Random-cluster analysis of a class of binary lattice gases

Olle Häggström*
Chalmers University of Technology

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Abstract

We introduce a class of binary lattice gases which can be viewed as a lattice analogue of the continuum Widom–Rowlinson model, and which also is related to the beach model of Burton and Steif. This new model is shown to exhibit phase transition for large particle intensities. Stochastic monotonicity results of varying strength are derived in various parts of the parameter space. The main tool is a random-cluster representation of the model, analogous to the Fortuin–Kasteleyn representation of the Potts model.

Key words: Phase transition, Gibbs measure, Widom–Rowlinson model, lattice gas, random-cluster representation.

1 Introduction

In 1970, Widom and Rowlinson [21] introduced a stochastic binary gas model which informally can be described as follows. We have two types of particles A and B living in $\mathbf{R}^d$, and the distribution of particles is that of two independent Poisson processes, with respective intensities λ_A and λ_B, conditioned on the event that no two particles of different type are within distance r from each other. By scaling, we can without loss of generality set $r = 1$. Widom and Rowlinson conjectured that for $d \geq 2$, this model would exhibit a phase transition in the symmetric high-intensity regime where $\lambda_A = \lambda_B = \lambda$ and λ is sufficiently large. That this indeed is the case was soon established by Ruelle [19], and more recently Chayes *et al.* [6] gave a modern stochastic-geometric proof of this result. By phase transition, we here mean the nonuniqueness of infinite-volume DLR (Dobrushin–Lanford–Ruelle) states with prescribed conditional distributions on compact subsets of $\mathbf{R}^d$. It is also known (see e.g. [6] or [15]) that phase transition does not occur when λ is small. This strongly suggests that the following conjecture should be true:

Conjecture 1.1: *For $d \geq 2$, there exists a critical value $\lambda_c = \lambda_c(d) \in (0, \infty)$ such that the Widom–Rowlinson model in $\mathbf{R}^d$ with $\lambda_A = \lambda_B = \lambda$ exhibits a phase transition for $\lambda > \lambda_c$, but not for $\lambda < \lambda_c$.*

What current rigorous knowledge about the Widom–Rowlinson model is lacking in this conjecture is the monotonicity property that if $\lambda_1 < \lambda_2$ and there is phase transition at $\lambda = \lambda_1$, then there should be phase transition also at $\lambda = \lambda_2$. The corresponding monotonicity is well known for several lattice models such as the Ising and Potts models

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(see e.g. [10] and [15]). At first, one might therefore think that the problem with proving the monotonicity in λ for the Widom–Rowlinson model would have something to do with the fact that the particles live in the continuum $\mathbf{R}^d$, rather than on a discrete lattice. This, however, is not the case. Lebowitz and Gallavotti [17] introduced a binary lattice gas closely analogous to the Widom–Rowlinson model, and established phase transition in the "high-intensity" regime, but also for this model the conjectured monotonicity in the intensity escapes a rigorous proof. The principal difficulty, which will be discussed in Section 6 of the present paper, appears to be essentially the same for the lattice analogue as for the original Widom–Rowlinson model; see also Sections 7 and 8 in Häggström [15].

The purpose of this paper is to find some other lattice model that (i) is close in spirit to the Widom–Rowlinson model, and (ii) is mathematically sufficiently tractable to admit a proof of a statement analogous to Conjecture 1.1. In Section 2 we present such a model, which we call **the volume-perturbed lattice Widom–Rowlinson (VLWR) model**, and state our main results. Section 3 and 4 provide additional motivation for the VLWR model by establishing intimate relations with other lattice models (the beach model of Burton and Steif [4], and a certain trinary lattice gas). In Section 5, we introduce a so-called random-cluster representation of the VLWR model. This representation is the key tool used to prove the main results; this is done in Section 6. Finally, in Section 7, we will discuss a multitype generalization of the VLWR model.

Before closing this introductory section, let us say a few general words about the method we exploit in order to prove our main results: the use of a random-cluster representation. This approach involves a geometric representation of the original model. The types of particles are unidentified, and this allows *a priori* difficult questions about phase transition to be translated into easier questions about percolation (i.e. into stochastic-geometric questions about long-range connectivities). The original random-cluster model can be used to represent Ising and Potts models, and was introduced by Fortuin and Kasteleyn [8]. Aizenman *et al.* [1] later demonstrated how this representation could be used to study the phase transition phenomenon in simple and elegant probabilistic terms. Subsequently, random-cluster representations of several other models were introduced and exploited. For instance, Chayes *et al.* [6] studied the phase transition behaviour of the Widom–Rowlinson model by such means, and Häggström and Georgii [11] used similar methods to study a wider class of models allowing e.g. soft-core interactions between particles. We refer to Häggström [15] for a general introduction to such methods, and to Grimmett [13] for a modern probabilistic discussion of the original (Fortuin–Kasteleyn) random-cluster model.

2 The model and the main results

We first describe the VLWR model on a finite graph. Let $G = (V, E)$ be a finite connected graph with vertex set V and edge set E. For $v, w \in V$, write $d(v, w)$ for the graph-theoretical distance between v and w, i.e. $d(v, w)$ is the number of edges in the shortest path between v and w. Each $v \in V$ can be in one of three states A, B and 0, where A and B represent two different types of particles, and 0 is void. The state space $\{A, 0, B\}$ is equipped with the ordering $A < 0 < B$. For each $W \subseteq V$ we write $\preceq$ for the induced coordinatewise partial order on $\{A, 0, B\}^W$, so that for $\xi, \eta \in \{A, 0, B\}^W$ we have $\xi \preceq \eta$ if and only if $\xi(v) \leq \eta(v), \forall v \in W$.

The VLWR model on G has two parameters λ (the activity parameter) and γ (the volume-interaction parameter). A particle configuration $\xi \in \{A, 0, B\}^V$ is called **feasible** if for no pair of vertices $v, w \in V$ with $d(v, w) \leq 2$ we have $\xi(v) = A$ and $\xi(w) = B$. In other words, for a configuration to be feasible we need that no two particles of different type sit within distance 2 from each other. Writing $|\cdot|$ for cardinality of a set, we define, for $\xi \in \{A, 0, B\}^V$,

$$\begin{cases} n_A(\xi) & = & |\{v \in V : \xi(v) = A\}| \\ n_B(\xi) & = & |\{v \in V : \xi(v) = B\}| \\ n_*(\xi) & = & |\{v \in V : \xi(w) = 0 \text{ for all } w \in V \text{ such that } d(v, w) \leq 1\}|. \end{cases} \tag{1}$$

Definition 2.1: *The VLWR measure $\nu_G^{\lambda,\gamma}$ on $\{A, 0, B\}^V$ with parameters $\lambda \geq 0$ and $\gamma \geq 0$ is the probability measure which to each $\xi \in \{A, 0, B\}^V$ assigns probability*

$$\nu_G^{\lambda,\gamma}(\xi) = \begin{cases} \frac{1}{Z_G^{\lambda,\gamma}} \lambda^{n_A(\xi)+n_B(\xi)} \gamma^{n_*(\xi)} & \textit{if } \xi \textit{ is feasible} \\ 0 & \textit{otherwise.} \end{cases}$$

Here $Z_G^{\lambda,\gamma}$ is a normalizing constant.

When $\gamma = 1$, the factor $\gamma^{n_*(\xi)}$ disappears and the VLWR measure $Z_G^{\lambda,1}$ arises by letting each site independently be in state A, 0 or B with respective probabilities $\frac{\lambda}{1+2\lambda}$, $\frac{1}{1+2\lambda}$ and $\frac{\lambda}{1+2\lambda}$, and then conditioning on the event that the arising configuration is feasible. It is natural to interpret the feasibility condition by thinking of each particle as having a nonzero radius which makes it occupy not only the site at which it is centered, but also all neighbouring vertices in G. The feasibility condition then says that particles can overlap only if they are of the same type. With this interpretation, n_* becomes the number of sites that are not covered by some particle. Taking $\gamma > 1$ then amounts to biasing $Z_G^{\lambda,1}$ in favour of configurations where the amount of such "vacuum" is large, so that particles tend to be packed closer together, whereas taking $\gamma < 1$ instead biases the measure towards spreading the particles more evenly over G. (The way in which the γ parameter perturbs $Z_G^{\lambda,1}$ is similar in spirit to a certain perturbation of the Poisson process known as the area-interaction process; see Baddeley and van Lieshout [2].)

Of course, it would be natural to allow the particles A and B to have two different intensities λ_A and λ_B, but in this paper we will deal exclusively with the symmetric case $\lambda_A = \lambda_B$, so we save some ink by writing simply λ for their common value.

