Interpolation of subspaces

Jörgen Löfström

Abstract

Let (A_0, A_1) be a Banach couple and $\vec{Y} = (Y_0, Y_1)$ a couple of closed subspaces. Given an interpolation functor F, what is the relation between $F(A_0, A_1)$ and $F(Y_0, Y_1)$? This question is investigated for the real interpolation method, in many general cases, for instance when $Y_0 = A_0$ and Y_1 has finite codimension or more generally is the kernel of bounded operators on A_1. Applications are given to L_p−spaces and Sobolev spaces.

Keywords

Interpolation spaces, real interpolation, subspaces, Sobolev spaces, Lebesgue spaces.

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Introduction

Consider a couple $\vec{A} = (A_0, A_1)$ and let $\vec{Y} = (Y_0, Y_1)$ be a pair of closed subspaces, (i.e. $Y_j,\ j = 0, 1$ is a closed subspace of A_j). As norm on Y_j we use the norm inherited from A_j. Then $\vec{Y}$ is a new Banach couple. The theme of this paper is to investigate the relation between the interpolation spaces relative to $\vec{Y}$ and the interpolation spaces relative to $\vec{A}$. Assuming that F denotes any interpolation functor, we have

$$F(\vec{Y}) \hookrightarrow F(\vec{A}) \cap \Sigma(\vec{Y}) \tag{1}$$

In order to say more we clearly need additional information on F and the couple $\vec{Y}$. In this paper we shall assume that F is defined by the real $K-$method. (See section 1.1).

The first natural question is the following one. When do we have equality in (1), i.e. when is

$$F(\vec{Y}) \cong F(\vec{A}) \cap \Sigma(\vec{Y}) \tag{2}$$

(We use the symbol $\cong$ to indicate that the spaces involved have equivalent norms!) There are many interesting situations when this holds but in general it is false. We give a brief discussion in the section 1.2. See Jansson [7].

What plausible conjectures can be made if (2) fails? In order to answer this question we shall first discuss different ways of describing a subspace. Thus let A be any Banach space. We can identify three natural methods of describing a subspace Y of A, using one or several operators. The first method is to define Y as the kernel of a bounded operator (or as the intersection of the kernels of several operators). Secondly we can define Y as the image of an operator (or intersection of several images). Finally, if A is the domain of an unbounded operator S, then we can define the subspace Y as the domain of a restriction T of S.

There are of course other methods of defining subspaces. For instance new subspaces can be constructed from a set of given subspaces, using intersections, direct sums and other abstract constructions. Such constructions will, however, not be systematically studied in this paper.

Let us now return to interpolation of subspaces. First let us assume that the subspaces are defined as kernels of given continuous linear operators T_j, so that $Y_j = A_j \cap \ker(T_j)$. Then it is natural to hope for a formula like

$$F(A_0 \cap \ker(T_0), A_1 \cap \ker(T_1)) \cong F(A_0, A_1) \cap \ker(T_F)$$

where T_F is some continuous operator on $F(A_0, A_1)$. To simplify the situation we shall assume that $T_0 = 0$. Then the formula above becomes

$$F(A_0, A_1 \cap \ker(T_1)) \cong F(A_0, A_1) \cap \ker(T_F) \tag{3}$$

We shall see that (3) is not always true. In fact we shall see that in general the interpolation space on the left hand side need not even be a closed subspace of $F(A_0, A_1)$. However there are many important situations when (3) holds. We shall also characterize the interpolation space on the left hand side of (3) even in cases when it is not true.

Next consider subspaces defined as images of operator $T_j : B_j \to A_j$. Then a natural formula to hope for is

$$F(\mathrm{im}(T_0), \mathrm{im}(T_1)) \cong \mathrm{im}(T_F)$$

Again T_F must be constructed in some way from T_0 and T_1 and there must be some connection between these two operators. We shall only consider the case when $T_0 = T_1 = T$. Then it is resonable to put $T_F = T|F(\vec{A})$. The formula above then reduces to the hypothetical formula

$$F(T(B_0), T(B_1)) \cong T(F(B_0, B_1)) \tag{4}$$

Finally, consider the case when the subspaces are defined as domains of unbounded operators. For simplicity assume that $Y_0 = A_0$ and that $A_1 = \mathrm{dom}(S)$ is the domain in A_0 of an unbounded closed and densely defined operator S. Assume that T is a closed, densely defined restriction of S, i.e. $T \subset S$. Then a natural formula to investigate is

$$F(A_0, \mathrm{dom}(T)) \cong \mathrm{dom}(T_F) \tag{5}$$

Here T_F would be a closed operator such that $T \subset T_F \subset S$.

The plan of the paper is the following. Chapter 1 contains standard background material on real interpolation. In chapter 1 we have also included an important extension theorem for general operators and some other new features.

The main part of the paper is devoted to interpolation of subspaces defined as kernels. In chapter 2 we study the subspaces of finite codimension. This section generalizes previous special cases considered by the author (see [10], [11], [12]). Applications are given to L_p−spaces and Sobolev spaces. In chapter 3 we study subspaces defined as kernels of more general operators.

Chapter 4 contains results on the interpolation of domains and ranges. We start with a general result on domains of operators in Hilbert spaces (section 4.1). In section 4.2 and 4.3 we give applications of our results to interpolation of boundary value problems. The proofs concerning boundary value problems are simplified compared to Löfstöm [9]. Finally, formula (4) is investigated in section 4.4.

The present paper contains and extends some results which are published earlier. Many results, however, are new. As a whole this paper presents what is hopefully a coherent theory.

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1 Real interpolation spaces

In this chapter we present basic facts on interpolation theory. For proofs and further details we refer the reader to the book by Brudnuĭ and Krugljak [3]. See also Bergh-Löfström [1] and Triebel [16]. We have also included here an important extension theorem for general operators and some related material.

1.1 Basic definitions

Two Banach spaces A_0 and A_1 form a *Banach couple* (or *couple* for short) if there exists a Hausdorff topological vector space $\mathcal{A}$ such that A_0 and A_1 are continuously embedded in $\mathcal{A}$. We then write $\vec{A}$ for the ordered pair (A_0, A_1).

Given a Banach couple $\vec{A}$ we can form two Banach spaces, the intersection $\Delta(\vec{A}) = A_0 \cap A_1$ and the sum $\Sigma(\vec{A}) = A_0 + A_1$, with norms

$$\|a\|_{\Delta(\vec{A})} = \max\left(\|a\|_{A_0}, \|a\|_{A_1}\right)$$

$$\|a\|_{\Sigma(\vec{A})} = \inf\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1\right\}$$

A Banach space A is an intermediate space for the couple $\vec{A}$ if

$$\Delta(\vec{A}) \hookrightarrow A \hookrightarrow \Sigma(\vec{A})$$

Here $\hookrightarrow$ denotes a continuous linear embedding.

A bounded linear operator from the Banach couple $\vec{A}$ into the Banach couple $\vec{B}$ is a bounded linear operator $T : \Sigma(\vec{A}) \to \Sigma(\vec{B})$ such that the restriction $T|A_j$ of T to A_j is a continuous mapping from A_j into B_j for $j = 0, 1$.

Let A and B are intermediate spaces for the couples $\vec{A}$ and $\vec{B}$, respectively. Then A and B are called interpolation spaces relative the couples $\vec{A}$ and $\vec{B}$, if T maps A into B, continuously, for every bounded linear operator T from $\vec{A}$ into $\vec{B}$.

The aim of interpolation theory is to construct and study interpolation spaces. There are several constructive methods. Any *functor* F from the category of all Banach couples to the category of Banach couples is called an *interpolation functor* if for any couples $\vec{A}$ and $\vec{B}$ the spaces $F(\vec{A})$ and $F(\vec{B})$ are interpolation spaces relative to the couples $\vec{A}$ and $\vec{B}$. In this paper we are studying so called $K-$functors. They are based on a scale of norms on $\Sigma(\vec{A})$, namely

$$K(t, a) = K(t, a; \vec{A}) = \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\right\}$$

To define intermediate spaces and interpolation spaces one imposes conditions on the behaviour of this scale of norms. The conditions are described by means of certain Banach function lattices Φ on the measure space $(\mathbb{R}_+, dt/t)$ i.e. a Banach space of measurable functions with the additional property that

$$|f| \leq |g|,\ g \in \Phi\ \Rightarrow\ f \in \Phi,\ \|f\|_\Phi \leq \|g\|_\Phi$$

Any such Banach function lattice will be called a *parameter* for the $K-$method if

$$\min(1, t) \in \Phi \tag{1}$$

The most widely used example of a parameter is denoted by $\Phi_{\theta,\rho}$. It is defined by the norm

$$\|f\|_{\theta,\rho} = \Big(\int_0^\infty \Big(t^{-\theta}|f(t)|\Big)^\rho \frac{dt}{t}\Big)^{1/\rho}$$

To quarante that (1) holds we must choose $0 < \theta < 1$ and $1 \leq \rho \leq \infty$ or $0 \leq \theta \leq 1$ and $\rho = \infty$.

Definition 1.1 *Given a couple $\vec{A}$ and a parameter Φ we let $K_\Phi(\vec{A})$ be the space of all $a \in \Sigma(\vec{A})$ such that*

$$\|a\|_{K_\Phi(\vec{A})} = \|K(\cdot, a; \vec{A})\|_\Phi$$

is finite. If $\Phi = \Phi_{\theta,\rho}$ we write $K_{\theta,\rho}(\vec{A})$ for the space $K_\Phi(\vec{A})$.

Theorem 1.1 *Let $\vec{A}$ and $\vec{B}$ be two Banach couples and Φ a parameter for the $K-$method. Then $A = K_\Phi(\vec{A})$ and $B = K_\Phi(\vec{B})$ are interpolation spaces for the couples $\vec{A}$ and $\vec{B}$. If T maps A_j into B_j with norm M_j $(j = 0, 1)$, then T maps A into B with norm*

$$M \leq \max(M_0, M_1)$$

If $\Phi = \Phi_{\theta,\rho}$ we have the sharper estimate

$$M \leq M_0^{1-\theta} M_1^\theta$$

The proof is based on the following lemma, which we shall use on several occassions later on.

Lemma 1.1 *The function $t \mapsto K(t, a; \vec{A})$ is non-negative, increasing and concave, i.e.*

$$K(t, a; \vec{A}) \leq \max(1, t/s) K(s, a; \vec{A})$$

or equivalentely

$$\min(1, s/t) K(t, a; \vec{A}) \leq K(s, a; \vec{A})$$

Consider now the couple $\vec{L}_\infty = (L_\infty, L_\infty^1)$, where L_∞ is the space of essentially bounded functions on the measure space $(\mathbb{R}_+, dt/t)$ and L_∞^1 is the space of all f such that $t^{-1}f(t) \in L_\infty$. For any parameter Φ, the space

$$\hat{\Phi} = K_\Phi(\vec{L}_\infty) \tag{2}$$

is a Banach function lattice and also a parameter. This is easily seen. In fact, assume that $|f| \leq |g|$ where $g \in \hat{\Phi}$. Put $T\phi = h\phi$ where $h = f/g$ (interpreted as zero where g vanishes). Then T maps L_∞ into L_∞ and L_∞^1 into L_∞^1, both with norm 1. Thus T maps $\hat{\Phi}$ into $\hat{\Phi}$ with norm at most 1 (by theorem 1.1). This implies that $f = Tg \in \hat{\Phi}$ and $\|f\|_{\hat{\Phi}} \leq \|g\|_{\hat{\Phi}}$.

It is an important fact that one needs only to consider parameters of the form $\hat{\Phi}$ (i.e. exact interpolation spaces for the couple $\vec{L}_\infty$). In fact, for any parameter Φ we have the following result. (See [3], corollary 3.3.6.)

Theorem 1.2 *Let $\vec{A}$ be an arbitrary couple and Φ a parameter for the $K-$method. Define $\hat{\Phi}$ by formula (2). Then*

$$K_\Phi(\vec{A}) = K_{\hat{\Phi}}(\vec{A})$$

1.2 $K-$subcouples

We shall start by considering the relation

$$F(\vec{Y}) \cong F(\vec{A}) \cap \Sigma(\vec{Y})$$

where $\vec{Y}$ is a subcouple of $\vec{A}$, (i.e. Y_j is a closed subspace of A_j and the norm of Y_j is inherited from A_j).

Definition 1.2 *The subcouple $\vec{Y}$ is called a $K-$subcouple of $\vec{A}$ if*

$$K(t, y; \vec{Y}) \leq CK(t, y; \vec{A}) \quad \textit{for all} \quad y \in \Sigma(\vec{Y})$$

See Jansson [7] , Cf Peetre [13]. Clearly this relation implies that

$$K_\Phi(\vec{Y}) \cong K_\Phi(\vec{A}) \cap \Sigma(\vec{Y}) \tag{1}$$

for all parameters Φ.

Let us give a two simple examples. Further examples can be found in Jansson [7]. We say that $\vec{Y}$ is a complemented subcouple of $\vec{A}$ if there is a bounded projection $P : \vec{A} \to \vec{Y}$ which maps A_j onto Y_j. Clearly $\vec{Y}$ is a $K-$subcouple.

As a second example we mention the Hardy space H^p on the unit disc, considered as a closed subspace of L^p on the circle. Then (H^p, H^q) is a $K-$subcouple of (L^p, L^q). See Pisier [15].

We shall now indicate a possible generalisation of the concept of $K-$subcouple. Let us consider estimates of the form

$$K(t, y; \vec{Y}) \leq C \sum_{k=-\infty}^{\infty} \gamma_k K(t2^k, y; \vec{A}) \tag{2}$$

where (γ_n) is a given non-negative sequaence. From this estimate we easily get

$$K(t, y; \vec{Y}) \leq C \max(1, 2^m) \sum_{k=-\infty}^{\infty} \gamma_{k+m} K(t2^k, y; \vec{A})$$

We may therefore assume that $\gamma_0 > 0$ in (2).

Definition 1.3 *Let $\gamma = (\gamma_k)$ be a non-negative sequence with $\gamma_0 > 0$ and let M be a subspace of $\Sigma(\vec{A})$. A subcouple $\vec{Y}$ of $\vec{A}$ will be called a $(\gamma, M)-$subcouple of $\vec{A}$ if the right hand side of (2) is finite and the estimate (2) holds, for all $t > 0$ and for all $y \in M$.*

Clearly $\vec{Y}$ is a $K-$subcouple if and only if it is a $(\gamma, M)-$subcouple with $\gamma = \delta_0$ (the Dirac sequence) and $M = \Sigma(\vec{Y})$.

Definition 1.4 *The multiplicative order of a parameter Φ is the function $s \mapsto \omega_\Phi(s)$ defined by*

$$\omega_\Phi(s) = \sup\{\|f(\cdot s)\|_\Phi : \|f\|_\Phi \leq 1 \;\; f \in \Phi\}$$

Note that the multiplicative order of $\Phi_{\theta,\rho}$ is $\omega_{\theta,\rho}(s) = s^\theta$. Note also that if f is non-negative, increasing and concave then

$$f(s) \leq C\omega_\Phi(s)\|f\|_\Phi$$

The following theorem is an immediate consequence of these definitions. The simple idea behind this result will however be used on many occassions in the sequel.

Theorem 1.3 *Let $\vec{Y}$ be a $(\gamma, M)-$subcouple of $\vec{A}$. Then*

$$K_\Phi(\vec{Y}) \cong K_\Phi(\vec{A}) \cap M$$

for all parameters Φ such that

$$K_\Phi(\vec{Y}) \subseteq M$$

and

$$\sum_{k=-\infty}^{\infty} \gamma_k \, \omega_\Phi(2^k) < \infty$$

Here ω_Φ is the multiplicative order of Φ.

1.3 Reiteration, density and duality

Let Φ_0 and Φ_1 be two parameters for the $K-$method. Then we can construct a new Banach couple from the couple $\vec{A}$, namely

$$(K_{\Phi_0}(\vec{A}), K_{\Phi_1}(\vec{A}))$$

What will happen if we apply the $K_\Phi-$ method on this couple. The answer is given in the following theorem, often called the *reiteration theorem.* For a proof of this result see [3] , theorem 3.3.11.

Theorem 1.4 *Put* $\hat{\Phi}_j = K_{\Phi_j}(\vec{L}_\infty)$ *and* $\Psi = K_\Phi(\hat{\Phi}_0, \hat{\Phi}_1)$. *Then*

$$K_\Phi(K_{\Phi_0}(\vec{A}), K_{\Phi_1}(\vec{A})) \cong K_\Psi(\vec{A})$$

In particular

$$K_{\theta,\rho}(K_{\theta_0,\rho_0}(\vec{A}), K_{\theta_1,\rho_1}(\vec{A})) \cong K_{\eta,\rho}(\vec{A})$$

where $0 < \theta_j < 1$, $\theta_0 \neq \theta_1$ *and* $\theta = (1-\eta)\theta_0 + \eta\theta_1$ *where* $0 < \eta < 1$, *(*ρ_0, ρ_1 *and* ρ *arbitrary).*

We next consider a result on density. An intermediate space A for the couple $\vec{A}$ is said to be *regular* if $\Delta(\vec{A})$ is dense in A. The couple $\vec{A}$ is regular if $\Delta(\vec{A})$ is dense in A_0 and in A_1. The following result gives a characterization of which parameters we get a regular interpolation space.

Theorem 1.5 *Assume that* $\hat{\Phi} = K_\Phi(\vec{L}_\infty)$. *Then* $K_\Phi(\vec{A})$ *is regular for all couples* $\vec{A}$ *if and only if* $\Delta(\vec{L}_\infty)$ *is dense in* $\hat{\Phi}$ *(i.e.* $\hat{\Phi}$ *is regular) and* $\hat{\Phi}$ *is non-degenerate i.e.* $\hat{\Phi} \setminus (L_\infty \cup L^1_\infty) \neq \emptyset$. *(For the* $K_{\theta,\rho}-$*method this means that* $0 < \theta < 1$ *and* $1 \leq \rho < \infty$.*)*

Finally we turn to questions on duality, which have to be treated with some care since we are dealing with couples of spaces, not just one space. Given a single Banach space A we let A^* denote its dual. However, if A is an intermediate space for the couple $\vec{A}$, we can also consider the space A' of all bounded linear functionals γ on $\Delta(\vec{A})$ which are bounded in the norm on A. Thus A' consists of all $\gamma \in \Delta(\vec{A})^*$ for which the norm

$$\|\gamma\|_{A'} = \sup\{|\gamma(a)| : \|a\|_A \leq 1\}$$

is finite. Note that the space A' depends not only on A, but also on the couple $\vec{A}$. If A is regular each $\gamma \in A'$ can be extended by continuity to a functional $\bar{\gamma} \in A^*$. If A is not regular we can still extend γ by the Hahn-Banach theorem, but not in a unique way. We let $\vec{A}'$ denote the couple (A'_0, A'_1).

Theorem 1.6 *We have the following relations (with equal norms)*

$$\Sigma(\vec{A}') = \Delta(\vec{A})' = \Delta(\vec{A})^*$$

$$\Delta(\vec{A}') = \Sigma(\vec{A})'$$

More precisely we have

$$K(1/t, \gamma; \vec{A}') = \sup_{a \in \Delta(\vec{A})} \frac{|\gamma(a)|}{J(t, a; \vec{A})} \tag{1}$$

$$J(1/t, \gamma; \vec{A}') = \sup_{a \in \Delta(\vec{A})} \frac{|\gamma(a)|}{K(t, a; \vec{A})} \tag{2}$$

Here

$$J(s, a; \vec{A}) = \max\left(\|a\|_{A_0}, s\|a\|_{A_1}\right)$$
$$J(s, \gamma; \vec{A}') = \max\left(\|\gamma\|_{A_0'}, s\|\gamma\|_{A_1'}\right)$$

This result is proved in [3], proposition 2.4.6. Several important duality results can be derived. We mention only the following special result, (see [3], corollary 3.7.5).

Theorem 1.7 *Put* $\frac{1}{\rho'} = 1 - \frac{1}{\rho}$ *and assume that* $0 < \theta < 1$. *Then*

$$K_{\theta,\rho}(\vec{A})' \cong K_{\theta,\rho'}(\vec{A}')$$

1.4 Good approximation and quasi-linearization

In order to describe the space $K_\Phi(\vec{A})$ in a concrete situation, one has to know the $K-$functional. However it is not necessary to compute $K(t, a)$ exactely. To begin with we can replace $K(t, a)$ by the equivalent functional

$$K_p(t, a; \vec{A}) = \inf\left\{(\|a_0\|_{A_0}^p + t^p\|a_1\|_{A_1}^p)^{1/p} : a = a_0 + a_1\right\}$$

Similarly we replace $J(t, a)$ by

$$J_q(t, a; \vec{A}) = (\|a\|_{A_0}^q + t^q\|a\|_{A_1}^q)^{1/q}$$

(with $1/q = 1 - 1/p$). We also introduce the following defintion.

