L^p AND H^p EXTENSIONS OF HOLOMORPHIC FUNCTIONS FROM SUBVARIETIES OF ANALYTIC POLYHEDRA

KENZŌ ADACHI, MATS ANDERSSON, AND HONG RAE CHO

ABSTRACT. Let V be a regular subvariety of a non-degenerate analytic polyhedron $\Omega \subset \mathbb{C}^n$. If V intersects $\partial\Omega$ transversally in a certain sense, then each bounded holomorphic function on V has a bounded holomorphic extension to Ω. Furthermore, a function in $H^p(V)$ has an extension in $H^p(\Omega)$. Under a weaker transversality condition each $f \in \mathcal{O}(V) \cap L^p(V)$ has an extension to a function in $\mathcal{O}(\Omega) \cap L^p(\Omega)$, $p < \infty$

1. Introduction and statement of results.

In this paper we study the problem of extending holomorphic functions from a regular subvariety of an analytic polyhedron. More precisely, we present some results that can be deduced from explicit extension formulas constructed along the lines in Berndtsson [5]. The estimation follows the ideas in [3] and [6].

Henkin [10] introduced methods of integral representations in order to obtain bounded extensions of holomorphic functions from submanifolds to strongly pseudoconvex domains. Since then, many works on regularity problems of extension functions have been done in various function spaces. In particular, Beatrous [4] obtained L^p extensions of holomorphic functions from submanifolds to strongly pseudoconvex domains, and Cumenge [8] and Amar [2] obtained H^p extensions on strongly pseudoconvex domains. Adachi obtained L^p and H^p extensions on real ellipsoids [1]. In general, it is not possible to obtain L^p extensions in pseudoconvex domains, see [4] and Section 5.

A bounded domain $\Omega \subset \mathbb{C}^n$ is an analytic polyhedron with defining functions ϕ_j if

$$\Omega = \{z \in \mathbb{C}^n;\ |\phi_j(z)| < 1, j = 1, \dots, N\},$$

where the defining functions ϕ_j are holomorphic in some neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$. For a multiindex $I \subset \{1, \dots, N\}$ we let $\sigma_I = \{z \in \overline{\Omega};\ |\phi_j(z)| = 1,\ j \in I\}$. The skeleton of Ω is the subset

$$\sigma = \bigcup_{|I|=n} \sigma_I$$

1991 *Mathematics Subject Classification.* 32A25, 32D15.

Key words and phrases. L^p and H^p extensions of holomorphic functions, subvariety, analytic polyhedron.

The second author was partly supported by the Swedish Natural Research Council and the third author was partly supported by KOSEF postdoctoral fellowship 1997.

of $\partial\Omega$. We say that Ω is non-degenerate if $\partial\phi_{I_1} \wedge \cdots \wedge \partial\phi_{I_k} \neq 0$ on σ_I for every multiindex $I = \{I_1, \dots, I_k\}$ such that $|I| = k \leq n$. In particular, then

$$(1.1) \qquad d|\phi_{I_1}| \wedge \cdots \wedge d|\phi_{I_n}|$$

is nonvanishing on σ_I for $|I| = n$, and hence σ_I is a submanifold of real codimension n which we can orientate by the form (1.1).

We say that the analytic polyhedron Ω is strongly non-degenerate if $\partial\phi_{I_1} \wedge \cdots \wedge \partial\phi_{I_k} \neq 0$ on σ_I for all multiindices I. In particular this means that not more than n of the functions ϕ_j can have modulii 1 at the same point. The polydisk D^n in $\mathbb{C}^n$ is a strongly non-degenerate analytic polyhedron with n defining functions and its skeleton is the torus T^n. It is easy to see that Ω being strongly non-degenerate is equivalent to that Ω is locally biholomorphic to a part of the polydisk D^n.

