Phase Transition in the Random Triangle Model

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June 17, 1997

Abstract

The random triangle model was recently introduced as a random graph model that captures the property of transitivity that is often found in social networks, i.e. the property that given that two vertices are second neighbors, they are more likely to be neighbors. For parameters $p \in [0,1]$ and $q \geq 1$, and a finite graph $G = (V, E)$, it assigns to elements η of $\{0,1\}^E$ probabilities which are proportional to $\prod_{e \in E} p^{\eta(e)}(1-p)^{1-\eta(e)} q^{t(\eta)}$, where $t(\eta)$ is the number of triangles in the open subgraph. In this paper the behavior of the random triangle model on the two-dimensional triangular lattice is studied. By mapping the system onto an Ising model with external field on the hexagonal lattice, it is shown that phase transition occurs if and only if $p = (q-1)^{-2/3}$ and $q > q_c$ for a critical value q_c which turns out to equal $27 + 15\sqrt{3} \approx 52.98$. It is furthermore demonstrated that phase transition cannot occur unless $p = p_c(q)$, the critical value for percolation of open edges for given q. This implies that for $q \geq q_c$, $p_c(q) = (q-1)^{-2/3}$.

1 Introduction

The study of random graphs began in the late 1950's with the pioneering papers by Broadbent and Hammersley [3] and Erdős and Rényi [6]. Since then, the subject has attracted enormous interest in mathematics (see e.g. the books by Grimmett [12] and Bollobás [2]) as well as in various applied disciplines. One class of such disciplines is the social sciences where random graphs are often used to describe social networks; see e.g. Faust and Wasserman [7] or Frank [9] for overviews. For instance, an open edge between two vertices could mean that the corresponding individuals are friends, family, colleagues etc. It has been well recognized that many types of social networks exhibit *transitivity*, i.e. the property that "friends of friends often make friends". In other words, an edge is more likely to be open if this would produce some triangle in the open subgraph (i.e. if its endpoints are known to be second neighbors) than if not.

Jonasson [17] introduced the *random triangle model* in order to capture and isolate the property of transitivity between second neighbors. As opposed to other models that have been used to capture the transitivity phenomenon, it has the Markov random field property that the conditional probability that an edge e is present given everything else depends only on the states of edges adjacent to e; see [17] for further comparison with other models. Another attractive property of the random triangle model is that the resulting distribution has maximal entropy in the class of random graphs with given edge and triangle probabilities.

Let G be a finite graph with vertex set V and edge set E. An element of $\{0,1\}^E$ is identified with a subgraph of G by letting the value 1 (resp. 0) indicate the presence (resp. absence) of an edge in the subgraph. Conforming to percolation terminology, we also call an edge with value 1 open, and an edge with value 0 closed. A *triangle* is a set $\{e_1, e_2, e_3\} \subseteq E$ of cardinality 3, with the property that any two edges in the set share a vertex.

DEFINITION 1.1 *For $p \in [0,1]$, $q \geq 1$, and a finite graph $G = (V, E)$, the random triangle measure $\mu_G^{p,q}$ is the probability measure on $\{0,1\}^E$ which to each $\eta \in \{0,1\}^E$ assigns probability*

$$\mu_G^{p,q}(\eta) = \frac{1}{Z_G^{p,q}} \prod_{e \in E} p^{\eta(e)}(1-p)^{1-\eta(e)} q^{t(\eta)} \tag{1}$$

where $Z_G^{p,q}$ is a normalizing constant and $t(\eta)$ is the number of triangles in η.

The definition can of course be extended to any $q > 0$, but we restrict to $q \geq 1$ for two reasons. Firstly, the social network interpretation makes more sense for $q \geq 1$. Secondly (and more importantly), a number of stochastic monotonicity properties (starting with Corollary 1.3) that are crucial for all our results hold for $q \geq 1$ but in general not for $q < 1$.

Thus, compared to the basic model with independent edges, the random triangle model rewards a frequent appearance of triangles. The desired transitivity follows from Lemma 1.2 below; the conditional probability that an edge is present given everything else is increasing in the number of "mutual friends" that its endpoints have.

LEMMA 1.2 *For $e \in E$ and a configuration $\eta \in \{0,1\}^{E\setminus\{e\}}$, let $\Delta(\eta, e) = t(\overline{\eta}) - t(\underline{\eta})$ where $\overline{\eta}$ and $\underline{\eta}$ are defined by $\overline{\eta}(e) = 1$, $\underline{\eta}(e) = 0$ and $\overline{\eta}(e') = \underline{\eta}(e') = \eta(e')$ for $e' \in E \setminus \{e\}$. Then*

$$\mu_G^{p,q}(Y(e) = 1 | Y(E\setminus\{e\}) = \eta) = \frac{pq^{\Delta(Y,e)}}{pq^{\Delta(Y,e)} + 1 - p}.$$

Here and in the sequel Y is understood to be a random element in $\{0,1\}^E$ distributed according to a random triangle measure. For any $E' \subseteq E$, we write $\leq$ for the usual (coordinatewise) partial order on $\{0,1\}^{E'}$.

COROLLARY 1.3 *For any $0 \leq p_1 \leq p_2 \leq 1$, any $1 \leq q_1 \leq q_2$ and any $\eta_1, \eta_2 \in \{0,1\}^{E\setminus\{e\}}$ such that $\eta_1 \leq \eta_2$, we have*

$$\mu_G^{p_1,q_1}(Y(e) = 1 | Y(E\setminus\{e\}) = \eta_1) \leq \mu_G^{p_2,q_2}(Y(e) = 1 | Y(E\setminus\{e\}) = \eta_2). \tag{2}$$

Lemma 1.2 is immediate from the definition of the random triangle measure, and Corollary 1.3 is then another immediate consequence.

In [17] the asymptotic behavior of the random triangle model on the complete graph on n vertices as $n \to \infty$ is analyzed for $p = n^{-\alpha}$, $\alpha > 0$. In this case it turns out that the model has a more or less degenerate behavior. If $q = 1 + (3+\epsilon)\alpha \log n / n$, then, for $\epsilon > 0$, almost all edges will be open, whereas for $\epsilon < 0$ the model becomes virtually indistinguishable from the ordinary Bernoulli model with independent edges. For $\epsilon = 0$, the probability mass divides between these two extremes. The reason for this degeneracy is that on the complete graph the number of triangles is so much larger than the number of edges. This suggests that one should study the model on graphs where the number of triangles is of the same order as the number of edges. A natural choice is to study what happens on the two-dimensional triangular lattice $\mathcal{L}_T$ (the more usual square lattice is of course useless for this purpose, since it does not contain any triangles). This line of study is also interesting from the point of view of percolation theory and statistical mechanics, as the random triangle model can be seen as a prototype for a "random-cluster-like" model with only local edge dependencies; the similarity between the random triangle model and the random-cluster model will be is discussed later in this section. Central questions for the random triangle model on $\mathcal{L}_T$ are:

(i) For what values of p and q will the set of open edges percolate (i.e. contain an infinite connected component)?