Definition 1.5 *We call* $a_1(t)$ *a good* $K(\vec{A})-$*approximation of* $a \in \Sigma(\vec{A})$ *if*

$$K(t, a; \vec{A}) \cong \|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1}$$

where $a_0(t) = a - a_1(t)$ *and* $0 < t < \infty$ *and if* $a_0(t)$ *and* $a_1(t)$ *depend continuously of* t *in* A_0 *and* A_1, *respectively.*

The definition of good $K(\vec{A})-$approximation is unsymmetric. It puts the focus on the space A_1, whoose elements are viewed as approximations of the elements in $\Sigma(\vec{A})$. Sometimes we shall need to switch the orders between A_0 and A_1 and then the following observation is useful.

Lemma 1.2 *Let $a_1(t)$ be a good $K(A_0, A_1)-$approximation of a. Then $a_0(1/t)$ is a good $K(A_1, A_0)-$approximation of a.*

Proof The result follows at once from the formula

$$K(t, a; A_0, A_1) = tK(1/t, a; A_1, A_0)$$

□

We finally turn to the concept of quasi-linearization. A strongly continuous family $\Lambda(t)$, $(0 < t < \infty)$ of bounded linear operators from $\Sigma(\vec{A})$ into $\Delta(\vec{A})$, is called a *quasi-linearization* of the couple $\vec{A}$, if $\Lambda(t)a$ is a good $K(\vec{A})-$approximation for every $a \in \Sigma(\vec{A})$.

Lemma 1.3 *The family $\Lambda(t)$ is a quasi-linearization of $\vec{A}$ if and only if*

$$\max\left(\|a - \Lambda(t)a\|_{A_0}, t\|\Lambda(t)a\|_{A_1}\right) \le C \begin{cases} \|a\|_{A_0} & \text{if } a \in A_0 \\ t\|a\|_{A_1} & \text{if } a \in A_1 \end{cases} \qquad (1)$$

Proof Clearly $K(t, a) \le \|a\|_{A_0}$ if $a \in A_0$ and $K(t, a) \le t\|a\|_{A_1}$ if $a \in A_1$. This implies that (1) holds. Conversely, assume (1) and let $a = a_0 + a_1$ be an arbitrary decomposition of a. Then

$$K(t, a) \le \|a - \Lambda(t)a\|_{A_0} + t\|\Lambda(t)a\|_{A_1} \le$$

$$\le \|a_0 - \Lambda(t)a_0\|_{A_0} + \|a_1 - \Lambda(t)a_1\|_{A_0} + t\|\Lambda(t)a_0\|_{A_1} + t\|\Lambda(t)a_1\|_{A_1} \le$$
$$\le C(\|a_0\|_{A_0} + t\|a_1\|_{A_1})$$

This implies that $\Lambda(t)$ is a quasi-linearization.

1.5 An extension theorem

We shall here consider the problem of extending a given operator. We start with the following general definition.

Definition 1.6 *Let T be a bounded linear operator from $\Delta(\vec{A})$ into a Banach space B. We then put*

$$q(\tau,T) = q(\tau,T;\vec{A}) = \sup\{\|Tu\|_B : J(1/\tau,u;\vec{A}) \leq 1\ ,\ u \in \Delta(\vec{A})\}$$

Note that if $T = \Gamma$, a bounded linear functional on A_1, the theorem 1.6 implies

$$q(\tau,\Gamma;\vec{A}) = K(\tau,\Gamma;\vec{A}')$$

In general we have the following result.

Lemma 1.4 *The function $\tau \mapsto q(\tau,T)$ is positive, increasing and concave, i.e. $q(\sigma,T) \leq \max(1,\sigma/\tau)q(\tau,T)$.*

Proof This follows at once from the inequality

$$J(1/\sigma,u) \leq \max(1,\tau/\sigma)J(1/\tau,u)$$

□

The following extension theorem will be used throughout the paper.

Theorem 1.8 *Assume that T_1 is a bounded linear operator from A_1 into a Banach space B and put $T = T_1|\Delta(\vec{A})$. Assume that*

$$\sum_{k=0}^{\infty} q(2^k,T;\vec{A})\,K(2^{-k},a;\vec{A}) \leq C\|a\|_{K_\Phi(\vec{A})} \tag{1}$$

for every $a \in K_\Phi(\vec{A})$. Then there exists a bounded extension $\bar{T}$ of T to the space $K_\Phi(\vec{A}) \cup A_1$.

Proof Consider the auxiliary Banach space X, defined by the norm

$$\|a\|_X = \inf\{\|a_1\|_{A_1} + \sum_{k=1}^{\infty} q(2^k,T)J(2^{-k},v_k) : a = a_1 + \sum_{k=1}^{\infty} v_k \text{ in } \Sigma(\vec{A})\}$$

Then (1) implies that $K_\Phi(\vec{A})$ is continuously embedded in X. In fact, let $a_1(t)$ be a good $K(\vec{A})-$approximation of $a \in K_\Phi(\vec{A})$. Let us write

$$u_k = a_0(2^{-k+1}) - a_0(2^{-k}) = a_1(2^{-k}) - a_1(2^{-k+1})$$

Then we have $a = a_1 + \sum_{k\geq 1} u_k$, where $a_1 = a_1(1)$. Since $\|a_1\|_{A_1} \leq K(1,a)$ and $J(2^{-k}, u_k) \leq CK(2^{-k}, a)$ we conclude that the $X-$norm of a is bounded by a constant times the left hand side of (1). Hence $K_\Phi(\vec{A}) \hookrightarrow X$. Note also that the series $\sum v_k$ in the definition of the space X, converges in A_0, since

$$\|v_k\|_{A_0} \leq J(2^{-k}, v_k) \leq Cq(2^k, T)J(2^{-k}, v_k) \text{ if } k \geq 1.$$

Thus $X \hookrightarrow \Sigma(\vec{A})$.

Next we observe that $\Delta(\vec{A})$ is dense in $X \cap A_0$. In fact if $a \in A_0$ we have $a_1 \in \Delta(\vec{A})$ and

$$\|a - a_1 - \sum_{k=1}^{K} u_k\|_X \leq \sum_{k>K} q(2^k, T)J(2^{-k}, u_k) \leq C \sum_{k>K} q(2^k, T)K(2^{-k}, a) \to 0$$

as $K \to \infty$. Moreover T is bounded on $\Delta(\vec{A})$ in the $X-$norm, because if $a = a_1 + \sum_{k\geq 1} v_k$ we have

$$\|Ta\|_B \leq \|Ta_1\|_B + \sum_{k\geq 1} \|Tv_k\|_B \leq C\Big(\|a_1\|_{A_1} + \sum_{k\geq 1} q(2^k, T)J(2^{-k}, v_k)\Big).$$

Hence T can be extended by continuity to $X \cap A_0$. If this extension is denoted by S we define $\bar{T}$ on $K_\Phi(\vec{A}) \cup A_1$ by writing $\bar{T}a = Sa_0 + T_1 a_1$ where $a \in K_\Phi(\vec{A})$ and $a = a_0 + a_1$ as usual. (Note that a_0 belongs to $K_\Phi(\vec{A})$.) This definition is unambigous and clearly defines a bounded operator on $K_\Phi(\vec{A})$ since

$$\|T_1 a_1\|_B \leq C\|a_1\|_{A_1} \leq CK(1,a) \leq C\|a\|_X.$$

This completes the proof.

Corollary 1.1 *With the assumptions and notations of theorem 1.8, let $a_1(t)$ be a good $K(\vec{A})-$approximation of $a \in K_\Phi(\vec{A})$. Then (with convergence in B)*

$$\bar{T}(a - a_1(t)) = \sum_{k=1}^{\infty} T(u_k(t))$$

where

$$u_k(t) = a_0(t2^{-k+1}) - a_0(t2^{-k}) = a_1(t2^{-k}) - a_1(t2^{-k+1})$$

1.6 The order of an operator

Definition 1.7 *Let q be a positive, increasing and concave function on the positive real line. Then the (multiplicative) order of q is the function*

$$\hat{q}(s) = \sup_{\tau \geq 1} \frac{q(\tau s)}{q(\tau)}$$

We also put

$$\theta_+(q) = \limsup_{s \to \infty} \frac{\log \hat{q}(s)}{\log s} \quad , \quad \theta_-(q) = \liminf_{s \to 0} \frac{\log \hat{q}(s)}{\log s}$$

The numbers $\theta_+(q)$ and $\theta_-(q)$ will be called the upper and lower break-points for the function q.

If T is a given bounded linear operator on $\Delta(\vec{A})$ and $q(t) = q(t, T; \vec{A})$ we call $\hat{q}$ the (multiplicative) order of T and the numbers $\theta_+(q)$ and $\theta_-(q)$ are called the upper and lower break-points of T.

Lemma 1.5 *Assume that q is positive, increasing and concave. Then function $\hat{q}$ is positive, increasing and concave. Moreover*

$$\min(1, s) \leq \hat{q}(s) \leq \max(1, s)$$

$$0 \leq \theta_-(q) \leq \theta_+(q) \leq 1$$

We also have the following alternative characterisation of the lower and upper break-points

$$\theta_-(q) = \sup\{\theta \in [0,1] : \sup_{\tau \geq 1} \int_0^1 s^{-\theta} \frac{q(\tau s)}{q(\tau)} \frac{ds}{s} < \infty\}$$

$$\theta_+(q) = \inf\{\theta \in [0,1] : \sup_{\tau \geq 1} \int_1^\infty s^{-\theta} \frac{q(\tau s)}{q(\tau)} \frac{ds}{s} < \infty\}$$

Proof A function q is positive, increasing and concave if and only if

$$q(s) \leq \max(1, s/\sigma) q(\sigma) \tag{1}$$

Therefore we have the estimate

$$\frac{q(\tau s)}{q(\tau)} \leq \max(1, \tau s/\tau\sigma) \frac{q(\tau\sigma)}{q(\tau)}$$

which implies that (1) holds for $\hat{q}$. We also get the first inequality of the lemma since (1) implies

$$\min(1, s) q(\tau) \leq q(\tau s) \leq \max(1, s) q(\tau)$$

Next we prove $\theta_-(q) \leq \theta_+(q)$. Note that for $s \geq 1$

$$\frac{1}{\hat{q}(1/s)} = \inf_{\tau \geq 1} \frac{q(\tau)}{q(\tau/s)} \leq \inf_{\tau \geq s} \frac{q(\tau)}{q(\tau/s)} \leq \hat{q}(s)$$

Therefore

$$\theta_-(q) = \liminf_{s \to \infty} \frac{\log 1/\hat{q}(1/s)}{\log (s)} \leq \liminf_{s \to \infty} \frac{\log \hat{q}(s)}{\log s} \leq \theta_+(q)$$

To prove the characterisation of θ_-, assume first that

$$A = \sup_{\tau \geq 1} \int_0^1 s^{-\theta} \frac{q(\tau s)}{q(\tau)} \frac{ds}{s} < \infty$$

Choose $\tau \geq 1$ so that $2q(\tau 2^{-n}) \geq \hat{q}(2^{-n}) q(\tau)$. Then it is easy to verify that $4q(\tau 2^{-n}) \geq \hat{q}(2^{-n}) q(s)$ if $2^{-n-1} \leq s \leq 2^{-n}$. From this estimate we get

$$4A \geq \tau^{-\theta} \int_{\tau 2^{-n-1}}^{\tau 2^{-n}} s^{-\theta} \frac{ds}{s} \hat{q}(2^{-n}) = C_\theta 2^{n\theta} \hat{q}(2^{-n}) \ , \ n \geq 0$$

It follows that $\hat{q}(2^{-n}) \leq C 2^{-n\theta}$ for $n \geq 0$. As a consequence we have $\hat{q}(s) \leq C s^\theta$ if $0 < s < 1$ which implies $\theta_- \geq \theta$.

Conversely, if $\theta < \theta_-$ we have $\hat{q}(s) \leq C s^{\theta'}$ where $s \leq 1$ and $\theta < \theta' < \theta_-$. Then

$$A \leq C \int_0^1 s^{\theta - \theta'} \frac{ds}{s} < \infty$$

The remaining part of the lemma is proved in a similar way.

2 Interpolation of subspaces of finite codimension

In this chapter we shall study interpolation of subspaces defined by finitely many constraints of the form $\Gamma_j(a) = 0$, where Γ_j is a bounded linear functional on the space A_1. The first section, however, is dealing with the general situation where the subspace is defined as kernel of a general operator.

2.1 General discussion

Let $\vec{A} = (A_0, A_1)$ be a Banach couple. Throughout this section we shall assume that T_1 a bounded linear operator from A_1 onto a Banach space B. We shall also assume that

$$T = T_1 | \Delta(\vec{A})$$

maps $\Delta(\vec{A})$ onto B. As usual Φ is a parameter for the $K-$method.

Consider the hypothetical formula

$$K_\Phi(A_0, A_1 \cap \ker(T_1)) \cong K_\Phi(\vec{A}) \cap \ker(T_\Phi). \tag{1}$$

Here T_Φ would be a bounded linear operator from $K_\Phi(\vec{A})$ into B. Assuming that (1) holds, what can be said about the operator T_Φ ? To answer this question we note that every $a \in \ker(T)$ belongs to the space on the left hand side of (1). Therefore (1) implies that every such a belongs to $\ker(T_\Phi)$, i.e. $\ker(T) \subseteq \ker(T_\Phi | \Delta(\vec{A}))$. It follows that

$$T_\Phi | \Delta(\vec{A}) = Q_\Phi T \tag{2}$$

for some linear operator Q_Φ on B. (We simply define Q_Φ by setting $Q_\Phi b = T_\Phi a$ if $b = Ta$.) Clearly $Q_\Phi T$ must be bounded in $K_\Phi(\vec{A})-$norm i.e.

$$\|Q_\Phi Ta\|_B \leq C\|a\|_{K_\Phi(\vec{A})} \quad , \; a \in \Delta(\vec{A}). \tag{3}$$

In order to see what is needed to guarantee (3), assume that Q_Φ is bounded and take $a \in \Delta(\vec{A})$. Then

$$\|Q_\Phi Ta\|_B = \sup_{\|\beta\|_{B^*}=1} |\beta(Q_\Phi Ta)| = \sup_{\|\beta\|_{B^*}=1} |Q_\Phi^* \beta(Ta)| \tag{4}$$

Therefore we introduce the following definition.

Definition 2.1 *Let T be a bounded linear operator of $\Delta(\vec{A})$ onto B and let Φ be a given parameter. Then $\mathcal{B}(\Phi) = \mathcal{B}_T(\Phi)$ denotes the subspace of all $\beta \in B^*$, such that*

$$\|\beta\|_{\mathcal{B}_T(\Phi)} = \sup\{|\beta(Ta)| : a \in \Delta(\vec{A}) \;\; , \; \|a\|_{K_\Phi(\vec{A})} = 1\} \tag{5}$$

is finite.

By (4) we have

$$\sup\{\|Q_\Phi Ta\|_B : a \in \Delta(\vec{A})\ ,\ \|a\|_{K_\Phi(\vec{A})} = 1\} = \sup_{\|\beta\|_{B^*}=1} \|Q_\Phi^* \beta\|_{\mathcal{B}_T(\Phi)}$$

Therefore we get the following result.

Proposition 2.1 *Assume that there is a bounded linear operator T_Φ on $K_\Phi(\vec{A})$, such that (1) holds and that $T_\Phi a = Q_\Phi Ta$ for all $a \in \Delta(\vec{A})$ where Q_Φ is a bounded linear operator on B. Then Q_Φ^* maps B^* into $\mathcal{B}_T(\Phi)$ and Q_Φ^* is bounded in the $\mathcal{B}_T(\Phi)-$norm, i.e.*

$$\sup_{\|\beta\|_{B^*}=1} \|Q_\Phi^* \beta\|_{\mathcal{B}_T(\Phi)} < \infty \tag{6}$$

In particular, if $T_\Phi = 0$ i.e.

$$K_\Phi(A_0, A_1 \cap \ker(T_1)) \cong K_\Phi(\vec{A}) \tag{7}$$

then T is unbounded on $\Delta(\vec{A})$ in the $K_\Phi(\vec{A})-$norm.

Proof The first part follows at once. To prove the second part we note that (7) implies that $\mathcal{B}(\Phi) = \{0\}$. For every non-zero $\beta \in B^*$ there must therefore exist a sequence $a_n \in \Delta(\vec{A})$ such that $\|a_n\|_{K_\Phi} = 1$ but $|\beta(Ta_n)| \to \infty$. It follows that $\|Ta_n\|_B$ is unbounded. Thus T is unbounded. □

Here is a partial converse.

Proposition 2.2 *Assume that Q_Φ is a bounded linear operator on B, such that Q_Φ^* maps B^* into $\mathcal{B}_T(\Phi)$ and Q_Φ^* is bounded in the $\mathcal{B}_T(\Phi)-$norm. Then $Q_\Phi T$ is bounded on $\Delta(\vec{A})$ in the $K_\Phi(\vec{A})-$norm. i.e. (3) holds. Moreover if Φ is a regular parameter, then*

$$K_\Phi(A_0, A_1 \cap \ker(T_1)) \subseteq K_\Phi(\vec{A}) \cap \ker(T_\Phi)$$

where T_Φ is the extension by continuity of $Q_\Phi T$ to $K_\Phi(\vec{A})$.

Proof We need only to prove the last statement. Take $a \in K_\Phi(A_0, A_1 \cap \ker(T_1))$. If Φ is regular, then there exist a sequence a_n such that $a_n \in \Delta(\vec{A}) \cap \ker(T_1)$ and $a_n \to a$ in $K_\Phi(\vec{A})$. Then $T_\Phi a_n = 0$ for all n, implying that $a \in \ker(T_\Phi)$.

Proposition 2.3 *Assume that B is finite dimensional and that*

$$K_{\Phi_2}(\vec{A}) \hookrightarrow K_{\Phi_1}(\vec{A})$$

then

$$\mathcal{B}_T(\Phi_2) \supseteq \mathcal{B}_T(\Phi_1) \ \ \textit{and} \ \ \ker(Q_{\Phi_2}) \subseteq \ker(Q_{\Phi_1})$$

Proof Assume that

$$\|a\|_{K_{\Phi_1}(\vec{A})} \leq C\|a\|_{K_{\Phi_2}(\vec{A})}$$

Then

$$\|\beta\|_{\mathcal{B}(\Phi_2)} = \sup_{a\in\Delta(\vec{A})} \frac{|\beta(Ta)|}{\|a\|_{K_{\Phi_2}(\vec{A})}} \leq C \sup_{a\in\Delta(\vec{A})} \frac{|\beta(Ta)|}{\|a\|_{K_{\Phi_1}(\vec{A})}} = C\|\beta\|_{\mathcal{B}(\Phi_1)}$$

which implies that $\mathcal{B}(\Phi_1) \subseteq \mathcal{B}(\Phi_2)$. The second inclusion follows from the formula

$$\ker(Q_\Phi) = {}^{\perp}\mathrm{im}(Q_\Phi^*) = {}^{\perp}\mathcal{B}(\Phi)$$

2.2 Subspaces of codimension one

In this section we shall consider the case $T_1 a = \Gamma(a)$ where Γ is a bounded linear functional on A_1 which does not vanish identically on $\Delta(\vec{A})$. Then the space B is the space of all complex numbers. Therefore there are only two possibilities for the space $\mathcal{B}(\Phi)$. Either $\mathcal{B}(\Phi) = \mathbb{C}$ or $\mathcal{B}(\Phi) = \{0\}$. The mapping Q_Φ is either the identity or the zero-mapping.

Proposition 2.4 *Suppose that $K_\Phi(A_0, A_1 \cap \ker(T_1))$ is a closed subspace of $K_\Phi(\vec{A})$ and that Φ is a regular parameter. Then there exists a bounded linear functional Γ_Φ on $K_\Phi(\vec{A})$, such that*

$$K_\Phi(A_0, A_1 \cap \ker(\Gamma)) \cong K_\Phi(\vec{A}) \cap \ker(\Gamma_\Phi) \tag{1}$$

More precisely, if Γ is bounded on $\Delta(\vec{A})$ in the $K_\Phi(\vec{A})-$norm then

$$K_\Phi(A_0, A_1 \cap \ker(\Gamma)) \cong K_\Phi(\vec{A}) \cap \ker(\bar{\Gamma})$$

where $\bar{\Gamma}$ is the extension of Γ to $K_\Phi(\vec{A})$ by continuity. If Γ is not bounded on $\Delta(\vec{A})$ in the $K_\Phi(\vec{A})-$norm then

$$K_\Phi(A_0, A_1 \cap \ker(\Gamma)) \cong K_\Phi(\vec{A})$$

Proof If Γ is bounded on $\Delta(\vec{A})$ in the $K_\Phi(\vec{A})-$norm we let $T_\Phi = \Gamma_\Phi$ be the extension of Γ by continuity. Otherwise we put $T_\Phi = \Gamma_\Phi = 0$. Assume that

$$a \in K_\Phi(\vec{A}) \cap \ker(T_\Phi)$$

We shall prove that $a \in K_\Phi(A_0, A_1 \cap \ker(T_1))$. In view of proposition 2.2 this will prove the result.

