DIAGONALIZATION OF HOMOGENEOUS LINEAR OPERATORS IN BIORTHOGONAL WAVELET BASES

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ABSTRACT. We show how it is possible to diagonalize a certain class of homogeneous linear operators in a biorthogonal wavelet basis. Given a linear operator and a biorthogonal wavelet basis we construct a new biorthogonal wavelet basis such that by analyzing a function in the new basis and multiplying the wavelet coefficients by a scale dependent factor we get the wavelet coefficients of the transformed function in the original wavelet basis. Differentiation and integration, the Riesz potential and the Hilbert transform belong to this class of operators. Finally we generalize the method to several dimensions including non-separable bases.

1. Introduction

In a biorthogonal wavelet basis we have a wavelet ψ and a dual wavelet $\widetilde{\psi}$ such that any $f \in L^2(\mathbb{R})$ can be written

$$f = \sum_{j,l} \langle f, \widetilde{\psi}_{j,l} \rangle \psi_{j,l}.$$

For a given linear operator K we would like to expand Kf in this basis. It turns out that for a certain class of operators we can find a new biorthogonal wavelet basis, with wavelets ψ^{K} and $\widetilde{\psi}^{\mathrm{K}}$, so that $\langle Kf, \widetilde{\psi}_{j,l} \rangle = \kappa^j \langle f, \widetilde{\psi}^{\mathrm{K}}_{j,l} \rangle$. By diagonalization we thus mean that the wavelet coefficients of Kf in the original wavelet basis equals, up to a scale dependent factor, the wavelet coefficients of f in the new wavelet basis, i.e.

$$Kf = \sum_{j,l} \kappa^j \langle f, \widetilde{\psi}^{\mathrm{K}}_{j,l} \rangle \psi_{j,l}.$$

These new wavelets are simply given by $\widetilde{\psi}^{\mathrm{K}} = K^* \widetilde{\psi}$ and $\psi^{\mathrm{K}} = K^{-1}\psi$. Both the original and the new wavelet basis originate from a multiresolution analysis. This is important because it means that both the analysis of f in the new wavelet basis and the synthesis of Kf in the original wavelet basis come with fast algorithms. We will consider convolution operators,

$$Kf = k * f \quad \text{or} \quad \widehat{Kf}(\omega) = \widehat{k}(\omega)\widehat{f}(\omega),$$

that preserves the characteristics of a wavelet. It turns out that the operator K has to satisfy the homogeneity condition,

$$\kappa = \widehat{k}(\omega)/\widehat{k}(\omega/2) \quad \text{independent of } \omega.$$

Differentiation and integration, the Riesz potential and the Hilbert transform are examples of such operators.

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This method has already been used to diagonalize the derivative operator as noted by Lemarié and Daubechies [5]. It is also interesting to compare this method with the wavelet-vaguelette decomposition by Donoho [6]. He starts with an orthogonal wavelet basis, applies the operator K as above, and ends up with a biorthogonal basis. He does not construct a multiresolution analysis from which this new basis originates though, which is a drawback since we can not use the fast wavelet transform in the new system any more. By starting with a biorthogonal system, properly adapted to the operator, we construct a new biorthogonal multiresolution analysis. We do this by constructing a new pair of scaling functions for the new biorthogonal wavelets. In [1] Beylkin, Coifman, and Rokhlin presents a more general approach for sparse representation and fast computation of a very large class of linear operators. Since we restrict ourselves to linear operators satisfying the above homogeneity condition we are able to diagonalize the operators as well as finding more explicit representations of these.

The paper is organized as follows. First we give the definition of a biorthogonal multiresolution analysis and discuss the approximation properties of wavelets and scaling functions. The number of vanishing moments of the wavelet is especially important since this must be chosen to match the operator. This background is more or less based on an overview paper by Jawerth and Sweldens [8]. For a more comprehensive treatment of wavelets we refer to the book by Strang and Nguyen [9].

Next, we construct the new wavelet basis and find the homogeneity condition on the linear operator so that the diagonalization property holds. Then we consider the conditions on the wavelets and the operator for the new wavelets to be well defined and whether these are compactly supported or not. The crucial step in our construction is taken in section 4 were we describe how to construct the new scaling functions and thus the new multiresolution analysis. In section 5 we consider the special and important examples of differentiation, the Hilbert transform, and ramp filtering. We conclude by generalizing the technique to several dimensions for both separable and non-separable bases in section 6.

2. Wavelets

In this section we review basic wavelet theory mainly to fix the notation and we refer to [4] for proofs and more details.

2.1. Multiresolution Analysis.

A *multiresolution analysis* (MRA) of $L^2(\mathbb{R})$ is a sequence of closed subspaces V_j of $L^2(\mathbb{R})$, $j \in \mathbb{Z}$, with the following properties:

1. $V_j \subset V_{j+1}$,
2. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$,
3. $f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$,
4. $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_j V_j = \{0\}$,
5. There exists a *scaling function* $\varphi \in V_0$ such that the collection $\{\varphi(x-l) \colon l \in \mathbb{Z}\}$ is a Riesz basis of V_0.

It is immediate that the collection of functions $\{\varphi_{j,l} : l \in \mathbb{Z}\}$, with $\varphi_{j,l}(x) = 2^{j/2}\varphi(2^j x - l)$, is a Riesz basis of V_j. From the definition of the MRA it follows that the scaling function must satisfy the *dilation equation*

$$\varphi(x) = 2\sum_l h_l \varphi(2x-l) \quad \text{or} \quad \widehat{\varphi}(\omega) = H(\omega/2)\widehat{\varphi}(\omega/2), \tag{1}$$

where H is a 2π-periodic function defined by $H(\omega) = \sum_l h_l e^{-il\omega}$. If the scaling function belongs to $L^1(\mathbb{R})$ it is, under very general conditions, uniquely defined by the dilation equation and the normalization

$$\int \varphi(x)dx = 1 \quad \Leftrightarrow \quad \widehat{\varphi}(0) = 1.$$

We will always assume that this is the case and from (1) we then have $H(0) = 1$.

2.2. **Approximation.** The spaces V_j will be used to approximate functions. This will be done by defining appropriate projections onto these spaces. Since the union of all the V_j is dense in $L^2(\mathbb{R})$, we are guaranteed that any function can be approximated arbitrarily close by such projections.

If we want to write any polynomial of degree smaller than N as a linear combination of the scaling function and its translates then the scaling function must satisfy the Strang-Fix conditions,

$$\begin{aligned} &\widehat{\varphi}(0) = 1, \quad \text{and} \\ &\widehat{\varphi}^{(p)}(2\pi k) = 0 \quad \text{for} \quad k \neq 0,\ 0 \leq p < N. \end{aligned} \tag{2}$$

From (1) it follows that $H(\omega)$ must have a root of multiplicity N at $\omega = \pi$.

2.3. **Wavelets.** By W_j we will denote a space complementing V_j in V_{j+1}, i.e.

$$V_{j+1} = V_j \oplus W_j,$$

and consequently

$$\bigoplus_j W_j = L^2(\mathbb{R}).$$

A function ψ is a *wavelet* if the collection of functions $\{\psi(x-l) : l \in \mathbb{Z}\}$ is a Riesz basis of W_0. The collection of functions $\{\psi_{j,l} : j, l \in \mathbb{Z}\}$ is then a Riesz basis of $L^2(\mathbb{R})$. We define $\mathcal{P}_j$ as the projection onto V_j parallel to V_j^c and $\mathcal{Q}_j$ as the projection onto W_j parallel to W_j^c. A function f can now be written as

$$f(x) = \sum_j \mathcal{Q}_j f(x) = \sum_{j,l} \gamma_{j,l}\psi_{j,l}(x),$$

or if we start from a coarsest scale J as

$$f(x) = \mathcal{P}_J f(x) + \sum_{j=J}^{\infty} \mathcal{Q}_j f(x) = \sum_l \lambda_{J,l}\varphi_{J,l}(x) + \sum_{j=J}^{\infty}\sum_l \gamma_{j,l}\psi_{j,l}(x).$$

Below we will describe how to find the coefficients $\lambda_{j,l}$ and $\gamma_{j,l}$. Since the wavelet $\psi \in W_0 \subset V_1$

$$\psi(x) = 2\sum_l g_l \varphi(2x-l) \quad \text{or} \quad \widehat{\psi}(\omega) = G(\omega/2)\widehat{\varphi}(\omega/2), \tag{3}$$

where G is a 2π-periodic function defined by $G(\omega) = \sum_l g_l e^{-il\omega}$.