We now go on to define the VLWR model on $\mathbf{Z}^d$ in accordance with the usual DLR formalism. We think of $\mathbf{Z}^d$ as a graph with edges connecting vertices whose (Euclidean) distance is 1. For a finite set $S \subset \mathbf{Z}^d$ and configurations $\xi \in \{A, 0, B\}^S$ and $\xi' \in \{A, 0, B\}^{\mathbf{Z}^d \setminus S}$, we define $(\xi \vee \xi') \in \{A, 0, B\}^{\mathbf{Z}^d}$ to be the configuration on $\mathbf{Z}^d$ which coincides with ξ on S and with ξ' on $\mathbf{Z}^d \setminus S$. For such S, ξ and ξ', we define

$$\begin{cases} n_A(S, \xi) & = |\{v \in S : \xi(v) = A\}| \\ n_B(S, \xi) & = |\{v \in S : \xi(v) = B\}| \\ n_*(S, \xi, \xi') & = |\{v \in S \cup \partial S : (\xi \vee \xi')(w) = 0 \text{ for all } w \in \mathbf{Z}^d \text{ such that } d(v, w) \leq 1\}|, \end{cases}$$

where ∂S denotes the "outer boundary" of S, i.e.

$$\partial S = \{v \in \mathbf{Z}^d \setminus S : \exists w \in S \text{ such that } d(v, w) = 1\}.$$

For a feasible configuration $\xi' \in \{A, 0, B\}^{\mathbf{Z}^d \setminus S}$, we define $\nu^{\lambda,\gamma}_{S,\xi'}$ to be the probability measure on $\{A, 0, B\}^S$ which to each $\xi \in \{A, 0, B\}^S$ assigns probability

$$\nu^{\lambda,\gamma}_{S,\xi'}(\xi) = \begin{cases} \frac{1}{Z^{\lambda,\gamma}_{S,\xi'}} \lambda^{n_A(S,\xi)+n_B(S,\xi)} \gamma^{n_*(S,\xi,\xi')} & \text{if } (\xi \vee \xi') \text{ is feasible} \\ 0 & \text{otherwise,} \end{cases}$$

where $Z^{\lambda,\gamma}_{S,\xi'}$ is a normalizing constant (Z with various sub- and superscripts will always denote normalizing constants).

Definition 2.2: *Let ν be a probability measure on $\{A, 0, B\}^{\mathbf{Z}^d}$, and let X be a $\{A, 0, B\}^{\mathbf{Z}^d}$-valued random object distributed according to ν. We say the ν is a Gibbs measure for the VLWR model on $\mathbf{Z}^d$ with parameters $\lambda \geq 0$ and $\gamma \geq 0$ if it is concentrated on feasible elements of $\{A, 0, B\}^{\mathbf{Z}^d}$ and for all finite $S \subset \mathbf{Z}^d$ admits conditional probabilities such that*

$$\nu(X(S) = \xi \mid X(\mathbf{Z}^d \setminus S) = \xi') \;=\; \nu^{\lambda,\gamma}_{S,\xi'}(\xi) \tag{2}$$

for all feasible $\xi' \in \{A, 0, B\}^{\mathbf{Z}^d \setminus S}$ and all $\xi \in \{A, 0, B\}^S$.

Note that the VLWR model on a finite graph has similar conditional distributions, and that for any nested sequence $S_1 \subset S_2 \subset \cdots \subset \mathbf{Z}^d$, (2) gives a consistent set of conditional distributions. Furthermore, if ν is a Gibbs measure for the VLWR model on $\mathbf{Z}^d$, then the conditional distribution of $X(S)$ given $X(\mathbf{Z}^d \setminus S)$ depends on $X(\mathbf{Z}^d \setminus S)$ only via its values on vertices $w \in \mathbf{Z}^d \setminus S$ sitting within distance 2 from some $v \in S$. In other words, X is a Markov random field with range 2, justifying the term "Gibbs measure" used in the definition. The existence of some Gibbs measure for the VLWR model on $\mathbf{Z}^d$ with the given parameter values λ and γ follows by standard compactness arguments (see e.g. [10]). Here we focus on the question of uniqueness (or nonuniqueness) of such measures. In particular, we are interested in how the multiplicity of Gibbs measures varies with λ when γ is kept fixed. For $d = 1$, there is a unique Gibbs measure for any λ and γ (as is the case for all finite state Gibbs models in one dimension satisfying a mild irreducibility condition) so we will focus on $d \geq 2$ only. It turns out that for general $\gamma > 0$, we can do no more than what has been done for the other gas models discussed in the introduction:

Theorem 2.3: *For fixed $d \geq 2$ and $\gamma > 0$, the VLWR model on $\mathbf{Z}^d$ with parameters λ and γ has a unique Gibbs measure if λ is taken to be sufficiently small. If instead λ is taken to be sufficiently large, then the model has more then one Gibbs measure.*

If we restrict to $\gamma \geq 2$ then the situation is somewhat more satisfactory, as we can prove the following analogue of Conjecture 1.1.

Theorem 2.4: *For fixed $d \geq 2$ and $\gamma \geq 2$, there exists a critical value $\lambda_c = \lambda_c(d, \gamma)$ such that for $\lambda < \lambda_c$, the VLWR model on $\mathbf{Z}^d$ with parameters λ and γ has a unique Gibbs measure, whereas for $\lambda > \lambda_c$ the model has more then one Gibbs measure.*

It seems reasonable to expect that the assertion in Theorem 2.4 should be true for any $\gamma > 0$, but the monotonicity arguments that we will use in Section 6 are not sufficiently strong to yield such a conclusion. (Another issue left open is that of whether or not there is a unique Gibbs measure at the critical value λ_c.) Hence, the model exhibits a kind of threshold at $\gamma = 2$ as far as amenability to certain monotonicity arguments is concerned. It turns out that there is another such threshold at $\gamma = 1$; see Proposition 2.5 below. We need some more preliminaries before we can state that result.

For V finite or infinite, and two probability measures ν_1 and ν_2 on $\{A,0,B\}^V$, we say that ν_1 is stochastically dominated by ν_2, writing $\nu_1 \preceq_d \nu_2$, if for all increasing (with respect to $\preceq$) local functions $f : \{A,0,B\}^V \to \mathbf{R}$ we have

$$\int_{\{A,0,B\}^V} f\, d\nu_1 \leq \int_{\{A,0,B\}^V} f\, d\nu_2.$$

By a celebrated theorem of Strassen (see [20] or [18]), this is equivalent to the existence of a coupling of two $\{A,0,B\}^V$-valued random elements X_1 and X_2 (i.e. of a joint construction of X_1 and X_2 on the same probability space) such that X_1 has distribution ν_1 and X_2 has distribution ν_2, and with the property that $X_1 \preceq X_2$ with probability 1.

We equip $\{A,0,B\}^{\mathbf{Z}^d}$ with the usual product topology, so that weak convergence of a sequence $\{\nu_i\}_{i=1}^{\infty}$ of probability measures on $\{A,0,B\}^{\mathbf{Z}^d}$ to a limting measure ν, is equivalent to having $\lim_{i\to\infty} \nu_i(C) = \nu(C)$ for any cylinder event C (a cylinder event is an event which depends on finitely many coordinates only).

For finite $S \subset \mathbf{Z}^d$, we let ξ'_A denote the configuration on $\mathbf{Z}^d \setminus S$ consisting of A's only, and define $\nu_{S,A}^{\lambda,\gamma}$ to be the probability measure on $\{A,0,B\}^{\mathbf{Z}^d}$ for which $X(\mathbf{Z}^d \setminus S) = \xi'_A$ almost surely, and $X(S)$ has distribution $\nu_{S,\xi'_A}^{\lambda,\gamma}$. We define ξ'_B and $\nu_{S,B}^{\lambda,\gamma}$ analogously.

Proposition 2.5: *Fix $d \geq 2$, $\lambda \geq 0$ and $\gamma \geq 1$, and let $\mathbf{S} = \{S_i\}_{i=1}^{\infty}$ be any sequence of subsets of $\mathbf{Z}^d$ which is increasing in the sense that $S_1 \subset S_2 \subset \cdots$, and which converges to $\mathbf{Z}^d$ in the sense that each $v \in \mathbf{Z}^d$ is in all but finitely many S_i's. Then the limiting measures*

$$\nu_A^{\lambda,\gamma} = \lim_{i\to\infty} \nu_{S_i,A}^{\lambda,\gamma} \tag{3}$$

and

$$\nu_B^{\lambda,\gamma} = \lim_{i\to\infty} \nu_{S_i,B}^{\lambda,\gamma} \tag{4}$$

on $\{A,0,B\}^{\mathbf{Z}^d}$ exist and are independent of the choice of $\mathbf{S}$ (and therefore also translation invariant). Both limits are Gibbs measures for the VLWR model with parameters λ and γ, and furthermore

$$\nu_A^{\lambda,\gamma} \preceq_d \nu^{\lambda,\gamma} \preceq_d \nu_B^{\lambda,\gamma} \tag{5}$$

for any other Gibbs measure $\nu^{\lambda,\gamma}$ for the VLWR model with the same parameters. This means, in particular, that the existence of more than one Gibbs measure for the VLWR model with the given parameters is equivalent to having

$$\nu_A^{\lambda,\gamma} \neq \nu_B^{\lambda,\gamma}. \tag{6}$$

All these results will be proved in Section 6.