Since Φ is regular we can choose a sequence $a_n \in \Delta(\vec{A})$ such that $a_n \to a$ in $K_\Phi(\vec{A})$. If Γ is bounded in $K_\Phi-$norm then $\Gamma(a_n) \to 0$. Choose $w \in \Delta(\vec{A})$ so that

$\Gamma(w) = 1$ and put $\tilde{a}_n = a_n - \Gamma(a_n)w$. Then $\tilde{a}_n \in A_0 \cap \ker(T_1)$ and $\tilde{a}_n \to a$ in $K_\Phi(\vec{A})$. By assumption we then conclude that $a \in K_\Phi(A_0, A_1 \cap \ker(T_1))$.

If Γ is not bounded in $K_\Phi-$norm, we can find $w_m \in \Delta(\vec{A})$ such that $\Gamma(w_m) = 1$ but $\|w_m\|_{K_\Phi} \to 0$. With $\tilde{a}_{n,m} = a_n - \Gamma(a_n)w_m$ we then have $\tilde{a}_{n,m} \in A_0 \cap \ker(T_1)$. Since $\tilde{a}_{n,m} \to a$ (in $K_\Phi(\vec{A})$) as $m \to \infty$ we conclude that a_n and hence a is in $K_\Phi(A_0, A_1 \cap \ker(T_1))$.

Theorem 2.1 *Let $a_1(t)$ be a good $K(\vec{A})-$approximation of a. Then*

$$a \in K_\Phi(A_0, A_1 \cap \ker(\Gamma))$$

if and only if $a \in K_\Phi(\vec{A})$ and

$$\left\| \frac{\Gamma(a_1(t))}{K(1/t, \Gamma; \vec{A}')} \right\|_\Phi < \infty \tag{2}$$

Proof Choose $w(t) \in \Delta(\vec{A})$ so that $\Gamma(w(t)) = 1$ and

$$J(t, w(t)) \leq 2 \inf\{J(t, w; \vec{A}) : \Gamma(w) = 1\}$$

Using a regularisation, we can assume that $w(t)$ depends continuously on t in $\Delta(\vec{A})$. Note that the infimum on the right hand side is equal to the inverted value of

$$K(1/t, \Gamma; \vec{A}') = \sup\{|\Gamma(u)| : J(t, u; \vec{A}) \leq 1\}$$

Let $a \in K_\Phi(\vec{A})$ be given. Put $\tilde{a}_0 = a_0 + \Gamma(a_1)w$, $\tilde{a}_1 = a_1 - \Gamma(a_1)w$. Then $\tilde{a}_1 \in \ker(\Gamma)$ and an immediate computation shows that

$$K(t, a; A_0, A_1 \cap \ker(\Gamma)) \leq C\Big(K(t, a; \vec{A}) + \frac{|\Gamma(a_1(t)|}{K(1/t, \Gamma)}\Big)$$

Thus (2) implies that $a \in K_\Phi(A_0, A_1 \cap \ker(\Gamma))$.

To prove the converse implication, we shall show that

$$\frac{|\Gamma(a_1(t))|}{K(1/t, \Gamma)} \leq CK(t, a; A_0, A_1 \cap \ker(\Gamma))$$

Assume that $a = \tilde{a}_0(t) + \tilde{a}_1(t)$ where $\Gamma(\tilde{a}_1(t)) = 0$ and $\tilde{a}_1$ is a good approximation of a relative the couple $(A_0, A_1 \cap \ker(\Gamma))$. Then we put $u(t) = \tilde{a}_0(t) - a_0(t) = a_1(t) - \tilde{a}_1(t)$. Then $\Gamma(u(t)) = \Gamma(a_1(t))$ and

$$\frac{|\Gamma(a_1(t))|}{K(1/t, \Gamma)} \leq J(t, u(t)) \leq C\Big(\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} + K(t, a; A_0, A_1 \cap \ker(\Gamma))\Big)$$

This implies the result since $K(t, a; \vec{A}) \leq K(t, a; A_0, A_1 \cap \ker(\Gamma))$.

Corollary 2.1 *Assume that* $\Lambda(t)$ *is a quasi-linearization of* $\vec{A}$. *Let* $w(t)$ *be a strongly continuous family of elements in* $\Delta(\vec{A})$ *such that* $\Gamma(w(t)) = 1$ *and*

$$J(t, w(t); \vec{A})) \leq C/K(1/t, \Gamma; \vec{A}')$$

Then

$$\Lambda_\Gamma(t) = \Lambda(t)a - w(t) \cdot \Gamma(\Lambda(t)a)$$

defines a quasi-linearization of $(A_0, A_1 \cap \ker(\Gamma))$. *As a consequence* $K_\Phi(A_0, A_1 \cap \ker(\Gamma))$ *consists of all* $a \in K_\Phi(\vec{A})$ *such that*

$$\left\| \frac{\Gamma(\Lambda(t)a)}{K(1/t, \Gamma; \vec{A}')} \right\|_\Phi < \infty$$

Proof Note that if $a \in A_0$ then

$$|\Gamma(\Lambda(t)a)| \leq CK(1/t, \Gamma) J(t, \Lambda(t)a) \leq CK(1/t, \Gamma) \|a\|_{A_0}$$

Morover if $a \in A_1$ and $\Gamma(a) = 0$ then

$$|\Gamma(\Lambda(t)a)| = |\Gamma(a - \Lambda(t)a)| \leq CK(1/t, \Gamma)\, t \|a\|_{A_1}$$

In view of theorem 2.1 this gives the result.

Theorem 2.2 *Assume that* $T_1 a = \Gamma(a)$, *where* Γ *is a bounded linear functional on* A_1 *which does not vanish identically on* $\Delta(\vec{A})$. *Let* $\hat{q}$ *be the multiplicative order of* Γ, *i.e.*

$$\hat{q}(s, \Gamma) = \hat{q}(s, \Gamma; \vec{A}) = \sup_{\tau \geq 1} \frac{K(\tau s, \Gamma; \vec{A}')}{K(\tau, \Gamma; \vec{A}')}$$

Then the following conclusions hold:
If

$$\sum_{k=0}^{\infty} \hat{q}(2^k, \Gamma)\, \omega_\Phi(2^{-k}) < \infty \tag{3}$$

then there is a bounded extension $\bar{\Gamma}$ *of* Γ *to the space* $K_\Phi(\vec{A})$ *such that*

$$K_\Phi(A_0, A_1 \cap \ker(\Gamma)) \cong K_\Phi(\vec{A}) \cap \ker(\bar{\Gamma})$$

If on the other hand

$$\sum_{k=0}^{\infty} \hat{q}(2^{-k}, \Gamma)\, \omega_\Phi(2^k) < \infty \tag{4}$$

then Γ *is not bounded in the* $K_\Phi(\vec{A})-$*norm and*

$$K_\Phi(A_0, A_1 \cap \ker(\Gamma)) \cong K_\Phi(\vec{A})$$

Proof First note that $q(s,\Gamma) = K(s,\Gamma;\vec{A}')$ and that $q(s,\Gamma) \le C\hat{q}(s,\Gamma)$. Moreover $K(2^{-k},a) \le \omega_\Phi(2^{-k})\|a\|_{K_\Phi(\vec{A})}$. The existence of an extension will therefore be delivered by theorem 1.8 in section 1.5.

We start by considering the second part of the theorem. Assuming (4) we shall prove that $K_\Phi(A_0, A_1 \cap \ker(\Gamma)) \supseteq K_\Phi(\vec{A})$. Thus take $a \in K_\Phi(\vec{A})$ and a good $K(\vec{A})-$approximation $a_1(t)$ of a. Since $2^{-k} \le K(2^{-k},\Gamma)$ for $k \ge 1$, we have

$$t\|a_1(t2^k)\|_{A_1} \le 2^{-k}K(t2^k,a;\vec{A}) \le q(2^{-k},\Gamma)K(t2^k,a;\vec{A})$$

From (4) we see that the right hand side tends to zero in $\Phi-$norm as $k \to \infty$. It follows that $a_1(t2^k) \to 0$ as $k \to \infty$ in A_1. Consequently we have

$$a_1(t) = -\sum_{k=0}^{\infty} v_k(t) \ \text{ with } \ v_k(t) = a_1(t2^{k+1}) - a_1(t2^k)$$

(convergence in A_1). This gives the estimate

$$|\Gamma(a_1(t))| \le \sum_{k=0}^{\infty} |\Gamma(v_k(t))| \le \sum_{k=0}^{\infty} K(1/(t2^k),\Gamma)J(t2^k,v_k(t))$$

Since $J(t2^k,v_k(t)) \le CK(t2^k,a)$ we conclude that if $t \le 1$ then

$$\frac{|\Gamma(a_1(t))|}{K(1/t,\Gamma)} \le C\sum_{k=0}^{\infty} \frac{K(1/(t2^k),\Gamma)}{K(1/t,\Gamma)}K(t2^k,a) \le C\sum_{k=0}^{\infty} \hat{q}(2^{-k},\Gamma)K(t2^k,a) \qquad (5)$$

For $t \ge 1$ it is enough to make the following estimate

$$\frac{|\Gamma(a_1(t))|}{K(1/t,\Gamma)} \le C\frac{\|a_1(t)\|_{A_1}}{K(1/t,\Gamma)} \le CK(t,a) \qquad (6)$$

This follows from $K(1,\Gamma) \le tK(1/t,\Gamma)$. Using theorem 2.1 and the assumption (4), we get the result in this case.

Now consider the first case. Take an arbitrary $a \in K_\Phi(\vec{A}) \cap \ker(\bar{\Gamma})$. Using corollary 1.1 we see that

$$\bar{\Gamma}(a_0(t)) = \sum_{k=1}^{\infty} \Gamma(u_k(t)) \ \text{ where } \ u_k(t) = a_1(t2^{-k}) - a_1(t2^{-k+1}),$$

implying that

$$\Gamma(a_1(t)) = -\bar{\Gamma}(a - a_1(t)) = -\bar{\Gamma}(a_0(t)) = -\sum_{k=1}^{\infty} \Gamma(u_k(t))$$

For $t \le 1$ we therefore conclude that

$$\frac{|\Gamma(a_1(t))|}{K(1/t,\Gamma)} \le C\sum_{k=1}^{\infty}\frac{K(2^k/t,\Gamma)}{K(1/t,\Gamma)}K(t2^{-k},a) \le C\sum_{k=1}^{\infty}\hat{q}(2^k,\Gamma)K(t2^{-k},a) \tag{7}$$

Now theorem 2.1, (3) and (6) imply that $a \in K_\Phi(A_0, A_1 \cap \ker(\Gamma))$.

To prove the converse inclusion, take $a \in K_\Phi(A_0, A_1 \cap \ker(\Gamma))$ and assume that $a = \tilde{a}_0 + \tilde{a}_1$ where $\Gamma(\tilde{a}_1) = 0$. Replacing a_1 by $\tilde{a}_1$ and correspondingly u_k by $\tilde{u}_k$ we get

$$\bar{\Gamma}(a) = \bar{\Gamma}(\tilde{a}_0(t)) = \sum_{k=1}^{\infty}\Gamma(\tilde{u}_k(t)) = 0$$

Thus $a \in \ker(\bar{\Gamma})$. The theorem now follows.

2.3 Applications to the $K_{\theta,\rho}$−method

We shall now apply the theory of the preceeding sections to the parameter $\Phi_{\theta,\rho}$.

Theorem 2.3 *Let Γ be a bounded linear functional on A_1 and let θ_+ and θ_- be the upper and lower break-points of Γ. Then if $\theta > \theta_+$, we can extend Γ to a bounded linear functional $\bar{\Gamma}$ on $K_{\theta,\rho}(\vec{A})$ and*

$$K_{\theta,\rho}(A_0, A_1 \cap \ker(\Gamma)) \cong K_{\theta,\rho}(\vec{A}) \cap \ker(\bar{\Gamma})$$

In case $\theta < \theta_-$ we have

$$K_{\theta,\rho}(A_0, A_1 \cap \ker(\Gamma)) \cong K_{\theta,\rho}(\vec{A})$$

If $\theta_- < \theta < \theta_+$ then $K_{\theta,\rho}(A_0, A_1 \cap \ker(\Gamma))$ is not a closed subspace of $K_{\theta,\rho}(\vec{A})$.
If $\Lambda(t)$ is a quasi-linearization of the couple $\vec{A}$ then $K_{\theta,\rho}(A_0, A_1 \cap \ker(\Gamma))$ consists of all $a \in K_{\theta,\rho}(\vec{A})$ such that

$$\Big(\int_0^\infty \Big(t^{-\theta}\frac{|\Gamma(\Lambda(t)a)|}{K(1/t,\Gamma;\vec{A}')}\Big)^\rho \frac{dt}{t}\Big)^{1/\rho} < \infty$$

Proof First recall that $\Phi = \Phi_{\theta,\rho}$ then $\omega_\Phi(s) = s^\theta$. Put $q(t) = K(t,\Gamma;\vec{A}')$. If $\theta < \theta' < \theta_- = \theta_-(q)$ we have $\hat{q}(s) < s^{\theta'}$ for small values of s. Similarly $\hat{q}(s) < s^{\theta''}$ if $\theta'' > \theta_+ = \theta_+(q)$, s large. We can now use theorem 2.2 to get the first part of the theorem. The last part follows from corollary 2.1.

Before we proceed with the remaining part of the theorem, we note that proposition 2.4 gives us two alternatives if $K_{\theta,\rho}(A_0, A_1 \cap \ker(\Gamma))$ is a closed subspace of $K_{\theta,\rho}(A_0, A_1)$ and $\rho < \infty$. Either the first space coincides with the second one or is the intersection of the second one with the kernel of $\bar{\Gamma}$. We shall prove that $\theta \leq \theta_-$ in the first case and $\theta \geq \theta_+$ in the second case.

Let us also review the last part of the proof of theorem 2.1, where we showed that

$$\frac{|\Gamma(a_1(t))|}{K(1/t,\Gamma)} \leq C\Big(\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} + K(t,a;A_0,A_1\cap\ker(\Gamma))\Big) \tag{1}$$

for any decomposition $a = a_0(t) + a_1(t)$.

Consider the case $\rho = 1$ and assume that $K_{\theta,1}(A_0, A_1 \cap \ker(\Gamma)) \cong K_{\theta,1}(\vec{A})$. Choose $v_k \in \Delta(\vec{A})$ so that $J(2^k, v_k) = 1$ and $2\Gamma(v_k) \geq K(2^{-k}, \Gamma; \vec{A}')$. Define a by means of the formula

$$a = \sum_{k=-\infty}^{\infty} \lambda_k v_k \text{ where } \sum_{k=-\infty}^{\infty} \lambda_k 2^{-k\theta} < \infty \ , \ \lambda_k \geq 0$$

If we put

$$a_1(t) = \sum_{2^k \geq t} \lambda_k v_k \ , \ a_0(t) = \sum_{2^k < t} \lambda_k v_k$$

we have that

$$\Big\| \|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} \Big\|_{\theta,1} \leq C \sum_{k=-\infty}^{\infty} \lambda_k 2^{-k\theta} \tag{2}$$

Therefore $a \in K_{\theta,1}(\vec{A})$ and thus (1) implies that

$$\Big\| \frac{\Gamma(a_1(t))}{K(1/t,\Gamma)} \Big\|_{\theta,1} \leq C \sum_{k=-\infty}^{\infty} \lambda_k 2^{-k\theta} \tag{3}$$

On the other hand,

$$\Big\| \frac{\Gamma(a_1(t))}{K(1/t,\Gamma)} \Big\|_{\theta,1} \geq C \sum_{m=-\infty}^{\infty} 2^{-m\theta} \sum_{k \geq m} \lambda_k \frac{K(2^{-k},\Gamma)}{K(2^{-m},\Gamma)} \geq$$

$$\geq C \sum_{k\geq 0} \Big(\sum_{m \leq 0} \lambda_{m+k} 2^{-(m+k)\theta} 2^{k\theta} \frac{K(2^{-m}2^{-k},\Gamma)}{K(2^{-m},\Gamma)} \Big)$$

From (3) we then get that

$$\sup_{m \leq 0} 2^{k\theta} \frac{K(2^{-m}2^{-k},\Gamma)}{K(2^{-m},\Gamma)} \leq C \quad \text{for } k \geq 0$$

which implies that $\hat{q}(2^{-k}) \leq C 2^{-k\theta}$, i.e. $\hat{q}(s) \leq C s^{\theta}$ for $s \leq 1$. Thus $\theta \leq \theta_-$.

Next assume that $K_{\theta,1}(A_0, A_1 \cap \ker(\Gamma)) \cong K_{\theta,1}(\vec{A}) \cap \ker(\bar{\Gamma})$. Then Γ is bounded on $\Delta(\vec{A})$ in the $K_{\theta,1}-$norm. Thus $\Gamma(v_k) \leq K(2^{-k}, \Gamma) \leq C2^{-k\theta}$. Define a by the formula

$$a = \sum_{k=-\infty}^{\infty} \lambda_k v_k - \gamma w \ \text{ where } \ \gamma = \sum_{k=-\infty}^{\infty} \lambda_k \Gamma(v_k)$$

Here $w \in \Delta(\vec{A})$ and $\Gamma(w) = 1$. From what we have already proved we know that $a \in K_{\theta,1}(A_0, A_1)$. It follows that $a \in K_{\theta,1}(A_0, A_1) \cap \ker(\bar{\Gamma})$ and hence $a \in K_{\theta,1}(A_0, A_1 \cap \ker(\Gamma))$. Define a_0 and a_1 by

$$a_1(t) = \sum_{2^k \geq t} \lambda_k v_k - \gamma w \chi_{(0,1)}(t) \ , \ \ a_0(t) = \sum_{2^k < t} \lambda_k v_k - \gamma w \chi_{(1,\infty)}(t)$$

Then

$$\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} \leq \sum_{k=-\infty}^{\infty} \lambda_k \min(1, t2^{-k}) + \gamma \min(1,t) J(1,w)$$

This implies (2) and hence (3) holds in this case, too. Now

$$\Gamma(a_1(t)) = -\bar{\Gamma}(a_0(t)) = -\sum_{2^k < t} \lambda_k \Gamma(v_k) \quad \text{if} \quad 0 < t < 1$$

Therefore

$$\sum_{m \leq 0} 2^{-m\theta} \sum_{n < m} \lambda_n \frac{K(2^{-k}, \Gamma)}{K(2^{-m}, \Gamma)} \leq C \Big\| \frac{\bar{\Gamma}(a_1(t))}{K(1/t, \Gamma)} \Big\|_{\theta,1} \leq C \sum_{k=-\infty}^{\infty} \lambda_k 2^{-k\theta}$$

In the same way as obove we conclude that

$$\sup_{m \leq 0} 2^{k\theta} \frac{K(2^{-m} 2^{-k})}{K(2^{-m})} \leq C \ \text{ for } k < 0$$

which implies that $\hat{q}(s) \leq Cs^{\theta}$ for $s \geq 1$. Thus $\theta \geq \theta_+$. This proves the remaining part of the theorem in the case $\rho = 1$.

Suppose now that $K_{\theta,\rho}(A_0, A_1 \cap \ker(\Gamma))$ is a closed subspace of $K_{\theta,\rho}(\vec{A})$ for some $\rho > 1$ and $\theta_- < \theta < \theta_+$. Using reiteration, we would then find another θ' in the same interval , such that $K_{\theta',1}(A_0, A_1 \cap \ker(\Gamma))$ is a closed subspace of $K_{\theta',1}(\vec{A})$. This would then be a contradiction. This argument completes the proof of the theorem.

2.4 Subspaces of finite codimension

In this section we shall generalize the results of section 2.2 to the case of several functionals $\Gamma_1, \ldots, \Gamma_N$. We shall of course assume that these functionals are linearly independent. However we shall need more. Let us write

$$K_0(\tau, \Gamma_n; \vec{A'}) = \sup\{|\Gamma_n(u)| : J(1/\tau, u; \vec{A}) \leq 1, \ \ \Gamma_m(u) = 0 \text{ for } m \neq n\}$$

Definition 2.2 *Let $\mathcal{G}$ be a given set of bounded linear functionals on A_1. Then $\mathcal{G}$ is said to be strongly independent if there exists a basis $\Gamma_1, \cdots, \Gamma_N$ for the span of $\mathcal{G}$ such that*

$$K_0(\tau, \Gamma_n; \vec{A'}) \cong K(\tau, \Gamma_n; \vec{A'})$$

for all n. The basis $\Gamma_1, \cdots, \Gamma_N$ is called a strongly independent basis for $\mathcal{G}$.

Definition 2.3 *A sequence $w_1(t), \cdots, w_N(t)$ in $\Delta(\vec{A})$ is called a supporting sequence for the set $\Gamma_1, \cdots, \Gamma_N$ if*

$$\Gamma_m(w_n(t)) = \delta_{m,n}$$

$$J(t, w_n(t); \vec{A}) \leq C/K(1/t, \Gamma_n; \vec{A'})$$

for all $t > 0$ and if $w_1(t), \cdots, w_N(t)$ depend continuously on t.

