A NOTE ON ROSENBLATT DISTRIBUTIONS

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Rosenblatt processes arise as functional limits in non-central limit theorems for strongly dependent Gaussian sequences. Using local central- and χ^2-limit techniques we show that the marginal distributions of these processes belong to the Type I-domain of attraction of extremes. This in turn makes it possible to obtain bounds on local extremes for Rosenblatt processes.

1. Introduction. A random variable Y is Rosenblatt distributed provided that

$$\mathbf{E}\{e^{i\theta Y}\} = \exp\Bigg\{\sum_{k=2}^{\infty} \frac{(2i\theta)^k}{2k} \int_{x\in[0,1]^k} |x_1 - x_k|^{-2\gamma} \prod_{j=2}^{k} |x_j - x_{j-1}|^{-2\gamma}\, dx\Bigg\} \quad \text{for} \quad \theta\in\mathbb{R},$$

where $\gamma\in(0,\frac{1}{2})$ is a parameter, and then we write $Y\in\Re(\gamma)$. These distributions were introduced by Rosenblatt (1961), and named after him by Taqqu (1975).

Let η be an $N(0,1)$-distributed random variable. Then the Hermite polynomials $H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$ form an ON-basis in $\mathbb{L}^2(\mathbb{R},\eta) \equiv \{G \in \mathbb{L}^0(\mathbb{R}) : \mathbf{E}\{G(\eta)^2\}<\infty\}$. Writing $J_n \equiv \mathbf{E}\{G(\eta)H_n(\eta)\}$ we thus have $\mathbf{E}\{[\sum_{n=m}^{N} J_n H_n(\eta) - G(\eta)]^2\} \to 0$ as $N\to\infty$ for $G \in \mathcal{G}_m \equiv \{G\in\mathbb{L}^2(\mathbb{R},\eta) : J_0=\ldots=J_{m-1}=0\}$.

Theorem A. [Taqqu (1975), Dobrushin & Major (1979)]. *Let $\{X_i\}_{i\in\mathbb{Z}}$ be a stationary zero-mean Gaussian sequence. Assume that the covariance*

$$r(N) \equiv \mathbf{E}\{X_N X_0\} \to 0 \quad \textit{and} \quad \textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} r(i-j)^2 \sim L(N)^2 \quad \textit{as} \quad N\to\infty,$$

where L is a regularly varying function with index γ. Writing $\{B(t)\}_{t\in\mathbb{R}}$ for standard Brownian motion and choosing a $G\in\mathcal{G}_2$, we then have

$$\frac{1}{L(N)}\sum_{i=1}^{[Nt]} G(X_i) \to \frac{J_2}{2K_\gamma}\int_{x\in\mathbb{R}^2}\int_{s=0}^{s=t} [(s-x_1)^+(s-x_2)^+]^{-(1+\gamma)/2} ds\, dB(x_1)dB(x_2)$$

$$\equiv \frac{J_2}{2K_\gamma}\,\xi(t) \qquad \textit{weakly in } \mathcal{D}([0,1]) \textit{ as } N\to\infty,$$

where $K_\gamma^2 = \mathbf{Var}\{\xi(1)\}$. Moreover $K_\gamma^{-1} t^{\gamma-1}\xi(t) \in \Re(\gamma)$ for each $t>0$.

In Sections 2 and 3 we prove the following theorems:

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Theorem 1. *The distribution function F of a Rosenblatt distributed random variable $Y \in \Re(\gamma)$ has a density function f such that*

$$\lim_{u\to-\infty} w(u)\, f(u-w(u)x)/F(u) = e^{-x} \quad \textit{for } x\in\mathbb{R}, \tag{1.1}$$

for some non-negative function w. Thus F belongs to the Type I-domain of attraction for minima with auxiliary function w, i.e.,

$$\lim_{u\to-\infty} F(u-w(u)x)/F(u) = e^{-x} \quad \textit{for } x\in\mathbb{R}. \tag{1.2}$$

Theorem 2. *The distribution function F of a Rosenblatt distributed random variable $Y \in \Re(\gamma)$ has a density function f such that*

$$\lim_{u\to\infty} W\, f(u+Wx) \,/\, (1-F(u)) = e^{-x} \quad \textit{for } x\in\mathbb{R}, \tag{1.3}$$

where $W \equiv \sup\{s \in \mathbb{R} : \mathbf{E}\{e^{sY}\} < \infty\} \in (0,\infty)$. Thus F belongs to the Type I-domain of attraction for maxima with constant auxiliary function W, i.e.,

$$\lim_{u\to\infty}(1-F(u+Wx)) \,/\, (1-F(u)) = e^{-x} \quad \textit{for } x\in\mathbb{R}. \tag{1.4}$$

The proof of (1.1) and (1.3) is an adaption of the method of Feigin & Yashchin (1983) and Davis & Resnick (1991). Their idea is to derive a so-called strong Tauberian result by proving a local limit theorem for a suitably normalized and Esscher-transformed version of the distribution under consideration.