Let $\widetilde{V}$ be a regular subvariety of $\widetilde{\Omega}$ of codimension m given as

$$\widetilde{V} = \{z \in \widetilde{\Omega};\ h_1(z) = \cdots = h_m(z) = 0\},$$

where $h_j \in \mathcal{O}(\widetilde{\Omega})$, and $\partial h_1 \wedge \cdots \wedge \partial h_m \neq 0$ on $\widetilde{V}$. Set $V = \widetilde{V} \cap \Omega$. If we impose the transversal assumption that

$$(1.2) \qquad \partial h_1 \wedge \cdots \wedge \partial h_m \wedge \partial\phi_{I_1} \wedge \cdots \wedge \partial\phi_{I_k} \neq 0 \quad \text{on} \quad \overline{V} \cap \sigma_I,$$

for every multiindex I such that $|I| = k \leq n - m$, then V is a non-degenerate analytic polyhedron on the manifold $\widetilde{V}$. If we assume that (1.2) holds for any I, then V is a strongly non-degenerate polyhedron on $\widetilde{V}$.

If Ω is a strongly non-degenerate polyhedron, then also $\Omega_\epsilon = \{z \in \widetilde{\Omega};\ |\phi_j(z)| \leq 1 - \epsilon, j = 1, \dots, N\}$ is, for all small enough ϵ. Let σ_ϵ be the skeleton of Ω_ϵ. For a strongly non-degenerate polyhedron Ω we can define the Hardy spaces

$$H^p(\Omega) = \{f \in \mathcal{O}(\Omega);\ \sup_{\epsilon > 0} \|f\|_{L^p(\sigma_\epsilon)} < \infty\}.$$

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a non-degenerate analytic polyhedron. Let V be a regular subvariety in Ω of codimension m. Assume that (1.2) holds for $|I| \leq n - m$. Then for each $f \in \mathcal{O}(V) \cap L^p(V)$, $1 \leq p < \infty$, there exists $F \in \mathcal{O}(\Omega) \cap L^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $\|F\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(V)}$. For the case $p = 1$, no assumption on the intersection of $\widetilde{V}$ and $\partial\Omega$ is needed.*

Under the extra hypothesis that the transversal assumption (1.2) holds for all I we even have a bounded extension if f is bounded. We also have a corresponding result for H^p.

Theorem 1.2. *If Ω is a strongly non-degenerate analytic polyhedron and (1.2) holds for all I, then for all $f \in H^p(V)$, $1 < p \leq \infty$, there exists $F \in H^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $\|F\|_{H^p(\Omega)} \lesssim \|f\|_{H^p(V)}$.*

Remark 1.3. For the bounded extension, only the strong assumption on the intersection is needed. The assumption of strong non-degeneracy of Ω is present only to have a nice definition of H^p for $p < \infty$. In fact, in this case not even the strong condition on the intersection is needed, provided that one makes some appropriate definition of H^p, e.g. by taking the closure with respect to the $L^p(\sigma)$ of the space of holomorphic functions that are smooth up to the boundary, cf. Remark 4.3. $\square$

2. Construction of the extension formula.

To make the extension formulas more transparent, let us first briefly discuss some known representation formulas for holomorphic functions in an analytic polyhedron Ω. Let $\phi_k^j(\zeta,z)$ be holomorphic functions in $\widetilde{\Omega}\times\widetilde{\Omega}$ such that

$$\sum_{k=1}^{n}\phi_k^j(\zeta,z)(\zeta_k-z_k)=\phi_j(\zeta)-\phi_j(z),$$

ϕ_k^j are so-called Hefer functions to ϕ_j, and define the $(1,0)$-forms $\Phi_j=\sum_{k=1}^n\phi_k^j d\zeta_k$. Then for any $r>0$ we have a representation formula

$$f(z)=\frac{1}{(2\pi i)^n}\int_\Omega f(\zeta)\sum_{|\alpha|=n}{}^{'}\prod_{j\notin\alpha}\left(\frac{1-|\phi_j(\zeta)|^2}{1-\overline{\phi_j(\zeta)}\phi_j(z)}\right)^r$$
$$\bigwedge_{j\in\alpha}\frac{(1-|\phi_j(\zeta)|^2)^{r-1}}{(1-\overline{\phi_j(\zeta)}\phi_j(z))^{r+1}}\overline{\partial\phi_j}\wedge\Phi_j,\quad z\in\Omega,$$