Lemma 2.1 *The functionals $\Gamma_1, \cdots, \Gamma_N$ form a strongly independent basis if and only if there exists a supporting sequence $w_1(t), \cdots, w_N(t)$.*

Proof Choose $u_n(t)$ so that

$$\Gamma_n(u_n(t)) \geq 2^{-1} K_0(1/t, \Gamma_n), \ \ J(t, u_n(t)) \leq 1 \ \text{ and } \ \Gamma_m(u_n(t)) = 0, \ \text{ if } \ m \neq n$$

and put $w_n(t) = u_n(\min(1,t))/\Gamma_n(u_n(\min(1,t)t))$. Observing that $K(1/t, \Gamma_n) \leq (1/t)\|\Gamma_n\|_{A_1'}$ if $t \geq 1$, we see that $w_1(t), \cdots, w_N(t)$ is a supporting sequence. The converse is clear since

$$1 = \Gamma_n(w_n) \leq K_0(1/t, \Gamma_n) J(t, w_n) \leq C K_0(1/t, \Gamma_n)/K(1/t, \Gamma_n)$$

□

The next result is a direct generalization of theorem 2.2 to the case of several linear functionals.

Theorem 2.4 *Assume that* $\Gamma_1, \cdots, \Gamma_N$ *are strongly independent, bounded linear functionals on* A_1. *Put* $T = (\Gamma_1, \cdots, \Gamma_N)$. *For a given parameter* Φ *let* I_Φ *be the set of all indices* $n \in \{1, \cdots, N\}$ *such that* Γ_n *is bounded in* $K_\Phi(\vec{A})-$*norm. Put* $T_\Phi = (\Gamma_{1,\Phi}, \cdots, \Gamma_{N,\Phi})$, *where* $\Gamma_{n,\Phi} = \Gamma_n$ *if* $n \in I_\Phi$ *and* $\Gamma_{n,\Phi} = 0$ *otherwise. Then* T_Φ *can be extendend to a bounded linear operator on* $K_\Phi(\vec{A})$ *so that*

$$K_\Phi(A_0, A_1 \cap \ker(T)) \cong K_\Phi(\vec{A}) \cap \ker(T_\Phi)$$

provided that

$$\sum_{k=0}^{\infty} \hat{q}(2^k, \Gamma_n)\, \omega_\Phi(2^{-k}) < \infty \quad \text{if } n \in I_\Phi \tag{1}$$

$$\sum_{k=0}^{\infty} \hat{q}(2^{-k}, \Gamma_n)\, \omega_\Phi(2^{k}) < \infty \quad \text{if } n \notin I_\Phi \tag{2}$$

Proof In this case the space B is of course the space $\mathbb{C}^N$. We start by calculating the space $\mathcal{B}(\Phi) = \mathcal{B}_T(\Phi)$. First assume that $\beta \in \mathcal{B}(\Phi) \subseteq B^* = \mathbb{C}^N$. Then the mapping $a \mapsto \beta(Ta)$ is bounded in $K_\Phi(\vec{A})-$norm i.e.

$$|\beta(Ta)| = |\sum_{n=1}^{N} \beta_n \Gamma_n(a)| \leq C\|a\|_{K_\Phi(\vec{A})}$$

If $n \notin I_\Phi$ we must have $\beta_n = 0$, since otherwise we could choose $\beta_m = \delta_{m,n}$. But then the estimate above would contradict the definition of I_Φ. Thus we have proved that if $\beta \in \mathcal{B}(\Phi)$ then $\beta_n = 0$ for all $n \notin I_\Phi$. The converse also holds. To see that, assume that $\beta_n = 0$ for all $n \notin I_\Phi$. Then

$$|\beta(Ta)| \leq \sum_{n \in I_\Phi} |\beta_n \Gamma_n(a)| \leq C \sum_{n \in I_\Phi} |\beta_n| \|a\|_{K_\Phi(\vec{X})}$$

Thus $\beta \in \mathcal{B}(\Phi)$. We have proved that

$$\mathcal{B}(\Phi) = \{\beta \in B^* : \beta_n = 0 \text{ for all } n \notin I_\Phi\} \tag{3}$$

Let Q_Φ^* be the projection of $\mathbb{C}^N$ onto $\mathcal{B}(\Phi)$, and let Q_Φ be the corresponding operator on B. Then $T_\Phi a = Q_\Phi T a$ if $a \in A_1$. Therefore condition (1) and theorem 1.8 will guarantee the existence of the extension of T_Φ.

Now let $a_1(t)$ be a good $K(\vec{A})-$ approximation of $a \in K_\Phi(\vec{A}) \cap \ker(T_\Phi)$. Choose a supporting sequence $w_1, \cdots, w_N$ for the functionals $\Gamma_1, \cdots, \Gamma_N$ and define $\tilde{a}_1(t)$ by means of the following formula

$$\tilde{a}_1(t) = a_1(t) - \sum_{n=1}^{N} w_n(t) \cdot \Gamma_n(a_1(t)) \tag{4}$$

Then $\tilde{a}_1 \in A_1 \cap \ker(T)$. If $Q^c_\Phi = I - Q_\Phi$ we can rewrite the definition of $\tilde{a}_1$ as

$$\tilde{a}_1(t) = a_1(t) - W_0(t)Q_\Phi T a_1(t) - W_1(t)Q^c_\Phi T a_1(t)$$

where (with $b = (b_1, \cdots, b_N) \in \mathbb{C}^N$)

$$W_0(t)b = \sum_{n \in I_\Phi} w_n(t) \cdot b_n \ , \quad W_1(t)b = \sum_{n \notin I_\Phi} w_n(t) \cdot b_n$$

Then , for $t \leq 1$,

$$J(t, W_0(t)Q_\Phi Tu; \vec{A}) \leq \sum_{n \in I_\Phi} J(t, w_n(t); \vec{A})|\Gamma_n u| \leq C \sum_{n \in I_\Phi} \frac{K(1/(ts), \Gamma_n; \vec{A}')}{K(1/t, \Gamma_n; \vec{A}')} J(ts, u; \vec{A})$$

In view of the definition of $\hat{q}$ this implies

$$J(t, W_0(t)Q_\Phi Tu; \vec{A}) \leq C\hat{q}_0(1/s)J(ts, u; \vec{A}) \ \text{ where } \ \hat{q}_0(1/s) = \sum_{n \in I_\Phi} \hat{q}(1/s, \Gamma_n)$$

In the same way we get

$$J(t, W_1(t)Q^c_\Phi Tu; \vec{A}) \leq C\hat{q}_1(1/s)J(ts, u; \vec{A}) \ \text{ where } \ \hat{q}_1(1/s) = \sum_{n \notin I_\Phi} \hat{q}(1/s, \Gamma_n)$$

It is also easy to see that

$$J(t, W_j(t)b; \vec{A}) \leq Ct\|b\|_{\mathbb{C}^N} \ \text{ if } \ t \geq 1$$

because $J(t, w_n(t); \vec{A}) \leq C/K(1/t, \Gamma_n) \leq Ct$.

To complete the proof we note that

$$K(t, a; A_0, A_1 \cap \ker(T)) \leq$$
$$\leq C\Big(K(t, a; \vec{A}) + J(t, W_0(t)Q_\Phi T a_1(t); \vec{A}) + J(t, W_1(t)Q^c_\Phi T a_1(t); \vec{A})\Big)$$

But if $a \in \ker(T_\Phi)$ we have $Q_\Phi T a_1(t) = -T_\Phi a_0(t)$. Defining $u_k(t)$ as in corollary 1.1, we get for $t \leq 1$

$$J(t, W_0(t)T_\Phi a_0(t)) \leq \sum_{k=1}^{\infty} \hat{q}_0(2^k)J(t2^{-k}, u_k(t)) \leq C\sum_{k=1}^{\infty} \hat{q}_0(2^k)K(t2^{-k}, a)$$

$$J(t, W_1(t)Q^c_\Phi T a_1(t)) \leq \sum_{k=0}^{\infty} \hat{q}_1(2^{-k})J(t2^k, u_{-k}(t)) \leq C\sum_{k=0}^{\infty} \hat{q}_1(2^{-k})K(t2^k, a)$$

For $t \geq 1$ we note that

$$J(t, W_1(t)Q^c_\Phi T a_1(t)) \leq Ct\|a_1(t)\|_{A_1} \leq CK(t, a)$$

and similarly for $J(t, W_0(t)Q_\Phi T a_1(t))$. It follows that $a \in K_\Phi(A_0, A_1 \cap \ker(T))$.

The converse is easily proved in the same way as in the proof of theorem 2.2. We leave the details to the reader.

Corollary 2.2 *Assume that $\Lambda(t)$ is a quasi-linearization of the couple $\vec{A}$ and let $\Gamma_1, \ldots, \Gamma_N$ be strongly independent, bounded linear functionals on A_1. Put $T = (\Gamma_1, \ldots, \Gamma_N)$. Then*

$$\Lambda_T(t)a = \Lambda(t)a - \sum_{n=1}^{N} w_n(t) \cdot \Gamma_n(\Lambda(t)a)$$

defines a quasi-linearization of $(A_0, A_1 \cap \ker(T))$ provided that $w_1, \cdots, w_N$ is a supporting sequence for the functionals $\Gamma_1, \cdots, \Gamma_N$.

Corollary 2.3 *Let θ_{n-} and θ_{n+} be the lower and upper break-points associated with Γ_n and make the same assumptions as in the previous corollary. Then*

$$K_{\theta,\rho}(A_0, A_1 \cap \ker(T))$$

consists of all $a \in K_{\theta,\rho}(A_0, A_1)$ for which

$$\begin{array}{rll} \Gamma_n(a) = 0 & \textit{if} & \theta_{n+} < \theta \\ \Big(\int_0^\infty \Big(t^{-\theta} \frac{|\Gamma_n(\Lambda(t)a)|}{K(1/t, \Gamma_n; \vec{A}')}\Big)^\rho \frac{dt}{t}\Big)^{1/\rho} < \infty & \textit{if} & \theta_{n-} \le \theta \le \theta_{n+} \\ \textit{no additional condition} & \textit{if} & \theta < \theta_{n-} \end{array}$$

Proof Use theorem 2.4, corollary 2.2 and induction over N. Note that

$$\Gamma_N\Big(\Lambda(t)a - \sum_{n=1}^{N-1} w_n(t)\Gamma_n(\Lambda(t))\Big) = \Gamma_N(\Lambda(t)a)$$

2.5 Strongly independent functionals

We shall here discuss the notion of strongly independent functionals (see definition 2.2). First we give a sufficient condition for strong independence.

Lemma 2.2 *Consider a Banach couple $\vec{A}$ and let $\Gamma_1, \cdots, \Gamma_N$ be bounded linear functionals on A_1. Assume that there exist $u_1(t), \cdots, u_N(t) \in \Delta(\vec{X})$ and positive constants C and D such that for all small values of $t > 0$*

$$\Gamma_n(u_n(t)) = 1 \ , \quad J(t, u_n(t); \vec{X}) \le C/K(1/t, \Gamma_n; \vec{X}')$$

$$|\det[\Gamma_n(u_m(t))]| \ge D$$

Then $\Gamma_1, \cdots, \Gamma_N$ are strongly independent. The same conclusion holds if

$$\lim_{t \to 0} \Gamma_n(u_m(t)) \frac{K(1/t, \Gamma_m; \vec{X}')}{K(1/t, \Gamma_n; \vec{X}')} = a_{n,m}$$

where $\det[a_{n,m}] \neq 0$.

Proof Suppose that the assumptions hold for $0 < t \le t_0$. For each m we put $w_m(t) = \sum_{k=1}^{N} c_{m,k}(t)u_k(t)$, where $0 < t \le t_0$ and $c_{m,k}$ are the solutions of the system

$$\Gamma_n(w_m) = \sum_{k=1}^{N} c_{m,k}\Gamma_n(u_k) = \delta_{n,m}$$

The coefficients $c_{m,k}$ can be calculated by means of Cramers rule and then estimated. It turns out that

$$|c_{m,k}(t)| \le CK(1/t, \Gamma_k; \vec{A}')/K(1/t, \Gamma_m; \vec{A}')$$

For details see Löfström [12]. This gives the desired estimate for w_m, namely

$$J(t, w_m(t); \vec{A}) \le C/K(1/t, \Gamma_m; \vec{A}') \tag{1}$$

For $t > t_0$ we simply put $w_m(t) = w_m(t_0)$. Then $\Gamma_n(w_m(t)) = \delta_{n,m}$ for all $t > 0$. Moreover if $t > t_0$ we have

$$J(t, w_m(t); \vec{A}) \le C(t/t_0)J(t_0, w_m(t_0); \vec{A}) \le C/K(1/t, \Gamma_m; \vec{A}')$$

since $K(1/t, \Gamma_m) \le (1/t)\|\Gamma_m\|_{A_1'}$. Therefore the estimate above implies that (1) holds for all $t > 0$.

Definition 2.4 *Let Γ and γ be two bounded linear functionals on A_1. Then we say that Γ dominates γ if*

$$\lim_{t\to 0} \frac{K(1/t, \gamma; \vec{A}')}{K(1/t, \Gamma; \vec{A}')} = 0$$

If this is the case we write $\gamma \prec \Gamma$.

Lemma 2.3 *Let $\Gamma_1, \cdots, \Gamma_N$ be strongly independent functionals and assume that $\gamma_n \prec \Gamma_n$ for $n = 1, \cdots, N$. Put $\tilde{\Gamma}_n = \Gamma_n + \gamma_n$. Then $\tilde{\Gamma}_1, \cdots, \tilde{\Gamma}_N$ are strongly independent and $K(1/t, \tilde{\Gamma}_n; \vec{A}) \cong K(1/t, \Gamma_n; \vec{A})$ for small t.*

Proof There is a supporting sequence $w_1, \cdots, w_N$ for the functionals $\Gamma_1, \cdots, \Gamma_N$. Put $u_n = a_n w_n$ where $a_n = 1/(1 + \gamma_n(w_n))$. Then $\tilde{\Gamma}_n(u_n) = 1$ and $a_n \to 1$ as $t \to 0$, because

$$|\gamma_n(w_n)| \le K(1/t, \gamma_n)J(t, w_n) \le CK(1/t, \gamma_n)/K(1/t, \Gamma_n) \to 0$$

It follows that $J(t, u_n) \le C/K(1/t, \Gamma_n)$ and $K(1/t, \Gamma_n) \le K(1/t, \tilde{\Gamma}_n)$. But we also have $K(1/t, \tilde{\Gamma}_n) \le K(1/t, \Gamma_n) + K(1/t, \gamma_n) \le CK(1/t, \Gamma_n)$. Thus $K(1/t, \tilde{\Gamma}_n) \cong K(1/t, \Gamma_n)$. Moreover

$$\det[\Gamma_n(u_m)] = a_1 \cdots a_n \det\left[\delta_{n,m} + \gamma_n(w_m)\frac{K(1/t, \Gamma_m)}{K(1/t, \Gamma_n)}\right] \to 1$$

because

$$|\gamma_n(w_m)|\frac{K(1/t,\Gamma_m)}{K(1/t,\Gamma_n)} \leq C\frac{K(1/t,\gamma_n)}{K(1/t,\Gamma_n)} \to 0$$

The result now follows from lemma 2.2

Proposition 2.5 *Let $\mathcal{G}$ be a strongly independent set and suppose that $\Gamma_1, \cdots, \Gamma_N$ and $\tilde{\Gamma}_1, \cdots, \tilde{\Gamma}_N$ are two strongly independent bases . Let $\theta_+(\Gamma_n)$ and $\theta_-(\Gamma_n)$ denote the upper and lower break-points associated with Γ_n, and similarly for $\tilde{\Gamma}_n$. Then*

$$\{\theta_+(\Gamma_1), \cdots, \theta_+(\Gamma_N)\} = \{\theta_+(\tilde{\Gamma}_1), \cdots, \theta_+(\tilde{\Gamma}_N)\}$$

$$\{\theta_-(\Gamma_1), \cdots, \theta_-(\Gamma_N)\} = \{\theta_-(\tilde{\Gamma}_1), \cdots, \theta_-(\tilde{\Gamma}_N)\}$$

Proof We shall prove the first equality. Suppose that

$$\theta_+(\tilde{\Gamma}_N) > \max\{\theta_+(\Gamma_n) : n = 1, \ldots, N\}$$

Choose θ so that $\theta_+(\tilde{\Gamma}_N) > \theta > \theta_+(\Gamma_n)$ for all n. Assume (for simplicity) that $\tilde{\Gamma}_N = \Gamma_N + \sum_{n<N} \lambda_n \Gamma_n$ and put

$$A_N = A_1 \cap \ker(\Gamma_1) \cap \cdots \cap \ker(\Gamma_{N-1})$$

Note that by assumptions we have $K(t, \Gamma_N; A_0, A_N) \cong K(t, \Gamma_N; A_0, A_1)$. Thus the upper break-points for Γ_N relative to the couple (A_0, A_N) is the same as the upper break-point $\theta_+(\Gamma_N)$ relative to the couple (A_0, A_1) and similarly for $\tilde{\Gamma}_N$.

Now $K_{\theta,\rho}(A_0, A_N \cap \ker(\tilde{\Gamma}_N)) \cong K_{\theta,\rho}(A_0, A_N \cap \ker(\Gamma_N))$. Since $\theta > \theta_+(\Gamma_N)$ we know that this space is a closed subspace of $K_{\theta,\rho}(A_0, A_N)$, and since $\theta > \theta_+(\Gamma_n)$, all Γ_n must be bounded in $K_{\theta,\rho}-$norm. This implies that $\tilde{\Gamma}_N$ is also bounded in $K_{\theta,\rho}-$norm, contradicting the assumption $\theta < \theta_+(\tilde{\Gamma}_N)$.

We conclude that $\theta_+(\tilde{\Gamma}_N) \leq \max\theta_+(\Gamma_n)$. More generally $\max\theta_+(\tilde{\Gamma}_n) \leq \max\theta_+(\Gamma_n)$. Switching the roles between the two sets of functionals we conclude that

$$\max\theta_+(\tilde{\Gamma}_n) = \max\theta_+(\Gamma_n)$$

Excluding one functional with largest possible upper break-point from each set and repeating the argument will now give the first equality. □

Remark It is an open problem to construct a finite linearly independent set $\mathcal{G}$ of bounded linear functionals having no strongly independent basis.

Using the duality map

We conclude this section a general discussion on the construction of good approximations and supporting sequences. Such constructions can sometimes be inspired by using the duality map. (Cf Peetre [14].) If X is a Banach space, the duality map $D_X x$ is a set-valued function on X, taking as values subsets of the dual X^*. The defining property is

$$\eta \in D_X x \quad \Leftrightarrow \quad \eta(x) = \|x\|_X^2 = \|\eta\|_{X^*}^2$$

The set $D_X x$ is always non-empty and convex. If the norm on X is differentiable then D_X is single-valued and $\eta = D_X x$ is defined by

$$\eta(y) = \frac{d}{ds}\frac{1}{2}\|x+sy\|_X^2\Big|_{s=0}$$

This can be used in the following informal way to find a good $K(\vec{X})-$approximation. Assume that

$$K_2(t,a;\vec{X}) = \left(\|a-a_1\|_{X_0}^2 + t^2\|a_1\|_{X_1}^2\right)^{1/2}$$

Then we must have (assuming differentiable norms)

$$\frac{d}{ds}\frac{1}{2}\left(\|a-a_1-sx\|_{X_0}^2 + t^2\|a_1+sx\|_{X_1}^2\right)^{1/2} = 0$$

for all $x \in \Delta(\vec{X})$. In terms of the duality maps D_{X_0} and D_{X_1} this means that $-D_{X_0}(a-a_1)(x) + t^2 D_{X_1}a_1(x) = 0$ for all $x \in \Delta(\vec{X})$, i.e.

$$(D_{X_0} + t^2 D_{X_1})a_1 = D_{X_0}a \tag{2}$$

It is resonable to try this as a definition of a good $K(\vec{X})-$approximation of a.

Note that $D_{\Delta_t(\vec{X})} = D_{X_0} + t^2 D_{X_1}$ (the left hand side of (2)) is the duality map of $\Delta_t(\vec{X}) = X_0 \cap tX_1$, if this space is normed by $J_2(t,\cdot;\vec{X})$. Observe also that $\Delta_t(\vec{X}) \subseteq \Delta_t(\vec{X})^* = \Sigma_{1/t}(\vec{X}')$.

The duality map $D'_{\Delta_t(\vec{X})}$ is defined by saying that $D'_{\Delta_t(\vec{X})}\Gamma$ consists of all $v = v(t) \in \Delta(\vec{X})$ such that

$$\Gamma(v) = J_2^2(t,v;\vec{X}) = K_2^2(1/t,\Gamma;\vec{X}')$$

If we put

$$u = u(t) = v(t)/K_2^2(1/t,\Gamma;\vec{X}')$$

we have $\Gamma(u(t)) = 1$ and $J_2(t,u(t);\vec{X}) = 1/K_2(1/t,\Gamma;\vec{X}')$. This idea can sometimes be used to construct functions u_n in lemma 2.2.