(ii) When does the model exhibit phase transition, i.e. for what p and q is there more than one possible random triangle measure?

We now proceed to make these questions more precise. Definition 1.1 requires that the graph is finite so the first thing we must do is to define what a random triangle measure on an infinite graph shall mean. The following definition is an adaption of the usual Dobrushin–Lanford–Ruelle definition of an infinite-volume Gibbs measure (see e.g. [11] or [14]) to the present setting.

DEFINITION 1.4 *Let $G = (V, E)$ be an infinite but locally finite graph. A probability measure μ on $\{0,1\}^E$ is said to be a random triangle measure with parameters $p \in [0,1]$ and $q \geq 1$ if, for all finite subsets S of E, all $\eta' \in \{0,1\}^S$ and μ-a.e. $\eta'' \in \{0,1\}^{E\setminus S}$*

$$\mu(Y(S) = \eta' | Y(E \setminus S) = \eta'') = Z^{-1} \prod_{e \in S} p^{\eta'(e)}(1-p)^{1-\eta'(e)} q^{t(\eta',\eta'')} \tag{3}$$

where Z is a normalizing constant (which may depend on η'' but not on η') and $t(\eta', \eta'')$ is the number of triangles in the open subgraph of G that have at least one edge in S.

Note that this definition produces a consistent set of conditional distributions, and that natural analogues of Lemma 1.2 and Corollary 1.3 go through for infinite-volume random triangle measures (the analogue of Lemma 1.2 is in fact contained in the definition: set $S = \{e\}$ in (3)).

The existence of random triangle measures on infinite locally finite graphs can be proved by standard compactness arguments, using that $\{0,1\}^E$ endowed with the natural product topology is compact so that the set of probability measures on $\{0,1\}^E$ is compact in its weak topology. Note that weak convergence of a sequence of probability measures on $\{0,1\}^E$ is equivalent to convergence of the corresponding probabilities for cylinder events, i.e. events depending on only finitely many coordinates.

The triangular lattice $\mathcal{L}_T$ is defined as the infinite locally finite graph with vertex set

$$V = \{(x,y) \in \mathbf{R}^2 : \frac{x}{\sqrt{3}} \in \mathbf{Z},\, y \in \mathbf{Z}\} \cup \{(x,y) \in \mathbf{R}^2 : \frac{x - \frac{1}{2}}{\sqrt{3}} \in \mathbf{Z},\, y - \frac{1}{2} \in \mathbf{Z}\}$$

and edge set E consisting of all pairs of vertices at Euclidean distance 1 from each other.

Let Λ_n be the closed hexagon with side length n centered at the origin and two of its sides parallel to the x-axis, and let E_{Λ_n} denote the set of edges with both endpoints in Λ_n. Define two probability measures, $\mu_{0,n}^{p,q}$ and $\mu_{1,n}^{p,q}$, on $\{0,1\}^E$ by letting $\mu_{i,n}^{p,q} = i$ for $e \in E \setminus E_{\Lambda_n}$ and then assigning values to the edges of E_{Λ_n} according to (3) with $S = E_{\Lambda_n}$ and $\eta'' \equiv i$. It turns out (see Section 2) that the weak limits $\mu_i^{p,q} = \lim_{n\to\infty} \mu_{i,n}^{p,q}$, $i = 0, 1$, exist and are translation invariant. Moreover, $\mu_0^{p,q}$ and $\mu_1^{p,q}$ are infinite-volume random triangle measures in the sense of Definition 1.4.

Let $\mathcal{C}$ denote the event that the random subset of E consisting of edges taking value 1 contains an infinite connected component. We will see that if $\mu^{p,q}(\mathcal{C}) > 0$ for some random triangle measure for $\mathcal{L}_T$ with parameter values p and q, then in particular $\mu_1^{p,q}(\mathcal{C}) > 0$ (in fact, even the stronger conclusion that $\mu_1^{p,q}(\mathcal{C}) = 1$ holds). Furthermore, $\mu_1^{p,q}(\mathcal{C})$ is nondecreasing in p. A natural way to make problem (i) above more precise is therefore:

(i′) Given q, determine the critical value $p_c(q)$, defined as

$$p_c(q) = \inf\{p : \mu_1^{p,q}(\mathcal{C}) > 0\}. \tag{4}$$

We will also see that there is a unique random triangle measure with parameter values p and q if and only if $\mu_0^{p,q} = \mu_1^{p,q}$ (the 'only if' direction is of course trivial), so that a simple way of formulating problem (ii) becomes:

(ii′) For what values of p and q do we have $\mu_0^{p,q} \neq \mu_1^{p,q}$?

A complete answer will be given to problem (ii′), whereas we are only able to give a partial answer to problem (i′).

At this stage let us make some comparisons between the random triangle model and the random-cluster model. If we modify (1) by replacing $t(\eta)$ with $k(\eta)$, defined as the number of connected components in η, we obtain the definition of the random-cluster model on a finite graph, and by making a similar replacement in (3) we get the infinite-volume random-cluster model. The random-cluster model was introduced in the early 1970's by Fortuin and Kasteleyn [8], and has gained great popularity in the 1990's (see e.g. [13] or [14]), mainly motivated by the striking and useful connections with Ising and Potts models. For $q \geq 1$, the random-cluster model satisfies an analogue of Corollary 1.3, and for this reason the random-cluster model and the random triangle model share several important monotonicity properties. A major difference between the two models is that whereas interaction between edges in the random triangle model is local, the random-cluster model exhibits highly non-local interactions, in the sense that the conditional probability that an edge e is open given the status of all other edges may depend on edges arbitrarily far away in the graph. The problem of phase transition in the $q \geq 1$ random-cluster model on the cubic lattice $\mathbf{Z}^d$, $d \geq 2$, has been treated e.g. by Grimmett [13]. Defining $p_c(q)$ similarly as in the random triangle model, the widely held belief is that there exists a $q_c = q_c(d) \in (1, \infty)$ such that phase transition occurs if and only if $q > q_c$ (or possibly $q \geq q_c$) and $p = p_c(q)$; this is conjectured in [13] and elsewhere. Partial progress has been made, but so far the full conjecture remains unsolved for all $d \geq 2$.

The following two theorems, which are the main results of this paper, prove an analogue of this conjecture for the random triangle model on $\mathcal{L}_T$, and even identify q_c and $p_c(q)$.