3 Relation to the beach model

In this section, we will show how the VLWR model with $\gamma = 2$ in a certain sense is equivalent to the so-called beach model, which was introduced by Burton and Steif [4] and further studied by Häggström [14].

Let M_1 and M_2 be positive integers such that $M_1 < M_2$. The beach model on a finite graph G with parameters M_1 and M_2 can be described as follows. Define the set F of attainable values at each vertex $v \in V$ as

$$F = F_1 \cup F_2 \cup F_3 \cup F_4$$

where

$$\begin{array}{rcl} F_1 & = & \{-M_2, -M_2+1, \ldots, -M_1-1\} \\ F_2 & = & \{-M_1, -M_1+1, \ldots, -1\} \\ F_3 & = & \{1, 2, \ldots, M_1\} \\ F_4 & = & \{M_1+1, M_1+2, \ldots, M_2\}. \end{array}$$

Call $f \in F$

$$\begin{cases} \text{negative} & \text{if} \quad f \in F_1 \cup F_2 \\ \text{positive} & \text{if} \quad f \in F_3 \cup F_4 \\ \text{unprivileged} & \text{if} \quad f \in F_1 \cup F_4 \\ \text{privileged} & \text{if} \quad f \in F_2 \cup F_3, \end{cases}$$

and call a configuration $\zeta \in F^V$ feasible if for all $u, v \in V$ such that $d(u,v) = 1$ we have that $\zeta(u)$ and $\zeta(v)$ are either both positive, both negative or both privileged. In other words, negatives and positives are not allowed to sit next to each other unless they are both privileged. The name "beach model" comes from the interpretation in two-dimensional lattices that if a symbol represents altitude above sea level, then the feasibility condition prevents shores from being too steep.

Definition 3.1: *The beach measure $\psi_G^{M_1,M_2}$ for the graph $G = (V, E)$ with parameters M_1 and M_2 is the probability measure on F^V which is equidistributed over all feasible elements of F^V.*

This definition extends in a natural way to infinite-volume Gibbs measures as follows.

Definition 3.2: *Let ψ be a probability measure on $F^{\mathbf{Z}^d}$, and let U be an $F^{\mathbf{Z}^d}$-valued random object with distribution ψ. We call ψ a Gibbs measure for the beach model on $\mathbf{Z}^d$ with parameters M_1 and M_2 if it is concentrated on feasible elements of $F^{\mathbf{Z}^d}$ and admits conditional probabilities such that for all finite $S \subset \mathbf{Z}^d$ and all feasible $\zeta' \in F^{\mathbf{Z}^d \setminus S}$ the conditional distribution of $U(S)$ given $U(\mathbf{Z}^d \setminus S) = \zeta'$ is uniform over all $\zeta \in F^S$ for which $(\zeta \vee \zeta')$ is feasible.*

The beach model in $d \geq 2$ dimensions with $\frac{M_2}{M_1}$ sufficiently large exhibits a phase transition, as shown in [4]. It was later shown in [14] that for $d \geq 2$, there is a critical value $M_c = M_c(d) \in (1, \infty)$ such that there is a unique Gibbs measure for $\frac{M_2}{M_1} < M_c$ and multiple Gibbs measures for $\frac{M_2}{M_1} > M_c$.

Here we shall demonstrate a certain equivalence between on one hand Gibbs measures for the VLWR model with $\lambda = \frac{M_2 - M_1}{M_1}$ and $\gamma = 2$, and on the other hand Gibbs measures for the beach model with parameters M_1 and M_2. One consequence of this equivalence is that

$$\lambda_c(d, 2) = M_c(d) - 1 \tag{7}$$

where λ_c is defined as in Theorem 2.4. The exact value of $M_c(d)$ is not known in any dimension; some very crude upper and lower bounds can be found in [4] and in [14], respectively.

We begin with the finite graph case. Consider the following way of picking a random configuration $U \in F^V$. First pick $X \in \{A, 0, B\}^V$ according to the VLWR measure for

G with $\lambda = \frac{M_2 - M_1}{M_1}$ and $\gamma = 2$. Then, for each $v \in V$ independently, pick $U(v)$ uniformly from

$$\begin{cases} F_1 & \text{if } X(v) = A \\ F_2 & \text{if } X(v) = 0 \text{ and } X(w) = A \text{ for some } w \text{ with } d(v,w) = 1 \\ F_2 \cup F_3 & \text{if } X(v) = 0 \text{ and } X(w) = 0 \text{ for all } w \text{ with } d(v,w) = 1 \\ F_3 & \text{if } X(v) = 0 \text{ and } X(w) = B \text{ for some } w \text{ with } d(v,w) = 1 \\ F_4 & \text{if } X(v) = B. \end{cases} \tag{8}$$

Write $\tilde{\psi}_G^{M_1,M_2}$ for the distribution of $U \in F^V$ obtained in such a way.

Proposition 3.3: *The above procedure yields an F^V-valued random object distributed according to the beach measure for G with parameters M_1 and M_2, i.e.*

$$\tilde{\psi}_G^{M_1,M_2} = \psi_G^{M_1,M_2}.$$

Proof: It is clear from the construction that $\tilde{\psi}_G^{M_1,M_2}$ assigns positive probability only to feasible elements of F^V. It is therefore sufficient to show that

$$\tilde{\psi}_G^{M_1,M_2}(\zeta_1) = \tilde{\psi}_G^{M_1,M_2}(\zeta_2) \tag{9}$$

for any two feasible configurations $\zeta_1, \zeta_2 \in F^V$. Write ξ_1 for the (unique) element of $\{A, 0, B\}$ from which ζ_1 can be obtained by the above procedure, and define ξ_2 similarly. Recall the definitions of $n_A(\xi)$, $n_B(\xi)$ and $n_*(\xi)$ in (1). In addition to these, define

$$\begin{cases} n_a = |\{v \in V : \xi(v) = 0, \exists w \in V \text{ such that } d(v,w) = 1 \text{ and } \xi(w) = A\}| \\ n_b = |\{v \in V : \xi(v) = 0, \exists w \in V \text{ such that } d(v,w) = 1 \text{ and } \xi(w) = B\}|, \end{cases}$$

and note that for ξ feasible, we have

$$n_A(\xi) + n_a(\xi) + n_*(\xi) + n_b(\xi) + n_B(\xi) = |V|.$$

We get

$$\begin{aligned}
\frac{\tilde{\psi}_G^{M_1,M_2}(\zeta_1)}{\tilde{\psi}_G^{M_1,M_2}(\zeta_2)} &= \\
&= \frac{\nu_G^{\lambda,2}(\xi_1)\left(\frac{1}{M_2-M_1}\right)^{n_A(\xi_1)+n_B(\xi_1)}\left(\frac{1}{M_1}\right)^{n_a(\xi_1)+n_b(\xi_1)}\left(\frac{1}{2M_1}\right)^{n_*(\xi_1)}}{\nu_G^{\lambda,2}(\xi_2)\left(\frac{1}{M_2-M_1}\right)^{n_A(\xi_2)+n_B(\xi_2)}\left(\frac{1}{M_1}\right)^{n_a(\xi_2)+n_b(\xi_2)}\left(\frac{1}{2M_1}\right)^{n_*(\xi_2)}} \\
&= \frac{\frac{1}{Z_G^{\lambda,2}}\lambda^{n_A(\xi_1)+n_B(\xi_1)}2^{n_*(\xi_1)}\left(\frac{1}{M_2-M_1}\right)^{n_A(\xi_1)+n_B(\xi_1)}\left(\frac{1}{M_1}\right)^{n_a(\xi_1)+n_b(\xi_1)}\left(\frac{1}{2M_1}\right)^{n_*(\xi_1)}}{\frac{1}{Z_G^{\lambda,2}}\lambda^{n_A(\xi_2)+n_B(\xi_2)}2^{n_*(\xi_2)}\left(\frac{1}{M_2-M_1}\right)^{n_A(\xi_2)+n_B(\xi_2)}\left(\frac{1}{M_1}\right)^{n_a(\xi_2)+n_b(\xi_2)}\left(\frac{1}{2M_1}\right)^{n_*(\xi_2)}} \\
&= \frac{\frac{1}{Z_G^{\lambda,2}}\left(\frac{1}{M_1}\right)^{n_A(\xi_1)+n_a(\xi_1)+n_*(\xi_1)+n_b(\xi_1)+n_B(\xi_1)}}{\frac{1}{Z_G^{\lambda,2}}\left(\frac{1}{M_1}\right)^{n_A(\xi_2)+n_a(\xi_2)+n_*(\xi_2)+n_b(\xi_2)+n_B(\xi_2)}} = 1
\end{aligned}$$

so (9) holds, and we are done. □

One can also go the other way (from the beach model to the VLWR model): Suppose we pick $U \in F^V$ according to the beach measure $\psi_G^{M_1,M_2}$, and then pick $X \in \{A,0,B\}^V$ by letting

$$X(v) = \begin{cases} A & \text{if } U(v) \in F_1 \\ 0 & \text{if } U(v) \in F_2 \cup F_3 \\ B & \text{if } U(v) \in F_4. \end{cases} \tag{10}$$

Then X has distribution $\nu_G^{\lambda,\gamma}$, with $\lambda = \frac{M_2-M_1}{M_1}$ and $\gamma = 2$; this follows immediately from Proposition 3.3.