2.6 Example: The couple (L_∞, L_1)

Given any measure space (Ω, μ) we consider L_1 and L_∞ as subspaces of the space of measurable functions. Thus (L_1, L_∞) and (L_∞, L_1) are Banach couples. It is well-known that a good $K(L_1, L_\infty)-$approximation of a is

$$\tilde{a}_1(t,x) = \begin{cases} a^*(t)\text{sign}(a(x)) & \text{if } |a(x)| > a^*(t) \\ a(x) & \text{otherwise} \end{cases}$$

where a^* is the decreasing rearrangement of A. In fact we even have

$$K(t, a; L_1, L_\infty) = \int_0^t a^*(s)\, ds$$

As a consequence we have $K_{\theta,\rho}(L_1, L_\infty) \cong L_{p,\rho}$ (Lorenz space) if $\theta = 1/p$.

We shall however consider the couple (L_∞, L_1). Then

$$K(t, a; L_\infty, L_1) = tK(1/t, a; L_1, L_\infty) = t\int_0^{1/t} a^*(s)\, ds$$

and a good $K(L_\infty, L_1)-$approximation of a is $a_1(t) = a - \tilde{a}_1(1/t)$ (see lemma 1.2), i.e.

$$a_1(t,x) = \begin{cases} a(x) - a^*(1/t)\text{sign}(a(x)) & \text{if } |a(x)| > a^*(1/t) \\ 0 & \text{otherwise} \end{cases}$$

A bounded linear functional Γ on L_1 is associated with a function $g \in L_\infty$, by the formula

$$\Gamma(a) = \int ga\, d\mu$$

Then

$$K(t, \Gamma; L'_\infty, L'_1) = K(t, g; L_1, L_\infty) = q(t) = \int_0^t g^*(s)\, ds$$

From theorem 2.1 we therefore get that $a \in K_\Phi(L_\infty, L_1 \cap \ker(\Gamma))$ if and only if $a \in K_\Phi(L_\infty, L_1)$ and

$$\left\| \frac{\int_0^{1/t} g^*(s)(a^*(s) - a^*(1/t))ds}{\int_0^{1/t} g^*(s)ds} \right\|_\Phi < \infty$$

Explicite examples

Consider the case $\Omega = (0, \infty)$ with the Lebesgue measure. We put

$$\Psi(t) = \Psi_\alpha(t) = \begin{cases} (1+t)^\alpha & \text{if } 0 < \alpha < 1 \\ \log(e+t) & \text{if } \alpha = 0 \end{cases}$$

and

$$g(t) = g_\alpha(t) = \frac{1}{\Psi_\alpha(t)}$$

Then it is easy to see that $g(ts)/g(t)$ is decreasing if $s \geq 1$ and increasing if $s \leq 1$, and that the same statements are true for $q(ts)/q(t)$. Since the limit of the last quotient is of the order $s^{1-\alpha}$ when $t \to \infty$ and $t \to 1$ we conclude that

$$\hat{q}(s) \cong s^{1-\alpha}, \text{ and thus } \theta_-(q) = \theta_+(q) = 1 - \alpha$$

Therefore we have

$$K_{\theta,\rho}(L_\infty, L_1 \cap \ker(\Gamma)) \cong K_{\theta,\rho}(L_\infty, L_1), \text{ if } \theta < 1 - \alpha$$

$$K_{\theta,\rho}(L_\infty, L_1 \cap \ker(\Gamma)) \cong K_{\theta,\rho}(L_\infty, L_1) \cap \ker(\Gamma), \text{ if } \theta > 1 - \alpha$$

As a second example we make the following construction (inspired by of Wallstén [17]). Define the sequence (t_n) by $t_{n+1} = \Psi^{-1}(t_n)$ where t_0 is a large number, and put

$$g(t) = \sup\{\frac{1}{\Psi(t_n)} : t_n \geq t\}$$

Then g is decreasing and

$$q(t_n) \geq t_n g(t_n) = \frac{t_n}{\Psi(t_n)}$$

$$q(\tau) \leq q(t_n) + \frac{\tau - t_n}{\Psi(t_{n+1})} \leq q(t_n) + \frac{\tau}{t_n}$$

if $t_n < \tau < t_{n+1}$. It follows that $\theta_-(q) = 0$. To see that put $\tau_n = t_n^2/\Psi(t_n)$ and $s_n = t_n/\tau_n$. Then $s_n \to 0$ and

$$\hat{q}(s_n) = \sup_{t \geq s_n} \frac{q(t)}{q(t/s_n)} \geq \frac{q(t_n)}{q(\tau_n)} \geq \frac{1}{1 + \tau_n \Psi(t_n)/t_n^2} = \frac{1}{2}$$

Thus $\theta_-(q) = 0$ for all $0 \leq \alpha < 1$. On the other hand Γ is bounded on $(L_\infty, L_1)_{\theta,1}$ if and only if $g \in (L_\infty, L_1)'_{\theta,1} = (L_1, L_\infty)_{1-\theta,\infty}$, i.e. $g \in L_{p,\infty}$ where $(1-\theta)p = 1$. This means that $\theta_+(q) = 1 - \alpha$. In conclusion we have

$$\theta_-(q) = 0, \;\; \theta_+(q) = 1 - \alpha$$

As a concequence the space $(L_\infty, L_1 \cap \ker(\Gamma))_{\theta,\rho}$ is not a closed subspace of $(L_\infty, L_1)_{\theta,\rho}$ if $0 < \theta < 1 - \alpha$. With the extreme choice $\alpha = 0$ we get an example where $(L_\infty, L_1 \cap \ker(\Gamma))_{\theta,\rho}$ is never closed.

2.7 Example: Weighted L_p−spaces

We consider a general measure space $(\Omega, d\mu)$ and a positive weight function w. Then let $L_p(w)$ denote the space defined by the norm $a \to \|wa\|_{L_p}$ where we shall assume $1 < p < \infty$. Then the duality maps on L_p and $L_p(w)$ are

$$D_{L_p} a = |a|^{p-1}\text{sign}(a)/\|a\|_{L_p}^{p-2}$$

$$D_{L_p(\omega)} a = |a|^{p-1}\text{sign}(a)\omega^p/\|\omega a\|_{L_p}^{p-2}$$

In the case $p = 2$ we get

$$(D_{L_2} + t^2 D_{L_2(w)}) = (1 + t^2 w^2)a$$

Therefore it is resonable to believe that

$$\Lambda(t)a = \frac{1}{1 + t^2 w^2} a$$

defines a quasi-linearization of the couple $(L_2, L_2(w))$. This is in fact easy to prove. Moreover the same family of operators is a quasi-linearization of the couple $(L_p, L_p(w))$. This follows easily from the fact that

$$\Big(\frac{t^2 w^2}{1 + t^2 w^2}\Big)^p + \Big(\frac{tw}{1 + t^2 w^2}\Big)^p \cong \min(1, tw)^p$$

Thus

$$K(t, a; L_p, L_p(w)) \cong \Big(\int_0^\infty (\min(1, tw)|a|)^p d\mu\Big)^{1/p} \tag{1}$$

This relation implies that there is a simpler alternative quasi-linearization, namely the family

$$\tilde{\Lambda}(t)a = \begin{cases} a & \text{if} \quad tw \leq 1 \\ 0 & \text{if} \quad tw < 1 \end{cases} \tag{2}$$

As a consequence of formula (1) we have that

$$K_{\theta,p}(L_p, L_p(w)) \cong L_p(w^\theta)$$

It is also easy to see that

$$J(t, a; L_p, L_p(w)) \cong \Big(\int_0^\infty (w_t|a|)^p d\mu\Big)^{1/p} \tag{3}$$

where $w_t = \max(1, tw)$.

A bounded linear functional Γ on $L_p(w)$ is defined by the formula

$$\Gamma(a) = \int ga \, d\mu$$

where g is a function in $L_q(1/w)$, $1/q = 1 - 1/p$. By (3) and theorem 1.6, we have that

$$K(1/t, \Gamma; L'_p, L'_p(w)) \cong \Big(\int \Big(\frac{|g|}{w_t} \Big)^q d\mu \Big)^{1/q}$$

Using the general construction discussed in the previous section, we put $u(t) = v(t)/K_2^2(1/t, \Gamma)$, where $v(t) \in D'_{\Delta_t(\vec{X})}\Gamma$. Here we simply get

$$u(t) = |g|^{q-1} \mathrm{sign}(g) w_t^{-q} / \|g/w_t\|^q_{L_q}$$

We shall use a modification of this formula in our forthcoming discussion on strongly independent functionals. First, however, we shall give a simple example where the break-points $\theta_-(\Gamma)$ and $\theta_+(\Gamma)$ are different.

A case where $0 < \theta_- < \theta_+ < 1$

Consider the couple $\vec{A} = (L_p, L_p(w))$ on the positive real line and with weight $w(x) = (1+x)$. Let the functional Γ be given by the formula $\Gamma(a) = \int g(x)\, a(x)\, dx$ where

$$g(x) = \begin{cases} (1+x)^{\alpha_0} & \text{if } x \in \Omega_0 \\ (1+x)^{\alpha_1} & \text{if } x \in \Omega_1 \end{cases}$$

Here Ω_0 is the union of all intervals of the form $[2^n, 2^{n+1/2})$ and Ω_1 is the union of $[2^{n+1/2}, 2^{n+1})$ where $n = 1, 2, \cdots$. We put $g(x) = 0$ outside the union of Ω_0 and Ω_1. The numbers $\theta_j = \alpha_j + 1/q$ are choosen arbitrarily in the open interval $(0,1)$, but we assume that $\theta_0 \leq \theta_1$. Then

$$K(1/t, \Gamma)^q \cong \int_0^\infty \Big(\frac{g(x)}{\max(1, t(1+x))} \Big)^q dx =$$

$$= \sum_{n=0}^\infty \Big(\int_{2^n}^{2^{n+1/2}} \Big(\frac{(1+x)^{\alpha_0}}{\max(1, t(1+x))} \Big)^q dx + \int_{2^{n+1/2}}^{2^{n+1}} \Big(\frac{(1+x)^{\alpha_1}}{\max(1, t(1+x))} \Big)^q dx \Big) \cong$$

$$\cong \sum_{n=0}^\infty \Big(\Big(\frac{(1+2^n)^{\alpha_0}}{\max(1, t(1+2^n))} \Big)^q + \Big(\frac{(1+2^n)^{\alpha_1}}{\max(1, t(1+2^n))} \Big)^q \Big) 2^n \cong$$

$$\cong \int_0^\infty \Big(\Big(\frac{(1+x)^{\alpha_0}}{\max(1, t(1+x))} \Big)^q + \Big(\frac{(1+x)^{\alpha_1}}{\max(1, t(1+x))} \Big)^q \Big) dx$$

It follows that

$$K(1/t, \Gamma; \vec{A}') \cong \Big(t^{-1-\alpha_0 q} + t^{-1-\alpha_1 q} \Big)^{1/q}$$

and thus we conclude that

$$\hat{q}(s, \Gamma) \cong \sup_{\tau \geq 1} \Big(\frac{(\tau s)^{\theta_0 q} + (\tau s)^{\theta_1 q}}{\tau^{\theta_0 q} + \tau^{\theta_1 q}} \Big)^{1/q} \cong \max(s^{\theta_0}, s^{\theta_1})$$

As a consequence the lower and upper break-points are θ_0 and θ_1, respectively. In particular, if $\theta_0 < \theta_1$, then $K_{\theta,p}(L_p, L_p(\omega) \cap \ker(\Gamma))$ is not closed in $L_p(\omega^\theta)$ when $\theta_0 < \theta < \theta_1$.

By theorem 2.3 we see that $K_{\theta,p}(L_p, L_p(w) \cap \ker(\Gamma))$ consists of all $a \in L_p(w^\theta)$ such that

$$\begin{array}{rll} \Gamma(a) = 0 & \text{if} & \theta > \theta_1 \\ \displaystyle\int_0^1 \left(t^{\theta_0-\theta}\Big|\int_{tw<1} g\, a\, d\mu\Big|\right)^p \frac{dt}{t} < \infty & \text{if} & \theta_0 \le \theta \le \theta_1 \\ \text{no additional condition} & \text{if} & \theta < \theta_0 \end{array}$$

This follows from the fact that $\Gamma(\tilde{\Lambda}(t)a) = \int_{tw<1} g\, a\, d\mu$.

A general case of strong independence

Definition 2.5 *Assume that $1 < p < \infty$ and put $1/q = 1 - 1/p$. We say that $g_1, \ldots, g_N \in L_q(1/w)$ are asymptotically disjoint if there are sets $\Omega_1, \ldots, \Omega_N$ (possibly depending on t) such that*

$$\int \left(\frac{|g_n|}{w_t}\right)^q d\mu \le C \int_{\Omega_n} \left(\frac{|g_n|}{w_t}\right)^q d\mu \tag{4}$$

$$\frac{\int_{\Omega_n} \frac{|g_m|}{w_t} \left(\frac{|g_n|}{w_t}\right)^{q-1} d\mu}{\left(\int_{\Omega_m} \left(\frac{|g_m|}{w_t}\right)^q d\mu\right)^{1/q} \left(\int_{\Omega_n} \left(\frac{|g_n|}{w_t}\right)^q d\mu\right)^{1/p}} \to 0 \quad as \quad t \to 0 \tag{5}$$

for all $m \neq n$.

Proposition 2.6 *If $g_1, \ldots, g_N$ are asymptotically disjoint then the corresponding functionals $\Gamma_1, \ldots, \Gamma_N$ are strongly independent.*

Proof Define $u_1, \ldots, u_N$ by the formula

$$u_n = \begin{cases} |g_n|^{q-1} \text{sign}(g_n) w_t^{-q} / \int_{\Omega_n} |g_n| w_t^{-q} d\mu & \text{on } \Omega_n \\ 0 & \text{outside } \Omega_n \end{cases}$$

Then $\Gamma_n(u_n) = \int g_n u_n \, d\mu = 1$ and

$$J(t, u_n; L_p, L_p(w)) \cong \left(\int_{\Omega_n} (w_t |u_n|)^p d\mu\right)^{1/p} = \frac{\left(\int_{\Omega_n} \left(|g_n| w_t^{-1}\right)^{(q-1)p} d\mu\right)^{1/p}}{\int_{\Omega_n} \left(|g_n| w_t^{-1}\right)^q d\mu}$$

Thus (4) implies that

$$J(t,u_n) \le C\Big(\int_{\Omega_n} \Big(|g_n|w_t^{-1}\Big)^q d\mu\Big)^{-1/q} \le C\Big(\int \Big(|g_n|w_t^{-1}\Big)^q d\mu\Big)^{-1/q} = C/K(1/t,\Gamma_n)$$

Moreover (5) implies that if $m \neq n$ then

$$\Gamma_m(u_n)\cdot \frac{\Big(\int_{\Omega_n}\Big(|g_n|w_t^{-1}\Big)^q d\mu\Big)^{1/q}}{\Big(\int_{\Omega_m}\Big(|g_m|w_t^{-1}\Big)^q d\mu\Big)^{1/q}} \to 0 \text{ as } t\to 0$$

This implies that $\det\left[\Gamma_m(u_n)\right] \to 1$ as $t \to 0$. Now the the result follows from lemma 2.2. □

From proposition 2.6 we see that $\Gamma_1,\ldots,\Gamma_N$ are strongly independent if the functions $g_1,\ldots,g_M$ have disjoints supports. This follows at once if we choose Ω_n to be the supports of g_n, because (4) is obvious and the left hand side of (5) vanishes. However it is enough to assume that

$$\frac{\int_{\Omega_n}\Big(|g_m|w_t^{-1}\Big)^q d\mu}{\int\Big(|g_m|w_t^{-1}\Big)^q d\mu} \to 0 \;\;,\;\; m\neq n,\; \Omega_n = \text{support of } g_n \tag{6}$$

because (5) follows from (6) and Hölders inequality. Thus (6) implies that $\Gamma_1,\ldots,\Gamma_M$ are strongly independent.

For further examples of strongly independent functionals in weighted L_p−spaces see Löfström[12] .

2.8 Example: Sobolev spaces

Consider the space $L_p = L_p(\mathbb{R})$ with Lebesgue-measure and let W_p^M be the corresponding Sobolev space of all $a \in L_p$ such that $D^\alpha a \in L_p$ for all $|\alpha| \leq M$. We shall restrict ourselves to the case $1 < p < \infty$. Then we can use the following norm on W_p^N

$$\|a\|_{W_p^N} = \|a\|_{L_p} + \||D|^M a\|_{L_p}$$

A quasi-linearization $\Lambda(t)$ can be found by using the Fourier transform $\mathcal{F}$ and the duality map on L_2. We define $\Lambda(t) = G_t * a$ where

$$(\mathcal{F}G_t)(\xi) = \lambda(\tau\xi) = \frac{1}{1 + |\tau\xi|^{2M}} \tag{1}$$

where $\tau = t^{1/M}$. Note that

$$G_t(s) = \tau^{-1} g(|s|/\tau),$$

where

$$|D^k g(r)| \leq Ce^{-\kappa|r|}, \quad k = 0, 1, \ldots, 2M - 1$$

for some number $\kappa > 0$.

We shall consider functionals of the form

$$\Gamma(a) = (D^m a)(0) + \sum_{\alpha < m} \int (D^\alpha a) d\mu_\alpha \tag{2}$$

where μ_α are bounded measures on $\mathbb{R}$ and $0 \leq n < M - 1/p$. The number m is called the order of Γ and the functional $a \to (D^m a)(0)$ is called the principal part of Γ.

Lemma 2.4 *Assume that Γ is given by (2). Then*

$$K(1/t, \Gamma; (L_p)', (W_p^M)') \cong t^{-\theta_m}$$

where

$$\theta_m = \frac{m + 1/p}{M}$$

Moreover the difference between Γ and its principal part is dominated by Γ.

Proof First we use Kolmogorofs inequality

$$|D^\alpha a(y)| \leq C\|a\|_{L_p}^{1-\theta_\alpha} \|a\|_{W_p^N}^{\theta_\alpha}$$

where $\theta_\alpha = (\alpha + 1/p)/M$. Thus

$$|\Gamma(a)| \leq C \sum_{\alpha \leq m} \|a\|_{L_p}^{1-\theta_\alpha} \|a\|_{W_p^M}^{\theta_\alpha} \leq Ct^{-\theta_m} J(t, a; L_p, W_p^M)$$

This proves that

$$K(1/t,\Gamma;L_p',(W_p^M)')\le Ct^{-\theta_m}$$

To prove the converse estimate, let $\tilde{\Gamma}$ be the principal part of Γ and put $\Gamma=\tilde{\Gamma}+\gamma$. It is enough to prove that $K(1/t,\tilde{\Gamma})\ge Ct^{-\theta_m}$, because then it follows that γ is dominated by $\tilde{\Gamma}$. (See lemma 2.3.)

Let us introduce the following functions:

$$\varphi(t,s)=\tilde{\Gamma}(G_t(\cdot-s))=(D^mG_t)(-s)$$

$$u(t)=(G_t*\bar{\varphi}(t))/\nu^2(t)$$

where the bar denotes complex conjugate and

$$\nu^2(t)=\int_{\mathbb{R}}|\varphi(t,s)|^2ds$$

Then $\tilde{\Gamma}(u(t))=1$. Parsevals formula gives $\nu^2(t)\cong\tau^{-1-2m}$. Moreover $\|\bar{\varphi}(t)\|_{L_p}\le C\tau^{-m-1-1/p}$. Therefore

$$\begin{aligned}\|G_t*\bar{\varphi}(t)\|_{L_p}&\le C\|\bar{\varphi}(t)\|_{L_p}\le C\tau^{-m-1+1/p}\\ \|G_t*\bar{\varphi}(t)\|_{W_p^M}&\le C\tau^{-M}\|\bar{\varphi}(t)\|_{L_p}\le Ct^{-1}\tau^{-m-1+1/p}\end{aligned}$$

It follows that

$$J(t,u(t);L_p,W_p^M)\le C\tau^{-m-1+1/p}/\nu^2(t)\le Ct^{\theta_m}$$

This proves that $K(1/t,\tilde{\Gamma})\ge Ct^{-\theta_m}$. The proof is complete. □

We shall now consider several functionals.

Lemma 2.5 *Assume that $\Gamma_1,\ldots,\Gamma_N$ have the form (2) with different orders m_n, $n=1,\ldots,N$. Then the set $\{\Gamma_1,\ldots,\Gamma_N\}$ is strongly independent.*

Proof Using lemma 2.3 and 2.4, we see that it is sufficient to consider the principal parts of $\Gamma_1,\ldots,\Gamma_N$. Thus put $\varphi_n(t,s)=(D^{m_n}G_t)(-s)$, $u_n(t)=(G_t*\bar{\varphi}_n(t))/\nu_n^2(t)$, where $\nu_n^2(t)=\int|\varphi_n(t,s)|^2ds$. Writing $\tilde{\Gamma}_n$ for the principal part of Γ_n, we have that $\tilde{\Gamma}_n(u_n(t))=1$ and $J(t,u_n(t))\le C/K(1/t,\tilde{\Gamma}_n)$. Moreover

$$\begin{aligned}\tilde{\Gamma}_k(u_n)\cdot\frac{K(1/t,\tilde{\Gamma}_n)}{K(1/t,\tilde{\Gamma}_k)}&\cong\frac{\int\tau^{m_k}\varphi_k(t,s)\tau^{m_n}\bar{\varphi}_n(t,s)ds}{\int\left|\tau^{m_n}\varphi_n(t,s)\right|^2ds}\\ &\to\frac{\int\xi^{m_k+m_n}\lambda^2(\xi)d\xi}{\int|\xi^{m_n}\lambda(\xi)|^2d\xi}=\frac{c_{nk}}{c_{nn}}\end{aligned}$$

Thus it is enough to prove that $\det[c_{nk}] \neq 0$. To see this let $z_1, \dots, z_N$ be arbitrary complex numbers. Then

$$Q = \sum_{n,k} c_{nk} z_n \bar{z}_k = \int |\sum_n \xi^{m_n} z_n|^2 \lambda^2(\xi) d\xi$$

Since the functions $\xi^{m_1}, \dots, \xi^{m_N}$ are linearly independent if the numbers $m_1, \dots, m_N$ are different, Q must be positive definite. Therefore $\det[c_{nk}] \neq 0$. This completes the proof.