(the sum is over increasing multiindices) for f that are holomorphic in a neighborhood of $\overline{\Omega}$, see [3]. If Ω is non-degenerate we can let $r\to 0$ and get the classical Weil formula

$$f(z)=\frac{1}{(2\pi i)^n}\sum_{|\alpha|=n}{}^{'}\int_{\sigma_\alpha}f(\zeta)\bigwedge_{j\in\alpha}\frac{\Phi_j}{\phi_j(\zeta)-\phi_j(z)},\ z\in\Omega. \tag{2.1}$$

If Ω is strongly non-degenerate and $Cf(z)$ denotes the holomorphic function obtained by plugging in an arbitrary function $f\in L^p(\sigma)$, $1<p<\infty$, in the integral in (2.1), then Cf is in H^p, see [6].

The extension formulas that we will discuss below are such Weil formulas for the polyhedron $V=\Omega\cap\widetilde{V}$, with the extra property that they provide extensions of holomorphic functions to the ambient domain Ω.

Let $h_1,\dots,h_m$ be holomorphic in $\widetilde{\Omega}$ as before, and choose Hefer functions $h_k^j(\zeta,z)$ in $\widetilde{\Omega}\times\widetilde{\Omega}$, i.e. such that $h_j(\zeta)-h_j(z)=\sum_{k=1}^n h_k^j(\zeta,z)(\zeta_k-z_k)$. Furthermore, let $H_j=\sum_1^n h_k^j d\zeta_k$, $|\partial h|$ be the Euclidean norm of the form $\partial h_1\wedge\cdots\wedge\partial h_m$ and dS the surface measure on V, induced by the Euclidean metric. Then

$$\mu=\frac{H_1\wedge\cdots\wedge H_m\wedge\overline{\partial h_1}\wedge\cdots\wedge\overline{\partial h_m}}{|\partial h|^2}dS$$

is a (m,m)-current whose coefficients are measures supported on V, and which depending holomorphically on $z\in\Omega$.

Theorem 2.1. *Let Ω be an analytic polyhedron and $\widetilde{V}$ as before. For any $r>0$ and f holomorpic in a neighborhood of $\overline{V}$ in $\widetilde{V}$, we have a holomorphic extension*

$$F(z)=\int_V f(\zeta)P^r(\zeta,z),\quad z\in\Omega, \tag{2.2}$$

to Ω*, where*

$$P^r(\zeta,z)=\sum_{|\alpha|=n-m} c\prod_{j\notin\alpha}\left(\frac{1-|\phi_j(\zeta)|^2}{1-\overline{\phi_j(\zeta)}\phi_j(z)}\right)^r \bigwedge_{j\in\alpha}\left(\frac{-r(1-|\phi_j(\zeta)|^2)^{r-1}}{(1-\overline{\phi_j(\zeta)}\phi_j(z))^{r+1}}\overline{\partial\phi_j(\zeta)}\wedge\Phi_j\right)\wedge\mu. \tag{2.3}$$

We write V in the integral in (2.2) to emphasize that the integration is performed only over V, even though it would be more correct to write Ω, since the kernel is a (n,n)-current.

Proof. Let $Q(\zeta,z)$ be an n-tuple of functions defined in $\widetilde{\Omega}\times\widetilde{\Omega}$ which are holomorpic in z, and suppose that $G(\lambda)$ is holomorpic on the image of $\langle Q_j,\zeta-z\rangle$ and that $G(0)=1$. As in [3] or [6], it follows from [5] that we have a representation $F(z)=\int_\Omega F(\zeta)P(\zeta,z)$ for F that are holomorphic in some neighborhood of $\overline{\Omega}$ for $r>0$, if

(2.4)

$$P(\zeta,z)=\sum_{k=0}^{n}\sum_{|\alpha|=n-k} c_k\prod_{j\notin\alpha}\left(\frac{1-|\phi_j(\zeta)|^2}{1-\overline{\phi_j(\zeta)}\phi_j(z)}\right)^r \bigwedge_{j\in\alpha}\left(\frac{-r(1-|\phi_j(\zeta)|^2)^{r-1}}{(1-\overline{\phi_j(\zeta)}\phi_j(z))^{r+1}}\overline{\partial\phi_j(\zeta)}\wedge\Phi_j\right)G^{(k)}(\langle Q,\zeta-z\rangle)(\overline{\partial}Q)^k.$$