2.4. **Biorthogonal Wavelets.** In a biorthogonal MRA we have a scaling function φ, a wavelet ψ, a dual scaling function $\widetilde{\varphi}$, and a dual wavelet $\widetilde{\psi}$ such that any function f can be written

$$f(x) = \sum_l \langle f, \widetilde{\varphi}_{J,l} \rangle \varphi_{J,l}(x) + \sum_{j=J}^{\infty} \sum_k \langle f, \widetilde{\psi}_{j,l} \rangle \psi_{j,l}(x). \tag{4}$$

For this to be true the scaling functions and wavelets must satisfy the biorthogonality conditions

$$\langle \widetilde{\varphi}, \psi(\cdot - l) \rangle = \langle \widetilde{\psi}, \varphi(\cdot - l) \rangle = 0 \quad \text{and}$$
$$\langle \widetilde{\varphi}, \varphi(\cdot - l) \rangle = \langle \widetilde{\psi}, \psi(\cdot - l) \rangle = \delta_l.$$

Expressed in the filter functions $H, G, \widetilde{H}$, and $\widetilde{G}$ necessary conditions are given by

$$\begin{array}{lcl} \widetilde{H}(\omega)\overline{H(\omega)} & + & \widetilde{H}(\omega+\pi)\overline{H(\omega+\pi)} & = & 1 \\ \widetilde{G}(\omega)\overline{G(\omega)} & + & \widetilde{G}(\omega+\pi)\overline{G(\omega+\pi)} & = & 1 \\ \widetilde{G}(\omega)\overline{H(\omega)} & + & \widetilde{G}(\omega+\pi)\overline{H(\omega+\pi)} & = & 0 \\ \widetilde{H}(\omega)\overline{G(\omega)} & + & \widetilde{H}(\omega+\pi)\overline{G(\omega+\pi)} & = & 0 \end{array} \tag{5}$$

Now, if we define the *modulation matrix* M by

$$M(\omega) = \begin{bmatrix} H(\omega) & H(\omega+\pi) \\ G(\omega) & G(\omega+\pi) \end{bmatrix}, \tag{6}$$

and similarly for $\widetilde{M}$, then

$$\widetilde{M}(\omega)\overline{M(\omega)}^{\mathrm{T}} = I.$$

Cramer's rule now states that

$$\widetilde{H}(\omega) = \frac{\overline{G(\omega+\pi)}}{\overline{\Delta(\omega)}} \qquad \widetilde{G}(\omega) = -\frac{\overline{H(\omega+\pi)}}{\overline{\Delta(\omega)}}, \tag{7}$$

where $\Delta(\omega) = \det M(\omega)$.

When constructing wavelets one often starts by defining the low-pass filters H and $\widetilde{H}$. Then one defines G and $\widetilde{G}$ through equation (7) where $\Delta(\omega)$ is chosen equal to $e^{-i\omega}$.

2.5. **Vanishing Moments.** The moments of the wavelet are defined by

$$\mathcal{N}_p = \int x^p \psi(x) dx \quad \text{with} \quad p \in \mathbb{N},$$

and similarly for the dual wavelet. We recall that if the scaling function reproduces any polynomial of degree smaller than N then $H(\omega)$ has a root of multiplicity N at $\omega = \pi$. From (7) we see that this is equivalent to $\widetilde{G}(\omega)$ having a root of multiplicity N at $\omega = 0$. Since $\mathcal{F}(\varphi)(0) = 1$ this is also equivalent to $\mathcal{F}(\widetilde{\psi})(\omega)$ having a root of multiplicity N at $\omega = 0$, i.e. the dual wavelet has N vanishing moments. By a similar argument the wavelet ψ will have $\widetilde{N}$ vanishing moments if the dual scaling function reproduces polynomials of degree smaller than $\widetilde{N}$.

3. The New Wavelet Basis

3.1. **Diagonalization.** In a biorthogonal wavelet basis we have a wavelet ψ and a dual wavelet $\widetilde{\psi}$ such that any $f \in L^2(\mathbb{R})$ can be written

$$f = \sum_{j,l} \langle f, \widetilde{\psi}_{j,l} \rangle \psi_{j,l}.$$

For a given linear operator K we would like to expand Kf in this basis. We will consider convolution operators,

$$Kf = k * f \quad \text{or} \quad \widehat{Kf}(\xi) = \widehat{k}(\xi)\widehat{f}(\xi),$$

that preserves the characteristics of a wavelet. If we denote the adjoint of K by K^* we can write Kf as

$$Kf = \sum_{j,l} \langle Kf, \widetilde{\psi}_{j,l} \rangle \psi_{j,l} = \sum_{j,l} \langle f, K^* \widetilde{\psi}_{j,l} \rangle \psi_{j,l}.$$

We will now describe how the coefficients $\langle f, K^* \widetilde{\psi}_{j,l} \rangle$ can be calculated in a fast and numerically stable way by analyzing f in a new biorthogonal wavelet basis. We define this new basis by the relations

$$\widetilde{\psi}^{\mathrm{K}} = K^* \widetilde{\psi} \quad \text{and} \quad \psi^{\mathrm{K}} = K^{-1} \psi, \tag{8}$$

and in the Fourier domain we have

$$\widehat{\widetilde{\psi}^{\mathrm{K}}}(\omega) = \overline{\widehat{k}(\omega)} \widehat{\widetilde{\psi}}(\omega) \quad \text{and} \quad \widehat{\psi^{\mathrm{K}}}(\omega) = \frac{1}{\widehat{k}(\omega)} \widehat{\psi}(\omega). \tag{9}$$

For the moment we assume that the wavelets, ψ and $\widetilde{\psi}$, and the operator K are such that these new functions are well defined and below we will show that under certain assumptions these functions in fact form a biorthogonal wavelet basis. Our goal is to find a condition on the operator K such that the following relation holds

$$K^* \widetilde{\psi}_{j,l} = \overline{\kappa}_j \widetilde{\psi}^{\mathrm{K}}_{j,l},$$

since then the wavelet coefficients of Kf, $\langle f, K^* \widetilde{\psi}_{j,l} \rangle = \kappa_j \langle f, \widetilde{\psi}^{\mathrm{K}}_{j,l} \rangle$. The constant κ_j is independent of l since K^* is translation invariant. So for this to hold true K^* must be invariant under dyadic dilations, up to the constant κ_j. Let us therefore define the dyadic dilation operator D_j as

$$D_j f(x) = 2^{j/2} f(2^j x) \quad \text{or} \quad \widehat{D_j f}(\omega) = 2^{-j/2} \widehat{f}(2^{-j}\omega).$$

The dilation invariance of K^* means that

$$K^* D_j f = \overline{\kappa}_j D_j K^* f$$

or in the Fourier domain

$$\overline{\widehat{k}(\omega)} 2^{-j/2} \widehat{f}(2^{-j}\omega) = \overline{\kappa}_j 2^{-j/2} \overline{\widehat{k}(2^{-j}\omega)} \widehat{f}(2^{-j}\omega).$$

From this we arrive at the following condition on K

$$\kappa_j = \frac{\widehat{k}(\omega)}{\widehat{k}(2^{-j}\omega)} \quad \text{is constant.} \tag{10}$$

We observe that $\kappa_j = \kappa_1^j$ so if we let $\kappa = \kappa_1$, we have $\kappa_j = \kappa^j$. It is now also clear that the new functions ψ^{K} and $\widetilde{\psi}^{\mathrm{K}}$ are biorthogonal. To conclude, by analyzing f in the new wavelet basis and multiplying the wavelet coefficients by κ^j, we get the wavelet coefficients of Kf in the original wavelet basis:

$$Kf = \sum_{j,l} \kappa^j \langle f, \widetilde{\psi}^{\mathrm{K}}_{j,l} \rangle \psi_{j,l}, \tag{11}$$

and this is what we refer to as a diagonalization of the operator K. Examples of operators satisfying (10) are

1. Differentiation and integration
   $\widehat{k}(\omega) = (i\omega)^\alpha, \alpha \in \mathbb{Z}$. Here $\kappa = 2^\alpha$.
2. The Riesz potential
   $\widehat{k}(\omega) = |\omega|^\alpha, \alpha \in \mathbb{R}$. Here $\kappa = 2^\alpha$.
3. The Hilbert transform
   $\widehat{k}(\omega) = -i \operatorname{sgn} \omega$. Here $\kappa = 1$.