The next result is an infinite-volume analogue of Proposition 3.3.

Proposition 3.4: *Suppose that we pick $X \in \{A,0,B\}^{\mathbf{Z}^d}$ according to a Gibbs measure $\nu^{\lambda,\gamma}$ for the VLWR model with $\lambda = \frac{M_2-M_1}{M_1}$ and $\gamma = 2$, and that we then obtain $U \in F^{\mathbf{Z}^d}$ from X by the procedure described in (8). Then the distribution of U is a Gibbs measure for the beach model with parameters M_1 and M_2.*

Proof: Write $\tilde{\psi}^{M_1,M_2}$ for the distribution of $U \in F^{\mathbf{Z}^d}$. It is clear from the construction that $\tilde{\psi}^{M_1,M_2}$ is concentrated on the set of feasible elements of F^V, so all we need to do is to show for any finite $S \subset \mathbf{Z}^d$ that $\tilde{\psi}^{M_1,M_2}$ satisfies the uniform conditional probability property of Definition 3.3. Define

$$S^* = \{v \in \mathbf{Z}^d : \exists w \in S \text{ such that } d(v,w) \leq 2\}.$$

Since X is a Markov random field with range 2, we have that U is a Markov random field with range at most 2. In other words, the conditional distribution of $U(S)$ given $U(\mathbf{Z}^d \setminus S)$ depends only on $U(S^* \setminus S)$. Let G be a finite graph whose vertex set V is some cubic portion $\{-k,\ldots,k\}^d$ of $\mathbf{Z}^d$, where k is large enough so that V contains S^*, and whose edge set E consists of Eucliedan nearest neighbours (just as in the $\mathbf{Z}^d$ lattice). Pick a $\{A,0,B\}^V$-valued random element X_G according to the VLWR measure $\nu_G^{\lambda,2}$, and pick $U_G \in F^V$ as in (8) using X_G. By the Markov random field properties of X and X_G, and the construction of U and U_G, we have for any $\zeta' \in F^{S^* \setminus S}$ that the conditional distribution of $U(S)$ given $U(S^*) = \zeta'$ must be the same as the conditional dostribution of $U_G(S)$ given $U_G(S^*) = \zeta'$. But the latter conditional distribution has the desired uniformity property by Proposition 3.3, so we are done. $\Box$

We thus have a simple way to create Gibbs measures for the beach model from Gibbs meaures for the VLWR model. The next result shows that we can also go the other way.

Proposition 3.5: *Suppose that we pick $U \in F^{\mathbf{Z}^d}$ according to a Gibbs measure for the beach model with parameters M_1 and M_2, and that we obtain $X \in \{A,0,B\}^{\mathbf{Z}^d}$ by pointwise application of (10). Then X is distributed according to some Gibbs measure for the VLWR model with $\lambda = \frac{M_2-M_1}{M_1}$ and $\gamma = 2$.*

Proof: Let $\bar{\nu}^{\lambda,2}$ be *any* Gibbs measure for the VLWR model on $\mathbf{Z}^d$ with the given parameter values, let $\bar{X} \in \{A,0,B\}^{\mathbf{Z}^d}$ have distribution $\bar{\nu}^{\lambda,2}$, and obtain $\bar{U} \in F^{\mathbf{Z}^d}$ from $\bar{X}$ as in Proposition 3.4. Then, by the same proposition, the distribution $\bar{\psi}^{M_1,M_2}$ of $\bar{X}$ is a Gibbs measure for the beach model. Let P be some probability measure supporting the random configurations U, X, $\bar{U}$ and $\bar{X}$. Define S^* as in the previous proof, and further define

$$S^{**} = \{v \in \mathbf{Z}^d : \exists w \in S \text{ such that } d(v,w) \leq 3\}.$$

In order to show that X is distributed according to a Gibbs measure for the VLWR model, we need to show

(i) that X is feasible with probability 1,

(ii) that for any finite $S \subset \mathbf{Z}^d$ the conditional distribution of $X(S)$ given $X(\mathbf{Z}^d \setminus S)$ depends only on $X(S^{**} \setminus S)$, and

(iii) that for any feasible $\xi' \in \{A, 0, B\}^{S^{**}\setminus S}$ the conditional distribution of $X(S)$ given $X(S^{**} \setminus S) = \xi'$ is the same as the conditional distribution of $\bar{X}(S)$ given $\bar{X}(S^{**} \setminus S) = \xi'$.

That (i) holds is immediate from the construction. To show that (ii) and (iii) hold, we let ζ'' be some feasible element of $F^{S^{**}\setminus S^*}$. By Definition 3.2, the conditional distribution of $U(S^*)$ given $U(\mathbf{Z}^d \setminus S^*)$ depends only on $U(S^{**}\setminus S^*)$, and the corresponding statement holds for $\bar{U}$. Furthermore, by the same definition, the conditional distribution of $U(S^*)$ given that $U(S^{**} \setminus S^*) = \zeta''$ is the same as the conditional distribution of $\bar{U}(S^*)$ given that $\bar{U}(S^{**} \setminus S^*) = \zeta''$. It follows that the conditional distribution of $X(S^*)$ given that $U(S^{**} \setminus S^*) = \zeta''$ is the same as the conditional distribution of $\bar{X}(S^*)$ given that $\bar{U}(S^{**} \setminus S^*) = \zeta''$. This, in turn, implies that the conditional distribution of $X(S)$ given $[X(S^* \setminus S) = \xi', U(S^{**} - S^*) = \zeta'']$ is the same as the conditional distribution of $\bar{X}(S)$ given $[\bar{X}(S^* \setminus S) = \xi', \bar{U}(S^{**} - S^*) = \zeta'']$. But the latter conditional distribution agrees with Definition 2.2, so the former must do so as well, whence (iii) is proved. By noting that the last two conditional distributions are independent of ζ'' and of any further information about configurations on $\mathbf{Z}^d \setminus S^*$, we also get (ii). $\square$

It is easy to see that the two mappings in Propositions 3.4 and 3.5 constitute a bijection between Gibbs measures for the VLWR model and Gibbs measures for the beach model. This implies, in particular, that uniqueness of Gibbs measures for the VLWR model with parameters M_1 and M_2 is equivalent to uniqueness of Gibbs measures for the beach model with $\lambda = \frac{M_2 - M_1}{M_1}$ and $\gamma = 2$. Hence, (7) is established.

Let us finally point out that there exists yet another Gibbs model which is equivalent to the other two; namely, the so-called site-centered ferromagnet which was introduced in [14]. The site-centered ferromagnet has state space $\{-1, 1\}$ at each vertex and is obtained from the beach model similarly as in (10) by setting all positives to $+1$ and all negatives to -1. While the beach model is a Markov random field with range 1, both the VLWR model and the site-centered ferromagnet are Markov random fields with range 2.

4 Relation to a trinary lattice gas

To readers who find the γ factor in the definition of the VLWR model unnatural, we here give an alternative representation of the $\gamma \geq 1$ VLWR model. We call this new model the **particle-perturbed lattice Widom–Rowlinson (PLWR) model**. In this alternative representation, the volume-interaction factor is disposed of, at the cost of having to introduce a third particle type C. The AB symmetry of the VLWR model is preserved in the PLWR setting, but the third particle type C plays a role which is different from the other two.

As usual, we begin with the case of a finite graph $G = (V, E)$. Each $v \in V$ will be in one of four states A, B, C and 0. A particle configuration $\zeta \in \{A, B, C, 0\}^V$ is said to be feasible if

(i) for no pair of vertices $v, w \in V$ with $d(v,w) \leq 2$ we have $\zeta(v) = A$ and $\zeta(w) = B$, and

(ii) for no pair of vertices $v, w \in V$ with $d(v,w) = 1$, we have $\zeta(v) = C$ and $\zeta(w) \in \{A, B\}$.