Theorem 2.5 *Assume that $\Gamma_1, \dots, \Gamma_N$ have the form (2) with different orders m_n, $n = 1, \dots, N$. Put $T = (\Gamma_1, \dots, \Gamma_N)$. Then the interpolation space*

$$K_{\theta,\rho}(L_p, W_p^M) \cap \ker(T)$$

consists of all $a \in K_{\theta,\rho}(L_p, W_p^M)$ such that

$$\Gamma_n(a) = 0 \ \textit{for all} \ n \ \textit{for which} \ M\theta > m_n + 1/p \tag{3}$$

$$\Big(\int_0^\infty \Big(\frac{1}{t}\int_{-t}^t \Big|\Gamma_n(a(\cdot - s)))\Big|^p ds\Big)^{\rho/p} \frac{dt}{t}\Big)^{1/\rho} < \infty \ \textit{if} \ M\theta = m_n + 1/p \tag{4}$$

Here $1 < p < \infty$.

Proof From lemma 2.5 we get that $\Gamma_1, \dots, \Gamma_N$ are strongly independent. By lemma 2.4 we see that the lower and upper break points of Γ_n is θ_n. Therefore corollary 2.3 implies that it is enough to show that (4) is equivalent to

$$\Big(\int_0^\infty |\Gamma_n(\Lambda(t)a|^\rho \frac{dt}{t}\Big)^{1/\rho} < \infty \tag{5}$$

if $a \in K_{\theta_k,\rho}(L_p, W_p^M)$. Here $\Lambda(t)a = G_t * a$. To prove this equivalence we note that

$$\Gamma_n(\Lambda(t)a) = (G_t * y)(0)$$

where $y(s) = \Gamma_n(a(\cdot - s)) \in K_{\eta,\rho}(L_p, W_p^M)$ and $M\eta = m_n + 1/p - m_n = 1/p$. Therefore it is enough to prove that the following two conditions are equivalent

$$\Big(\int_0^\infty |(G_t * y)(0)|^\rho \frac{dt}{t}\Big)^{1/\rho} < \infty \tag{6}$$

$$\Big(\int_0^\infty \Big(\frac{1}{t}\int_{-t}^t |y(s)|^p ds\Big)^{\rho/p} \frac{dt}{t}\Big)^{1/\rho} < \infty \tag{7}$$

provided that $y \in K_{\eta,\rho}(L_p, W_p^M) = K_{1/p,\rho}(L_p, W_p^1)$. Let W_p^0 be the subspace of $W_p = W_p^1$ of all u such that $u(0) = 0$, and assume that (6) holds. Then corollary

2.3 implies that $y \in K_{1/p,\rho}(L_p, W_p^0)$. Therefore we can write $y = y_0 + y_1$, where $y_0 \in L_p, y_1 \in W_p^0$. Let us put $\tilde{y}_1(s) = 0$ if $s < 0$ and similarly for y and y_0. Then $\tilde{y}_0 \in L_p$ and $\tilde{y}_1 \in W_p$. It follows that $\tilde{y} \in K_{1/p,\rho}(L_p, W_p^1)$, implying that

$$\Big(\int_0^1 \Big(\frac{1}{t}\int_{\mathbb{R}} |\tilde{y}(s+t) - \tilde{y}(s)|^p ds\Big)^{\rho/p} \frac{dt}{t}\Big)^{1/\rho} < \infty$$

Restricting the domain of integration to the interval $-t < s < 0$ we get (half of) (7). (The other half is proved in the same way.)

Conversely assume that (7) holds. Then we note that

$$\begin{aligned}(G_t * y)(0) &= \int_{-\infty}^{\infty} G_t(-s)y(s)ds = \int_{-\infty}^{\infty} g(|s|)y(s\tau)ds \\ &= \int_{-\infty}^{\infty} sDg(|s|)\Big(\frac{1}{s\tau}\int_0^{s\tau} y(\sigma)d\sigma\Big)ds\end{aligned}$$

Therefore the left hand side of (6) is bounded by

$$\int_0^{\infty} |s||Dg(|s|)|\Big(\int_0^1 \frac{1}{s\tau}\Big(\int_0^{s\tau} |y(\sigma)|d\sigma\Big)^{\rho}\frac{d\tau}{\tau}\Big)^{1/\rho} \le C\Big(\int_0^{\infty}\Big(\frac{1}{t}\int_0^t |y(\sigma)|^p d\sigma\Big)^{\rho/p}\frac{dt}{t}$$

Using (7) and the fact that

$$\Big(\int_1^{\infty}\Big(\frac{1}{t}\int_0^t |y(\sigma)|^p d\sigma\Big)^{\rho/p}\frac{dt}{t}\Big)^{1/\rho} \le C\|y\|_{L_p}$$

we get (6). The proof is complete.

3 Interpolation of kernels

3.1 Discussion on general operators

We want to generalize theorem 2.4 by replacing the functionals $\Gamma_1, \ldots, \Gamma_N$ by more general operators $T_1, \ldots, T_N$. A simple way of getting a generalization is to go through the proof of theorem 2.4 and list all properties needed to make the proof work. This is essentially what we shall do in this section. The reader is asked to check the details.

Definition 3.1 *Let $T_1, \ldots, T_N$ be bounded linear operators from A_1 into a Banach space B. We say that $T_1, \ldots, T_N$ are strongly independent if there exist bounded linear and strongly continuous operators $V_1(t), \ldots, V_N(t)$ from A_1 into $\Delta(\vec{A})$, such that for all m and n*

$$T_m(\sum_{n=1}^{N} V_n(t)(a_1)) = T_m(a_1) \ , \ a_1 \in A_1 \tag{1}$$

$$J(t, V_n(t)u; \vec{A}) \leq C \frac{q(1/(ts), T_n; \vec{A})}{q(1/t, T_n; \vec{A})} \cdot J(ts, u; \vec{A}) \ , \ u \in \Delta(\vec{A}) \tag{2}$$

for $0 < t \leq 1, 0 < s < \infty$, and

$$J(t, V_n(t)u; \vec{A}) \leq Ct \|a_1\|_{A_1} \ , \ a_1 \in A_1 \tag{3}$$

for $t \geq 1$.

The following theorem is a direct generalization of theorem 2.4.

Theorem 3.1 *Let $T_1, \ldots, T_N$ be strongly independent, bounded linear operators from A_1 into a Banach space B and put $T = (T_1, \ldots, T_N)$. For a given parameter Φ let I_Φ be the set of all n such that T_n is bounded in $K_\Phi(\vec{A})-$norm and put $T_\Phi = (T_{1\Phi}, \ldots T_{N\phi})$, where $T_{n\Phi} = T_n$ if $n \in I_\Phi$ and $T_{n\Phi} = 0$ otherwise. Then T_Φ can be extended to a bounded linear operator on $K_\Phi(\vec{A})$, so that*

$$K_\Phi(A_0, A_1 \cap \ker(T)) \cong K_\Phi(\vec{A}) \cap \ker(T_\Phi)$$

provided that

$$\sum_{k=0}^{\infty} \hat{q}(2^k, T_n)\, \omega_\Phi(2^{-k}) < \infty \quad if \quad n \in I_\Phi$$

$$\sum_{k=0}^{\infty} \hat{q}(2^{-k}, T_n)\, \omega_\Phi(2^{k}) < \infty \quad if \quad n \notin I_\Phi$$

Proof Let $a_1(t)$ be a good $K(\vec{A})-$ approximation of $a \in K_\Phi(\vec{A}) \cap \ker(T_\Phi))$. Choose operators $V_n(t)$ according to definiton 3.1 and define $\tilde{a}_1(t)$ by means of the formula

$$\tilde{a}_1(t) = a_1(t) - \sum_{n=1}^{N} V_n(t)(a_1(t)) \tag{4}$$

Now repeat the proof of theorem 2.4.

Corollary 3.1 *Assume that $\Lambda(t)$ is a quasi-linearization of the couple $\vec{A}$. Make the same assumptions as in the previous theorem and let $V_n(t)$ given by definition 3.1. Then*

$$\Lambda_T(t)a = \Lambda(t)a - \sum_{n=1}^{N} V_n(t)(\Lambda(t)a))$$

defines a quasi-linearization of $(A_0, A_1 \cap \ker(T))$.

Corollary 3.2 *Let θ_{n-} and θ_{n+} be the lower and upper break-points associated with T_n. Make the same assumptions as in the previous corollary. Then*

$$K_{\theta,\rho}(A_0, A_1 \cap \ker(T))$$

consists of all $a \in K_{\theta,\rho}(A_0, A_1)$ for which

$$\begin{array}{rcl} T_n a = 0 & \textit{if} & \theta > \theta_{n+} \\ \textit{no additional condition} & \textit{if} & \theta < \theta_{n-} \end{array}$$

For any θ we have that $a \in K_{\theta,\rho}(A_0, A_1 \cap \ker(T))$ if and only if $a \in K_{\theta,\rho}(A_0, A_1)$ and

$$\Big(\int_0^\infty \Big(t^{-\theta} \frac{\|T_n(\Lambda(t)a)\|_B}{q(1/t, T_n; \vec{A})}\Big)^\rho \frac{dt}{t}\Big)^{1/\rho} < \infty$$

An example

We give a simple example with $N = 1$. Consider two measure spaces (Ω_1, μ_1) and (Ω_2, μ_2) and the product measure space $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$. Let $v \geq 1$ be a given weight function on Ω_1 and put $w(x, y) = v(x)$. Let A_0 be the space $L_p = L_p(\mu_1 \times \mu_2)$ and let A_1 be the space $L_p(w) = L_p(w; \mu_1 \times \mu_2)$ where $1 < p < \infty$. Assume that g is a given function in $L_q(1/v; \mu_1)$ where $1/q = 1 - 1/p$. Put

$$q(t) = \Big(\int \Big(\frac{|g(x)|}{\max(1, tv(x))}\Big)^q d\mu_1(x)\Big)^{1/q}$$

and assume (for simplicity) that $q(t) \cong t^{-\theta_0}$. Finally let h_m be a given sequence of funstions in $L_q(\mu_2)$ such that

$$\kappa_m = (\int |h_m(y)|^q \, d\mu_2)^{1/q} > 0$$

and $h_n \cdot h_m = 0$ if $n \neq m$. Define $T : A_1 \to \ell_\infty = B$ by the formula

$$(Ta)_m = \frac{1}{q(1)\kappa_m} \int\int g(x)h_m(y)a(x,y)\, d(\mu_1 \times \mu_2)$$

Then we clearly have $q(t,T) \cong q(t) \cong t^{-\theta_0}$. We can now define the family $V(t)$ by

$$V(t)a = \sum_m \frac{|gh_m|^{q-1}\mathrm{sign}(gh_m)}{(q(t)\kappa_m)^q \max(1,tv)^q} \int\int gh_m a\, d(\mu_1 \times \mu_2)$$

Clearly $T(V(t)a) = T(a)$ and an easy computation shows that (2) and (3) hold. We conclude that $a \in K_{\theta,\rho}(L_p, L_p(w) \cap \ker(T))$ if and only if $a \in K_{\theta,\rho}(L_p, L_p(w))$ and (in case $\theta > \theta_0$)

$$\int\int g(x)h_m(y)a(x,y)\, d(\mu_1 \times \mu_2) = 0 \;\text{ for all }\; m$$

3.2 Operators defined by linear functionals

In this section we shall consider subspaces defined by a kind of "vector-valued" linear functionals.

We shall work within the following general setting. Let $\vec{A} = (A_0, A_1)$ be a given couple, B a Banach space and X a linear subspace of a second Banach space $\mathcal{X}$. Let $X \otimes B$ denote the algebraic tensor product of X and B. We make the following assumptions

$$X \otimes B \;\text{ is a subspace of }\; \Delta(\vec{A}) \tag{1}$$

$$X \otimes B \;\text{ is dense in }\; A_1 \tag{2}$$

$$\|x\|_{\mathcal{X}} \leq C \sup_{\|b\|_B = 1} \|x \otimes b\|_{A_j} \quad x \in X,\; j = 0,1 \tag{3}$$

$$\|\beta(u) \otimes b\|_{A_j} \leq C\|u\|_{A_j}\|b\|_B\|\beta\|_{B^*}\,, \quad u \in X \otimes B,\; j = 0,1 \tag{4}$$

Note that if $u = \sum x_k \otimes b_k$ and $\beta \in B^*$, we define $\beta(u) = \sum x_k\beta(b_k)$.

Given these properties we let X_j be the closure of X in the norm

$$\|x\|_{X_j} = \sup_{\|b\|_B = 1} \|x \otimes b\|_{A_j} \tag{5}$$

By (3) we have $X_j \hookrightarrow \mathcal{X}$ so that $\vec{X} = (X_0, X_1)$ is a Banach couple and

$$\|x \otimes b\|_{A_j} \leq \|x\|_{X_j}\|b\|_B \;\text{ if } x \in X, b \in B \tag{6}$$

Thus we can extend the mapping $x \mapsto x \otimes b$ to X_j by continuity. Writing $x \otimes b$ for the value of this extension, (6) holds for all $x \in X_j$.

In the situation described above we shall write $\vec{A} = \vec{X}\vec{\otimes}_X B$ or (if the space X is understood) simply

$$\vec{A} = \vec{X}\vec{\otimes}B$$

Note also that (4) and (5) imply

$$\|\beta(u)\|_{X_j} \leq C\|u\|_{A_j}\|\beta\|_{B^*} \text{ if } u \in X \otimes B \tag{7}$$

We shall now define the type of operators T we shall work with. To begin with we have a bounded linear functional Γ on X_1. Moreover we have a bounded linear operator L on B. Define $L(u)$ for $u \in X \otimes B$ in the obvious way, i.e. $L(\sum x_k \otimes b_k) = \sum x_k \otimes L(b_k)$. Then

$$\|\beta(L(u))\|_{X_j} = \|(L^*\beta)(u)\|_{X_j} \leq C\|\beta\|_{B^*}\|u\|_{A_j} \tag{8}$$

In this situation we define the operator $T : X \otimes B \to B$ by the formula

$$T(\sum x_j \otimes b_j) = \sum \Gamma(x_j) \cdot L(b_j)$$

We shall see that T is bounded in A_1-norm. This follows from (7). In fact, if $u \in X \otimes B$, we have

$$\begin{aligned} \|T(u)\|_B &= \sup_{\|\beta\|_{B^*}=1} |\beta(T(u))| = \sup_{\|\beta\|_{B^*}=1} |\Gamma(\beta(L(u)))| \leq \\ &\leq C \sup_{\|\beta\|_{B^*}=1} \|\beta(L(u))\|_{X_1} \leq C\|u\|_{A_1} \end{aligned}$$

Since $X \otimes B$ is assumed to be dense in A_1, we can extend T to a bounded linear operator T_1 from A_1 into B. This operator will be denoted by

$$T_1 = \Gamma \otimes L \tag{9}$$

Lemma 3.1 *Let T_1 be defined by (9). Then*

$$q(\tau, T; \vec{A}) = \sup\{\|T(u)\|_B : J(1/\tau, u; \vec{A}) \leq 1\} \leq CK(\tau, \Gamma; \vec{X}')$$

If L has a bounded inverse, then

$$q(\tau, T; \vec{A}) \cong K(\tau, \Gamma; \vec{X}')$$

Proof Observe that

$$\|T(u)\|_B = \sup_{\|\beta\|_{B^*}=1} |\Gamma(\beta(L(u)))| \leq CK(\tau, \Gamma; \vec{X}')J(1/\tau, u; \vec{A})$$

which gives the first part of the lemma. For the second part choose $b \in B$ and x so that $\|b\|_B = 1$, $J(1/\tau, x; \vec{X}) \leq 1$ and $|\Gamma(x)| \geq 2^{-1}K(\tau, \Gamma)$. Then $J(1/\tau,\, x \otimes b\,; \vec{A}) \leq 1$ and

$$\|T(x \otimes b)\|_B = |\Gamma(x)|\|L(b)\|_B \geq CK(\tau, \Gamma; \vec{X}')$$

This gives the result.

Lemma 3.2 *Assume that $\vec{A} = \vec{X}\vec{\otimes}B$ and let $\Gamma_1, \ldots, \Gamma_N$ be strongly independent, bounded linear functionals on X_1. Moreover let $L_1, \ldots, L_N$ be bounded linear operators on B with bounded inverses. Put $T_n = \Gamma_n \otimes L_n$. Then $T_1, \ldots, T_N$ are strongly independent. The linear operators in definition 3.1 are given by*

$$V_n(t)(a_1) = w_n(t) \otimes (\Gamma_n \otimes \mathrm{id}_B)(a_1)$$

where $w_1(t), \ldots, w_N(t)$ is a supporting sequence for $\Gamma_1, \ldots, \Gamma_N$.

Proof First note that

$$L_m((\Gamma_m \otimes \mathrm{id}_B)(x \otimes b)) = \Gamma_m(x)L_m(b) = T_m(x \otimes b)$$

Thus $L_m((\Gamma_m \otimes \mathrm{id}_B)a_1) = T_m(a_1)$ for all $a_1 \in A_1$. It follows that

$$T_m(V_n(t)a_1) = \Gamma_m(w_n(t))L_m(\Gamma_m \otimes \mathrm{id}_B)a_1) = \delta_{n,m}T_m(a_1)$$

This gives formula (1) in definition 3.1. To prove (2), we note that

$$\|(\Gamma_m \otimes \mathrm{id}_B)(u)\|_B \leq CK(1/(ts), \Gamma_n; \vec{X}')J(ts, u; \vec{A})$$

Thus

$$J(t, V_n(t)u; \vec{A}) \leq J(t, w_n(t); \vec{X})\|(\Gamma_m \otimes \mathrm{id}_B)(u)\|_B \leq C\frac{K(1/(ts), \Gamma_n; \vec{X}')}{K(1/t, \Gamma_n; \vec{X}')}J(ts, u; \vec{A})$$

By lemma 3.1 we get formula (2) in definition 3.1. Since $(\Gamma_n \otimes \mathrm{id}_B)$ is bounded on A_1 and $K(1/t, \Gamma_n) \geq C1/t$ we also get (3). This proves the result. □

Next we consider operators which are pertubations of the operators used in lemma 3.2.