These three types of operators are essentially exhaustive. This follows if we consider continuous solutions of (10) for positive and negative ω separately since then we must have $\widehat{k}(\omega) = C\omega^\alpha$, for some constant C.

3.2. **Admissibility Conditions.** Let us now return to the question of whether the new wavelets as given by (9) are well defined. Given the operator K we would like to find suitable conditions on the original wavelets. First we assume that $\widehat{k}(\omega) = |\omega|^\alpha$ and that $\alpha > 0$, since any other choices of $\widehat{k}$ and α are treated similarly. We know that the new wavelets must have at least one vanishing moment each and from (9) we then see that ψ must have at least $\lfloor \alpha + 1 \rfloor$ vanishing moments. In this case there is no additional requirement on the number of vanishing moments on $\widetilde{\psi}$. On the other hand if $\widetilde{\psi} \in W^s(\mathbb{R})$ we realize that $\widetilde{\psi}^{\mathrm{K}} \in W^{s-\alpha}(\mathbb{R})$ so we must have $s \geq \alpha$ to have $\widetilde{\psi}^{\mathrm{K}} \in L^2(\mathbb{R})$. If $\alpha < 0$ the roles of ψ^{K} and $\widetilde{\psi}^{\mathrm{K}}$ are simply interchanged.

3.3. **Decay and Compact Support.** Finally, let us discuss the rate of decay of the new wavelets. In most applications we are interested in compactly supported wavelets and this corresponds to transfer functions that are finite impulse response filters. Non-compactly supported wavelets are also useful in practice if they have rapid decay.

If the original wavelets have compact support we know that their Fourier transforms are smooth. Looking at the definition of the new wavelets in the Fourier domain (9) we conclude that a necessary condition for the new wavelets to be compactly supported is that $\widehat{k}(\omega)$ is smooth for $\omega = 0$. This will only be the case when the operator is differentiation or integration. Indeed when this is the case it is obvious that the new wavelets are also compactly supported.

When $\alpha > 0$ is not an even integer we have, after a moment's consideration,

$$\widehat{\widetilde{\psi}^{\mathrm{K}}} \in C^{N-1+\lceil \alpha \rceil}(\mathbb{R}), \tag{12}$$

where N is the number of vanishing moments of the dual wavelet. Now, if the Fourier transform of $\widetilde{\psi}^{\mathrm{K}}$ and its derivatives were also in $L^1(\mathbb{R})$ the Riemann-Lebesgue lemma would give us the following estimate on the rate of decay of $\widetilde{\psi}^{\mathrm{K}}$

$$x^n \widetilde{\psi}^{\mathrm{K}}(x) \to 0 \quad \text{as} \quad |x| \to \infty, \qquad \text{for} \quad 0 \le n \le N - 1 + \lceil\alpha\rceil. \tag{13}$$

Let us therefore find out when this is actually the case. We assume that the dual wavelet $\widetilde{\psi} \in C^{\widetilde{M}}(\mathbb{R})$ and that it is compactly supported. It follows that $\widetilde{\psi}^{(m)} \in L^1$ for $0 \le m \le \widetilde{M}$ and

$$\omega^m \widehat{\widetilde{\psi}}(\omega) \to 0 \quad \text{as} \quad |\omega| \to \infty, \qquad \text{for} \quad 0 \le m \le \widetilde{M}.$$

From this it follows that there is a constant C such that

$$|\widehat{\widetilde{\psi}}(\omega)| \le C(1 + |\omega|)^{-\widetilde{M}}.$$

Actually, this holds for all derivatives of the Fourier transform of $\widetilde{\psi}$ since $x^p\widetilde{\psi}(x)$ and its $\widetilde{M}$ first derivatives are in $L^1(\mathbb{R})$ for all $p \in \mathbb{N}$, i.e.

$$|\frac{\partial^p}{\partial\omega^p}\widehat{\widetilde{\psi}}(\omega)| \le C(1 + |\omega|)^{-\widetilde{M}} \qquad \text{for} \quad p \in \mathbb{N}.$$

From the definition of $\widetilde{\psi}^{\mathrm{K}}$ and (12) we get

$$|\frac{\partial^p}{\partial\omega^p}\widehat{\widetilde{\psi}^{\mathrm{K}}}(\omega)| \le C(1 + |\omega|)^{\alpha-\widetilde{M}} \qquad \text{for} \quad 0 \le p \le N - 1 + \lceil\alpha\rceil.$$

That is, $\partial^p\widehat{\widetilde{\psi}^{\mathrm{K}}} \in L^1(\mathbb{R})$ if $\alpha - \widetilde{M} < -1$. So the decay estimate (13) holds when $\widetilde{M} > 1 + \alpha$. Rewriting the decay estimate we have thus arrived at

$$|\widetilde{\psi}^{\mathrm{K}}(x)| \le C(1 + |x|)^{1-N-\lceil\alpha\rceil} \qquad \text{if} \quad \widetilde{M} > 1 + \alpha \quad \text{where} \quad \widetilde{\psi} \in C^{\widetilde{M}}. \tag{14}$$

For the wavelets ψ and ψ^{K} a similar argument as above gives $\widehat{\psi^{\mathrm{K}}} \in C^{\widetilde{N}-1-\lfloor\alpha\rfloor}(\mathbb{R})$, where $\widetilde{N}$ is the number of vanishing moments of the wavelet. The rate of decay of the wavelet ψ^{K} is then given by

$$|\psi^{\mathrm{K}}(x)| \le C(1 + |x|)^{1-\widetilde{N}+\lfloor\alpha\rfloor} \qquad \text{if} \quad M > 1 - \alpha \quad \text{where} \quad \psi \in C^M. \tag{15}$$

We see that the number of vanishing moments of the original wavelets determines the rate of decay of the new wavelets. Again, if $\alpha < 0$ the roles of ψ^{K} and $\widetilde{\psi}^{\mathrm{K}}$ are interchanged.