THEOREM 1.5 *The random triangle model on the triangular lattice with parameters p and q exhibits a phase transition if and only if $q > q_c$ and $p = (q-1)^{-2/3}$. Here $q_c = 27 + 15\sqrt{3} \approx 52.98$.*

THEOREM 1.6 *The critical value $p_c(q)$ (defined in (4)) for percolation in the random triangle model on the triangular lattice is a decreasing continuous function of q on $[1, \infty)$, satisfying $p_c(1) = 2\sin(\pi/18)$ and $p_c(q) = (q-1)^{-2/3}$ for $q \geq 27 + 15\sqrt{3}$.*

Here, decreasing means nonincreasing, although presumably $p_c(q)$ is even strictly decreasing throughout $[1, \infty)$. The result that $p_c(1) = 2\sin(\pi/18)$ is due to Wierman [22]; this is simply the critical probability for independent bond percolation on $\mathcal{L}_T$.

Here is a brief outline of the rest of this paper. After some preliminary results in Section 2 on stochastic domination and monotonicity, we show in Section 3 that phase transition for the random triangle model on $\mathcal{L}_T$ is equivalent to phase transition for a certain probability model on $\{0,1\}^T$, where T is the set of triangles of $\mathcal{L}_T$. This model turns out to be nothing but the Ising model with external field on the hexagonal lattice $\mathcal{L}_H$. Once this correspondence is established, Theorem 1.5 follows from well known results about the phase transition behavior of the Ising model on $\mathcal{L}_H$. In Section 4 we use the Burton–Keane theorem on the uniqueness of infinite clusters in percolation models to show that there can be phase transition only if $p = p_c(q)$. In combination with Theorem 1.5, this proves the hardest part of Theorem 1.6. Finally, Section 5 contains a short discussion on some variants of the model, including extensions to higher dimensions.

2 Preliminaries on stochastic domination

In this section, we obtain some results on stochastic domination and monotonicity in the random triangle model, which are central to the methods in subsequent sections. The results rely critically on Corollary 1.3, and thus on the assumption that $q \geq 1$. All the proofs are standard, and the results have well-known analogues for the Ising model and the random-cluster model (see e.g. [19] and [14]).

We first need to introduce the concept of stochastic domination. If P and P' are two probability measures on some partially ordered measure space A, then we say that P is stochastically dominated by P' (or that P' stochastically dominates P) and write $P \leq_d P'$ (or $P' \geq_d P$) if $\int_A f dP \leq \int_A f dP'$ for all bounded and increasing functions, $f : A \to \mathbf{R}$. By Strassen's Theorem (see [20]), this is equivalent to the existence of a pair of A-valued random objects Y and Y' with respective distributions P and P' and the additional property that $Y \leq Y'$ with probability 1. We call such a pair a *monotone coupling* of P and P'. Recall that we endow $\{0,1\}^E$ with its usual coordinatewise partial order.

By applying Holley's Theorem (proved in [16]; see [14] for a formulation which is adapted to the present setting) to the random triangle model, using Corollary 1.3, we obtain the following result.

PROPOSITION 2.1 *Let $G = (V, E)$ be a finite or infinite locally finite graph, let S be an arbitrary finite subset of E, and let $\mu^{p,q}$ be a random triangle measure for G with parameters p and q. For an edge configuration $\eta \in \{0,1\}^{E \setminus S}$, write $\mu^{p,q}_{S,\eta}$ for the conditional distribution under $\mu^{p,q}$ of $Y(S)$. Then $\mu^{p,q}$ admits conditional probabilities such that*

$$\mu^{p,q}_{S,\eta_1} \leq_d \mu^{p,q}_{S,\eta_2} \tag{5}$$

whenever $\eta_1 \leq \eta_2$. *More generally, if* $p_1 \leq p_2$, $q_1 \leq q_2$ *and* $\eta_1 \leq \eta_2$, *then*

$$\mu^{p_1,q_1}_{S,\eta_1} \leq_d \mu^{p_2,q_2}_{S,\eta_2}. \tag{6}$$

We now specialize to the random triangle model on $\mathcal{L}_T$. Recall from the introduction the definitions of $\mu^{p,q}_{i,j}$ for $i = 0, 1$ and $j = 1, 2, \ldots$. As a first application of Proposition 2.1, we claim that

$$\mu^{p,q}_{0,j} \leq_d \mu^{p,q}_{0,j+1} \tag{7}$$

for any $j \geq 1$. To see this, let (Y, Y') be a coupling of $\mu^{p,q}_{0,j}$ and $\mu^{p,q}_{0,j+1}$ obtained as follows. First let $Y \equiv 0$ on Λ_j, let $Y' \equiv 0$ on Λ_{j+1}, and pick $Y'(E_{\Lambda_{j+1}} \setminus E_{\Lambda_j})$ according to the projection of $\mu^{p,q}_{0,j+1}$ on $\{0,1\}^{E_{\Lambda_{j+1}} \setminus E_{\Lambda_j}}$. Next, pick $Y(E_{\Lambda_j})$ and $Y'(E_{\Lambda_j})$ according to their respective conditional distributions, in such a way that $Y(E_{\Lambda_j}) \leq Y'(E_{\Lambda_j})$; this is possible by (5). This gives a monotone coupling of Y and Y', so (7) is established. Similarly,

$$\mu^{p,q}_{1,j} \geq_d \mu^{p,q}_{1,j+1}.$$

In other words, $\mu^{p,q}_{0,1} \leq_d \mu^{p,q}_{0,2} \leq_d \ldots$ and $\mu^{p,q}_{1,1} \geq_d \mu^{p,q}_{1,2} \geq_d \ldots$ so that by monotonicity the limits $\mu^{p,q}_0$ and $\mu^{p,q}_1$ exist, as claimed in the introduction. That $\mu^{p,q}_0$ and $\mu^{p,q}_1$ are random triangle measures in the sense of Definition 1.4 follows from the general theory of Gibbs measures (see [11]) since the interactions are local.

Next, let $\mu^{p,q}$ be *any* random triangle measure for $\mathcal{L}_T$ with parameter values p and q. The same argument as the one used to prove (7) shows that $\mu^{p,q}_{0,j} \leq_d \mu^{p,q} \leq_d \mu^{p,q}_{1,j}$, and taking limits implies the next result.

COROLLARY 2.2 *For any random triangle measure* $\mu^{p,q}$ *for* $\mathcal{L}_T$ *with parameter values* p *and* q, *we have*

$$\mu^{p,q}_0 \leq_d \mu^{p,q} \leq_d \mu^{p,q}_1.$$

This has several important consequences. Firstly, it shows that $\mu^{p,q}_0$ and $\mu^{p,q}_1$ are invariant under graph automorphisms of $\mathcal{L}_T$ (i.e. under translations, rotations and reflections). Secondly, it implies that phase transition is equivalent to having $\mu^{p,q}_0 \neq \mu^{p,q}_1$. Further consequences concern the percolation behavior: since the event $\mathcal{C}$ of having an infinite cluster of open edges is increasing, we have

$$\mu^{p,q}_0(\mathcal{C}) \leq \mu^{p,q}(\mathcal{C}) \leq \mu^{p,q}_1(\mathcal{C}).$$

It is also not hard to see that $\mu^{p,q}_0$ and $\mu^{p,q}_1$ have to be extremal in the set of random triangle measures with the given parameter values, and that this implies tail triviality of $\mu^{p,q}_0$ and $\mu^{p,q}_1$. Hence

$$\mu^{p,q}_i(\mathcal{C}) \in \{0,1\} \tag{8}$$

for $i = 0, 1$.