Strong Tauberian theorems origin in the asymptotic treatment of convolution kernels by Hirschmann & Widder (1955, Chapter V). Feigin & Yashchin (1983) gave a generalization to more general Laplace transforms (than convolution kernels), but under rather restrictive technical conditions. As did Davis & Resnick (1991) in their study of sums of non-negative random variables, we found that these conditions are not met by our framework. Thus a treatment adapted to the specific situation is required. But the scheme of Feigin & Yashchin (1983) remains the main inspiration.

Application of results from extreme value theory requires belongance to a domain of attraction: Methods in statistics for extremes are directly linked to these domains [e.g., Resnick (1987)], and extremal theory for stochastic processes also relies on attraction [e.g., Berman (1982) and Albin (1990, 1998)]. In Section 4 we show how Theorems 1 and 2 combine with a results of Albin (1998) to yield bounds on the probability of a local extrema for the Rosenblatt process $\xi(t)$ in Theorem A.

Theorem A suggests that $\Re(\gamma)$-distributions are related to χ^2-distributions: Letting $\eta_1, \eta_2, \ldots$ denote independent $N(0,1)$-variables the precise statement becomes [Taqqu (1975, Section 6), Dobrushin & Major (1979, Proposition 2)]

$$Y =_{\mathcal{L}} \sum_{j=1}^{\infty} \lambda_j(\eta_j^2 - 1) \quad \text{where} \quad \lambda_1 = \ldots = \lambda_{j_0} > \lambda_{j_0+1} \geq \ldots > 0 \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_j^2 < \infty. \tag{1.5}$$

In (1.5) we have $\sum_{j=1}^{\infty} \lambda_j = \infty$. This is important since otherwise we could write $Y =_{\mathcal{L}} \|\hat{Y}\|^2 - \sum_{j=1}^{\infty} \lambda_j$ where $\hat{Y} = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \eta_j e_j$ is Gaussian with values in the Hilbert space spanned by an ON-basis $\{e_j\}_{j=1}^{\infty}$. But then the fact that Y belongs to the Type I-domain of attraction would follow from the corresponding result for $\|\hat{Y}\|^2$ [e.g., Albin (1992, pp. 139-140) and Albin (1996, Proposition 1)].

Now note that in view of (1.5) the Laplace transform ϕ of Y is given by

$$\phi(s) = \mathbf{E}\{\exp[-sY]\} = \exp\Big\{-\sum_{j=1}^{\infty} \big(\tfrac{1}{2}\ln(1+2\lambda_j s) - \lambda_j s\big)\Big\} \quad \text{for } s \in (-(2\lambda_1)^{-1}, \infty).$$

2. Proof of Theorem 1. The Esscher-transform Y_s of Y at $s \in (-(2\lambda_1)^{-1}, \infty)$ is a random variable with distribution $dF_{Y_s}(x) = \mathrm{e}^{-sx} dF(x)/\phi(s)$. Here we have

$$m(s) \equiv \mathbf{E}\{Y_s\} = -\sum_{j=1}^{\infty} \frac{2\lambda_j^2 s}{1+2\lambda_j s} \quad \text{and} \quad \sigma(s)^2 \equiv \mathbf{Var}\{Y_s\} = \sum_{j=1}^{\infty} \frac{2\lambda_j^2}{(1+2\lambda_j s)^2}.$$

The normalized variable $Z_s \equiv (Y_s - m(s))/\sigma(s)$ has characteristic function

$$\begin{aligned} \mu_s(x) &\equiv \mathbf{E}\{\exp[ixZ_s]\} \\ &= \phi\big(s - ix/\sigma(s)\big) \exp\{-ix\, m(s)/\sigma(s)\} / \phi(s) \\ &= \exp\bigg\{-\sum_{j=1}^{\infty}\bigg[\frac{1}{2}\ln\bigg(1 - \frac{2\lambda_j ix}{(1+2\lambda_j s)\sigma(s)}\bigg) + \frac{\lambda_j ix}{(1+2\lambda_j s)\sigma(s)}\bigg]\bigg\} \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Since $\lim_{s\to\infty} s\,\sigma(s) = \infty$ it follows readily that

$$\lim_{s\to\infty} \mu_s(x) = \mathrm{e}^{-x^2/2} \quad \text{for } x \in \mathbb{R}, \quad \text{so that} \quad Z_s \to_{\mathcal{L}} N(0,1) \quad \text{as } s \to \infty. \tag{2.1}$$