Following [5], pages 409-411, we choose $G(\lambda)=\lambda^m$ and

$$Q(\zeta,z)=\frac{\sum_j\overline{h_j(\zeta)}H_j}{\sum_j|h_j(\zeta)|^2+\epsilon},$$

and let $\epsilon\to 0$. Then the terms in (2.4) that correspond to $|\alpha|=n-m$, tend to P^r in the theorem, whereas the other terms tend to an integrable kernel that vanishes for $z\in V$. Therefore, $F(z)=\int_V F(\zeta)P^r(\zeta,z)$ for $z\in V$ if F is holomorphic in a neighborhood of $\overline{\Omega}$. However, since any f that is holomorphic in a neighborhood of $\overline{V}$ in $\widetilde{V}$ is the restriction to V of such an F, the theorem is proved. □

3. L^p estimates of the extension function.

Proposition 3.1. *Let* Ω *be a non-degenerate analytic polyhedron, and let* P^r *be the extension kernel from Theorem 2.1. Furthermore, assume that (1.2) holds for all* I *of length less than or equal to* $n-m$*. For large enough* r *(*$r>1$ *will do) and* $1\le p<\infty$ *we have the estimate*

$$\int_\Omega\int_V|f(\zeta)P^r(\zeta,z)|^p dV(z)\lesssim\int_V|f(\zeta)|^p dS(\zeta),$$

for functions f *that are holomorphic in a neighborhood of* $\overline{V}$ *in* $\widetilde{V}$*.*

Once we have this proposition, we can apply it to the smaller polyhedra $\Omega_\epsilon=\{|\phi_j|<1-\epsilon\}$. Then we get, for each $\epsilon>0$, an extension F_ϵ in Ω_ϵ of f. It will be clear from the proof of Proposition 3.1 that the constant in $\lesssim$ is uniform in ϵ, and hence Theorem 1.1 follows by a normal family argument.

Proof of Proposition 3.1. Since

$$|P^r| \lesssim \sum_{|\alpha|=n-m} \prod_{j\notin\alpha} \frac{(1-|\phi_j(\zeta)|^2)^r}{|1-\overline{\phi_j(\zeta)}\phi_j(z)|^r} \prod_{j\in\alpha} \frac{(1-|\phi_j(\zeta)|^2)^{r-1}}{|1-\overline{\phi_j(\zeta)}\phi_j(z)|^{r+1}}$$

and

$$1-|\phi_j(\zeta)|^2 = (1+|\phi_j(\zeta)|)(1-|\phi_j(\zeta)|) \le 2(1-|\phi_j(\zeta)|) \le 2|1-\overline{\phi_j(\zeta)}\phi_j(z)|$$

in $\Omega\times\Omega$, we get the estimate

$$|P^r| \lesssim \sum_{|\alpha|=n-m} \prod_{j\in\alpha} \frac{(1-|\phi_j(\zeta)|^2)^{r-1}}{|1-\overline{\phi_j(\zeta)}\phi_j(z)|^{r+1}} \quad \text{in} \quad V\times\Omega.$$

Let P^r_α denote the term in the expression (2.3) for the kernel P^r that corresponds to the multiindex α.

We begin with the L^1 estimate. Since for all j,

$$|1-\overline{\phi_j(\zeta)}\phi_j(z)| \ge c > 0$$

uniformly for $\zeta\in V$ and z in any compact subset of Ω, it is sufficient to find a neighborhood $U^{z^0}\subset\overline{\Omega}$ to each point $z^0\in\partial\Omega$ and each α such that

$$\int_{U^{z^0}}\int_V |P^r_\alpha f|dV(z) \lesssim \int_V |f|dS(\zeta). \tag{3.1}$$