So far, we have only used part (5) of Proposition 2.1. We can also use part (6) to show e.g. that $\mu^{p_1,q_1}_{i,j} \leq_d \mu^{p_2,q_2}_{i,j}$ for $i = 0, 1$ and $j = 1, 2, \ldots$ whenever $p_1 \leq p_2$ and $q_1 \leq q_2$. By letting $j \to \infty$, we obtain the next result:

COROLLARY 2.3 *For any* $p_1, p_2 \in [0,1]$ *and any* $q_1, q_2 \geq 1$ *such that* $p_1 \leq p_2$ *and* $q_1 \leq q_2$, *we have for* $i = 0, 1$ *that*

$$\mu^{p_1,q_1}_i \leq_d \mu^{p_2,q_2}_i. \tag{9}$$

This shows e.g. that for fixed q, the percolation probability $\mu_1^{p,q}(\mathcal{C})$ is nondecreasing in q, as claimed in the introduction. In conjunction with (8), this implies that $\mu_1^{p,q}(\mathcal{C})$ is a step function, i.e.

$$\mu_1^{p,q}(\mathcal{C}) = \begin{cases} 0 & \text{for } p < p_c(q) \\ 1 & \text{for } p > p_c(q). \end{cases}$$

The value of $\mu_1^{p,q}(\mathcal{C})$ at $p = p_c(q)$ cannot be deduced using these elementary monotonicity arguments. It is shown in [22] that $\mu_1^{p_c(q),q}(\mathcal{C}) = 0$ for $q = 1$, whereas for $q \geq 27 + 15\sqrt{3}$ we shall see in Section 4 that $\mu_1^{p_c(q),q}(\mathcal{C}) = 1$.

Finally, another standard consequence of Corollary 1.3 is the FKG inequality (see [19]) which, specialized to the random triangle model, becomes

Corollary 2.4 *Let $G = (V, E)$ be a finite graph, let $f, g : \{0,1\}^E \to \mathbf{R}$ be increasing functions, and let $\mu_G^{p,q}$ be the random triangle measure for G with parameters $p \in [0,1]$ and $q \geq 1$. Then*

$$\int fg \, d\mu_G^{p,q} \geq \int f \, d\mu_G^{p,q} \int g \, d\mu_G^{p,q}.$$

For the random triangle model on $\mathcal{L}_T$, the same conclusion extends easily to the measures $\mu_{0,n}^{p,q}$ and $\mu_{1,n}^{p,q}$, and by standard limiting arguments we obtain the FKG inequality also for $\mu_0^{p,q}$ and $\mu_1^{p,q}$.

3 Phase transition

In this section, we analyze the phase transition behavior of the random triangle model on $\mathcal{L}_T$. A key tool is the following alternative representation, introduced in [17].

Definition 3.1 *Let $G = (V, E)$ be a finite graph and let T be the set of triangles of G. For parameters $p \in [0,1]$ and $q \geq 1$, we define the probability measure $\nu_G^{p,q}$ on $\{0,1\}^T$ by letting it assign, to each $\omega \in \{0,1\}^T$, probability*

$$\nu_G^{p,q}(\omega) = \frac{1}{\tilde{Z}_G^{p,q}} p^{f(\omega)} (q-1)^{|\omega|} \tag{10}$$

where $|\omega| = \sum_{t \in T} \omega(t)$, and $f(\omega)$ is the number of edges that are part of a triangle t with $\omega(t) = 1$.

The relation between $\nu_G^{p,q}$ and the random triangle measure $\mu_G^{p,q}$ is best understood in terms of the following coupling.

Proposition 3.2 *Let $P_G^{p,q}$ be a probability measure on $\{0,1\}^E \times \{0,1\}^T$ given by*

$$P_G^{p,q}(\eta, \omega) = Z^{-1} \prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)} (q-1)^{|\omega|} \mathbf{1}_B(\eta, \omega) \tag{11}$$

where B is the set of outcomes (η, ω) such that $\omega(t) = 1$ implies that $\eta(e) = 1$ for all three edges of t. Then the first and second marginals of $P_G^{p,q}$ are $\mu_G^{p,q}$ and $\nu_G^{p,q}$, respectively.

This is Theorem 2.2 of [17], where a proof can be found (the proof is only a matter of summing out the marginals). It is analogous to the correspondence between random-cluster and Ising/Potts models (see e.g. [14]). We shall see that for a certain class of graphs, including the triangular lattice, the alternative representation of the random triangle model can be described as an Ising model with external field (for this reason, the $\omega(t)$'s will sometimes be called "spins"). It will turn out that the connection between the random triangle model on $\mathcal{L}_T$ and the Ising model is different from the connection between random-cluster and Ising/Potts models in the following respect: There is a direct correspondence between phase transitions in the random triangle and Ising models, whereas (see [14] again) phase transition in the Ising/Potts models does not correspond to phase transition in the random-cluster model, but rather to *percolation* in the random-cluster model.

Just like for random triangle measures one can define an infinite-volume version of $\nu_G^{p,q}$. Let us for convenience introduce a name for such measures: ν-measures. Here and in the sequel X is understood to be a random element in $\{0,1\}^T$ distributed according to a ν-measure.

DEFINITION 3.3 *A ν-measure with parameters $p \in [0,1]$ and $q \geq 1$ for an infinite locally finite graph $G = (V,E)$ with triangle set T is a probability measure on $\{0,1\}^T$ such that for all finite subsets S of T, all $\omega' \in \{0,1\}^S$ and ν-a.e. $\omega'' \in \{0,1\}^{T\setminus S}$ we have*

$$\nu(X(S) = \omega' | X(T \setminus S) = \omega'') = Z^{-1}(q-1)^{|\omega'|} p^{f(\omega',\omega'')} \tag{12}$$

where $f(\omega',\omega'')$ is the number of edges that are part of a triangle $s \in S$ with $\omega'(s) = 1$ but no triangle $t \in T \setminus S$ with $\omega''(t) = 1$. The normalizing constant Z may depend on ω'' but not on ω'.