As in the VLWR model, the feasibility condition should be thought of as preventing particles of different type from overlapping each other. We then have to think of A and B particles as having the same (large) radius, and of C as having a different (smaller) radius.

For $\zeta \in \{A, B, C, 0\}^V$, define $n_A(\zeta)$ and $n_B(\zeta)$ as in Section 2, and define $n_C(\zeta)$ analogously (i.e. $n_C(\zeta)$ is the number of vertices $v \in V$ for which $\zeta(v) = C$).

Definition 4.1: *The PLWR measure $\pi_G^{\lambda,\lambda_C}$ on $\{A, B, C, 0\}^V$ with parameters $\lambda \geq 0$ and $\lambda_C \geq 0$ is the probability measure which to each $\zeta \in \{A, B, C, 0\}^V$ assigns probability*

$$\pi_G^{\lambda,\lambda_C} = \begin{cases} \frac{1}{\hat{Z}_G^{\lambda,\lambda_C}} \lambda^{n_A(\zeta)+n_B(\zeta)} \lambda_C^{n_C(\zeta)} & \text{if } \zeta \text{ is feasible} \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the following way of obtaining a random element $W \in \{A, B, C, 0\}^V$. First pick $X \in \{A, 0, B\}^V$ according to the VLWR measure $\nu_G^{\lambda,\gamma}$ with $\gamma \geq 1$. Then let $W(v) = X(v)$ for every $v \in V$ with the property that $X(v) \in \{A, B\}$ or $X(w) \in \{A, B\}$ for some $w \in V$ with $d(v,w) = 1$. Finally, let $\lambda_C = \gamma - 1$, and let all remaining vertices get value 0 or C independently, with respective probabilities $\frac{1}{\lambda_C+1}$ and $\frac{\lambda_C}{\lambda_C+1}$.

Proposition 4.2: *The random element $W \in \{A, B, C, 0\}^V$, obtained as above, is distributed according to the PLWR measure $\pi_G^{\lambda,\lambda_C}$.*

We omit the proof of this result, which is an easy adaption of the proof of Proposition 3.3. An immediate consequence of the result is that if we pick $W \in \{A, B, C, 0\}^V$ according to $\pi_G^{\lambda,\lambda_C}$ and then just delete all the C-particles (i.e. turn them into 0's), then the arising $\{A, 0, B\}^V$-valued random element has distribution $\mu_G^{\lambda,\gamma}$ with $\gamma = \lambda_C + 1$.

The PLWR model may be extended from finite graphs G to the lattice $\mathbf{Z}^d$ in the same way that such extensions of the VLWR and beach models were made in Definitions 2.2 and 3.2. A straightforward adaption of the proofs of Propositions 3.4 and 3.5 then give us an equivalence between Gibbs measures for the VLWR model on $\mathbf{Z}^d$ with parameters λ and γ on one hand, and Gibbs measures for the PLWR model on $\mathbf{Z}^d$ with parameters λ and $\lambda_C = \gamma - 1$ on the other, completely analogous to the equivalence between Gibbs measures for VLWR and beach models in Section 3. This means e.g. that once we have proved Theorems 2.3 and 2.4, we can immediately claim the corresponding results for the PLWR model.

5 The random-cluster representation

We now introduce the random-cluster representation of the VLWR model on a finite graph $G = (V, E)$. In this representation, each site can only be in one of two states, 0 and 1, where a 1 signifies the presence of a particle (of either type A or type B), and a 0 means that no particle is present.

Given a random-cluster configuration $\eta \in \{0,1\}^V$, we define an edge configuration $\omega = \omega(\eta) \in \{0,1\}^E$ by letting each $e \in E$ take value

$$\omega(e) = \begin{cases} 1 & \text{if } \eta(v) = 1 \text{ for at least one of the endpoints } v \text{ of } e, \\ 0 & \text{otherwise.} \end{cases}$$

We then consider the connected components in the random graph obtained from G by deleting each edge $e \in E$ with $\omega(e) = 0$. We write $k_1(\eta)$ for the number of such connected components that contain only a single vertex $v \in V$, and $k_2(\eta)$ for the number of connected components that contain at least two vertices (note that $k_2(\eta)$ counts exactly those connected components that contain some $v \in V$ such that $\eta(v) = 1$). We also let $n(\eta)$ denote the number of vertices $v \in V$ for which $\eta(v) = 1$.

Definition 5.1: *For $p \in [0,1]$ and $\gamma \geq 0$, we define the VLWR random-cluster measure $\mu_G^{p,\gamma}$ on $\{0,1\}^V$ to be the probability measure which, to each $\eta \in \{0,1\}^V$, assigns probability*

$$\mu_G^{p,\gamma}(\eta) = \frac{1}{\tilde{Z}_G^{p,\gamma}} p^{n(\eta)}(1-p)^{|V|-n(\eta)} \gamma^{k_1(\eta)} 2^{k_2(\eta)}. \tag{11}$$

For $\lambda \geq 0$ and $\gamma \geq 0$, consider the following way of picking a random configuration $X \in \{A,0,B\}^V$. First pick $Y \in \{0,1\}^V$ according to the VLWR random-cluster measure $\mu_G^{p,\gamma}$, where $p = \frac{\lambda}{\lambda+1}$. Then let $X(v) = 0$ for each $v \in V$ for which $Y(v) = 0$. Finally, for each connected component C of $\omega(Y)$, we flip an independent fair coin, and if it comes up heads, then we let $X(v) = A$ for every vertex v in C such that $Y(v) = 1$, while if tails, then we let $X(v) = B$ for every v in C with $Y(v) = 1$. Write $\tilde{\nu}_G^{\lambda,\gamma}$ for the distribution of $X \in \{A,0,B\}^V$ obtained in this way.

Proposition 5.2: *The above procedure yields an $\{A,0,B\}^V$-valued random element distributed according to the VLWR measure for G with parameters λ and γ, i.e.*

$$\tilde{\nu}_G^{\lambda,\gamma} = \nu_G^{\lambda,\gamma}.$$

Proof: This proof resembles closely the proof of Proposition 3.3. It is immediate from the construction that $\tilde{\nu}_G^{\lambda,\gamma}(\xi) > 0$ if and only if ξ is feasible. Therefore, it suffices to show that

$$\frac{\tilde{\nu}_G^{\lambda,\gamma}(\xi_1)}{\tilde{\nu}_G^{\lambda,\gamma}(\xi_2)} = \frac{\nu_G^{\lambda,\gamma}(\xi_1)}{\nu_G^{\lambda,\gamma}(\xi_2)}$$

for any two feasible configurations $\xi_1, \xi_2 \in \{A,0,B\}^V$. Pick two such configurations ξ_1 and ξ_2, and define $\eta_1, \eta_2 \in \{0,1\}^V$ to be the corresponding random-cluster configurations, i.e.

$$\eta_i(v) = \begin{cases} 1 & \text{if } \xi_i(v) \in \{A,B\} \\ 0 & \text{if } \xi_i(v) = 0 \end{cases}$$

for $i = 1,2$ and each $v \in V$. Note that $n(\eta_i) = n_A(\xi_i) + n_B(\xi_i)$, and that $k_1(\eta_i) = n_*(\xi_i)$. We get

$$\frac{\tilde{\nu}_G^{\lambda,\gamma}(\xi_1)}{\tilde{\nu}_G^{\lambda,\gamma}(\xi_2)} = \frac{\mu_G^{p,\gamma}(\eta_1) 2^{-k_2(\eta_1)}}{\mu_G^{p,\gamma}(\eta_2) 2^{-k_2(\eta_2)}} = \frac{\frac{(1-p)^{|V|}}{\tilde{Z}_G^{p,\gamma}} (\frac{p}{1-p})^{n(\eta_1)} \gamma^{k_1(\eta_1)}}{\frac{(1-p)^{|V|}}{\tilde{Z}_G^{p,\gamma}} (\frac{p}{1-p})^{n(\eta_2)} \gamma^{k_1(\eta_2)}}$$

$$
\begin{aligned}
&= \frac{\lambda^{n(\eta_1)}\gamma^{k_1(\eta_1)}}{\lambda^{n(\eta_2)}\gamma^{k_1(\eta_2)}} = \frac{\lambda^{n_A(\xi_1)+n_B(\xi_1)}\gamma^{n_*(\xi_1)}}{\lambda^{n_A(\xi_2)+n_B(\xi_2)}\gamma^{n_*(\xi_2)}} \\
&= \frac{\nu_G^{\lambda,\gamma}(\xi_1)}{\nu_G^{\lambda,\gamma}(\xi_2)}
\end{aligned}
$$

as desired. $\square$

In the next section, we will use the random-cluster representation to study the VLWR model on $\mathbf{Z}^d$. Fortunately, this does not require an extension of the VLWR random-cluster model to the case of infinite graphs (for the Fortuin–Kasteleyn random-cluster model, such an extension is not entirely elementary (see e.g. [13]) and although a similar extension for the VLWR random-cluster model is possible, it does invlove certain technicalities). However, we will need a random-cluster representation of the measure $\nu_{S,A}^{\lambda,\gamma}$ defined prior to Proposition 2.5. This turns out to be a straightforward extension of the finite graph case, because $\nu_{S,A}^{\lambda,\gamma}$ is concentrated on a finite subset of $\{A,0,B\}^{\mathbf{Z}^d}$.