4. The New Multiresolution Analysis

We will now describe how to associate a multiresolution analysis with the new biorthogonal wavelet basis. We will do this by defining a new pair of scaling functions. It is natural to try with

$$\widehat{\widetilde{\varphi}^{\mathrm{K}}}(\omega) = \overline{\widehat{\ell}(\omega)}\widehat{\widetilde{\varphi}}(\omega) \quad \text{and} \quad \widehat{\varphi^{\mathrm{K}}}(\omega) = \frac{1}{\widehat{\ell}(\omega)}\widehat{\varphi}(\omega), \tag{16}$$

where $\widehat{\ell}$ is an unknown function. Biorthogonality of the original scaling functions then implies biorthogonality of the new scaling functions φ^{K} and $\widetilde{\varphi}^{\mathrm{K}}$

$$\langle\widetilde{\varphi}^{\mathrm{K}}, \varphi^{\mathrm{K}}(\cdot - l)\rangle = \langle\widetilde{\varphi}, \varphi(\cdot - l)\rangle = \delta_l.$$

The scaling functions must also be biorthogonal to the wavelets

$$\langle\widetilde{\varphi}^{\mathrm{K}}, \psi^{\mathrm{K}}(\cdot - l)\rangle = \langle\widetilde{\psi}^{\mathrm{K}}, \varphi^{\mathrm{K}}(\cdot - l)\rangle = 0,$$

which, expressed in the filter functions, amounts to

$$\begin{array}{rcl} \widetilde{H}^{\mathrm{K}}(\omega)\overline{G^{\mathrm{K}}(\omega)} &+& \widetilde{H}^{\mathrm{K}}(\omega+\pi)\overline{G^{\mathrm{K}}(\omega+\pi)} = 0 \quad \text{and} \\ \widetilde{G}^{\mathrm{K}}(\omega)\overline{H^{\mathrm{K}}(\omega)} &+& \widetilde{G}^{\mathrm{K}}(\omega+\pi)\overline{H^{\mathrm{K}}(\omega+\pi)} = 0. \end{array}$$

From the dilation equation for $\widetilde{\varphi}$ and (16) we get

$$\widehat{\widetilde{\varphi}^{\mathrm{K}}}(\omega) = \frac{\overline{\widehat{\ell}(\omega)}}{\overline{\widehat{\ell}(\omega/2)}}\widetilde{H}(\omega/2)\widehat{\widetilde{\varphi}^{\mathrm{K}}}(\omega/2) = \widetilde{H}^{\mathrm{K}}(\omega/2)\widehat{\widetilde{\varphi}^{\mathrm{K}}}(\omega/2),$$

and by similar arguments we get the following expressions for the new filter functions

$$H^{\mathrm{K}}(\omega) = \frac{\widehat{\ell}(\omega)}{\widehat{\ell}(2\omega)}H(\omega), \qquad G^{\mathrm{K}}(\omega) = \frac{\widehat{\ell}(\omega)}{\widehat{k}(2\omega)}G(\omega),$$

$$\widetilde{H}^{\mathrm{K}}(\omega) = \frac{\overline{\widehat{\ell}(2\omega)}}{\overline{\widehat{\ell}(\omega)}}\widetilde{H}(\omega), \qquad \widetilde{G}^{\mathrm{K}}(\omega) = \frac{\overline{\widehat{k}(2\omega)}}{\overline{\widehat{\ell}(\omega)}}\widetilde{G}(\omega).$$

Biorthogonality between $\widetilde{\varphi}^{\mathrm{K}}$ and $\psi^{\mathrm{K}}(\cdot - l)$ is then equivalent to

$$\frac{\overline{\widehat{\ell}(2\omega)}}{\overline{\widehat{\ell}(\omega)}}\widetilde{H}(\omega)\overline{\frac{\widehat{\ell}(\omega)}{\widehat{k}(2\omega)}G(\omega)} + \frac{\overline{\widehat{\ell}(2\omega+2\pi)}}{\overline{\widehat{\ell}(\omega+\pi)}}\widetilde{H}(\omega+\pi)\overline{\frac{\widehat{\ell}(\omega+\pi)}{\widehat{k}(2\omega+2\pi)}G(\omega+\pi)} = 0.$$

Since $\widetilde{H}(\omega)\overline{G(\omega)} + \widetilde{H}(\omega+\pi)\overline{G(\omega+\pi)} = 0$ this is equivalent to

$$\frac{\widehat{\ell}(2\omega)}{\widehat{k}(2\omega)} = \frac{\widehat{\ell}(2\omega+2\pi)}{\widehat{k}(2\omega+2\pi)}.$$

This means that $\widehat{\ell}(\omega)$ must be chosen so that

$$m(\omega) = \frac{\widehat{k}(\omega)}{\widehat{\ell}(\omega)} \quad \text{is } 2\pi\text{-periodic.} \tag{17}$$

If we can find such an $\widehat{\ell}(\omega)$ all of the biorthogonality conditions will be satisfied. It is still not clear how we should define this function though. However, we have not considered the approximation properties, or the Strang-Fix conditions, of the new scaling functions. If we substitute (17) into (16) we get

$$\overline{m(\omega)}\widehat{\widetilde{\varphi}^{\mathrm{K}}}(\omega) = \overline{\widehat{k}(\omega)}\widehat{\widetilde{\varphi}}(\omega) \quad \text{and} \quad \widehat{k}(\omega)\widehat{\varphi^{\mathrm{K}}}(\omega) = m(\omega)\widehat{\varphi}(\omega). \tag{18}$$

Since we know that $\widehat{\widetilde{\varphi}^{\mathrm{K}}}(0) = \widehat{\varphi^{\mathrm{K}}}(0) = 1$ we must have

$$\frac{m(\omega)}{\widehat{k}(\omega)} \to 1 \text{ as } \omega \to 0.$$

From this we see that we must find a 2π-periodic function $m(\omega)$ that matches $\widehat{k}(\omega)$ at $\omega = 0$, i.e. it should have the same number of zeros at $\omega = 0$.

We make the following more or less canonical choice

$$m(\omega) = \widehat{k}(-i(e^{i\omega} - 1)), \tag{19}$$

since $-i(e^{i\omega}-1)=\omega+o(\omega)$ as $\omega\to 0$. This is indeed a natural choice because the determinants of the modulation matrices of the original and new system will only differ by a multiplicative constant

$$\Delta(\omega)=\widehat{k}(i)\kappa^2\Delta^{\mathrm{K}}(\omega). \tag{20}$$

This choice implies that $m(\omega)$ is a sort of discretized version of $\widehat{k}(\omega)$. From (18) we then see, via the Strang-Fix conditions, that the approximation properties of the new scaling functions are exactly related, as they should be, to the number of vanishing moments of the new wavelets. If we write the Fourier series of $m(\omega)$ as

$$m(\omega)=\sum_l m_l e^{-il\omega}$$

we see that

$$K^*\widetilde{\varphi}(x)=\sum_l \overline{m_l}\widetilde{\varphi}^{\mathrm{K}}(x+l). \tag{21}$$

For K being the derivative operator this becomes

$$\widetilde{\varphi}'(x)=\widetilde{\varphi}^{\mathrm{K}}(x)-\widetilde{\varphi}^{\mathrm{K}}(x-1),$$

i.e. differentiation in the original system corresponds to a finite difference in the new system.

At this point we have constructed a new biorthogonal multiresolution analysis with the wavelets ψ^{K} and $\widetilde{\psi}^{\mathrm{K}}$ and with the scaling functions φ^{K} and $\widetilde{\varphi}^{\mathrm{K}}$. This means that we can decompose any function in this new basis using the fast wavelet transform. We also know the relation between the wavelet coefficients of f and Kf in the new and original basis, respectively. In a numerical computation we always stop the decomposition at a coarsest scale and we are thus also interested in finding a relation between the scaling function coefficients of Kf and f. Expanding Kf in the original basis we get

$$Kf(x)=\sum_l\langle f,K^*\widetilde{\varphi}_{J,l}\rangle\varphi_{J,l}(x)+\sum_{j=J}^{\infty}\sum_l\langle f,K^*\widetilde{\psi}_{j,l}\rangle\psi_{j,l}(x).$$

Using (21) it is easy to verify that

$$\langle f,K^*\widetilde{\varphi}_{j,l}\rangle=\kappa^j\sum_n m_n\langle f,\widetilde{\varphi}^{\mathrm{K}}_{j,l-n}\rangle. \tag{22}$$

We note that this formula can be seen as a discretized version of the operator K acting on the subspace V_j.