Existence of ν-measures follows in the same way as for random triangle measures. Let us explicitly construct ν-measures for $\mathcal{L}_T$ in the same fashion as we did in Section 2 for the random triangle model. Let T_{Λ_n} be the set of triangles of $\mathcal{L}_T$ that have at least one edge in E_{Λ_n}. Define, for $i = 0,1$, the measure $\nu_{i,n}^{p,q}$ by assigning to all triangles in $T \setminus T_{\Lambda_n}$ spin i and then assigning spins to the triangles of T_{Λ_n} according to (12) with $\omega'' \equiv i$. The monotonicity arguments in Section 2 are easily modified to show that $\nu_{0,1}^{p,q} \leq_d \nu_{0,2}^{p,q} \leq_d \ldots$ and that $\nu_{1,1}^{p,q} \geq_d \nu_{1,2}^{p,q} \geq_d \ldots$ so that the limiting measures, denoted $\nu_0^{p,q}$ and $\nu_1^{p,q}$, exist and are automorphism invariant and also extremal in the class of ν-measures. We also get, in analogy with Corollary 2.2, that

$$\nu_0^{p,q} \leq_d \nu^{p,q} \leq_d \nu_1^{p,q} \tag{13}$$

for any ν-measure $\nu^{p,q}$ with the prescribed parameter values. The following theorem thus tells us that phase transitions for the random triangle model and its corresponding ν-model are equivalent.

THEOREM 3.4 *For $p \in [0,1]$ and $q \geq 1$, we have $\mu_0^{p,q} = \mu_1^{p,q}$ if and only if $\nu_0^{p,q} = \nu_1^{p,q}$.*

To prove this, we first need the following elementary modification of Proposition 3.2. The succeeding corollary is an immediate consequence.

PROPOSITION 3.5 *Let, for $i = 0, 1$, the probability measure $P_{i,n}^{p,q}$ on $\{0,1\}^E \times \{0,1\}^T$ be given by*

$$Z^{-1} \prod_{e \in E_{\Lambda_n}} p^{\eta(e)}(1-p)^{1-\eta(e)}(q-1)^{|\omega(T_{\Lambda_n})|}\mathbf{1}_{B_i}(\eta, \omega)$$

where B_i is the set of outcomes (η, ω), such that $\omega(t) = 1$ implies that $\eta(e) = 1$ for all three edges e of t, and $\omega(t) = \eta(e) = i$ for all $e \in E \setminus E_{\Lambda_n}$ and all $t \in T \setminus T_{\Lambda_n}$. Then the marginal distributions of $P_{i,n}^{p,q}$ are $\mu_{i,n}^{p,q}$ and $\nu_{i,n}^{p,q}$ respectively.

COROLLARY 3.6 *For $i = 0, 1$ the following statements hold.*

(a) *Choose a random element Y in $\{0,1\}^E$ by*

  (i) *choosing a random element X according to $\nu_{i,n}^{p,q}$,*
  (ii) *letting each edge $e \in E \setminus E_{\Lambda_n}$ take value i, and*
  (iii) *for each $e \in E_{\Lambda_n}$ letting e be open with probability 1 if e part of a triangle with spin 1 and with probability p if not, independently of other edges.*

  Then Y is distributed according to $\mu_{i,n}^{p,q}$.

(b) *Choose a random element X in $\{0,1\}^T$ by*

  (i) *choosing a random element Y according to $\mu_{i,n}^{p,q}$,*
  (ii) *letting each triangle $t \in T \setminus T_{\Lambda_n}$ take value i, and*
  (iii) *for each $t \in T_{\Lambda_n}$, letting t have spin 1 with probability $\frac{q-1}{q}$ if all three edges of t are open and with probability 0 if not, independently of other triangles.*

  Then X is distributed according to $\nu_{i,n}^{p,q}$.

Proof of Theorem 3.4. Let e_0 be any edge of $\mathcal{L}_T$. We claim that

$$\mu_0^{p,q} = \mu_1^{p,q} \quad \text{if and only if} \quad \mu_0^{p,q}(Y(e_0) = 1) = \mu_1^{p,q}(Y(e_0) = 1). \tag{14}$$

The 'only if' part of the claim is trivial so let us focus on the 'if' part. If $\mu_0^{p,q}(Y(e_0) = 1) = \mu_1^{p,q}(Y(e_0) = 1)$, then by automorphism invariance $\mu_0^{p,q}(Y(e) = 1) = \mu_1^{p,q}(Y(e) = 1)$ for every $e \in E$. Since $\mu_0^{p,q} \leq_d \mu_1^{p,q}$ we can invoke Strassen's Theorem to produce a coupling (Y, Y') such that $Y \sim \mu_0^{p,q}$, $Y' \sim \mu_1^{p,q}$ and $Y \leq Y'$ a.s. Letting P be the underlying probability measure of this coupling, we have

$$P(Y(e) \neq Y'(e)) = P(Y'(e) = 1) - P(Y(e) = 1) = 0$$

by assumption. This holds for any $e \in E$, so by countable additivity we have $P(Y = Y') = 1$, and and the claim (14) is proved.

Now, for $i = 0, 1$, weak convergence implies that

$$\mu_i^{p,q}(Y(e_0) = 1) = \lim_{n \to \infty} \mu_{i,n}^{p,q}(Y(e_0) = 1)$$

and by Corollary 3.6(a) this limit in turn equals

$$\lim_{n \to \infty} (\nu_{i,n}^{p,q}(C) + p(1 - \nu_{i,n}^{p,q}(C)))$$

where C is the event that at least one of the two triangles of which e_0 is part has spin 1. Again by weak convergence this limit equals

$$\nu_i^{p,q}(C) + p(1 - \nu_i^{p,q}(C)).$$

Finally, by modifying the coupling argument above it is readily shown that $\nu_0^{p,q} = \nu_1^{p,q}$ if and only if $\nu_0^{p,q}(C) = \nu_1^{p,q}(C)$. This completes the proof. □

We now proceed to show how ν-measures in some cases coincide with Gibbs measures for the Ising model. First, we define the Ising model. For a finite graph $G = (V, E)$ and a configuration $\omega \in \{0,1\}^V$, we denote, for $i = 0, 1$,

$$\begin{aligned} a_i(\omega) &= \text{the number of vertices with spin } i, \\ a_{ii}(\omega) &= \text{the number of edges both of whose endpoints have spin } i, \text{ and} \\ a_{01}(\omega) &= \text{the number of edges whose endpoints have different spin.} \end{aligned}$$

DEFINITION 3.7 *The Gibbs measure for the Ising model with coupling constant J and external field h on a finite graph $G = (V, E)$ is the probability measure $\pi_G^{J,h}$ on $\{0,1\}^V$ which to each $\omega \in \{0,1\}^V$ assigns probability*

$$\pi_G^{J,h}(\omega) = \frac{1}{\bar{Z}_G^{p,q}} \exp(-2ha_0(\omega) - 2Ja_{01}(\omega)).$$

(In the literature, the state space of the Ising model is often taken to be $\{-1,1\}^V$ rather than $\{0,1\}^V$.)