With no loss of generality we may assume that $\alpha=(1,\dots,n-m)$. We may also assume that there is a k (possibly $k=n-m$) such that $|\phi_j(z^0)|<1$ for $k<j\le n-m$ and $|\phi_j(z^0)|=1$ for $1\le j\le k$. By the assumption on Ω, $\partial\phi_1\wedge\cdots\wedge\partial\phi_k\neq 0$ at z^0. Therefore, we can choose a local holomorphic coordinate system w at z^0 such that $w_j=\phi_j$ for $1\le j\le k$. For w in a small neighborhood U^{z^0} of z^0, and $\zeta\in V$ we then have the estimate

$$|P^r_\alpha f| \lesssim \prod_1^k \frac{(1-|\phi_j(\zeta)|^2)^{r-1}|f(\zeta)|}{|1-\overline{\phi_j(\zeta)}w_j|^{r+1}}.$$

Since the Lebesgue measure with respect to w is equivalent to the volume measure in Ω, it follows by a standard estimate, see e.g. [3], that

$$\int_{U^{z^0}} |P^r_\alpha f|dV(z) \lesssim |f(\zeta)|dS(\zeta),$$

from which (3.1) follows. This concludes the proof of the case $p=1$. Notice that no assumption on the intersection of $\widetilde{V}$ and $\partial\Omega$ is needed.

For the L^p estimate we have to localize also in the ζ variable. For each pair of points $z^0 \in \partial\Omega$ and $\zeta^0 \in \partial V$ we must find neighborhoods $U^{z^0} \subset \overline{\Omega}$ and $U^{\zeta^0} \subset \overline{V}$ such that

$$\int_{U^{z^0}} \int_{V^{\zeta^0}} |P^r_\alpha f|^p dV(z) \lesssim \int_{U^{\zeta^0}} |f|^p dS(\zeta). \tag{3.2}$$

To this end, again assume that $\alpha = (1, \dots, n-m)$ and moreover, that both $|\phi_j(z^0)| = 1$ and $|\phi_j(\zeta^0)| = 1$ for $1 \le j \le k$ and either $|\phi_j(z^0)| < 1$ or $|\phi_j(\zeta^0)| < 1$ for $k < j \le n-m$. For $(\zeta, z) \in V \times \Omega$ close to (ζ^0, z^0) we then have the estimate

$$|P^r_\alpha f| \lesssim \prod_1^k \frac{(1 - |\phi_j(\zeta)|^2)^{r-1}}{|1 - \overline{\phi_j(\zeta)}\phi_j(z)|^{r+1}} |f(\zeta)|.$$

At z^0 we can choose local coordinates w as before. By assumption,

$$\partial h_1 \wedge \dots \partial h_m \wedge \partial \phi_1 \wedge \dots \wedge \partial \phi_k \neq 0$$

at ζ^0. This means that we have local coordinates $\xi = (\xi_1, \dots, \xi_{n-m})$ at ζ^0 on $\widetilde{V}$, such that $\xi_j = \phi_j(\zeta)$ for $1 \le j \le k$. For small enough neighborhoods U^{ζ^0} and U^{z^0} we thus have that

$$|P^r_\alpha f| \lesssim \prod_1^k \frac{(1 - |\xi_j|^2)^{r-1}}{|1 - \bar{\xi}_j w_j|^{r+1}} |f(\xi)|.$$

Notice that the Lebesgue measure with respect to ξ is equivalent to the surface measure dS on V. The desired estimate (3.2) now follows by standard technique, see e.g. [3]. $\square$

4. H^p estimates for the extension function.

To handle the H^∞ and H^p estimates, it is natural to let $r \to 0$ in (2.2), in order to get a formula similar to Weil's formula (2.1).

Proposition 4.1. *If (1.2) holds for all I of length $\le n-m$, then one can let $r \to 0$ in (2.2) and obtain the extension formula*

$$F(z) = c \sum_{|\alpha| = n-m} \int_{\sigma_\alpha} f(\zeta) \frac{\omega_\alpha(\zeta, z)}{\prod_{j \in \alpha} (\phi_j(\zeta) - \phi_j(z))}, \quad z \in \Omega, \tag{4.1}$$

for f holomorphic in some neighborhood of $\overline{V}$ in $\widetilde{V}$, where ω_α are $(n-m, 0)$-forms in $d\zeta$ which are smooth in a neighborhood of $\sigma_\alpha \times \overline{\Omega}$ and holomorphic in $z \in \Omega$. Here σ_α refers to the polyhedron V, i.e. $\sigma_\alpha = \{\zeta \in \overline{V};\ |\phi_j(\zeta)| = 1,\ j \in \alpha\}$.