To proceed we observe the easy fact that

$$|\mu_s(x)| = \exp\bigg\{-\frac{1}{4}\sum_{j=1}^{\infty} \ln\bigg(1 + \frac{4\lambda_j^2 x^2}{(1+2\lambda_j s)^2 \sigma(s)^2}\bigg)\bigg\} \quad \text{for } x \in \mathbb{R}. \tag{2.2}$$

Choosing a $c > 0$ such that $\ln(1+y^2) \geq cy^2$ for $|y| \leq 1$, we therefore get

$$\begin{aligned} &\overline{\lim_{s\to\infty}} \int_{K<|x|\leq s\sigma(s)} |\mu_s(x)|\, dx \\ &\leq \overline{\lim_{s\to\infty}} \int_K^{s\sigma(s)} 2\exp\bigg\{-\frac{1}{4}\sum_{j=1}^{\infty} \frac{c\lambda_j^2 x^2}{(1+2\lambda_j s)^2\sigma(s)^2}\bigg\}\, dx \leq \int_K^{\infty} 2\exp\{-\tfrac{1}{8}cx^2\}\, dx \to 0 \end{aligned} \tag{2.3}$$

as $K \to \infty$ [recall that $s\sigma(s) \to \infty$]. Invoking the trivial facts that

$$\int_1^\infty \big(1+\tfrac{4}{9}x^2\big)^{-\nu}\, dx \le (1+\tfrac{4}{9})^{1-\nu} \int_0^\infty \big(1+\tfrac{4}{9}x^2\big)^{-1}\, dx = \tfrac{\pi}{3}\, \big(\tfrac{13}{9}\big)^{1-\nu} \quad \text{for } \nu > 1,$$

and that $n(s) \equiv \#\{j : \lambda_j s > 1\} \to \infty$ as $s \to \infty$, we further obtain

$$\begin{aligned}
&\int_{|x| > s\sigma(s)} |\mu_s(x)|\, dx \\
&= 2\int_{s\sigma(s)}^\infty \exp\Big\{-\frac{n(s)}{4}\ln\Big(1+\frac{4\,x^2}{9\,s^2\sigma(s)^2}\Big) - \frac{1}{4}\sum_{\{j\,:\,\lambda_j s \le 1\}} \ln\Big(1+\frac{4\lambda_j^2 x^2}{9\,\sigma(s)^2}\Big)\Big\}\, dx \\
&\le 2\, s\sigma(s) \int_1^\infty \big(1+\tfrac{4}{9}x^2\big)^{-n(s)/4} \exp\Big\{-\tfrac{1}{4}\sum_{\{j\,:\,\lambda_j s \le 1\}} \ln\big(1+4\lambda_j^2 s^2\big)\Big\}\, dx \\
&\le 2\Big(\sum_{j=1}^\infty \frac{2\lambda_j^2 s^2}{(1+2\lambda_j s)^2}\Big)^{1/2} \tfrac{\pi}{3}\, \big(\tfrac{13}{9}\big)^{1-n(s)/4} \exp\Big\{-\tfrac{1}{4}c \sum_{\{j\,:\,\lambda_j s \le 1\}} \lambda_j^2 s^2\Big\} \\
&\le \frac{\sqrt{8}\,\pi}{3}\, \big(\tfrac{13}{9}\big)^{1-n(s)/4} \Big(\sqrt{\tfrac{1}{2}n(s)} + \sqrt{\sum_{\{j\,:\,\lambda_j s \le 1\}} \lambda_j^2 s^2}\, \exp\Big\{-\tfrac{1}{4}c \sum_{\{j\,:\,\lambda_j s \le 1\}} \lambda_j^2 s^2\Big\}\Big) \\
&\to 0 \quad \text{as } s \to \infty.
\end{aligned} \tag{2.4}$$