Suppose now that $G = (V, E)$ is a finite graph with triangle set T and the property each edge is contained in exactly two triangles. (There are plently of graphs with this property: the simplest example is the complete graph on four vertices, and others can e.g. be obtained from $\mathcal{L}_T$ by restricting to a rectangular subset of the vertex set and using a torus boundary condition.) It is then natural to define the *dual* graph of G. This is the graph $G^* = (V^*, E^*)$ obtained by letting $V^* = T$ and letting E^* consist of those pairs of triangles that share an edge in G.

PROPOSITION 3.8 *For $p \in (0,1)$, $q > 1$, and G and G^* as above, we have*

$$\nu_G^{p,q} = \pi_{G^*}^{J,h} \tag{15}$$

where

$$J = \frac{-\log p}{4} \tag{16}$$

and

$$h = \frac{\log(p^3(q-1)^2)}{4}. \tag{17}$$

Proof. Obviously, each vertex of G^* is the endpoint of exactly three edges. It follows that for any $\omega \in \{0,1\}^{V^*}$ the relations

$$\begin{cases} a_1(\omega) = \frac{2}{3}a_{11}(\omega) + \frac{1}{3}a_{01}(\omega) \\ a_0(\omega) = \frac{2}{3}a_{00}(\omega) + \frac{1}{3}a_{01}(\omega) \end{cases}$$

hold. Pick J and h according to (16) and (17). Writing $|V^*|$ for the cardinality of V^*, we get

$$\begin{aligned}\frac{\nu_G^{p,q}(\omega)}{\pi_{G^*}^{J,h}(\omega)} &= \frac{\frac{1}{\bar{Z}_G^{p,q}} p^{a_{11}(\omega)}(q-1)^{a_1(\omega)}}{\frac{1}{\tilde{Z}_G^{p,q}}(e^{-2h})^{a_0(\omega)}(e^{-2J})^{a_{01}(\omega)}} \\ &= \frac{\bar{Z}_G^{p,q} e^{2h|V^*|}}{\tilde{Z}_G^{p,q}} \left(\frac{e^{4h}}{(q-1)^2 p^3}\right)^{a_{11}(\omega)/3} \left(\frac{e^{2h}e^{-6J}}{(q-1)p^3}\right)^{a_{01}(\omega)/3} \\ &= \frac{\bar{Z}_G^{p,q} e^{2h|V^*|}}{\tilde{Z}_G^{p,q}} 1^{a_{11}(\omega)/3} 1^{a_{01}(\omega)/3} \\ &= \frac{\bar{Z}_G^{p,q} e^{2h|V^*|}}{\tilde{Z}_G^{p,q}}\end{aligned}$$

which does not depend on ω, so (15) follows by normalization. □

The property that each edge is contained in exactly two triangles holds for the triangular lattice $\mathcal{L}_T$, and the dual of $\mathcal{L}_T$ is the hexagonal lattice $\mathcal{L}_H$ (see Figure 1). The Ising model of course extends to infinite graphs in the same way as the random triangle model and its ν-representations (Definitions 1.4 and 3.3), and it posseses monotonicity properties similar to those discussed in Section 2 (see e.g. [19]). By the same argument as for Proposition 3.8, it is readily shown that the measures $\nu_{0,n}^{p,q}$ and $\nu_{1,n}^{p,q}$ coincide with the Gibbs measures for the Ising model on the appropriate subsets of $\mathcal{L}_H$ with 'minus' resp. 'plus' boundary conditions and parameter values given by (16) and (17). Taking limits, we get that $\nu_0^{p,q}$ and $\nu_1^{p,q}$ are nothing but the two stochastically extreme (minimal resp. maximal) Gibbs measures $\pi_0^{J,h}$ and $\pi_1^{J,h}$ for the Ising model on $\mathcal{L}_H$. Hence, the question of phase transition for the random triangle model on $\mathcal{L}_T$ has been reduced to that of determining whether or not

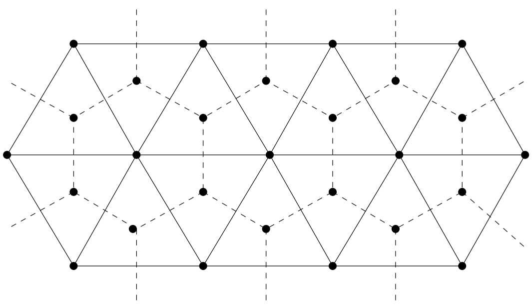

Figure 1: The planar duality of the triangular and hexagonal lattices.

$$\pi_0^{J,h} = \pi_1^{J,h}. \tag{18}$$

The first thing to observe is that (18) holds whenever $h \neq 0$; this is well-known in the case where $\mathcal{L}_H$ is replaced by the cubic lattice $\mathbf{Z}^d$, and the proof (see e.g. [5]) extends in a straightforward manner to $\mathcal{L}_H$ (although note that the result does not generalize to arbitrary infinite locally finite graphs due to non-amenable counterexamples such as the regular tree discussed in Section 5). For $h = 0$, it is known (see [1]) that (18) holds if and only if

$$J \leq \frac{\log(2+\sqrt{3})}{2}.$$

Theorem 1.5 follows by solving for p and q in (16) and (17).

4 The percolation threshold

The purpose of this section is to prove Theorem 1.6. The main part consists of showing that if $p \neq p_c(q)$, then the random triangle model cannot exhibit phase transition. The proof of this is easiest in the subcritical regime $p < p_c(q)$, so we begin with this case.

PROPOSITION 4.1 *If $p < p_c(q)$, then $\mu_0^{p,q} = \mu_1^{p,q}$.*

Proof. Let e_0 be some edge incident to the origin, and write $e_0 \leftrightarrow \partial\Lambda_n$ for the event that there exists an open path starting with e_0 and ending somewhere at $\partial\Lambda_n$. Here $\partial\Lambda_n$ is the set of vertices of Λ_n having at least one neighbor in $V \setminus \Lambda_n$. Fix $\epsilon > 0$. Since $p < p_c(q)$, we can find n large enough so that $\mu_1^{p,q}(e_0 \leftrightarrow \partial\Lambda_n) < \epsilon$. Then

$$\mu_1^{p,q}(Y(e_0) = 1) \leq \epsilon + (1-\epsilon)\mu_1^{p,q}(Y(e_0) = 1 \mid \{e_0 \leftrightarrow \partial\Lambda_n\}^c).$$