As usual, we let S be a finite subset of $\mathbf{Z}^d$. Let $\mathcal{Y}_S$ denote the set of configurations $\eta \in \{0,1\}^{\mathbf{Z}^d}$ with the property that $\eta(v) = 1$ for each $v \in \mathbf{Z}^d \setminus S$ (note that $\mathcal{Y}_S$ is finite). For $\eta \in \mathcal{Y}_S$, define $\omega(\eta)$, $k_1(\eta)$ and $k_2(\eta)$ as in the case of a finite graph G. Note that both $k_1(\eta)$ and $k_2(\eta)$ are finite for each $\eta \in \mathcal{Y}_S$, because the random graph corresponding to $\omega(\eta)$ will contain a single infinite connected component, and a finite number of finite connected components, all of which are contained in S. Also define $n(S,\eta)$ to be the number of vertices $v \in S$ for which $\eta(v) = 1$.

Definition 5.3: *For $p \in [0,1]$ and $\gamma \geq 0$, we define the measure $\mu_{S,1}^{p,\gamma}$ on $\{0,1\}^{\mathbf{Z}^d}$ to be the probability measure which is concentrated on $\mathcal{Y}_S$ and which to each $\eta \in \mathcal{Y}_S$ assigns probability*

$$
\mu_{S,1}^{p,\gamma} = \frac{1}{\tilde{Z}_{S,1}^{p,\gamma}} p^{n(S,\eta)}(1-p)^{|V|-n(S,\eta)}\gamma^{k_1(\eta)}2^{k_2(\eta)}.
$$

An $\{A,0,B\}^{\mathbf{Z}^d}$-valued random element X with distribution $\nu_{S,A}^{\lambda,\gamma}$ can now be obtained by a procedure analogous to the one described prior to Proposition 5.2. First let $p = \frac{\lambda}{1+\lambda}$ and pick the random element $Y \in \{0,1\}^V$ according to $\mu_{S,1}^{p,\gamma}$. Then let $X(v) = 0$ for each v for which $Y(v) = 0$, and let $X(v) = A$ for each $v \in \mathbf{Z}^d$ which sits in the infinite connected component of $\omega(Y)$ and which has $Y(v) = 1$. Finally, for each finite connected component C of $\omega(Y)$, flip an independent fair coin to determine whether all vertices v in C with $Y(v) = 1$ should take value A or B in X.

Lemma 5.4: *The random configuration $X \in \{A,0,B\}^{\mathbf{Z}^d}$, picked as above, has distribution $\nu_{S,A}^{\lambda,\gamma}$.*

We omit the proof, as it is completely analogous to the proof of Proposition 5.2. Of course, $\mu_{S,1}^{p,\gamma}$ can also be used to obtain a random element $X \in \{A,0,B\}^{\mathbf{Z}^d}$ with distribution $\nu_{S,B}^{\lambda,\gamma}$; just modify the above construction by assigning value B instead of A to vertices in the infinite connected component of $\omega(Y)$.

The next simple lemma plays a key role in the analysis of phase transitions in the VLWR model. It relates the effect that the "boundary condition" $X(\mathbf{Z}^d \setminus S)$ has on the distribution of $X(v)$ to a certain connectivity probability in the random-cluster representation. Given the finite set $S \subset \mathbf{Z}^d$, a vertex $v \in S$, and a random configuration

$Y \in \{0,1\}^{\mathbf{Z}^d}$, write $v \leftrightarrow \mathbf{Z}^d \setminus S$ for the event that v is in a connected component of $\omega(Y)$ which intersects $\mathbf{Z}^d \setminus S$.

Lemma 5.5: *With S and v as above, and $\lambda, \gamma \geq 0$, we have*

$$\nu_{S,A}^{\lambda,\gamma}(X(v) = A) - \nu_{S,A}^{\lambda,\gamma}(X(v) = B) \ = \ \mu_{S,1}^{p,\gamma}(Y(v) = 1,\, v \leftrightarrow \mathbf{Z}^d \setminus S)$$

where $p = \frac{\lambda}{1+\lambda}$.

Proof: Write $D_{S,v}$ for the event that $Y(v) = 1$ and v is in a connected component of $\omega(Y)$ which does not intersect $\mathbf{Z}^d \setminus S$. By Lemma 5.4, we have that

$$\nu_{S,A}^{\gamma,\lambda}(X(v) = A) \ = \ \mu_{S,1}^{p,\gamma}(Y(v) = 1,\, v \leftrightarrow \mathbf{Z}^d \setminus S) + \tfrac{1}{2}\mu_{S,1}^{p,\gamma}(D_{S,v})$$

and that

$$\nu_{S,A}^{\gamma,\lambda}(X(v) = B) \ = \ \tfrac{1}{2}\mu_{S,1}^{p,\gamma}(D_{S,v}).$$

The desired equality follows. □

We end this section by remarking that in the $\gamma = 2$ case, the random-cluster representation presented here reduces to the random-cluster representation of the beach model introduced in [15].

6 Proofs of main results

In addition to the random-cluster representation given in the previous section, the other basic ingredients in the proofs of the main results of this paper are (i) a simple percolation model and (ii) a comparison result (Lemma 6.1) which is essentially due to Holley [16] and which is closely related to the celebrated FKG inequality [9].

For a finite set V, we generalize the partial order $\preceq$ defined in Section 2 to the coordinatewise partial order on $\mathbf{R}^V$. For a finite set T and a probability measure P on T^V, we say that P is **irreducible** if the set $\{\eta \in T^V : P(\eta) > 0\}$ is connected in the sense that any element of T^V with positive P-probability can be reached from any other via successive coordinate changes without passing through elements with zero P-probability.

Lemma 6.1 (Holley): *Let V be a finite set and let T be a finite subset of* $\mathbf{R}$. *Let P and P' be two probability measures on S^V, and let X and X' be random elements with respective distributions P and P'. Assume that P' is irreducible and that it assigns positive probability to the maximal element of T^V. Suppose furthermore that for every $v \in V$, every $t \in T$, and every $\xi, \eta \in T^{V \setminus \{v\}}$ such that $\xi \preceq \eta$, $P(X(V \setminus \{v\}) = \xi) > 0$ and $P'(X'(V \setminus \{v\}) = \eta) > 0$, we have*

$$P(X(v) \geq t \,|\, X(V \setminus \{v\}) = \xi) \ \leq \ P'(X'(v) \geq t \,|\, X'(V \setminus \{v\}) = \eta).$$

Then $P \preceq_d P'$.

Proof: The result follows by copying (almost verbatim) Holley's [16] original proof, a variant of which can also be found in [15]. □

We next introduce a percolation model on $\mathbf{Z}^d$ as follows. Let $\bar{\mu}^p_{\mathbf{Z}^d}$ denote the probability measure on $\{0,1\}^{\mathbf{Z}^d}$ corresponding to letting each $v \in \mathbf{Z}^d$ independently take value 1 or 0 with respective probabilities p and $1-p$. Let $\bar{Y}$ be a $\{0,1\}^{\mathbf{Z}^d}$-valued random configuration with distribution $\bar{\mu}^p_{\mathbf{Z}^d}$, define the edge configuration $\omega(\bar{Y})$ as in Section 5, and write $v \leftrightarrow \infty$ for the event that $v \in \mathbf{Z}^d$ is in an infinite connected component of $\omega(\bar{Y})$. Of course, $\bar{\mu}^p_{\mathbf{Z}^d}(v \leftrightarrow \infty)$ is independent of the choice of v.