Summary. Before looking at some examples we summarize our results. Given a function f such that

$$f(x)=\sum_k\lambda_{J,l}\varphi^{\mathrm{K}}_{J,l}(x)+\sum_{j=J}^{\infty}\sum_k\gamma_{j,l}\psi^{\mathrm{K}}_{j,l}(x),$$

we can find the expansion of Kf

$$Kf(x)=\sum_k\lambda^{\mathrm{K}}_{J,l}\varphi_{J,l}(x)+\sum_{j=J}^{\infty}\sum_k\gamma^{\mathrm{K}}_{j,l}\psi_{j,l}(x),$$

by the relations

$$\gamma^{\mathrm{K}}_{j,l} = \kappa^j \gamma_{j,l},$$
$$\lambda^{\mathrm{K}}_{J,l} = \kappa^J \sum_n m_n \lambda_{J,l-n}.$$

With our choice of $m(\omega)$ the new filter functions become

$$H^{\mathrm{K}}(\omega) = \frac{\widehat{k}(e^{i\omega}+1)}{\kappa} H(\omega), \qquad G^{\mathrm{K}}(\omega) = \frac{1}{\kappa \widehat{k}(-i(e^{i\omega}-1))} G(\omega),$$
$$\widetilde{H}^{\mathrm{K}}(\omega) = \frac{\overline{\kappa}}{\overline{\widehat{k}(e^{i\omega}+1)}} \widetilde{H}(\omega), \qquad \widetilde{G}^{\mathrm{K}}(\omega) = \overline{\kappa}\overline{\widehat{k}(-i(e^{i\omega}-1))}\widetilde{G}(\omega).$$

Remark. A more elegant way to derive the choice of $m(\omega)$ was pointed out to us by Patrik Andersson. We start with a biorthogonal system and first find a new pair of scaling functions as follows. Begin with the identity

$$\frac{e^{i\omega}-1}{i\omega} = \prod_{j=1}^{\infty} \frac{e^{i2^{-j}\omega}+1}{2}.$$

Our operators satisfy $\widehat{k}(\omega_1\omega_2) = \widehat{k}(\omega_1)\widehat{k}(\omega_2)$ and if we apply $\widehat{k}$ to the left and right hand side of the identity we get

$$\frac{\widehat{k}(-i(e^{i\omega}-1))}{\widehat{k}(\omega)} = \prod_{j=1}^{\infty} \frac{\widehat{k}(e^{i2^{-j}\omega}+1)}{\kappa},$$

since $\widehat{k}(2) = \kappa \widehat{k}(1)$ and where we have assumed that $\widehat{k}(1) = 1$. By a repeated application of (1) we can write the scaling function as the infinite product

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega),$$

and it follows that

$$\frac{\widehat{k}(-i(e^{i\omega}-1))}{\widehat{k}(\omega)} \widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} \frac{\widehat{k}(e^{i2^{-j}\omega}+1)}{\kappa} H(2^{-j}\omega).$$

That is, if we define a new scaling function by

$$\widehat{\varphi^{\mathrm{K}}}(\omega) = \frac{\widehat{k}(-i(e^{i\omega}-1))}{\widehat{k}(\omega)} \widehat{\varphi}(\omega),$$

it will be associated with the filter

$$H^{\mathrm{K}}(\omega) = \frac{\widehat{k}(e^{i\omega}+1)}{\kappa} H(\omega),$$

and this is exactly the filter we got with our choice of $m(\omega)$ above. Similarly, we get the same filter for the dual scaling function. Now we can define the new wavelets from equation (7) and instead of the standard choice of the determinant of the modulation matrix we define Δ^{K} through equation (20). Then it is easy to verify that new wavelets are the same as before.

5. Examples

5.1. **Differentiation.** For the derivative operator we have

$$K = \frac{d}{dx}, \quad K^* = -\frac{d}{dx}, \quad K^{-1} = \int_{-\infty}^{x} dy,$$

and the new wavelets are thus given by

$$\widetilde{\psi}^{\mathrm{K}}(x) = -\widetilde{\psi}'(x), \qquad \psi^{\mathrm{K}}(x) = \int_{-\infty}^{x} \psi(y)dy.$$

Since $\widehat{k}(\omega) = i\omega$ we have $m(\omega) = e^{i\omega} - 1$ and

$$\widetilde{\varphi}'(x) = \widetilde{\varphi}^{\mathrm{K}}(x) - \widetilde{\varphi}^{\mathrm{K}}(x-1).$$

We also note that the new wavelets and scaling functions are compactly supported if the original wavelets and scaling functions are.

5.2. **The Ramp Filter.** We consider the Riesz potential operator with $\alpha = 1$ as an example, i.e. the ramp filter. Now

$$\widehat{Kf}(\omega) = |\omega|\widehat{f}(\omega) \quad \text{and} \quad \widehat{K^{-1}f}(\omega) = \frac{1}{|\omega|}\widehat{f}(\omega).$$

Since $\widehat{k}(\omega) = |\omega|$ we have

$$m(\omega) = |e^{-i\omega} - 1| \quad \text{and} \quad m_l = \frac{4}{\pi(1-4l^2)}.$$

In this case $\widehat{k}(\omega)$ is not smooth for $\omega = 0$ so the new wavelets and scaling functions will not have compact support. If we start with a biorthogonal basis where the wavelets have several vanishing moments the new wavelets will decay fast though.

5.3. **The Hilbert Transform.** In the case of the Hilbert transform we have

$$\widehat{Kf}(\omega) = -i\,\mathrm{sgn}\,\omega\;\widehat{f}(\omega) \quad \text{or} \quad Kf(x) = \frac{1}{\pi}\;\text{p.v.}\int \frac{f(y)}{x-y}dy.$$

We note that $\widehat{k}(\omega) = -i\;\mathrm{sgn}\,\omega = -i\frac{w}{|\omega|}$ so

$$m(\omega) = -\frac{e^{-i\omega}-1}{|e^{-i\omega}-1|} \quad \text{and} \quad m_l = \frac{1}{\pi(l+1/2)}.$$

We make the interesting observation that we have a convolution with $m_l = 1/(\pi(l+1/2))$ acting on the V_j spaces, i.e. a discretized version of the Hilbert transform. Just as for the ramp filter the new wavelets and scaling functions will not have compact support but if the original wavelets have several vanishing moments the new wavelets will decay fast. See figure 1 and 2 for an example where the original scaling functions and wavelets where chosen from the 6/10 factorization of the maxflat Daubechies halfband filters.

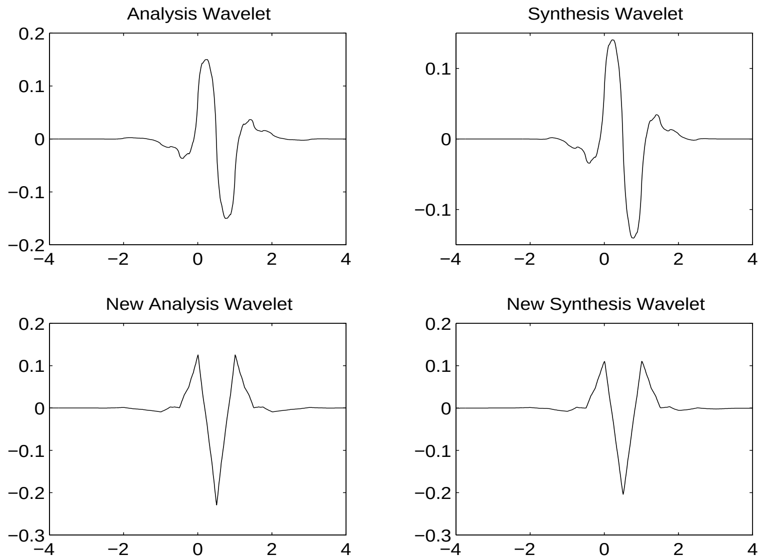

FIGURE 1. The wavelets for the Hilbert transform, (see sect. 5.3).