The event $\{e_0 \leftrightarrow \partial\Lambda_n\}^c$ implies that e_0 is cut off from the boundary of Λ_n by some "contour" of closed edges inside Λ_n. (To be exact, we mean by such a contour an unbroken path of open edges in the dual hexagonal lattice $\mathcal{L}_H$, where an edge in $\mathcal{L}_H$ is regarded as open if the edge in $\mathcal{L}_T$ that crosses it is closed.) Provided such a contour exists, then we can consider the outermost such contour contained in Λ_n. Since the event that the outermost contour is found at a given location is measurable with respect to all edges on and outside it (for a careful proof of this fact, one can e.g. mimic the sequential construction in the final section of [15]), the conditional distribution of the edge configuration inside the contour is, by the Markov random field structure of random triangle measures, the same as with all edges outside it closed. By Proposition 2.1, all such distributions are stochastically dominated by $\mu_0^{p,q}$. Thus

$$\begin{aligned}\mu_1^{p,q}(Y(e_0) = 1) &\leq \epsilon + (1-\epsilon)\mu_0^{p,q}(Y(e_0) = 1) \\ &\leq \epsilon + \mu_0^{p,q}(Y(e_0) = 1).\end{aligned} \tag{19}$$

By Corollary 2.2, we have $\mu_1^{p,q}(Y(e_0) = 1) \geq \mu_0^{p,q}(Y(e_0) = 1)$, and since ϵ was arbitrary in (19) it follows that

$$\mu_1^{p,q}(Y(e_0) = 1) = \mu_0^{p,q}(Y(e_0) = 1).$$

By (14), this implies $\mu_0^{p,q} = \mu_1^{p,q}$. $\Box$

The above proof relies on the fact that if there is no unbroken path of open edges from the origin to infinity, then there is somewhere a contour of closed edges surrounding the origin. A completely analogous argument would work for $p > p_c(q)$ if it could be shown for such p that there will for $\mu_0^{p,q}$ with certainty be an unbroken path of open edges surrounding the origin. Lemma 4.3 below states that this is indeed true. We first need an adaption to the present setting of a well-known result of Burton and Keane [4].

LEMMA 4.2 *Let P be a translation invariant probability measure on $\{0,1\}^E$, where E is the edge set of $\mathcal{L}_T$, with the property that*

$$0 < P(Y(e) = 1 \mid Y(E \setminus \{e\}) = \eta) < 1 \tag{20}$$

for each $e \in E$ and P-a.e. $\eta \in \{0,1\}^{E\setminus\{e\}}$. Then P-a.s. Y contains at most one infinite cluster of open edges. The same thing holds with $\mathcal{L}_T$ replaced by $\mathcal{L}_H$.

The property in (20) is usually referred to as the finite energy condition. Burton and Keane proved the analogous result for site percolation on $\mathbf{Z}^d$, but their proof goes through unchanged to prove Lemma 4.2. Lemma 4.2 is a key ingredient in the proof of the next lemma. Define, for each element $\eta \in \{0,1\}^E$, the dual element $\tilde{\eta} \in \{0,1\}^{E_H}$, where E_H is the set of edges of the dual hexagonal lattice $\mathcal{L}_H$ by, for each $e \in E$, letting $\tilde{\eta}(\tilde{e}) = 1 - \eta(e)$; here $\tilde{e} \in E_H$ is the unique edge crossing e.

LEMMA 4.3 *Let $p > p_c(q)$, pick a random element $Y \in \{0,1\}^E$ according to $\mu_0^{p,q}$, and consider the dual configuration $\tilde{Y} \in \{0,1\}^{E_H}$. Then $\mu_0^{p,q}(\tilde{\mathcal{C}}) = 0$, where $\tilde{\mathcal{C}}$ is the event that $\tilde{Y}$ contains an infinite connected component of open edges.*

Proof. We use the ideas of [15, Section 4] (alternatively, it is also possible to adapt the arguments of [10] in order to prove the lemma). Since $\mu_0^{p,q}$ is extremal in the class of random triangle measures on $\mathcal{L}_T$, the tail σ-field is trivial under $\mu_0^{p,q}$. Thus $\mu_0^{p,q}(\tilde{\mathcal{C}})$ is either 0 or 1. Assume for contradiction that

$$\mu_0^{p,q}(\tilde{\mathcal{C}}) = 1. \tag{21}$$

From Theorem 1.5 we know that provided that q stays fixed, $\mu_0^{p,q} \neq \mu_1^{p,q}$ for at most one value of p. Therefore we can find p' strictly between $p_c(q)$ and p such that $\mu_0^{p',q} = \mu_1^{p',q}$. Since $p' > p_c(q)$ and $\mu_0^{p',q} \leq_d \mu_0^{p,q}$ it follows that $\mu_0^{p,q}(0 \leftrightarrow \infty) > 0$. By tail triviality we thus have $\mu_0^{p,q}(\mathcal{C}) = 1$, so we a.s. have an infinite cluster of open edges both in Y and in $\tilde{Y}$. We saw in Section 2 that $\mu_0^{p,q}$ is translation invariant. Furthermore, by Lemma 1.2, it satisfies (20), whence by Lemma 4.2 the infinite open cluster in Y is almost surely unique. The same properties hold for $\tilde{Y}$, so we know that Y and $\tilde{Y}$ must a.s. contain exactly one infinite open cluster each.

Now let n be large enough to ensure that

$$\mu_0^{p,q}(A_n) > 0.9999$$

and

$$\mu_0^{p,q}(B_n) > 0.9999$$

where A_n is the event that the infinite connected component of Y intersects Λ_n and B_n is the corresponding event concerning $\tilde{Y}$. Enumerate the six sides of Λ_n, in clockwise order, $1, 2, \ldots, 6$. Write $A_n = \cup_{i=1}^6 A_n^i$ and $B = \cup_{i=1}^6 B_n^i$ where A_n^i is the event that the infinite connected component of Y intersects side i of Λ_n and that there is a path from side i going off to infinity without intersecting the interior of Λ_n. The B_n^i's are defined analogously for $\tilde{Y}$. The events A_n^i are increasing in the partial order on $\{0,1\}^E$ so by the FKG inequality (Corollary 2.4) they are positively correlated. This implies that

$$\mu_0^{p,q}(A_n) \leq 1 - \prod_{i=1}^{6} \mu_0^{p,q}(A_n^i).$$

By symmetry the A_n^i's all have the same probabiltity so that

$$\mu_0^{p,q}(A_n^i) \geq 1 - 0.0001^{1/6} > 0.78.$$

The events B_n^i are decreasing so exactly the same arguments apply to show that

$$\mu_0^{p,q}(B_n^i) > 0.78.$$

By Bonferroni's inequality it follows that for any $\{i,j,k,l\} \subseteq \{1,2,3,4,5,6\}$ we have

$$\mu_0^{p,q}(A_n^i \cap A_n^j \cap B_n^k \cap B_n^l) > 1 - 4 \cdot 0.22 = 0.12 > 0. \tag{22}$$

Suppose now that the event in (22) occurs with $(i,j,k,l) = (1,4,2,5)$. Then the infinite open clusters of Y that intersect sides 1 and 4 must be connected to each other by uniqueness of the infinite cluster. Since an open path in $\tilde{Y}$ cannot cross an open path in Y, the geometry of the situation now prevents the infinite open clusters of $\tilde{Y}$ intersecting sides 2 and 5 from being connected to each other. This contradicts the uniqueness of the infinite cluster property of $\tilde{Y}$, so the assumed (21) must be false, and the proof is complete. □

As argued prior to Lemma 4.2, the following result is implied by Lemma 4.3.