Proof. We consider a fixed term P^r_α in (2.3). Again we may assume that $\alpha = (1, \dots, n-m)$. Recall that so far, strictly speaking, P^r_α is a (n, n)-current, supported on V; more precisely it is a smooth form times the surface measure dS. Let ζ^0 be a fixed point on $\overline{V}$. We may assume that $|\phi_j(\zeta^0)| = 1$ for say $j \le k$ and $|\phi_j(\zeta^0)| < 1$ for $k < j \le n-m$. Then, by assumption, $\xi_1 = \phi_1, \dots, \xi_k = \phi_k$ is part of a local coordinate system $\xi_1, \dots, \xi_{n-m}$ for $\widetilde{V}$ at ζ^0, and hence there is a smooth $(n-m, n-m-k)$-form ω, such that

$$\int_\Omega \chi\mu \bigwedge_1^{n-m} \overline{\partial\phi_j} \wedge \Phi_j = \int_V \chi \bigwedge_1^k \overline{\partial\phi_j} \wedge \omega$$

for all test functions χ with support near ζ^0. Therefore,

$$\int_\Omega P_\alpha^r \chi = \int_V (1+o(1))\omega_\alpha \chi \wedge \mathcal{O}(r^{n-m-k}) \bigwedge_1^k \frac{-r(1-|\xi_j|^2)^{r-1} d\bar{\xi}_j}{(1-\bar{\xi}_j\phi_j(z))^{r+1}}, \tag{4.2}$$

for χ with support near ζ^0. If $\zeta^0 \in \sigma_\alpha$ (i.e. $n-m-k=0$), then this integral tends to

$$\int_{\sigma_\alpha} \frac{\omega_\alpha \chi}{\prod_1^{n-m}(\xi_j - \phi_j(z))}$$

when $r \to 0$. If ζ^0 is outside σ_α, then (4.2) tends to zero when $r \to 0$. The various $(n-m, 0)$ forms ω, corresponding to points on σ_α can be pieced together to a global form ω_α defined in a neighborhood of σ_α, and thus $\int P_\alpha^r f$ tends to the term $\int P_\alpha f$ corresponding to α in (4.1). Hence the proposition is proved. $\square$

Notice that so far we have only assumed that (1.2) holds for all I of length $\leq n-m$. Therefore it might happen that $\sigma_\alpha \cap \operatorname{supp}\chi$ is a proper subset of $\{|\xi_1| = \cdots = |\xi_{n-m}| = 1\} \cap \operatorname{supp}\chi$.

Proof of Theorem 1.2. Since now (1.2) holds for all I, the skeleton of V is stable under small perturbations. Therefore, it is enough to prove an a priori estimate for functions f that are holomorphic in a neighborhood of $\overline{V}$ in the manifold $\widetilde{V}$.

We concentrate on the case $p = \infty$. The H^p-estimate is obtained in a similar way. Consider a fixed P_α. It is enough to prove that for each pair of points $z^0 \in \overline{\Omega}$ and $\zeta^0 \in \sigma_\alpha$, we can find neighborhoods U^{z^0} and U^{ζ^0} such that if χ is a smooth cutoff function with support in U^{ζ^0}, then the estimate

$$\left| \int \chi f P_\alpha(\cdot, z) \right| \lesssim \|f\|_{H^\infty(V)}, \tag{4.3}$$

holds uniformly for all $z \in U^{z^0} \cap \Omega$ and all f which are holomorphic in any neighborhood of $\overline{V}$. (For the H^p-estimate, one has to show instead that the function on the left hand side of (4.3) is in L^p on the skeleton of Ω near z^0.)