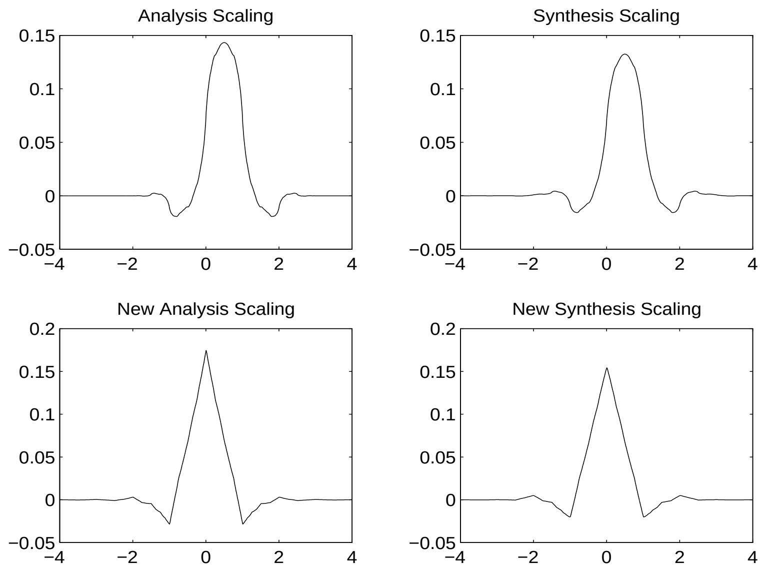

FIGURE 2. The scaling functions for the Hilbert transform.

6. Diagonalization in Several Dimensions

In this section we will generalize our method to higher dimensional spaces and convolution operators. If both the basis and operator are separable it is easy and straightforward to extend our results of the previous sections. What is interesting though is that we can generalize the diagonalization technique to the case of non-separable wavelet bases.

6.1. Separable Bases and Operators. For a separable multidimensional wavelet basis in $\mathbb{R}^n$ it is easy to see that the previous ideas can be generalized in a straightforward way if the convolution kernel k is also separable, i.e. if

$$\widehat{k}(\omega) = \widehat{k}_1(\omega_1) \cdots \widehat{k}_n(\omega_n), \qquad \text{where} \quad \omega \in \mathbb{R}^n,$$

and all the k_i:s satisfies the diagonalization condition (10). In a two-dimensional separable wavelet basis we form the scaling function Φ and the wavelets Ψ_ν by tensor products of a one-dimensional scaling function φ and wavelet ψ

$$\Phi = \varphi \otimes \varphi, \quad \Psi_1 = \varphi \otimes \psi, \quad \Psi_2 = \psi \otimes \varphi, \quad \text{and} \quad \Psi_3 = \psi \otimes \psi.$$

Similarly, we define a dual scaling function $\widetilde{\Phi}$ and dual wavelets $\widetilde{\Psi}_\nu$ if we want a biorthogonal two-dimensional wavelet basis. Starting from such a basis we define the new wavelets analogously with the one-dimensional case

$$\widetilde{\Psi}_\nu^{\mathrm{K}} = K^* \widetilde{\Psi}_\nu \quad \text{and} \quad \Psi_\nu^{\mathrm{K}} = K^{-1} \Psi_\nu.$$

We will form the new scaling functions by defining new one-dimensional scaling functions for each coordinate direction, $i = 1, 2$,

$$\widehat{\widetilde{\varphi}_i^{\mathrm{K}}}(\omega_i) = \overline{\widehat{\ell}_i(\omega_i)} \widehat{\widetilde{\varphi}}(\omega_i) \quad \text{and} \quad \widehat{\varphi_i^{\mathrm{K}}}(\omega_i) = \frac{1}{\widehat{\ell}_i(\omega_i)} \widehat{\varphi}(\omega_i),$$

where each $\widehat{\ell}_i(\omega)$ is derived from $\widehat{k}_i$ trough (17) and (19) as before. The new scaling functions Φ^{K} and $\widetilde{\Phi}^{\mathrm{K}}$ are formed by taking tensor products of the new one-dimensional scaling functions

$$\widetilde{\Phi}^{\mathrm{K}} = \widetilde{\varphi}_1^{\mathrm{K}} \otimes \widetilde{\varphi}_2^{\mathrm{K}} \quad \text{and} \quad \Phi^{\mathrm{K}} = \varphi_1^{\mathrm{K}} \otimes \varphi_2^{\mathrm{K}}.$$

6.2. **Non-separable Bases.** It is possible to construct non-separable wavelet bases in several dimensions although fairly few such bases have actually been constructed. Non-separable bases are of interest in for example image processing since they are more isotropic than a separable basis, which is strongly oriented along the coordinate axes. For a separable basis in $\mathbb{R}^n$ the underlying structure is the integer lattice $\mathbb{Z}^n$ and the dilation is the same along all coordinate axes. This is not the case for a non-separable basis where we have some other lattice and/or another dilation. Two examples for which non-separable wavelet bases have been constructed are the hexagonal and the Quincunx lattice. Cohen and Daubechies [2] have constructed symmetric biorthogonal wavelets with compact support and arbitrarily high regularity on the Quincunx lattice. The two-dimensional biorthogonal wavelets on the hexagonal lattice by Cohen and Schlenker [3] have symmetry under 30^0 rotations, compact support and some regularity. For a class of lattices with certain tiling properties Strichartz [10] constructs n-dimensional orthogonal wavelets with arbitrarily high regularity but not with compact support. In a forthcoming paper Jawerth and Mao [7] present a general method for the construction of wavelets on lattices.

6.3. **Lattices.** For a standard separable basis in $\mathbb{R}^n$ the underlying structure is the integer lattice $\mathbb{Z}^n$ and the wavelets are generated from $2^n - 1$ mother wavelets ψ_ν, $\nu = 1, 2, \ldots, 2^n - 1$,

$$\psi_{\nu,j,\gamma}(x) = 2^{jn/2} \psi_\nu(D^j x - \gamma), \qquad j \in \mathbb{Z},\ \gamma \in \mathbb{Z}^n,$$

where $D = 2I$ is the dilation matrix.

To construct non-separable wavelet bases we start with a lattice $\mathbf{\Gamma}$ and dilation matrix D. A *lattice* in $\mathbb{R}^n$ is defined as $\mathbf{\Gamma} = \Gamma \mathbb{Z}^n$, where Γ is a nonsingular n-by-n matrix. With

$$\Gamma = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

we get the hexagonal lattice in $\mathbb{R}^2$, see figure 3. The integer lattice has $\Gamma = I$ and so has the Quincunx lattice. As we will see it is the dilation matrix that distinguishes these lattices from each other. Given a lattice $\mathbf{\Gamma}$ the *dilation matrix* D has to satisfy the requirement $D\mathbf{\Gamma} \subset \mathbf{\Gamma}$. Moreover, all eigenvalues of D must have modulus greater then one so that we are expanding in all directions. We will refer to the lattice $D\mathbf{\Gamma} = D\Gamma\mathbb{Z}^n$ as the *subsampling lattice* of $\mathbf{\Gamma}$. The subsampling lattice of the Quincunx lattice, see figure 4, is defined by

$$D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

From the conditions on D it follows that m is an integer and if we let $m = |\det D|$ we will see that we get $m-1$ different mother wavelets. For a detailed treatment of the interrelation between the matrices Γ and D, and the wavelet construction see [7].

6.4. **Multiresolution Analysis.** We define a multiresolution analysis of $L^2(\mathbb{R}^n)$, associated with the lattice $\mathbf{\Gamma}$ and the dilation matrix D, as a sequence of closed subspaces V_j of $L^2(\mathbb{R}^n)$, $j \in \mathbb{Z}$, such that

1. $V_j \subset V_{j+1}$,
2. $f(x) \in V_j \Leftrightarrow f(Dx) \in V_{j+1}$,
3. $f(x) \in V_0 \Leftrightarrow f(x+\gamma) \in V_0,\ \forall\gamma \in \mathbf{\Gamma}$,
4. $\bigcup_j V_j$ is dense in $L^2(\mathbb{R}^n)$ and $\bigcap_j V_j = \{0\}$,
5. There exists a scaling function $\varphi \in V_0$ such that the collection $\{\varphi(x-\gamma) : \gamma \in \mathbf{\Gamma}\}$ is a Riesz basis of V_0.