Proposition 4.4 *If $p > p_c(q)$, then $\mu_0^{p,q} = \mu_1^{p,q}$.*

Theorem 1.5 and Propositions 4.1 and 4.4 together imply that $p_c(q) = (q-1)^{-2/3}$ for $q > 27 + 15\sqrt{3}$. What remains in order to prove the full statement of Theorem 1.6 is to show that $p_c(q)$ is decreasing and continuous.

Proposition 4.5 *The function $p_c(q)$ is decreasing and continuous on $[1, \infty)$.*

Proof. The event $\mathcal{C}$ that there exists an infinite open cluster is increasing, so Corollary 2.3 implies that

$$\mu_1^{p,q_1}(\mathcal{C}) \le \mu_1^{p,q_2}(\mathcal{C})$$

whenever $q_1 \le q_2$. It follows that $p_c(q)$ is a decreasing function of q. To show continuity, we need a slight sharpening of Corollary 1.3. With the notation of Lemma 1.2 in force, we have for the random triangle model on $\mathcal{L}_T$ that $\Delta(\eta, e) \in \{0,1,2\}$ for any $e \in E$ and any $\eta \in \{0,1\}^E$. It follows that (2) holds whenever

$$\frac{p_1 q_1^i}{p_1 q_1^i + 1 - p_1} \le \frac{p_2 q_2^i}{p_2 q_2^i + 1 - p_2} \quad \text{for } i = 0,1,2. \tag{23}$$

Fixing p_1, q_1 and $\epsilon > 0$, it is clear that (23) holds with $p_2 = p_1 + \epsilon$ and $q_2 = q_1 - \delta$ provided that $\delta > 0$ is sufficiently small. From this, we can deduce, using the same arguments as those used to prove Corollary 2.3, that (9) holds for such a choice of p_1, q_1, p_2 and q_2. We thus have for any $q \ge 1$ and any $\epsilon > 0$ that

$$\mu_1^{p+\epsilon, q-\delta}(\mathcal{C}) \ge \mu_1^{p,q}(\mathcal{C})$$

for small enough $\delta > 0$. Since $p_c(q)$ is decreasing, this shows left continuity of $p_c(q)$. Right continuity follows similarly. □

5 Extensions and variants

A natural direction for further research is to study the behavior of the random triangle model on triangular lattices in $d \ge 3$ dimensions. Unfortunately, most of what we do in Sections 3 and 4 breaks down as we pass to higher dimensions. In particular, the ν-measures will no longer be equivalent to Gibbs measures for the Ising model, and furthermore Lemma 4.3 relies crucially on the planar geometry of

$\mathcal{L}_T$. We nevertheless conjecture that, for reasonable choices of d-dimensional lattices, the random triangle model will exhibit phase transition if and only if $p = p_c(q)$ and q is sufficiently large. One result that does go through in higher dimensions is Proposition 4.1, i.e. the result that there is a unique random triangle measure whenever $p < p_c(q)$. It is also not hard to show that for fixed q, phase transition occurs for at most countably many values of p; this can be done by proving convexity in p of the so called pressure, following Grimmett's [13] proof of the analogous result for the random-cluster model. Perhaps Pirogov–Sinai theory (see e.g. [18]) can be exploited to make further progress towards proving the full conjecture.

Let us next give an example of an infinite graph G for which the behavior of the random triangle model is qualitatively different from that on $\mathcal{L}_T$. Its vertex and edge sets are defined as follows. Start with three edges e_1, e_2 and e_3 forming a triangle. We say that these edges constitute the first generation of edges. The rest of G is defined recursively: To each edge e in the ith generation, we associate a new vertex v, and form edges between v and the two endpoints of e. These edges belong to the $(i+1)$st generation. See Figure 2. Like $\mathcal{L}_T$, this graph has the property that every

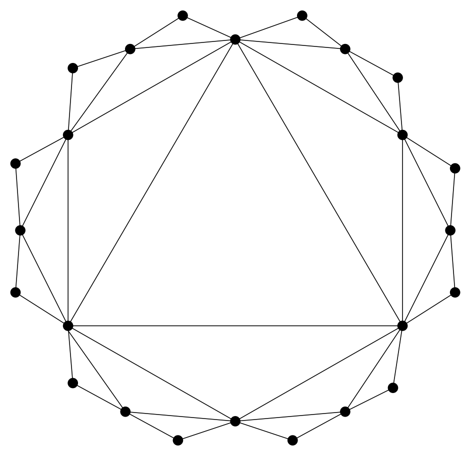

Figure 2: The first four generations of G.

edge is contained in exactly two triangles. Furthermore, its dual G^* is the regular tree in which every vertex is incident to exactly three edges. Using arguments in Section 3, results about the Ising model on regular trees (see e.g. [21] or [11]) can thus be translated into results about the random triangle model on G, and it turns out that for q sufficiently large, there is an entire interval (of length greater than 0) of values of p for which phase transition occurs. This contrasts sharply with the random triangle model in d-dimensional lattices. Note that the random triangle model on G is well-defined despite the fact that G is rather badly behaved in other repects (for instance, every vertex of G is incident to infinitely many edges, which implies that $p_c(q) = 0$ for each q).

Finally, we point out that there are various natural variants of the random triangle model for which the methods in Sections 2–4 also work. Instead of rewarding triangles as in Definitions 1.1 and 1.4, one may reward the appearance of other structures such as squares on the square lattice and hexagons on the hexagonal lattice. Key properties are that the rewarded structures are such that each edge of the lattice is part of exactly two such structures (in order for the Ising model translation to work), and that the structures are sufficiently "localized" to make the contour arguments used in the proofs of Propositions 4.1 and 4.4 work. For instance, suppose that we are on the square lattice, and that we replace $t(\eta', \eta'')$ in Definition 1.4 by $s(\eta', \eta'')$, defined as the number of open squares (of size 1×1) that have at least one edge in S. Let us call this the "random square model". Adapting the arguments of Sections 2 and 3 and using the well-known result that the Ising model on the square lattice has a phase transition if and only if $J > \frac{1}{2}\log(1+\sqrt{2})$ and $h = 0$ (see e.g. [11] or [5]), we can deduce that the random square model on the square lattice with parameters p and q exhibits phase transition if and only if $q > 18 + 12\sqrt{2} \approx 34.97$ and $p = (q-1)^{-1/2}$. Furthermore, the arguments in Section 4 show that $p_c(q) = (q-1)^{-1/2}$ for all $q \geq 18 + 12\sqrt{2}$.

ACKNOWLEDGMENT. We thank Jeff Steif for stimulating discussions.

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