Since (2.2) shows that $|\mu_s(\cdot)| \in \mathbb{L}^1(\mathbb{R})$, Z_s has a density f_s given by the invers Fourier transform of $\mu_s(\cdot)$. Combining (2.1) and (2.3)-(2.4) we thus conclude

$$\begin{aligned}
\sup_{y\in\mathbb{R}} \Big| f_s(y) - \frac{\exp\{-\frac{1}{2}y^2\}}{\sqrt{2\pi}} \Big| &= \sup_{y\in\mathbb{R}} \Big| \frac{1}{2\pi} \int_{-\infty}^\infty e^{-iyx} \big(\mu_s(x) - \exp\{-\tfrac{1}{2}x^2\}\big)\, dx \Big| \\
&\le \frac{1}{2\pi} \int_{-\infty}^\infty \big|\mu_s(x) - \exp\{-\tfrac{1}{2}x^2\}\big|\, dx \\
&\to 0 \quad \text{as } s \to \infty.
\end{aligned} \tag{2.5}$$

Since Z_s has a density, Y has a density f that satisfies

$$f\big(m(s) + x\,\sigma(s)\big) = \sigma(s)^{-1}\, \phi(s)\, \exp\big\{s\big(m(s) + x\,\sigma(s)\big)\big\}\, f_s(x). \tag{2.6}$$

Choosing $x = 0$ and using (2.5) we now obtain

$$f(m(s)) \sim (2\pi)^{-1/2}\, \sigma(s)^{-1}\, \phi(s)\, \exp\{s\, m(s)\} \quad \text{as } s \to \infty. \tag{2.7}$$

Combining (2.5)-(2.7) we get [again recalling that $s\sigma(s) \to \infty$]

$$f(m(s) - x/s) = \sigma(s)^{-1}\, \phi(s)\, \exp\{s\, m(s)\}\, e^{-x}\, f_s\big(x/(s\sigma(s))\big) \sim e^{-x}\, f(m(s))$$

as $s \to \infty$. Another application of (2.5) therefore yields

$$\frac{e^{-x}\, sF(m(s))}{f(m(s) - x/s)} \sim \frac{sF(m(s))}{f(m(s))} = \int_0^\infty \frac{f(m(s) - y/s)}{f(m(s))}\, dy = \int_0^\infty e^{-y}\, \frac{f_s\big(y/(s\sigma(s))\big)}{f_s(0)}\, dy \to 1$$

as $s \to \infty$. Consequently (1.1) holds with $w(u) = (m^{-1}(u))^{-1}$.

The fact that (1.1) implies (1.2) is well-known and follows e.g., from an applications of the theorem by Scheffé (1947). $\square$

3. Proof of Theorem 2. Let $\hat{Y}_s$ be a variable with density $f_{\hat{Y}_s}(x) = \mathrm{e}^{-sx}\big(x-m(s)\big)^2 f(x) \,/\, \big(V(s)\phi(s)\big)$ for $s\in(-(2\lambda_1)^{-1},\infty)$. Then we have

$$\hat{m}(s) \equiv \mathbf{E}\{\hat{Y}_s\} \quad = m(s) - V'(s)/V(s) \qquad\qquad \sim \frac{j_0/2+2}{(2\lambda_1)^{-1}+s} \tag{3.1}$$

$$\hat{\sigma}(s)^2 \equiv \mathbf{Var}\{\hat{Y}_s\} = V(s) + V''(s)/V(s) - V'(s)^2/V(s)^2 \sim \frac{j_0/2+2}{((2\lambda_1)^{-1}+s)^2} \tag{3.2}$$

as $s{\downarrow}-(2\lambda_1)^{-1}$, where $V(s)\equiv\sigma(s)^2$. It follows that

$$V(s-ix/\hat{\sigma}(s)) \sim (j_0/2)\,/\,\big(1-ix/\sqrt{j_0/2+2}\big) \quad \text{as } s{\downarrow}-(2\lambda_1)^{-1}.$$