It is immediate that the collection of functions $\{\varphi_{j,\gamma} : \gamma \in \mathbf{\Gamma}\}$, with $\varphi_{j,\gamma}(x) = m^{j/2}\varphi(D^j x - \gamma)$, is a Riesz basis of V_j. As usual, the scaling function satisfies a dilation equation

$$\varphi(x) = m \sum_{\gamma\in\mathbf{\Gamma}} h_\gamma \varphi(Dx - \gamma), \tag{23}$$

and the transfer function of h_γ is defined by $H(\omega) = \sum_{\gamma\in\mathbf{\Gamma}} h_\gamma e^{-i\gamma\cdot\omega}$. To proceed we have to be able to do Fourier analysis on the lattice group $\mathbf{\Gamma}$. We then need to define the dual group of $\mathbf{\Gamma}$. The *dual lattice* of $\mathbf{\Gamma}$ is defined as

$$\mathbf{\Gamma}^* = \{\gamma^* \in \mathbb{R}^n : \gamma\cdot\gamma^* \in \mathbb{Z},\ \ \forall\gamma \in \mathbf{\Gamma}\}, \tag{24}$$

and the dual group of $\mathbf{\Gamma}$ is then the quotient group $\mathbb{R}^n/2\pi\mathbf{\Gamma}^*$. From this definition it easy to verify that $\mathbf{\Gamma}^* = \Gamma^{-\mathrm{T}}\mathbb{Z}^n$, where $\Gamma^{-\mathrm{T}}$ denotes the transpose of the inverse of the matrix Γ. Since $D^{\mathrm{T}}\gamma^*\cdot\gamma = \gamma^*\cdot D\gamma$ for every $\gamma \in \mathbf{\Gamma}$ and $\gamma^* \in \mathbf{\Gamma}^*$ we have $D^{\mathrm{T}}\mathbf{\Gamma}^* \subset \mathbf{\Gamma}^*$. From the definition of H we notice that for any $\gamma^* \in \mathbf{\Gamma}^*$

$$H(\omega + 2\pi\gamma^*) = \sum_{\gamma\in\mathbf{\Gamma}} h_\gamma e^{-i\gamma\cdot\omega} e^{-i2\pi\gamma\cdot\gamma^*} = \sum_{\gamma\in\mathbf{\Gamma}} h_\gamma e^{-i\gamma\cdot\omega} = H(\omega),$$

since $\gamma\cdot\gamma^* \in \mathbb{Z}$ for any $\gamma \in \mathbf{\Gamma}$. In other words, $H(\omega)$ is $2\pi\mathbf{\Gamma}^*$-periodic.

For the standard integer lattice $\mathbb{Z}^n$ the dual lattice is also $\mathbb{Z}^n$. The dual lattice of the hexagonal lattice is the hexagonal lattice rotated 30^0 and it is shown in figure 3, where also the *Voronoi cell* of $2\pi\mathbf{\Gamma}^*$ around 0, i.e. the set of points closer to 0 than any other $\gamma^* \in 2\pi\mathbf{\Gamma}^*$, is shown. This is the smallest possible set on which $H(\omega)$ is completely determined since it is a representative of the quotient group $\mathbb{R}^n/2\pi\mathbf{\Gamma}^*$.

Having defined a multiresolution analysis let us now introduce wavelets. In the separable case we need $2^n - 1$ different W_j spaces, and the same number of mother wavelets, to complement V_j in V_{j+1}. In the non-separable case we need $m-1$ different mother wavelets which follows from the fact that the order of the quotient group $\mathbf{\Gamma}/D\mathbf{\Gamma}$ is m. Each mother wavelet ψ_ν will satisfy a scaling equation

$$\psi_\nu(x) = m\sum_{\gamma\in\mathbf{\Gamma}} g_{\nu,\gamma}\varphi(Dx-\gamma), \quad \nu = 1,\ldots,m-1,$$

where the transfer function of each $g_{\nu,\gamma}$ is defined by $G_\nu(\omega) = \sum_{\gamma\in\mathbf{\Gamma}} g_{\nu,\gamma}e^{-i\gamma\cdot\omega}$. The wavelet basis is obtained by taking D-dilates and $\mathbf{\Gamma}$-translates of these mother wavelets $\psi_{\nu,j,\gamma}(x) = m^{j/2}\psi_\nu(D^jx-\gamma)$ for $j\in\mathbb{Z}$ and $\gamma\in\mathbf{\Gamma}$. Let us find the conditions on the filters in order to obtain an orthogonal MRA. First φ and its $\mathbf{\Gamma}$-translates have to be orthogonal

$$\begin{aligned}
\delta_{0,\gamma} = \langle\varphi,\varphi(\cdot-\gamma)\rangle &= \frac{1}{2\pi}\int|\widehat{\varphi}(\omega)|^2e^{-i\gamma\cdot\omega}d\omega \\
&= \frac{1}{2\pi}\sum_{\gamma^*\in\mathbf{\Gamma}^*}\int_{V(2\pi\gamma^*)}|\widehat{\varphi}(\omega)|^2e^{-i\gamma\cdot\omega}d\omega \\
&= \frac{1}{2\pi}\sum_{\gamma^*\in\mathbf{\Gamma}^*}\int_{V(0)}|\widehat{\varphi}(\omega+2\pi\gamma^*)|^2e^{-i\gamma\cdot\omega}d\omega \\
&= \frac{1}{2\pi}\int_{V(0)}\sum_{\gamma^*\in\mathbf{\Gamma}^*}|\widehat{\varphi}(\omega+2\pi\gamma^*)|^2e^{-i\gamma\cdot\omega}d\omega
\end{aligned}$$

where $V(2\pi\gamma^*)$ are the Voronoi cells of the lattice $2\pi\mathbf{\Gamma}^*$. This gives us a necessary orthogonality condition on the scaling function

$$\sum_{\gamma^*\in\mathbf{\Gamma}^*}|\widehat{\varphi}(\omega+2\pi\gamma^*)|^2 = \frac{1}{|V(0)|}. \tag{25}$$

We will now find out the corresponding condition on the filter function H. In the Fourier domain the dilation equation (23) becomes

$$\widehat{\varphi}(\omega) = H(D^{-\mathrm{T}}\omega)\widehat{\varphi}(D^{-\mathrm{T}}\omega). \tag{26}$$

Let $\mathbf{\Gamma}_0^* = \{r_1^*,\ldots,r_m^*\}$ be a representative of the quotient group $\mathbf{\Gamma}^*/D^{\mathrm{T}}\mathbf{\Gamma}^*$ so that every $\gamma^*\in\mathbf{\Gamma}^*$ can be written uniquely as $\gamma^* = \gamma_0^*+\gamma_1^*$, where $\gamma_0^*\in\mathbf{\Gamma}_0^*$

and $\gamma_1^* \in D^{\mathrm{T}}\mathbf{\Gamma}^*$. We then have

$$
\begin{aligned}
\sum_{\gamma^* \in \mathbf{\Gamma}^*} |\widehat{\varphi}(\omega + 2\pi\gamma^*)|^2 &= \sum_{\gamma_0^* \in \mathbf{\Gamma}_0^*} \sum_{\gamma^* \in D^{\mathrm{T}}\mathbf{\Gamma}^* + \gamma_0^*} |\widehat{\varphi}(\omega + 2\pi\gamma^*)|^2 \\
&= \sum_{\gamma_0^* \in \mathbf{\Gamma}_0^*} \sum_{\gamma^* \in \mathbf{\Gamma}^*} |\widehat{\varphi}(\omega + 2\pi D^{\mathrm{T}}\gamma^* + 2\pi\gamma_0^*)|^2 \\
&= \sum_{\gamma_0^* \in \mathbf{\Gamma}_0^*} |H(D^{-\mathrm{T}}\omega + 2\pi D^{-\mathrm{T}}\gamma_0^*)|^2 \\
&\qquad \times \sum_{\gamma^* \in \mathbf{\Gamma}^*} |\varphi(D^{-\mathrm{T}}\omega + 2\pi\gamma^* + 2\pi D^{-\mathrm{T}}\gamma_0^*)|^2 \\
&= \frac{1}{|V(0)|} \sum_{\gamma_0^* \in \mathbf{\Gamma}_0^*} |H(D^{-\mathrm{T}}\omega + 2\pi D^{-\mathrm{T}}\gamma_0^*)|^2
\end{aligned}
$$

by (25), (26) and the $2\pi\mathbf{\Gamma}^*$-periodicity of $H(\omega)$. Just as in dimension one this leads to the orthogonality condition

$$
\sum_{\gamma_0^* \in \mathbf{\Gamma}_0^*} |H(D^{-\mathrm{T}}\omega + 2\pi D^{-\mathrm{T}}\gamma_0^*)|^2 = 1 \tag{27}
$$