Lemma 3.3 *Assume that $\vec{A} = \vec{X}\vec{\otimes}B$ and that $\Gamma_1, \ldots, \Gamma_N$ are strongly independent, bounded linear functionals on X_1. Let $L_1, \ldots, L_N$ be bounded on B with bounded inverses. Put*

$$T_n = \Gamma_n \otimes L_n + \sum_{k=1}^{K_n} \Gamma_{n,k} \otimes L_{n,k}$$

where $\Gamma_{n,k}$ are bounded linear functionals on X_1 dominated by Γ_n and $L_{n,k}$ are bounded on B. Then $T_1, \ldots, T_N$ are strongly independent. Moreover

$$q(t, T_n; \vec{A}) \cong K(t, \Gamma_n; \vec{X}')$$

Proof In the proof we consider only small values of t. For large values of t we can define $V_n(t)$ to be independent of t. See the proof of lemma 2.2. Let $w_1, \ldots, w_N$ be a supporting sequence for $\Gamma_1, \ldots, \Gamma_N$. We then introduce the following families of operators on B:

$$Q_{m,n}(t) = \Gamma_m(w_n(t))L_m + \sum_{k \leq K_m} \Gamma_{m,k}(w_n(t))L_{m,k}$$

$$\tilde{Q}_{m,n}(t) = \frac{K(1/t, \Gamma_n)}{K(1/t, \Gamma_m)} Q_{m,n}(t)$$

Then

$$\tilde{Q}_{m,n}(t) = \delta_{m,n} L_m + \sum_{k \leq K_m} \tilde{\Gamma}_{m,k}(w_n(t))L_{m,k}$$

where

$$\tilde{\Gamma}_{m,k} = \frac{K(1/t, \Gamma_n)}{K(1/t, \Gamma_m)} \Gamma_{m,k}$$

From the assumptions we conclude that $\tilde{\Gamma}_{m,k}(w_n(t)) \to 0$ as $t \to 0$. Therefore we can find operators $\tilde{P}_n(t)$, from A_1 into B such that for sufficiently small t we have

$$\sum_{n=1}^{N} \tilde{Q}_{m,n} \tilde{P}_n(t) = \tilde{T}_m(t) = \frac{1}{K(1/t, \Gamma_m)} T_m \quad , \quad m = 1, \ldots, N$$

and

$$\|\tilde{P}_n(t)a_1\|_B \leq C \frac{1}{K(1/t, \Gamma_n)} \|T_n a_1\|_B$$

Writing $P_n(t) = K(1/t, \Gamma_n)\tilde{P}_n(t)$ we have

$$\sum_{n=1}^{N} Q_{m,n}(t)P_n(t) = T_m$$

and

$$\|P_n(t)a_1\|_B \leq C\|T_n a_1\|_B$$

Now put

$$V_n(t) = w_n(t) \otimes P_n(t)$$

Since $Q_{m,n}(t)b = T_m(w_n(t) \otimes b)$ we have

$$T_m(\sum_{n=1}^{N} V_n(t)(a_1)) = \sum_{n=1}^{N} Q_{m,n}(t)P_n(t)a_1 = T_m(a_1)$$

Moreover

$$J(t, V_n(t)a_1) \leq C \frac{\|P_n(t)a_1\|_B}{K(1/t, \Gamma_n)} \leq C \frac{\|T_n a_1\|_B}{K(1/t, \Gamma_n)} \leq C \frac{q(1/(ts), T_n)}{q(1/t, T_n)} J(ts, u)$$

Finally we note that lemma 3.1 implies

$$\|T_n u\|_B \le CK(1/t,\Gamma_n)\Big(1+\sum_{k=1}^{K_n}\frac{K(1/t,\Gamma_{n,k})}{K(1/t,\Gamma_n)}\Big)J(t,u) \le CK(1/t,\Gamma_n)J(t,u)$$

Thus $q(1/t,T_n) \le CK(1/t,\Gamma_n)$. To prove the converse inequality, we immitate the proof of lemma 3.1. Thus choose b and x so that $\|b\|_B = 1$, $J(1/t,x;\vec{X}) \le 1$ and $|\Gamma_n(x)| \ge 2^{-1}K(t,\Gamma_n)$. Then

$$\|T_n(x\otimes b)\|_B \ge |\Gamma_n(x)|\|\|L_n b\|_B - \sum_{k=1}^{K_n}|\Gamma_{n,k}|\|L_{n,k}b\|_B \ge$$

$$\ge CK(1/t,\Gamma_n)\Big(1-\sum_{k=1}^{K_n}\frac{K(1/t,\Gamma_{n,k})}{K(1/t,\Gamma_n)}\Big) \ge CK(1/t,\Gamma_n)$$

This proves the lemma.

Application

The general theory above can be used to extend the examples of chapter 2 to vector-valued L_p- and Sobolev spaces. We indicate this extension in the case of interpolation of Sobolev spaces only.

Consider the Lebesgue space $L_p(B)$ of $B-$valued L_p-functions on the real line. Let $W_p^M(B)$ be the corresponding Sobolev space of all $a \in L_p(B)$ such that $D^m a \in L_p(B)$ for all $m \le M$. Since we shall restrict ouerselves to the case $1 < p < \infty$, we can use the following norm on W_p^M:

$$\|a\|_{W_p^M(B)} = \|a\|_{L_p(B)} + \||D|^M a\|_{L_p(B)}$$

Put $X_0 = L_p = L_p(\mathbb{C})$ and $X = X_1 = W_p^M = W_p^M(\mathbb{C})$ and define $x\otimes b = x\cdot b$. Then the couple $(A_0,A_1) = (L_p(B), W_p^M(B))$ can be written as $\vec{X}\vec{\otimes}B$. In this situation we can use the same quasi-linearization $\Lambda(t)$ as in section 2.8. We shall consider operators of the form

$$T_n(a) = (D^{m_n}a)(0) + \sum_{k<m_n}(D^k L_{n,k}a)(0) \tag{10}$$

where $L_{n,k}$ are bounded linear operators on B. In the formalism used above we have

$$T_n = \Gamma_{m_n}\otimes \mathrm{id}_B + \sum_{k<m_n}\Gamma_k\otimes L_{k,n}$$

where $\Gamma_k(x) = (D^k x)(0)$. (More general operators could be used in the pertubation term. See section 2.8.) We leave to the reader to formulate the generalization of theorem 2.5 in this situation. See also theorem 4.2 and 4.3 below.

4 Interpolation of domains and ranges

4.1 Domains of operators in Hilbert spaces

Let H be a Hilbert space with inner product (a, b). We shall consider an unbounded linear operator S, which is closed and densely defined. We let $H_1 = \mathrm{dom}(S)$ be the domain of S, equipped with the (semi-)norm

$$\|a\|_{H_1} = \|Sa\|_H$$

We shall now find a quasi-linearization of the couple (H, H_1) using duality maps as described in section 2.5. It is easy to see that $(D_H b)(x) = \mathrm{Re}(b, x)$ and $(D_{H_1} b)(x) = \mathrm{Re}(Sb, Sx)$. Therefore one could guess that the equation

$$\mathrm{Re}((b, x) + t^2(Sb, Sx)) = \mathrm{Re}(a, x),\ x \in H_1$$

defines a good approximation b of a. Assuming that $b \in \mathrm{dom}(S^*S)$ we get

$$(I + t^2 S^* S)b = a$$

Therefore we introduce the family of operators

$$\Lambda_S(t) = (I + t^2 S^* S)^{-1} \tag{1}$$

For each $t > 0$ this is a bounded linear operator on H which maps H into H_1. We have the following result which (essentially) is given in Löfström [8].

Theorem 4.1 *The family $\Lambda_S(t)$ is a quasi-linearization of the couple (H, H_1) where $H_1 = \mathrm{dom}(S)$. More precisely*

$$K_2(t, a; H, H_1) = \Big(\|a - \Lambda_S(t)a\|_H^2 + t^2\|S\Lambda_S(t)a\|_H^2)\Big)^{1/2} = \Big(a - \Lambda_S(t)a, a\Big)^{1/2}$$

Proof The second equality is a direct consequence of the definition of $\Lambda_S(t)$. Therefore it is enough to show that

$$(a - \Lambda_S(t)a, a) \le K_2(t, a; H, H_1)^2 \tag{2}$$

Assume that $a = a_0 + a_1$. Since $I - \Lambda_S(t) = t^2 S^* S \Lambda_S(t)$ we have

$$(a - \Lambda_S(t)a, a) = (a_0 - \Lambda_S(t)a_0, a_0) + t^2(\Lambda_S(t)Sa_1, Sa_1) + 2\mathrm{Re}(a_0 - \Lambda_S(t)a_0, a_1)$$

Now put $Q(t) = \sqrt{I - \Lambda_S(t)}, R = \sqrt{\Lambda_S(t)}$. Then $Q(t) = t\sqrt{S^*S}R(t)$ and

$$\begin{aligned} 2\mathrm{Re}(a_0 - \Lambda_S(t)a), a_1) &= 2\mathrm{Re}(Q(t)a_0, Q(t)a_1) = 2t\mathrm{Re}(R(t)a_0, \sqrt{S^*S}Q(t)a_1) \\ &\le 2t\|R(t)a_0\|\|\sqrt{S^*S}Q(t)a_1\| \le \|R(t)a_0\|^2 + t^2\|\sqrt{S^*S}Q(t)a_1\|^2 \\ &= (\Lambda_S(t)a_0, a_0) + t^2((I - \Lambda_S(t))Sa_1, Sa_1) \end{aligned}$$

It follows that

$$(a - \Lambda_S(t)a, a) \le (a_0, a_0) + t^2(Sa_1, Sa_1)$$

This implies (2).

Corollary 4.1 *Let S be a closed, densely defined operator and define $\Lambda_S(t)$ by means of (1). Then the interpolation space $K_{\theta,2}(H, \mathrm{dom}(S))$ is a Hilbert space with the inner product*

$$(a,b)_\theta = \int_0^\infty t^{-2\theta}\Big(a - \Lambda_S(t)a, b\Big)\frac{dt}{t}$$

Corollary 4.2 *Consider two closed and densely defined operators T and S and assume that $T \subseteq S$ (i.e. $\mathrm{dom}(T) \subseteq \mathrm{dom}(S)$ and $Ta = Sa$ for all $a \in \mathrm{dom}(T)$). Then $a \in K_\Phi(H, \mathrm{dom}(T))$ if and only if $a \in K_\Phi(H, \mathrm{dom}(S))$ and*

$$\Phi\Big((\Lambda_T(\cdot)a - \Lambda_S(\cdot)a, a)^{1/2}\Big) < \infty$$

Proof This is an immediate consequence of theoremm 4.1 since

$$K_2(t,a;H,\mathrm{dom}(S))^2 - K_2(t,a;H,\mathrm{dom}(T))^2 = (\Lambda_T(\cdot)a - \Lambda_S(\cdot)a, a)$$

4.2 Sobolev spaces on $\mathbb{R}_+$

Let L_{p+} denote the Lebesgue space on $\mathbb{R}_+ = (0,\infty)$ with norm

$$\|u\|_p = \Big(\int_0^\infty |u(\sigma)|^p d\sigma\Big)^{1/p}$$

Let W^M_{p+} be the corresponding Sobolev space defined by the norm

$$\|u\|_{p,M} = \|u\|_p + \|D^M u\|_p \quad , \quad \text{where } \; D = i\cdot\frac{d}{d\sigma} \quad \text{and} \quad 1 < p < \infty$$

To prepare for later applications, we shall work with $B-$valued functions, where B is an arbitrary Banach space. Thus we shall consider the couple (A_0, A_1), where $A_0 = L_{p+}(B)$ and $A_1 = W^M_{p+}(B)$. Clearly the setting of section 3.2 can be used here, with the tensor product defined by $(x\otimes b)(s) = x(s)\cdot b$ and with $X_0 = L_{p+}, X_1 = W^M_{p+}$.

The space A_1 described above can be considered as the domain in A_0 of the operator $S = D^M$. To describe the K_2-functional for the couple $(L_{2+}(B), W^M_{2+}(B))$, we can therefore use theorem 4.1. Thus we have to find the operators $\Lambda_S(t)$, which amounts to solving the equation $(I + t^2 S^* S)y = a$. Now it is easy to compute the adjoint S^*. It turns out that $S^* \subseteq D^M$, with domain given by the conditions $y^{(M-r-1)}(0) = 0, r = 0,1,\ldots,M-1$. Thus we shall consider the problem

$$\begin{cases} y + t^2 D^{2M} y = a & , \; y \in W^M_{2+} \\ y^{(2M-r-1)}(0) = 0 & , r = 0,1,\ldots,M-1 \end{cases} \tag{1}$$

Let $G_M = G_M(t,s,\sigma)$ be the Green function for (1). Then we put $\Lambda_M(t) = \Lambda_S(t)$ i.e.

$$(\Lambda_M(t)a)(s) = \int_0^\infty G_M(t,s,\sigma)a(\sigma)d\sigma \tag{2}$$

We shall now use the family $\Lambda_M(t)$ not only in the case $p = 2$ but also for general $p \in (1, \infty)$.

Lemma 4.1 *Put* $A_0 = L_{p+}(B), A_1 = W^M_{p+}(B)$, *where* $1 < p < \infty$, *and define* $\Lambda_M(t)$ *by the formulas (1) and (2). Then* $\Lambda_M(t)$ *is a quasi-linearization of the couple* (A_0, A_1).

Proof Let $G(t, s, \sigma)$ be the Green function for the problem $y + t^2 D^{2M} y = x$ where $y \in W_2^{2M}(\mathbb{R})$. Then

$$G(t, s, \sigma) = \frac{i}{2M\tau} \sum_{k=0}^{M-1} (-i\mu_k) e^{-\mu_k |s-\sigma|/\tau} \ , \quad \mu_k = e^{(2k+1-M)\pi i/2M}$$

where $\tau = t^{1/M}$.

Clearly G_M has the form

$$G_M(t, s, \sigma) = G(t, s, \sigma) + \frac{i}{2M\tau} \sum_{j,k=0}^{M-1} g_{j,k} e^{-(\mu_j s + \mu_k \sigma)/\tau}$$

where the coefficients $g_{j,k}$ should be choosen so that G_M satisfies the boundary conditions of the problem (1). Since the real parts of μ_k are all positive if $k < M$ we have that

$$|D_s^k D_\sigma^j G_M(t, s, \sigma)| \leq C\tau^{-k-j-1} e^{-\mu|s-\sigma|/\tau} \ , \ k + j < 2M$$

for some positive constant μ. Note also that $\tau G_M(t, s, \sigma)$ is a function of s/τ and σ/τ. Therefore

$$\|\Lambda_M(t)a\|_p \leq C\|a\|_p \ , \ t\|D^M \Lambda_M(t)a\|_p \leq C\|a\|_p$$

Using the differential equation for G_M, its symmetry and the boundary conditions we also get

$$a(s) - (\Lambda_M(t)a)(s) = \tau^{2M} \int_0^\infty D_\sigma^M G_M(t, s, \sigma) D_\sigma^M a(\sigma) d\sigma$$

This implies that

$$\|(I - \Lambda_M(t))a\|_p \leq Ct\|D^M a\|_p$$

Similarly

$$\|D^M \Lambda_M(t)a\|_p \leq C\|D^M a\|_p$$

This proves that $\Lambda_M(t)$ is a quasi-linearization of (A_0, A_1). The proof is complete. □

We shall now consider functionals $\tilde{\Gamma}$ on W^M_{p+} of the simple form

$$\tilde{\Gamma}(a) = (D^m a)(0) \tag{3}$$

Lemma 4.2 *Let* $\tilde{\Gamma}$ *be defined by (3). Then*

$$K(1/t, \tilde{\Gamma}; (L_{p+})', (W^M_{p+})') \cong t^{-\theta_m} \text{ , } \textit{where} \quad \theta_m = \frac{m + 1/p}{N}$$

Proof First we use Kolmogorofs inequality to show that

$$K(1/t, \tilde{\Gamma}; (L_{p+})', (W^M_{p+})') \leq Ct^{-\theta_m}$$

This is done as in the proof of lemma 2.4

To prove the converse inequality we note that

$$\tilde{\Gamma}(\Lambda_M(t)a) = \int_0^\infty a(\sigma)\bar{\phi}(t,\sigma)d\sigma,$$

where

$$\bar{\phi}(t,\sigma) = \int_0^\infty D^m_s G_M(t,s,\sigma)ds = \tau^{-m-1}\psi_m(\sigma/\tau).$$

Here $\psi = \psi_m$ solves the differential equation $\psi + D^{2M}\psi = 0$, with boundary conditions

$$\begin{cases} \psi^{(2M-r-1)}(0) & = & 0 \quad \text{if} \quad r = 0, 1, \ldots, M-1, r \neq m \\ \psi^{(2M-r-1)}(0) & = & 1 \quad \text{if} \quad r = m \end{cases} \tag{4}$$

Note that $\psi \in L_{q+}$ for all q. Now put

$$u(t) = \nu(t)^{-1}\Lambda_M(t)\phi(t)$$

where

$$\nu(t)^2 = \int_0^\infty |\phi(t,\sigma)|^2 d\sigma = \tau^{-2m-1}\int_0^\infty |\psi(\sigma)|^2 d\sigma = c\tau^{-2m-1}$$

Then $\tilde{\Gamma}(u(t)) = 1$ and

$$\|u(t)\|_p \leq C\nu(t)^{-1}\|\phi(t)\|_p \leq C\tau^{m+1/p} = Ct^{\theta_m}$$

$$\|D^M u(t)\|_p \leq C\nu(t)^{-1}\|D^M\phi(t)\|_p \leq Ct^{-1+\theta_m}$$

This implies that $J(t, u(t)) \leq Ct^{\theta_m}$ proving the lemma.

Corollary 4.3 *Assume that* $\gamma(a) = (D^k a)(0)$ *and put* $\tilde{\Gamma}(a) = (D^m a)(0)$. *Then* $\tilde{\Gamma}$ *dominates* γ *if* $k < m$.

Lemma 4.3 *Assume that* $m_1, \ldots, m_N$ *are different integers smaller than* M. *Then*

$$\tilde{\Gamma}_n(a) = (D^{m_n}a)(0)$$

defines a set of strongly independent functionals for the couple (L_{p+}, W^M_{p+}).

Proof As in the proof of lemma 4.2 we introduce the functions

$$\bar{\phi}_{m_n}(t,\sigma) = \int_0^\infty D_s^{m_n} G_M(t,s,\sigma)d\sigma = \tau^{-m_n-1}\psi_{m_n}(\sigma/\tau)$$

$$u_n(t) = \nu_n(t)^{-2}\Lambda_M(t)\phi_{m_n}(t)$$

$$\nu_n(t)^2 = \int_0^\infty |\phi_{m_n}(t,\sigma)|^2 d\sigma = c_n\tau^{-2m_n-1}$$

Then we have $\tilde{\Gamma}_n(u_n) = 1$. To prove that $|\det[\tilde{\Gamma}_n(u_k(t))]| \geq B$ for some positive constant B, we first note that

$$\tilde{\Gamma}_n(u_n(t)) = \nu_n(t)^{-2}\int_0^\infty \phi_{m_n}(t,\sigma)\bar{\phi}_{m_n}(t,\sigma)d\sigma$$

Therefore

$$\sum_{n,m}\tilde{\Gamma}_n(u_k(t))\frac{\nu_k(t)}{\nu_n(t)}z_k\bar{z}_n = \int_0^\infty \Big|\sum_n \frac{\psi_{m_n}(\sigma)}{\sqrt{c_n}}z_n\Big|^2 d\sigma$$

The right hand side is non-vanishing for all z_n with $\sum|z_n|^2 = 1$, if and only if $\psi_{m_1},\ldots,\psi_{m_N}$ are linearly independent. This is the case if all the orders $m_1,\ldots,m_N$ are different, since $D^{m_n}\psi_{m_k}$ vanishes at the origin except when $n = k$. The result now follows. □

We now return to the couple $(A_0, A_1) = (L_{p+}(B), W^M_{p+}(B))$. Define

$$(T_n a)(\cdot) = (D_s^{m_n}a)(0) + \sum_{k<m_n}(D_s^k L_{n,k}a)(0,\cdot) \tag{5}$$

where $L_{n,k}$ are bounded linear operators on B. Using the results of section 3.2 and corollary 3.2 we shall now prove

Theorem 4.2 *Let T_n be defined by (5) for $k = 1,\ldots,N$ and assume that the non-negative integers $m_1,\ldots,m_N$ are all different and less than M. Then*

$$K_{\theta,\rho}(L_{p+}(B), W^M_{p+}(B)\cap \ker(T_1)\cap\cdots\cap\ker(T_N))$$

consists of all $a \in K_{\theta,\rho}(L_{p+}(B), W^M_{p+}(B))$ for which

$$T_n a = 0 \quad \text{if } M\theta > m_n + 1/p \tag{6}$$

$$\Big(\int_0^1\Big(\frac{1}{t}\int_0^t \|T_n a(\cdot+\sigma)\|_B^p d\sigma\Big)^{\rho/p}\frac{dt}{t}\Big)^{1/\rho} < \infty \quad \text{if } M\theta = m_n + 1/p \tag{7}$$

Corollary 4.4 *Let T be the restriction of the operator D^M defined on the subspace of $W^M_{p+}(B)$ defined by the conditions $T_1 a = \ldots = T_N a = 0$. For a given $\theta \in (0,1)$ let T_θ be the restriction to the subspace defined by $T_n a = 0$ for all n with $\theta_n < \theta$. Then*

$$K_{\theta,\rho}(L_{p+}(B), \mathrm{dom}(T)) \cong \mathrm{dom}(T_\theta).$$

Proof of theorem 4.2. We need only to prove that if $a \in K_{\theta,\rho}(L_{p+}(B), W^M_{p+}(B))$ then the following conditions are equivalent

$$\Big(\int_0^\infty \|T_n(\Lambda_M(t)a\|^\rho_B \frac{dt}{t}\Big)^{1/\rho} < \infty \tag{8}$$

$$\Big(\int_0^1 \Big(\frac{1}{t}\int_0^t \|T_n a(\cdot+\sigma)\|^p_B d\sigma\Big)^{\rho/p} \frac{dt}{t}\Big)^{1/\rho} < \infty \tag{9}$$

This can be proved in essentially the same way as in the proof of theorem 2.5. In fact

$$T_n(\Lambda_M(t)a) = (\Lambda_M(t)y)(0) \tag{10}$$

where

$$y(s) = (D_s^{m_n} a)(s) + \sum_{k<m_n} (D_s^k L_{n,k} a)(s)$$

To see this we note that $(D^m \Lambda_M(t)a)(0) = \int_0^\infty a(\sigma)\phi_m(t,\sigma)d\sigma$, where $\phi_m(t,\sigma) = (D_s^m G_M)(t,0,\sigma) = \tau^{-m-1}\psi_m(\sigma/\tau)$, with ψ_m defined by (4). Partial integraton gives

$$(D^m \Lambda_M(t)a)(0) = \int_0^\infty h_\tau(\sigma) D^m a(\sigma) d\sigma$$

where $h_\tau(\sigma) = -\tau^{2M-m-1} D_\sigma^{2M-m}\psi_m(\sigma/\tau) = G(t,0,\sigma)$. This implies (10). Now we have only to prove that if $y \in K_{1/(Np),\rho}(L_{p+}(B), W^M_{p+}(B)) = K_{1/p,\rho}(L_{p+}(B), W^M_{p+}(B))$ then the following conditions are equivalent

$$\Big(\int_0^\infty \|\int_0^\infty h_\tau(\sigma)y(\sigma)d\sigma\|^\rho_B \frac{d\tau}{\tau}\Big)^{1/\rho} < \infty \tag{11}$$

$$\Big(\int_0^1 \Big(\frac{1}{t}\int_0^t \|y(\sigma)\|^p_B d\sigma\Big)^{\rho/p} \frac{dt}{t}\Big)^{1/\rho} < \infty \tag{12}$$

This follows however in the same way as in the proof of theorem 2.5. We leave the details to the reader. See also Löfström [9] and [12] .