As usual, we assume that $\alpha = (1, \dots, n-m)$ and that $|\phi_j(z^0)| = 1$ for $1 \leq j \leq k$ and $|\phi_j(z^0)| < 1$ for $k < j \leq n-m$. Near z^0, $w_1 = \phi_1, \dots, w_k = \phi_k$ are part of a local coordinate system $w_1, \dots, w_n$ and moreover $\xi_1 = \phi_1, \dots, \xi_{n-m} = \phi_{n-m}$ are local coordinates near ζ^0. Here, we made use of the strong transversality condition. In these coordinates, the integral to estimate is

$$\int_{\xi \in T^{n-m}} \frac{f(\xi)\chi(\xi)\omega(\xi, w)}{\prod_1^k(\xi_j - w_j)}. \tag{4.4}$$

Knowing that f is holomorphic in some fixed neighborhood U of ζ^0 in $\overline{D}^{n-m}$ (and ω is smooth on T^{n-m} in this neighborhood), we have to show that if χ is chosen with sufficiently small support, then (4.4) is bounded by a constant times $\|f\|_{H^\infty(U)}$. However, since we are in a genuine product situation, this estimate follows immediately from the following one variable lemma. Thus Theorem 1.2 is proved for $p=\infty$. The corresponding H^p version of this lemma is also true, and follows immediately from the fact that the Cauchy integral is bounded on L^p. From this the case $p<\infty$ of Theorem 1.2 follows. $\square$

Lemma 4.2. *Let $U\subset\overline{D}$ be a neighborhood of 1 in the closed unit disk $\overline{D}$, and assume that $\omega(\xi)$ is smooth in $T\cap U$. If χ is a smooth cutoff function with sufficiently small support near 1, then*

$$\left|\int_T \frac{f(\xi)\omega(\xi)\chi(\xi)d\xi}{\xi-w}\right| \lesssim \|f\|_{H^\infty(U)},$$

uniformly in w, and for all functions f that are holomorphic in U. Moreover, the constant only depends on the sup norm of ω and χ and their first order derivatives.

Proof. If the support of χ is small, then one can replace T by a closed curve γ that is contained in U and which coincides with T on the support of χ. It is enough to consider w inside this curve, since when w is outside, then the kernel is bounded. Let $\psi=\omega\chi$. Then the integral is equal to

$$\int_\gamma \frac{\psi(\xi)-\psi(w)}{\xi-w}f(\xi)d\xi+\int_\gamma\frac{\psi(w)f(\xi)}{\xi-w}d\xi.$$

The first term is bounded since the integrand is bounded, and the constant only depends on (the size of γ and) the sup norm of ψ and its first order derivatives. The second integral is just $\psi(w)f(w)$ by virtue of Cauchy's formula. $\square$

Remark 4.3. If we only assume that (1.2) holds for I of length at most $n-m$, then the integration in (4.4) is restricted to $\overline{V}\cap\{|\xi_1|=\cdots=|\xi_{n-m}|=1\}$ i.e. there may be some extra restriction because of some additional functions ϕ_ℓ. However, this corresponds to multiplying the integrand with a characteristic function, and since, for the H^p-estimate, we only need that the integrand is in $L^p(T^k)$, the desired estimate still holds. This means that Theorem 1.2 is true for the case $p<\infty$ under this weaker transversality condition, provided that H^p is given a reasonable definition. $\square$

5. A concluding remark.

It is natural to ask whether the non-degeneracy condition really is necessary in our theorems. We have no counterexamples in analytic polyhedra, but it is worth pointing out that the general L^p extension problem is not solvable in pseudoconvex domains, see also [4].

Let F be a compact subset of the unit disc D, Hausdorff dimension 0 which is regular for the Dirichlet problem. Such a compact subset is constructed in [7]. Then there exists a positive subharmonic function s in $U=D-F$ such that [9]:

(1) $e^{-s(z)}$ has a continuous extension to $\overline{D}$;
(2) $e^{-s(z)}|_{\partial D}=1$, $e^{-s(z)}|_F=0$;
(3) For $\nu\in\mathbb{N}$, $\int_U e^{\nu s(z)}<\infty$.

Let Ω be the Hartogs domain defined by

$$\Omega = \{(z,w) \in D^2; z \in U, \ |w| < e^{-s(z)}\},$$

where D^2 is the unit polydisc in $\mathbb{C}^2$.