Further the normalized variable $\hat{Z}_s\equiv(\hat{Y}_s-\hat{m}(s))/\hat{\sigma}(s)$ has characteristic function

$$\begin{aligned}
\hat{\mu}_s(x) &\equiv \mathbf{E}\{\exp[ixZ_s]\} \\
&= \big(V(s-ix/\hat{\sigma}(s))/V(s)\big)\,\big(\phi(s-ix/\hat{\sigma}(s)/\phi(s)\big)\,\exp\{-ix\,\hat{m}(s)/\hat{\sigma}(s)\} \\
&= \big(V(s-ix/\hat{\sigma}(s))/V(s)\big) \\
&\qquad \times\exp\bigg\{\frac{ix(m(s)-\hat{m}(s))}{\hat{\sigma}(s)} - \frac{1}{2}\sum_{j=1}^{\infty}\bigg[\ln\bigg(1-\frac{ix/\sigma(s)}{(2\lambda_j)^{-1}+s}\bigg)+\frac{ix/\hat{\sigma}(s)}{(2\lambda_j)^{-1}+s}\bigg]\bigg\} \\
&\to \big(1-ix/\sqrt{j_0/2+2}\big)^{-(j_0/2+2)}\exp\big\{-ix\sqrt{j_0/2+2}\big\} \quad \text{as } s{\downarrow}-(2\lambda_1)^{-1}
\end{aligned} \tag{3.3}$$

for $x\in\mathbb{R}$. Consequently $\hat{Z}_s \to_{\mathcal{L}} \big(\chi^2(j_0+4)-(j_0+4)\big)/\sqrt{2j_0+8}\equiv\chi$.

To proceed we observe the easy fact that

$$\begin{aligned}
&|\hat{\mu}_s(x)| \\
&= \bigg(\sum_{j=1}^{\infty}\frac{1/(2V(s))}{((2\lambda_j)^{-1}+s)^2+x^2/\hat{\sigma}(s)^2}\bigg)\exp\bigg\{-\frac{1}{4}\sum_{j=1}^{\infty}\ln\bigg(1+\frac{x^2/\hat{\sigma}(s)^2}{((2\lambda_j)^{-1}+s)^2}\bigg)\bigg\} \\
&\le \frac{j_0/(2V(s))}{((2\lambda_1)^{-1}+s)^2+x^2/\hat{\sigma}(s)^2} \\
&\quad + \bigg(\sum_{j>j_0}\frac{1/(2V(s))}{((2\lambda_j)^{-1}+s)^2+x^2/\hat{\sigma}(s)^2}\bigg)\exp\bigg\{-\frac{1}{4}\sum_{j>j_0}\ln\bigg(1+\frac{x^2/\hat{\sigma}(s)^2}{((2\lambda_j)^{-1}+s)^2}\bigg)\bigg\}.
\end{aligned} \tag{3.4}$$

Here (3.2) combines with the fact that $V(s)\sim(j_0/2)\,/\,((2\lambda_1)^{-1}+s)^2$ to give

$$\begin{aligned}
&\overline{\lim_{s\downarrow-(2\lambda_1)^{-1}}}\int_K^{\infty}\frac{j_0/(2V(s))}{((2\lambda_1)^{-1}+s)^2+x^2/\hat{\sigma}(s)^2}\,dx \\
&= \overline{\lim_{s\downarrow-(2\lambda_1)^{-1}}}\frac{j_0\hat{\sigma}(s)/(2V(s))}{(2\lambda_1)^{-1}+s}\bigg[\frac{\pi}{2}-\arctan\bigg(\frac{K/\hat{\sigma}(s)}{(2\lambda_1)^{-1}+s}\bigg)\bigg] \to 0 \quad \text{as } K\to\infty.
\end{aligned} \tag{3.5}$$

As $s{\downarrow}-(2\lambda_1)^{-1}$ we further have

$$\int_K^{\hat{\sigma}(s)} \sum_{j>j_0} \frac{1/(2V(s))}{((2\lambda_j)^{-1}+s)^2+x^2/\hat{\sigma}(s)^2} \leq \frac{\hat{\sigma}(s)}{V(s)} \sum_{j>j_0} \frac{2\lambda_j^2}{(1+2\lambda_j s)^2} \to 0 \tag{3.6}$$

and

$$\begin{aligned}
&\int_{\hat{\sigma}(s)}^{\infty} \biggl(\sum_{j>j_0} \frac{1/(2V(s))}{((2\lambda_j)^{-1}+s)^2+x^2/\hat{\sigma}(s)^2} \biggr) \exp\biggl\{ -\frac{1}{4} \sum_{j>j_0} \ln\biggl(1+\frac{x^2/\hat{\sigma}(s)^2}{((2\lambda_j)^{-1}+s)^2}\biggr)\biggr\}\, dx \\
&\leq \int_1^{\infty} \biggl(\sum_{j>j_0} \frac{2\lambda_j^2 \hat{\sigma}(s)/V(s)}{(1+2\lambda_j s)^2} \biggr) \exp\biggl\{ -\frac{1}{4} \sum_{j=j_0+1}^{j_0+5} \ln\biggl(1+\frac{4\lambda_j^2 x^2}{(1+2\lambda_j s)^2}\biggr)\biggr\}\, dx \\
&\leq \frac{\hat{\sigma}(s)}{V(s)} \biggl(\sum_{j>j_0} \frac{2\lambda_j^2}{(1-\lambda_j/\lambda_1)^2} \biggr) \int_1^{\infty} \biggl(1+\frac{4\lambda_{j_0+5}^2 x^2}{(1+2\lambda_{j_0+1} s)^2}\biggr)^{-1} dx \\
&\to 0.
\end{aligned} \tag{3.7}$$