Lemma 6.2: *For $d \geq 2$, there exists a critical value $p_c = p_c(d) \in (0,1)$ such that*

$$\bar{\mu}^p_{\mathbf{Z}^d}(\bar{Y}(v) = 1, v \leftrightarrow \infty) \begin{cases} = 0 & \text{if } p < p_c \\ > 0 & \text{if } p > p_c. \end{cases}$$

Proof: Consider standard independent site percolation with retention parameter p on the graph $\mathbf{Z}^{d*}$ whose vertex set is $\mathbf{Z}^d$ and whose edge set consists of all pairs of vertices within L_1-distance 2 from each other. Write $\theta(p)$ for the probability in this model that a given vertex v is in an infinite connected component. Then there exists a $p_c^* = p_c^*(d) \in (0,1)$ such that

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c^* \\ > 0 & \text{if } p > p_c^* \end{cases}$$

as follows from a completely straightforward adaption of the usual proofs of the corresponding result for site or bond percolation on the nearest neighbour graph of $\mathbf{Z}^d$; see e.g. Grimmett [12]. But a moment's thought reveals that $\theta(p) = \bar{\mu}^p_{\mathbf{Z}^d}(\bar{Y}(v) = 1, v \leftrightarrow \infty)$, so the lemma follows with $p_c = p_c^*$. $\Box$

Proof of Theorem 2.3: The first assertion (that sufficiently small λ implies a unique Gibbs measure) is easily shown by applying either of two standard techniques: Dobrushin's uniqueness theorem (see [7] or [10]) or the disagreement percolation approach of van den Berg and Maes [3]. We omit the details.

Instead, we go on to prove the second assertion (that sufficiently large λ implies non-uniqueness of Gibbs measures). Let $\mathbf{S} = \{S_i\}_{i=1}^\infty$ be an increasing sequence of finite subsets of $\mathbf{Z}^d$ converging to $\mathbf{Z}^d$ as in Proposition 2.5. Consider the sequence $\{\nu^{\lambda,\gamma}_{S_i,A}\}_{i=1}^\infty$ of probability measures on $\{A, 0, B\}^{\mathbf{Z}^d}$. By compactness, we can find some subsequential weak limit of these measures; write $\tilde{\nu}^{\lambda,\gamma}_A$ for some such limiting measure. By general Gibbs theory (see [10]), $\tilde{\nu}^{\lambda,\gamma}_A$ is a Gibbs measure for the VLWR model with the given parameter values. Pick a vertex $v \in S_1$. In order to prove non-uniqueness of Gibbs measures for large λ, it is sufficient to show that

$$\tilde{\nu}^{\lambda,\gamma}_A(X(v) = A) - \tilde{\nu}^{\lambda,\gamma}_A(X(v) = B) \; > \; 0 \tag{12}$$

because if we had a unique Gibbs measure, then by the symmetry of the VLWR model (with respect to interchange of A and B) the left-hand side of (12) would have to equal 0. To show that (12) holds for λ large, it suffices to show that

$$\liminf_{i \to \infty} \left(\nu^{\lambda,\gamma}_{S_i,A}(X(v) = A) - \nu^{\lambda,\gamma}_{S_i,A}(X(v) = B) \right) \; > \; 0$$

for λ large. By Lemma 5.5, this is equivalent to showing that

$$\liminf_{i \to \infty} \mu^{p,\gamma}_{S_i,1}(Y(v) = 1, \, v \leftrightarrow \mathbf{Z}^d \setminus S_i) > 0 \tag{13}$$

for p sufficiently close to 1. The strategy for proving (13) will be to compare $\mu^{p,\gamma}_{S_i,1}$ to the percolation measure $\bar{\mu}^{p^*}_{\mathbf{Z}^d}$ for a certain choice of p^*, using Lemma 6.1. To this end, we need to compute the single-site conditional distributions for $\bar{\mu}^{p^*}_{\mathbf{Z}^d}$ and for $\mu^{p,\gamma}_{S_i,1}$. Obviously,

$$\bar{\mu}^{p^*}_{\mathbf{Z}^d}(\bar{Y}(w) = 1 \,|\, \bar{Y}(\mathbf{Z}^d \setminus \{w\}) = \eta') \;=\; p^*$$

for any $w \in \mathbf{Z}^d$ and any $\eta' \in \{0,1\}^{\mathbf{Z}^d \setminus \{w\}}$. The corresponding relation for $\mu^{p,\gamma}_{S_i,1}$ is of course more complicated: For $w \in S_i$ and $\eta' \in \{0,1\}^{\mathbf{Z}^d \setminus \{w\}}$, define $\eta'_0 = (\eta' \vee 0)$, i.e. η'_0 is the element of $\{0,1\}^{\mathbf{Z}^d}$ which is 0 at w and agrees with η' elsewhere. Furthermore, define $\kappa_1(w,\eta'_0)$ (resp. $\kappa_2(w,\eta'_0)$) to be the number of connected components in $\omega(\eta'_0)$ which intersect the set $\{u \in \mathbf{Z}^d : d(u,w) \leq 1\}$, and which contain exactly one (resp. more than one) vertex of $\mathbf{Z}^d$. We only need to consider those η' for which $\eta'_0 \in \mathcal{Y}_S$. For such η', a direct application of Definition 5.3 yields

$$\mu^{p,\gamma}_{S_i,1}(Y(w) = 1 \,|\, Y(\mathbf{Z}^d \setminus \{w\}) = \eta') \;=\; \frac{p\gamma^{-\kappa_1(w,\eta'_0)} 2^{1-\kappa_2(w,\eta'_0)}}{p\gamma^{-\kappa_1(w,\eta'_0)} 2^{1-\kappa_2(w,\eta'_0)} + 1 - p}. \tag{14}$$

Since κ_1 and κ_2 both take their values in $\{0,\ldots,2d+1\}$, we have for $\gamma \leq 1$ that the right-hand side of (14) is bounded from below by

$$\frac{p2^{-2d}}{p2^{-2d} + 1 - p}$$

while for $\gamma \geq 1$ it is bounded from below by

$$\frac{p\gamma^{-2d-1}2^{-2d}}{p\gamma^{-2d-1}2^{-2d} + 1 - p}.$$

Combining these observations, we thus have, for any γ, any $w \in S_i$ and any η' chosen as above, that

$$\mu^{p,\gamma}_{S_i,1}(Y(w) = 1 \,|\, Y(\mathbf{Z}^d \setminus \{w\}) = \eta') \;\geq\; \min\left\{ \frac{p\gamma^{-2d-1}2^{-2d}}{p\gamma^{-2d-1}2^{-2d} + 1 - p}, \frac{p2^{-2d}}{p2^{-2d} + 1 - p} \right\}. \tag{15}$$

Note that the right-hand side of (15) tends to 1 as $p \nearrow 1$. Pick $p^* \in (p_c, 1)$, where p_c is defined as in Lemma 6.2, and then pick $p < 1$ close enough to 1 so that the right-hand side of (15) is at least p^*. We can then apply Lemma 6.1 to the projections on $\{0,1\}^{S_i}$ of $\bar{\mu}^{p^*}_{\mathbf{Z}^d}$ and $\mu^{p,\gamma}_{S_i,1}$ to show that the latter dominates the former stochastically. Since $\mu^{p,\gamma}_{S_i,1}(Y(\mathbf{Z}^d \setminus S_i) \equiv 1) = 1$, this extends trivially to

$$\bar{\mu}^{p^*}_{\mathbf{Z}^d} \preceq_d \mu^{p,\gamma}_{S_i,1}.$$

Since $(Y(v) = 1, v \leftrightarrow \mathbf{Z}^d \setminus S_i)$ is an increasing event which furthermore is implied by $(Y(v) = 1, v \leftrightarrow \infty)$, we get that

$$\begin{aligned}
\liminf_{i\to\infty} \mu^{p,\gamma}_{S_i,1}(Y(v) = 1,\, v \leftrightarrow \mathbf{Z}^d \setminus S_i) \;&\geq\; \liminf_{i\to\infty} \bar{\mu}^{p^*}_{\mathbf{Z}^d}(\bar{Y}(v) = 1,\, v \leftrightarrow \mathbf{Z}^d \setminus S_i) \\
&\geq\; \bar{\mu}^{p^*}_{\mathbf{Z}^d}(\bar{Y}(v) = 1,\, v \leftrightarrow \infty) \\
&>\; 0
\end{aligned}$$

so (15) is established, and the proof is complete. □

The next task is to prove Proposition 2.5. To this end, it is convenient to isolate a couple of lemmas.