By similar calculations as above we see that orthogonality for the whole multiresolution is equivalent to the m-by-m modulation matrix $M(\omega)$ being unitary, where

$$
\begin{aligned}
M(\omega)_{1,l} &= H(\omega + 2\pi D^{-\mathrm{T}} r_l^*) &\qquad l &= 1, \dots, m, \\
M(\omega)_{\nu+1,l} &= G_\nu(\omega + 2\pi D^{-\mathrm{T}} r_l^*) &\qquad \nu &= 1, \dots, m-1.
\end{aligned}
$$

In the same way, biorthogonality requires that

$$
\widetilde{M}(\omega)\overline{M(\omega)}^{\mathrm{T}} = I.
$$

6.5. **Diagonalization.** Now when we have introduced the appropriate framework and notation for non-separable multidimensional wavelets it is fairly straightforward to generalize the previous diagonalization technique. Let us therefore assume that we are given a biorthogonal wavelet system in $\mathbb{R}^n$ with scaling functions φ and $\widetilde{\varphi}$, m mother wavelets ψ_ν and $\widetilde{\psi}_\nu$, and an n-dimensional convolution operator K. Just as before we define the new wavelets as

$$
\widetilde{\psi}_\nu^{\mathrm{K}} = K^* \widetilde{\psi}_\nu \quad \text{and} \quad \psi_\nu^{\mathrm{K}} = K^{-1}\psi_\nu. \tag{28}
$$

As before a necessary condition on the operator K to obtain diagonalization is that K commutes with dilations

$$
\kappa = \frac{\widehat{k}(\omega)}{\widehat{k}(D^{-\mathrm{T}}\omega)} \quad \text{independent of } \omega. \tag{29}
$$

The wavelet expansion of Kf is then given by

$$
Kf = \sum_{\nu=1}^{m} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbf{\Gamma}} \kappa^j \langle f, \widetilde{\psi}_{\nu,j,l}^{\mathrm{K}} \rangle \psi_{\nu,j,l}. \tag{30}
$$

Again we try to find new scaling functions by setting

$$\widehat{\widetilde{\varphi}^{\mathrm{K}}}(\omega) = \overline{\widehat{\ell}(\omega)}\widehat{\widetilde{\varphi}}(\omega) \quad \text{and} \quad \widehat{\varphi^{\mathrm{K}}}(\omega) = \frac{1}{\widehat{\ell}(\omega)}\widehat{\varphi}(\omega),$$

which gives us the new filter functions

$$H^{\mathrm{K}}(\omega) = \frac{\widehat{\ell}(\omega)}{\widehat{\ell}(D^{\mathrm{T}}\omega)}H(\omega), \qquad G^{\mathrm{K}}_{\nu}(\omega) = \frac{\widehat{\ell}(\omega)}{\widehat{k}(D^{\mathrm{T}}\omega)}G_{\nu}(\omega),$$

$$\widetilde{H}^{\mathrm{K}}(\omega) = \frac{\overline{\widehat{\ell}(D^{\mathrm{T}}\omega)}}{\overline{\widehat{\ell}(\omega)}}\widetilde{H}(\omega), \qquad \widetilde{G}^{\mathrm{K}}_{\nu}(\omega) = \frac{\overline{\widehat{k}(D^{\mathrm{T}}\omega)}}{\overline{\widehat{\ell}(\omega)}}\widetilde{G}_{\nu}(\omega).$$

By calculations identical with those in the one-dimensional case we see that we get biorthogonal filters if $m(\omega)$ is $2\pi\mathbf{\Gamma}^*$-periodic where, $m(\omega) = \widehat{k}(\omega)/\widehat{\ell}(\omega)$. In order to make the right choice of $m(\omega)$ we state the Strang-Fix conditions for a scaling function defined on a lattice $\mathbf{\Gamma}$. That is, if we want to write any polynomial of degree smaller than N as a linear combination of the scaling function φ and its $\mathbf{\Gamma}$-translates then

$$\widehat{\varphi}^{(p)}(2\pi\gamma^*) = 0 \quad \text{for} \quad \gamma^* \in \mathbf{\Gamma}^* \setminus \{0\} \quad \text{and} \quad 0 \le p < N.$$

As in one dimension $m(\omega)$ must be chosen so that

$$\frac{\widehat{k}(\omega)}{m(\omega)} \to 1 \quad \text{as} \quad \omega \to 0.$$

For simplicity we consider the two-dimensional case only. In analogy with the one-dimensional case we try to write

$$m(\omega) = \widehat{k}(-iA(\omega), -iB(\omega)) \tag{31}$$

where $A(\omega)$ and $B(\omega)$ are $2\pi\mathbf{\Gamma}^*$-periodic functions such that $A(\omega) = i\omega_1 + o(|\omega|)$ and $B(\omega) = i\omega_2 + o(|\omega|)$ as $\omega \to 0$. Let γ_1 and γ_2 be the column vectors of Γ and try with

$$\begin{aligned} A(\omega) &= a_1(e^{i\gamma_1\cdot\omega} - 1) + a_2(e^{i\gamma_2\cdot\omega} - 1) \\ &= i(a_1\gamma_1 + a_2\gamma_2)\cdot\omega + o(|\omega|) \qquad \text{as } \omega \to 0. \end{aligned}$$

We now chose a_1 and a_2 such that $(a_1\gamma_1 + a_2\gamma_2)\cdot\omega = \omega_1$, i.e such that

$$\Gamma\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

and similarly for $B(\omega)$. This can be written in the condensed form

$$m(\omega) = \widehat{k}(-i\Gamma^{-\mathrm{T}}(e^{\Gamma^{\mathrm{T}}\omega} - \mathbf{1})), \tag{32}$$

where the exponential of the vector $\Gamma^{\mathrm{T}}\omega$ is taken element wise. This choice can be thought of as a one-sided difference approximation of k in the directions γ_1 and γ_2. That is, if k is a directional derivative in one of the directions γ_i, then m will be a one-sided difference approximation in that direction.

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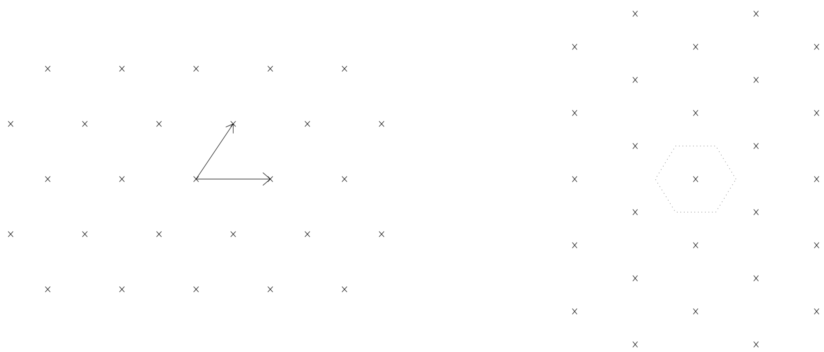

FIGURE 3. The hexagonal lattice and its dual with a Voronoi cell.

FIGURE 4. The Quincunx lattice and sublattice.

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