Since (3.4) shows that $|\hat{\mu}_s(\cdot)| \in \mathbb{L}^1(\mathbb{R})$, $\hat{Z}_s$ has a density $\hat{f}_s$ given by the invers Fourier transform of $\hat{\mu}_s(\cdot)$. Combining (3.3) and (3.5)-(3.7) we thus get [cf. (2.5)]

$$\hat{f}_s(0) - f_\chi(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bigl(\hat{\mu}_s(x) - \mathbf{E}\{\exp[-ix\chi]\}\bigr)\, dx \to 0 \quad \text{as } s{\downarrow}-(2\lambda_1)^{-1}.$$

Since

$$f(\hat{m}(s)) \;=\; \hat{f}_s(0)\, \phi(s)\, \mathrm{e}^{s\,\hat{m}(s)}\, (\hat{m}(s)-m(s))^2\, (V(s)/\hat{\sigma}(s)),$$

and [cf. (3.1)-(3.2)]

$$\tfrac{1}{2}(j_0/2+2)\, (\hat{m}(s)-m(s)) \;\sim\; \hat{m}(s) \;\sim\; \sqrt{j_0/2+2}\, V(s)/\hat{\sigma}(s),$$

it follows that

$$f(\hat{m}(s)) \sim f_\chi(0)\, \phi(s)\, \mathrm{e}^{j_0/2+2}\, \mathrm{e}^{-\hat{m}(s)/(2\lambda_1)}\, ((j_0/2+2)/2)^2\, \hat{m}(s)^{-2}\, (j_0/2+2)^{-1/2}\, \hat{m}(s).$$

But here $\phi(s) \sim C\, \hat{m}(s)^{j_0/2}$ for some constant $C>0$, and thus we conclude

$$f(u) \sim f_\chi(0)\, C\, u^{j_0/2}\, \mathrm{e}^{j_0/2+2}\, \mathrm{e}^{-u/(2\lambda_1)}\, ((j_0/2+2)/2)^2\, u^{-2}\, (j_0/2+2)^{-1/2}\, u \quad \text{as } u\to\infty.$$

Now (1.3) and (1.4) follow from elementary computations. □

References

Albin, J.M.P. (1990). On extremal theory for stationary processes. *Ann. Probab.* **18** 92-128.

Albin, J.M.P. (1992). Extremes and crossings for differentiable stationary processes with application to Gaussian processes in $\Re^m$ and Hilbert space. *Stochastic Process. Appl.* **42** 119-147.

Albin, J.M.P. (1996). Minima of H-valued Gaussian processes. *Ann. Probab.* **24** 788-824.

Albin, J.M.P. (1998). On extremal theory for self-similar processes. Accepted for publication in *Ann. Probab.*

Berman, S.M. (1982). Sojourns and extremes of stationary processes. *Ann. Probab.* **10** 1-46.

Davis, R.A. & Resnick, S.I. (1991). Extremes of moving averages of random variables with finite endpoint. *Ann. Probab.* **19** 312-328.

Dobrushin, R.L. & Major, P. (1979). Non-central limit theorems for non-linear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** 27-52.

Feigin, P.D. & Yashchin, E. (1983). On a strong Tauberian result. *Z. Wahrsch. Verw. Gebiete* **65** 35-48.

Hirschmann, I.I. & Widder, D.V. (1955). *The Convolution Transform.* Princeton Univ. Press.

Resnick, S.I. (1987). *Extreme Values, Regular Variation, and Point Processes.* Springer, New York.

Rosenblatt, M. (1961). Independence and dependence. *Proc. 4th Berkeley Sympos. Math. Statist.* 431-443. Univ. of California Press.

Scheffé, H. (1947). A useful convergence theorem for probability distributions. *Ann. Math. Statist.* **18** 434-438.

Taqqu, M.S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* **31** 287-302.