4.3 Smooth boundary value problems

Consider a bounded domain Ω in $\mathbb{R}^d$ with C^M-boundary and let $\mathcal{B}_1, \ldots, \mathcal{B}_N$ be boundary operators of Neumann type, i.e.

$$\mathcal{B}_n = D_\nu^{m_n} + \sum_{k<m_n} \varphi_{n,k} D_\nu^k$$

Here D_ν denotes the interior normal derivative on the boundary $\delta\Omega$. We shall assume that the orders m_n are all different, that $\varphi_{n,k}$ are C^M-functions on the boundary. Let $W_{p,\mathcal{B}}^M(\Omega)$ be the Sobolev space of all $a \in W_p^M(\Omega)$, such that $\mathcal{B}_n a = 0$ on the boundary for $n = 1, \ldots, N$. We are interested in finding the interpolation space

$$K_{\theta,\rho}(L_p(\Omega), W_{p,\mathcal{B}}^M(\Omega))$$

Theorem 4.3 *Assume that Ω has C^M-boundary and let $\mathcal{B}_1, \ldots, \mathcal{B}_N$ be boundary operators of Neumann type on Ω with orders $m_1 < m_2 < \cdots m_N < M$. Then the interpolation space*

$$K_{\theta,\rho}(L_p(\Omega), W_{p,\mathcal{B}}^M(\Omega))$$

consists of all $a \in K_{\theta,\rho}(L_p(\Omega), W_p^M(\Omega))$ for which the following conditions hold

$$\mathcal{B}_n a = 0 \quad on \ \delta\Omega \quad if \ \ M\theta > m_n + 1/p$$

$$\Big(\int_0^\epsilon \Big(\frac{1}{t} \int_{\Omega(t)} |\mathcal{B}_n a(s)|^p ds \Big)^{\rho/p} \frac{dt}{t} \Big)^{1/\rho} < \infty \quad if \ \ M\theta = m_n + 1/p$$

Here $1 < p < \infty$ and $\Omega(t)$ is the set of points in Ω with distance at most t to the boundary of Ω. The number ϵ is some sufficiently small number.

Proof Since we have smooth boundary we can use local mappings and reduce the problem to the corresponding problem on $\mathbb{R}_+ \times \mathbb{R}^d$. Therefore we write

$$A_0 = L_p(\mathbb{R}_+ \times \mathbb{R}^d), \quad A_1 = W_p^M(\mathbb{R}_+ \times \mathbb{R}^d)$$

$$B_0 = L_p(\mathbb{R}^d), \quad B_1 = W_p^M(\mathbb{R}^d)$$

Then

$$A_0 = L_{p+}(B_0), \quad A_1 = W_{p+}^M(B_1)$$

Note that if $1 < p < \infty$ we have

$$A_1 \cong L_{p+}(B_1) \cap W_{p+}^M(B_0)$$

This relation will now be extended to the following one:

$$K_{\theta,\rho}(A_0, A_1) \cong K_{\theta,\rho}(L_{p+}(B_0), L_{p+}(B_1)) \cap K_{\theta,\rho}(L_{p+}(B_0), W_{p+}^M(B_0)) \qquad (1)$$

Let $W^M_{p+,\mathcal{B}}$ be the subspace of W^M_{p+} consisting of all a satisfying the boundary conditions $\mathcal{B}_1 a = \ldots = \mathcal{B}_N a = 0$ and put

$$A_{1,\mathcal{B}} = W^M_{p+,\mathcal{B}}$$

Then we also have

$$K_{\theta,\rho}(A_0, A_{1,\mathcal{B}}) \cong K_{\theta,\rho}(L_{p+}(B_0), L_{p+}(B_1)) \cap K_{\theta,\rho}(L_{p+}(B_0), W^M_{p+,\mathcal{B}}(B_0)) \qquad (2)$$

Using the results of the previous section we clearly get then the theorem from (1) and (2).

To prove (1) we first note that the couple (B_0, B_1) is quasi linearizable by means of the family $b \to G_t * b$, where G_t is defined in the beginning of section 2.8. Therefore the couple $(L_{p+}(B_0), L_{p+}(B_1))$ is quasi-linearizable by means of the family $\Lambda_0(t)$, where

$$(\Lambda_0(t)a)(s, \cdot) = (G_t * a(s))(\cdot)$$

In section 4.2 we constructed a quasi-linearization $\Lambda_M(t)$ of the couple $(L_{p+}(B_0), W^M_{p+}(B_0))$. We now claim that $\Lambda(t) = \Lambda_M(t)\Lambda_0(t)$ defines a quasi-linearization of the couple (A_0, A_1). To prove this we note that $I - \Lambda = -(I - \Lambda_M)(I - \Lambda_0) + (I - \Lambda_M)$. Thus

$$\|a - \Lambda(t)a\|_{A_0} \leq C\Big(\|a - \Lambda_M(t)a\|_{A_0} + \|a - \Lambda_0(t)a\|_{A_0}\Big)$$

But since $\Lambda = \Lambda_M - \Lambda_M(I - \Lambda_0)$ we also have

$$t\|\Lambda(t)a\|_{W^M_{p+}(B_0)} \leq C\Big(t\|\Lambda_M(t)a\|_{W^M_{p+}(B_0)} + \|a - \Lambda_0(t)a\|_{A_0}\Big)$$

and similarly

$$t\|\Lambda(t)a\|_{L_{p+}(B_1)} \leq Ct\|\Lambda_0(t)a\|_{L_{p+}(B_1)}$$

It follows that

$$\|a - \Lambda(t)a\|_{A_0} + t\|\Lambda(t)a\|_{A_1} \leq$$

$$\leq C\Big(\|a - \Lambda_M(t)\|_{A_0} + t\|\Lambda_M(t)a\|_{W^M_{p+}(B_0)} + \|a - \Lambda_0(t)\|_{A_0} + t\|\Lambda_0(t)a\|_{L_{p+}(B_1)}\Big)$$

This implies

$$K_{\theta,\rho}(A_0, A_1) \hookrightarrow K_{\theta,\rho}(L_{p+}(B_0), L_{p+}(B_1)) \cap K_{\theta,\rho}(L_{p+}(B_0), W^M_{p+}(B_0))$$

This gives (1), since the converse inclusion is obvious. The proof of (2) is quite the same. In fact, using the theory of section 3.1 and section 3.2 we can construct a quasi-linearization of the couple $(L_{p+}(B_0), W^M_{p+,\mathcal{B}}(B_0))$.

Remark The proof given above is similar to the one given in Löfström [9], but the present proof is considerably simpler. Cf Löfström [11]. For another approach to the interpolation of boundary value problems see Grisvard [4] and [5]. For non-smooth boundary value problems in the plane see Zolesio [18].

4.4 A theorem on ranges

Assume that $T:\vec{A}\to\vec{B}$. We shall consider the following relation

$$K_\Phi(T(A_0),T(A_1))\cong T(K_\Phi(A_0,A_1)) \tag{1}$$

First we note that the inclusion $T(K_\Phi(A_0,A_1))\subseteq K_\Phi(T(A_0),T(A_1))$ follows directely from the interpolation property. Therefore we have only to concentrate on the converse inclusion. Of course some conditions must be satisfied. One of them is

$$T(A_0)\cap T(A_1)=T(A_0\cap A_1) \tag{2}$$

First we assume that T is injective.

Lemma 4.4 *Suppose that T is injective on A_j, that $T(A_j)$ is closed in B_j, ($j=0,1$) and that (2) holds. Then (1) follows.*

Proof Put $T_j=T|A_j$ for $j=0,1$. By the open mapping theorem we can find a positive constant c such that

$$\|a_j\|_{A_j}\le c\|b_j\|_{B_j}\ \text{ if }\ b_j=T_j(a_j)\in T_j(A_j)$$

Thus assume that $b=b_0(t)+b_1(t)\in K_\Phi(T(A_0),T(A_1))$ where $b_j(t)=T_j(a_j(t))\in T_j(A_j)$. Then put $a=a_0+a_1$ where $a_j=a_j(1)$. Clearly $b=T_0(a_0)+T_1(a_1)=T_0(a_0(t))+T_1(a_1(t))$. Therefore $T_0(a_0-a_0(t))=T_1(a_1(t)-a_1)$ is an element of $T(A_0)\cap T(A_1)$. Thus (1) implies that $T_0(a_0-a_0(t))=T_1(a_1(t)-a_1)=T(\tilde{a})$ for some $\tilde{a}\in A_0\cap A_1$. By the injectivity we conclude that $a_0-a_0(t)=a_1(t)-a_1=\tilde{a}$. Hence $a_0(t)+a_1(t)=a_0+a_1=a$. It follows that

$$K(t,a;A_0,A_1)\le cK(t,b;T(A_0),T(A_1)),\ \text{ if }\ b=T(a)$$

which gives the inclusion. □

We now intend to reduce the general situation to the case considered in the lemma. To that end we shall write

$$N_j=\ker T_j,\ j=0,1,\ \text{ and }\ N=\ker T$$

Lemma 4.5 *A necessary and sufficient condition for (2) is that $N=N_0+N_1$.*

Proof The inclusion $N_0+N_1\subseteq N$ is obvious. Conversely assume that $n\in N$. Then $n=a_0+a_1$, where $a_j\in A_j$. Since $T(n)=0$ we have $T(a_0)=T(-a_1)$. By (2) we get that $T(a_0)=T(-a_1)=T(\tilde{a})$, where $\tilde{a}\in A_0\cap A_1$. Thus $T(a_0-\tilde{a})=T(-a_1-\tilde{a})=0$, so that $a_0-\tilde{a}\in N_0$ and $a_1+\tilde{a}\in N_1$. This implies that $n=a_0+a_1=(a_0-\tilde{a})+(a_1+\tilde{a})\in N_0+N_1$.

Conversely assume that $N = N_0 + N_1$ and let b be an element in $T(A_0) \cap T(A_1)$. Then $b = T(a_0) = T(a_1), a_j \in A_j$. Therefore $a_0 - a_1$, being an element of N, can be written as $a_0 - a_1 = -n_0 + n_1, n_j \in N_j$. It follows that $a_0 + n_0 = a_1 + n_1 = \tilde{a} \in A_0 \cap A_1$. Since $b = T(\tilde{a})$ we conclude that $b \in T(A_0 \cap A_1)$. This proves the lemma. $\square$

Now put

$$\Sigma = A_0 + A_1, \quad \bar{\Sigma} = \Sigma / N$$

where $\bar{\Sigma}$ is equipped with the quotient norm. Similarly we put

$$\bar{A}_j = A_j / N_j$$

again with quotient norm. Then $\bar{A}_j$ is contionuously embedded in $\bar{\Sigma}$, and if (2) holds

$$\bar{\Sigma} = \bar{A}_0 + \bar{A}_1$$

The proof of this fact uses the previous lemma. We leave the simple details to the reader. Next we define a linear mapping $\bar{T} : \bar{A}_0 + \bar{A}_1 \to B_0 + B_1$ by writing

$$\bar{T}(\bar{a}_0 + \bar{a}_1) = T_0(a_0) + T_1(a_1) \tag{3}$$

Lemma 4.6 *Assume that (2) holds. Then $\bar{T}$ is well defined and continuous and $\bar{T}_j = \bar{T}|\bar{A}_j$ is contionuous with the same norm as T_j. Moreover $\bar{T}_j$ is injective, $\bar{T}_j(\bar{A}_j) = T_j(A_j)$ and*

$$\bar{T}(\bar{A}_0 \cap \bar{A}_1) = \bar{T}_0(\bar{A}_0) \cap \bar{T}_1(\bar{A}_1) \tag{4}$$

As a consequence we have

$$K_\Phi(T(A_0), T(A_1)) = K_\Phi(\bar{T}(\bar{A}_0), \bar{T}(\bar{A}_1)) = \bar{T}(K_\Phi(\bar{A}_0, \bar{A}_1))$$

Proof If $\bar{a}_0 + \bar{a}_1 + \bar{a}'_0 = \bar{a}'_1$ we have $(a'_0 + a'_1) - (a_0 + a_1) \in N$. Thus lemma 4.5 implies that $(a'_0 + a'_1) - (a_0 + a_1) = n_0 + n_1$, where $n_j \in N_j$. Thus $T_0(a'_0 - a_0) = T(a'_0 - a_0 - n_0) = T(a_1 - a'_1 + n_1) = T_1(a_1 - a'_1)$ i.e. $T_0(a'_0) + T_1(a'_1) = T_0(a_0) + T_1(a_1)$. This proves that $\bar{T}$ is well defined.

To prove (4) , let $b \in \bar{T}_0(\bar{A}_0) \cap \bar{T}_1(\bar{A}_1)$. Then $b = T(a_0) = T(a_1)$ fore some $a_0 \in A_0$ and $a_1 \in A_1$. Thus (1) implies that $b \in T(A_0 \cap A_1)$, i.e. $b = T(a)$ where $a \in A_0 \cap A_1$. But then $b = \bar{T}(\bar{a}) \in \bar{T}(\bar{A}_0 \cap \bar{A}_1)$. This proves (4). The remaining part of the lemma is left to the reader.

We shall now give the main result of this section. First recall that a subcouple $\vec{N}$ is a complemented couple of $\vec{A}$, if there exists a projection $P : \Sigma(\vec{A}) \to \Sigma(\vec{N})$, such that $P : A_j \to N_j, j = 0, 1$.

Theorem 4.4 *Let T be a bounded linear operator from the couple $\vec{A}$ into the couple $\vec{B}$. Assume that the couple $T(A_j)$ is a closed subspace of B_j, and that the couple* $(\ker(T|A_0), \ker(T|A_1)$ *is a complemented couple in* (A_0, A_1). *Then*

$$K_\Phi(T(A_0), T(A_1)) = T(K_\Phi(A_0, A_1)) \tag{5}$$

Proof Using the notations introduced above we have $N_0 + N_1 = N$. Thus (2) holds. Therefore we are done if we can prove

$$\bar{T}(K_\Phi(\bar{A}_0, \bar{A}_1)) \subseteq T(K_\Phi(A_0, A_1)) \tag{6}$$

Assume that $b \in \bar{T}(\bar{a})$, where $\bar{a} = \bar{a}_0(t) + \bar{a}_1(t) \in K_\Phi(\bar{A}_0, \bar{A}_1)$. Now $P_j = P|A_j$ is a projection into N_j. Put $Q_j = I - P_j$ and $Q = I - P$. Then

$$\|Q_j a_j(t)\|_{A_j} \leq c\|\bar{a}_j(t)\|_{\bar{A}_j}$$

since $Q_j n_j = 0$ for each $n_j \in N_j$. Thus we have

$$K(t, Qa; A_0, A_1) \leq c(\|\bar{a}_0(t)\|_{\bar{A}_0} + t\|\bar{a}_1(t)\|_{\bar{A}_1})$$

It follows that $K(t, Qa) \leq cK(t, \bar{a})$. Thus $Qa \in K_\Phi(A_0, A_1)$. But

$$T(Qa) = T(Qa + Pa) = T(a) = \bar{T}(\bar{a}) = b$$

Therefore $b \in T(K_\Phi(A_0, A_1))$.

A counterexample

We shall now give an example of an injective operator T, such that the conclusion of theorem 4.4 (or lemma 4.4) fails. Clearly the relation (2) will not be satisfied in this example. (The example as well as the whole of this section is inspired by a construction by Berkson, Doust, Gillespie, personal communication by I. Doust. See [2].)

Consider the Hilbert space H_r of all sequences $a = (a_n)_{-\infty}^{\infty}$ such that

$$\|a\|_r = \left(\sum_{-\infty}^{\infty} (2^{nr}|a_n|)^2\right)^{1/2} < \infty$$

Here r is an arbitrary real number. Let S be the backwards shift operator

$$(Sa)_n = a_{n-1}$$

Lemma 4.7 *The operator $\lambda I - S$ is injective on H_r. Its range is the entire space H_r if and only if $|\lambda| \neq 2^r$.*

Proof Assume that $(\lambda I - S)a = 0, a \in H_r$. If $\lambda = 0$ we clearly have $a = 0$. If $\lambda \neq 0$ we have $a_n = \lambda^{-n} a_0$ for all n. Since

$$\|a\|_r = |a_0| \Big(\sum_{-\infty}^{\infty} \Big| \lambda^{-1} 2^r \Big|^{2n} \Big)^{1/2} < \infty$$

we must have $a_0 = 0$. Hence $a = 0$, proving that $\lambda I - S$ is injective.

Assume now that $|\lambda| > 2^r$. Suppose that $b \in H_r$ and that $b_n = 0$ for $n < -N$. Put $a_n = 0$ if $n < -N$, $a_{-N} = \lambda^{-1} b_{-N}$ and define a_n for $n > -N$ recursively by the formula $a_n = \lambda^{-1} b_n + \lambda^{-1} a_{n-1}$. Then $(\lambda I - S)a = b$ and

$$a_n = \frac{1}{\lambda} \sum_{k=0}^{n+N} \frac{1}{\lambda^k} b_{n-k} \ , \ n \geq -N$$

Thus

$$\|a\|_r \leq \frac{1}{|\lambda| - 2^r} \|b\|_r$$

If b is an arbitrary element of H_r, we let b^N be the sequence defined by $b_n^N = 0$ if $n < -N$ and $b_n^N = b_n$ if $n \geq -N$. Then $b^N \to b$ in H_r. Therefore the corresponding sequence a^N converges in H_r to an element a. Since $\lambda I - S$ is continuous on H_r we conclude that $(\lambda I - S)a = b$. It follows that $\lambda I - S$ is surjective if $|\lambda| > 2^r$.

In the case $|\lambda| < 2^r$, we can use the same idea, but instead of cutting away small indices, we cut away large ones. Thus if $b_n = 0, n > M$ we define $a_n = 0$ if $n \geq M$, $a_{M-1} = -b_M$ and $a_{n-1} = \lambda a_n - b_n$ if $n < M$. Then $(\lambda I - S)a = b$ and

$$a_n = - \sum_{k=0}^{M-n-1} \lambda^k b_{n+k+1} \ , n < M$$

This implies

$$\|a\|_r \leq \frac{1}{2^r - |\lambda|} \|b\|_r$$

From this we see that $\lambda I - S$ is surjective if $|\lambda| < 2^r$.

It remains to show that $\lambda I - S$ is not injective in the case $|\lambda| = 2^r$. Suppose the contrary. Let (c_n) be any sequence in H_0 and put $b_n = \lambda^{-n} c_n, n \geq 1$ and $b_n = 0, n \leq 0$. Then the relation $(\lambda I - S)a = b$ gives

$$\lambda^n a_n = a_0 + \frac{1}{\lambda} \sum_{k=1}^{n} c_k \ , \ n \geq 1 \text{ and } \lambda^n a_n = a_0 \ , \ n \leq 0$$

For each sequence $(c_n) \in H_0$ we would have $b \in H_r$. Therefore we $a \in H_r$, implying that $a_0 = 0$ and that

$$\sum_{n=0}^{\infty} (2^{nr} |a_n|)^2 = \sum_{n=0}^{\infty} |\frac{1}{\lambda} \sum_{k=1}^{n} c_k|^2 < \infty$$

for every $(c_n) \in H_0$. This is clearly impossible. Thus $\Lambda I - S$ is not surjective. The proof of the lemma is complete. $\square$

We shall now construct our counter-example. We chose $r_0 < r_1 < r_2$ and put $T = 2^{r_1} I - S$. Then we claim that

$$K_\Phi(T(H_{r_0}), T(H_{r_2})) \ncong T(K_\Phi(H_{r_0}, H_{r_1}))$$

if $\Phi = \Phi_{\theta,2}$ and $r_1 = (1-\theta)r_0 + \theta r_2$. This is easy to see since $T(H_{r_0}) = H_{r_0}, T(H_{r_2}) = H_{r_2}$ and $K_\Phi(H_{r_0}, H_{r_2}) \cong H_{r_1}$ but $T(H_{r_1}) \ncong H_{r_1}$. Note also that $T(H_{r_0}) \cap T(H_{r_2}) \ncong T(H_{r_0} \cap H_{r_2})$.

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