(u)=c^{-1}$ and $F(x)=1-\mathrm{e}^{-x}$, and so $G\in\mathcal{D}(\mathrm{I})$. Taking $q(u)\equiv(1\vee u)^{-1/(1-\gamma)}$ we further obtain

$$\begin{aligned}\mathbf{P}\big\{R(1)>u+(\lambda+\nu)w,\, R(1-qt)\leq u+\nu w\big\} &\leq \mathbf{P}\big\{R(1)-R(1-qt)>\lambda w\big\}\\ &= \mathbf{P}\big\{R(1)>c^{-1}\lambda\, t^{\gamma-1}u\big\}.\end{aligned}$$

Hence Theorem A shows that (8.1) holds, so that Proposition 3 proves Assumption 3. Consequently Theorem 3 combines with (2.1) to give the upper bound. □

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References

Acosta, A. de (1977). Asymptotic behaviour of stable measures. *Ann. Probab.* **5** 494-499.
Albin, J.M.P. (1990). On extremal theory for stationary processes. *Ann. Probab.* **18** 92-128.

Albin, J.M.P. (1992a). On the general law of iterated logarithm with application to selfsimilar processes and to Gaussian processes in $\Re^n$ and Hilbert space. *Stochastic Process. Appl.* **41** 1-31.

Albin, J.M.P. (1992b). Extremes and crossings for differentiable stationary processes with application to Gaussian processes in $\Re^m$ and Hilbert space. *Stochastic Process. Appl.* **42** 119-147.

Albin, J.M.P. (1995). Upper and lower classes for $\mathbb{L}^2$- and $\mathbb{L}^p$-norms of Brownian motion and norms of α-stable motion. *Stochastic Process. Appl.* **58** 91-103.

Albin, J.M.P. (1997). Extremes for non-anticipating moving averages of totally skewed α-stable motion. To appear in *Statistics & Probability Letters.*

Albin, J.M.P. (1998^+). A note on Rosenblatt distributions. Submitted for publication in *Statistics & Probability Letters.*

Berman, S.M. (1982). Sojourns and extremes of stationary processes. *Ann. Probab.* **10** 1-46.

Berman, S.M. (1992). *Sojourns and Extremes of Stochastic Processes.* Wadsworth & Brooks/Cole, Belmont.

Boylan, E. (1964). Local times for a class of Markoff processes. *Illinois J. Math.* **8** 19-39.

Cameron, R.H. and Martin, W.T. (1944). The Wiener measure of Hilbert neighborhoods in the space of continuous functions. *J. Math. Phys.* **23** 195-209.

Chan, T., Dean, D.S., Jansons, K.M. and Rogers, L.C.G. (1994). On polymer conformations in elongational flows. *Commun. Math. Phys.* **160** 239-257.

Dobrushin, R.L. & Major, P. (1979). Non-central limit theorems for non-linear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** 27-52.

Doob, J.L. (1953). *Stochastic Processes* (Wiley, New York).

Duplantier, B. (1989). Areas of planar Brownian curves. *J. Phys. A Math. Gen.* **22** 3033-3048.

Kasahara, Y., Maejima. M. and Vervaat, W. (1988). Log-fractional stable processes. *Stochastic Process. Appl.* **30** 329-339.

Kesten, H. and Spitzer, F. (1979). A limit theorem related to a new class of self similar processes. *Z. Wahrsch. verw. Gebiete* **50** 5-25.

Konstant, D.G. and Pitebarg, V.I. (1993). Extreme values of the cyclostationary Gaussian random process. *J. Appl. Probab.* **30** 82-97.

Maejima, M. (1983). On a class of self-similar processes. *Z. Wahrsch. verw. Gebiete* **62** 235-245.

Li, W.V. (1992). Limit theorems for the square integral of Brownian motion and its increments. *Stochastic Process. Appl.* **41** 223-239.

Pickands, J. III (1968). Moment convergence of sample extremes. *Ann. Math. Statist.* **39** 881-889.

Pickands, J. III (1969). Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.* **145** 51-73.

Resnick, S.I. (1987). *Extreme Values, Regular Variation, and Point Processes.* Springer, New York.

Rosenblatt, M. (1961). Independence and dependence. *Proc. 4th Berkeley Sympos. Math. Statist.* 431-443. Univ. of California Press.

Samorodnitsky, G. (1988). Extrema of skewed stable processes. *Stochastic Process. Appl.* **30** 17-39.

([**S&T**]) Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, London.

Scheffé, H. (1947). A useful convergence theorem for probability distributions. *Ann. Math. Statist.* **18** 434-438.

Sharpe, K. (1978). Some properties of the crossings process generated by a stationary χ^2 process. *Adv. in Appl. Probab.* **10** 373-391.

Taqqu, M.S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* **31** 287-302.

Taqqu, M.S. (1986). A bibliographical guide to self-similar processes and long-range dependence, in: E. Eberlein and M.S. Taqqu , eds., *Dependence in Probability and Statistics, Progress in Probability and Statistics Vol. 11.* Birkhäuser, Boston.

Taqqu, M.S. and Czado, C. (1985). A survey of functional laws of the iterated logarithm for self-similar processes. *Stochastic Models* **1** 77-115.

Taqqu, M.S. and Wolpert, R. (1983). Infinite variance self-similar processes subordinate to a Poisson measure. *Z. Wahrsch. verw. Gebiete* **62** 53-72.

Vervaat, W. (1985). Sample path properties of self-similar processes with stationary increments. *Ann. Probab.* **13** 1-27.
Yor, M. (1992). *Some Aspects of Brownian Motion. Part I: Some Special Functionals*. Birkhäuser, Basel.

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