Lemma 5.1 ([9]). *Ω is a pseudoconvex domain such that:*

(1) $\Omega = \overset{\circ}{\overline{\Omega}} \subset D^2$;
(2) $\Omega \neq D^2$;
(3) *For $p > 2$, functions in $\mathcal{O}(\Omega) \cap L^p(\Omega)$ extend holomorphically to D^2.*

Ohsawa-Takegoshi [11] proved that L^2 holomorphic functions on a complex linear subspace of codimension one of a pseudoconvex domain Ω can be extended to L^2 holomorphic functions on Ω. For $p > 2$, it is false in general.

Theorem 5.2. *Choose $z^0 \in U$ such that $0 < e^{-s(z^0)} < 1$. Let $V = \{(z,w); z = z^0\}$. For $e^{-s(z^0)} < a < 1$,*

$$f(z,w) = \frac{1}{w-a}$$

is an L^p holomorphic function on $V \cap \Omega$. For $p > 2$, there exist no L^p holomorphic functions $F(z,w)$ on Ω such that $F|_{V\cap\Omega} \equiv f$ on $V \cap \Omega$.

Proof. Suppose that there is a function $F \in \mathcal{O}(\Omega) \cap L^p(\Omega)$ such that $F|_{V\cap\Omega} \equiv f$. Then, by Lemma 5.1, F extends to a holomorphic function $\hat{F}$ in D^2 such that $\hat{F}|_\Omega \equiv F$ on Ω. Thus

$$\hat{F}(z^0, w) = \frac{1}{w-a} \quad \text{in} \quad V \cap \Omega.$$

Since $\frac{1}{w-a}$ is holomorphic in $V - \{(z^0, a)\}$, by the identity theorem,

$$\hat{F}(z^0, w) = \frac{1}{w-a} \quad \text{in} \quad (V - \{(z^0,a)\}) \cap D^2.$$

Now, $\hat{F}$ is holomorphic in $V \cap D^2$ and $\frac{1}{w-a}$ has a singularity at a, this is a contraction. $\square$

References

1. K. Adachi, *Extending H^p functions from subvarieties to real ellipsoids*, Trans. A.M.S. **317** (1990), 351–359.
2. E. Amar, *Extension de fonctions holomorphes et courants*, Bull. Sci. Math. **107-2** (1983), 25–48.
3. M. Andersson, *L_p estimates for the $\overline{\partial}$ equation in analytic polyhedra in Stein manifolds*, Proc. of Mittag–Leffler 87/88 Math. Notes 38, Princeton Univ. Press, 34–47.
4. F. Beatrous, *L^p estimates for extensions of holomorphic functions*, Michigan Math. J. **32** (1985), 361–380.
5. B. Berndtsson, *A formula for interpolation and division in $\mathbb{C}^n$*, Math. Ann. **263** (1983), 399–418.
6. J. Boo, *The H^p corona theorem in analytic polyhedra,* Dissertation, Arkiv för mat., 1995 (to appear).

7. L. Carleson, *Selected problems on exceptional sets*, Van Nostrand, Princeton, N. J., 1987.
8. A. Cumenge, *Extension dans des classes de Hardy de fonctions holomorphes et estimation de type "measures de Carleson" pour l'equation $\bar{\partial}$*, Ann. Inst. Fourier **33** (1983), 59–97.
9. J. E. Fornaess and N. Sibony, *On L^p estimates for $\overline{\partial}$*, Proc. Symp. Pure Math. **52** (1991), 129–163.
10. G. M. Henkin, *Continuation of bounded holomorphic functions from submanifolds in general position to strictly pseudoconvex domains*, Math. USSR Izv. **6** (1972), 536–563.
11. T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), 197-204.

Kenzō Adachi
Department of Mathematics, Nagasaki University, Nagasaki 852, Japan

Mats Andersson
Department of Mathematics, Chalmers University of Technology and Göteborg University, S-41296 Göteborg, Sweden

Hong Rae Cho
Department of Mathematics, Pohang University of Science and Technology, Pohang 790–784, South Korea