Lemma 6.3: *Let $S \subset \mathbf{Z}^d$ be finite, and let ξ' and ξ'' be two feasible elements of $\{A, 0, B\}^{\mathbf{Z}^d \setminus S}$ satisfying $\xi' \preceq \xi''$. For $\lambda \geq 0$ and $\gamma \geq 1$, we then have*

$$\nu_{S,\xi'}^{\lambda,\gamma} \preceq_d \nu_{S,\xi''}^{\lambda,\gamma}. \tag{16}$$

Proof: By Lemma 6.1, it is sufficient to prove (16) in the case where S consists of a single vertex v. For this, it suffices to show that

$$\nu_{\{v\},\xi'}^{\lambda,\gamma}(X(v) = A) \geq \nu_{\{v\},\xi''}^{\lambda,\gamma}(X(v) = A) \tag{17}$$

and that

$$\nu_{\{v\},\xi'}^{\lambda,\gamma}(X(v) = B) \leq \nu_{\{v\},\xi''}^{\lambda,\gamma}(X(v) = B). \tag{18}$$

For $\xi \in \{A, 0, B\}^{\mathbf{Z}^d \setminus \{v\}}$, define $m_A(\xi)$ to be the number of vertices w with $d(v, w) \leq 1$ that either have value A or have some nearest neighbour with value A, i.e.

$$m_A(\xi) = |\{w \in \mathbf{Z}^d : d(v, w) \leq 1, \exists u \in \mathbf{Z}^d \setminus \{v\} \text{ such that } d(w, u) \leq 1 \text{ and } \xi(u) = A\}|$$

and define $m_B(\xi)$ analogously. A direct calculation using the definition of $\nu_{S,\xi}^{\lambda,\gamma}$ shows that

$$\nu_{\{v\},\xi}^{\lambda,\gamma}(X(v) = A) = \begin{cases} 0 & \text{if } m_B(\xi) > 0 \\ \frac{\lambda\gamma^{-2d-1}}{2\lambda\gamma^{-2d-1}+1} & \text{if } m_A(\xi) = m_B(\xi) = 0 \\ \frac{\lambda\gamma^{-2d-1+m_A(\xi)}}{\lambda\gamma^{-2d-1+m_A(\xi)}+1} & \text{otherwise.} \end{cases}$$

A key observation now is that $\nu_{\{v\},\xi}^{\lambda,\gamma}(X(v) = A)$ is increasing in $m_A(\xi)$ and decreasing in $m_B(\xi)$ (this is where the assumption $\gamma \geq 1$ is needed). Moreover, $m_A(\xi)$ is decreasing in ξ and $m_B(\xi)$ is increasing in ξ. Combining these observations, we get that $\nu_{\{v\},\xi}^{\lambda,\gamma}(X(v) = A)$ is decreasing in ξ, so we can conclude that (17) holds. The other inequality (18) follows similarly. $\square$

Lemma 6.4: *Let S_1 and S_2 be finite subsets of $\mathbf{Z}^d$ such that $S_1 \subseteq S_2$. For $\lambda \geq 0$ and $\gamma \geq 1$, we then have*

$$\nu_{S_1,A}^{\lambda,\gamma} \preceq_d \nu_{S_2,A}^{\lambda,\gamma}. \tag{19}$$

Proof: Let X_1 and X_2 be random elements of $\{A, 0, B\}^{\mathbf{Z}^d}$ with respective distribution $\nu_{S_1,A}^{\lambda,\gamma}$ and $\nu_{S_2,A}^{\lambda,\gamma}$. By conditioning X_2 on its values off S_1 and applying Lemma 6.3, we get that the projection on $\{A, 0, B\}^{S_1}$ of $\nu_{S_1,A}^{\lambda,\gamma}$ is stochastically dominated by the projection on $\{A, 0, B\}^{S_1}$ of $\nu_{S_2,A}^{\lambda,\gamma}$. Since $\nu_{S_1,A}^{\lambda,\gamma}(X(\mathbf{Z}^d \setminus S_1) \equiv A) = 1$, we get (19) as an immediate consequence. $\square$

Proof of Proposition 2.5: By Lemma 6.4, we have

$$\nu_{S_1,A}^{\lambda,\gamma} \preceq_d \nu_{S_2,A}^{\lambda,\gamma} \preceq_d \cdots$$

so that the limiting measure $\nu_A^{\lambda,\gamma}$ exists by monotonicity. By the same reference to general Gibbs theory as in the proof of Theorem 2.3, this limiting measure is a Gibbs measure for the VLWR model with the given parameter values. Let $\nu^{\lambda,\gamma}$ be *any* Gibbs measure for VLWR model with the same parameters. By conditioning on the configuration off S_i, as in the proof of Lemma 6.4, we get that

$$\nu^{\lambda,\gamma}_{S_i,A} \preceq_d \nu^{\lambda,\gamma},$$

and this inequality is preserved under limits, so that

$$\nu^{\lambda,\gamma}_A \preceq_d \nu^{\lambda,\gamma}. \tag{20}$$

This implies that $\nu^{\lambda,\gamma}_A$ is independent of the choice of $\mathbf{S} = \{S_i\}_{i=1}^\infty$ by the following argument: If $\nu^{\lambda,\gamma}_A$ and $\tilde{\nu}^{\lambda,\gamma}_A$ are two limiting measures arising with two different choices of $\mathbf{S}$, then $\nu^{\lambda,\gamma}_A \preceq_d \tilde{\nu}^{\lambda,\gamma}_A$ and $\tilde{\nu}^{\lambda,\gamma}_A \preceq_d \nu^{\lambda,\gamma}_A$ by two applications of (20). This of course implies that $\nu^{\lambda,\gamma}_A = \tilde{\nu}^{\lambda,\gamma}_A$.

We have thus shown (3) and the first half of (5). To show (4) and the second half of (5) we use the exact same arguments with the roles of A and B interchanged. The equivalence between nonuniqueness of Gibbs measures and (6) follows immediately from (5). $\square$

It remains to prove Theorem 2.4. Again, it is convenient to isolate two lemmas.

Lemma 6.5: *For $\lambda \geq 0$, $\gamma \geq 1$ and any $v \in \mathbf{Z}^d$, we have*

$$\nu^{\lambda,\gamma}_A = \nu^{\lambda,\gamma}_B$$

if and only if

$$\nu^{\lambda,\gamma}_A(X(v) = A) = \nu^{\lambda,\gamma}_A(X(v) = B). \tag{21}$$

Proof: The 'only if' direction is immediate from the AB symmetry of the VLWR model, so we proceed to prove the 'if' direction. Suppose that (21) holds. Then, by translation invariance of $\nu^{\lambda,\gamma}_A$, we have

$$\nu^{\lambda,\gamma}_A(X(w) = A) = \nu^{\lambda,\gamma}_A(X(w) = B) \tag{22}$$

for all $w \in \mathbf{Z}^d$. Further application of the AB symmetry of the model yields

$$\nu^{\lambda,\gamma}_A(X(w) = A) = \nu^{\lambda,\gamma}_B(X(w) = B) \tag{23}$$

and

$$\nu^{\lambda,\gamma}_A(X(w) = B) = \nu^{\lambda,\gamma}_B(X(w) = A) \tag{24}$$

so that all four probabilities considered in (22), (23) and (24) are equal. By (5), we have $\nu^{\lambda,\gamma}_A \preceq_d \nu^{\lambda,\gamma}_B$, so by Strassen's Theorem we can construct a coupling of two $\{A,0,B\}^{\mathbf{Z}^d}$-valued random objects X and X' with respective distributions $\nu^{\lambda,\gamma}_A$ and $\nu^{\lambda,\gamma}_B$ such that $X \preceq X'$ almost surely. Writing P for some probability measure supporting such a coupling, we get

$$\begin{aligned} P(X(w) = A,\, X'(w) \geq 0) &= \nu^{\lambda,\gamma}_A(X(v) = A) - \nu^{\lambda,\gamma}_B(X(v) = A) \\ &= 0 \end{aligned}$$

and similarly $P(X(w) \leq 0, X'(w) = B) = 0$. Hence $P(X(w) \neq X'(w)) = 0$. Countable additivity implies $P(X \neq X') = 0$, so $\nu^{\lambda,\gamma}_A = \nu^{\lambda,\gamma}_B$ as desired. $\square$

Lemma 6.6: *Let S be a finite subset of $\mathbf{Z}^d$, and pick $\gamma \geq 2$ and $0 \leq p_1 \leq p_2 \leq 1$. Then*

$$\mu_{S,1}^{p_1,\gamma} \preceq_d \mu_{S,1}^{p_2,\gamma}. \tag{25}$$

Proof: In view of Lemma 6.1, it is sufficient to prove that the right-hand side of (14) is increasing in p and η'. That it is increasing in p is immediate, and to see that it is increasing in η' it is sufficient to show that the expression

$$\gamma^{-\kappa_1(w,\eta_0')} 2^{1-\kappa_2(w,\eta_0')} \tag{26}$$

is increasing in η_0'. To do this, we define $\kappa(w,\eta_0') = \kappa_1(w,\eta_0') + \kappa_2(w,\eta_0')$ and rewrite (26) as

$$\left(\frac{\gamma}{2}\right)^{-\kappa_1(w,\eta_0')} 2^{1-\kappa(w,\eta_0')} \tag{27}$$

Since $\frac{\gamma}{2} \geq 1$, we have that this expression is decreasing both in $\kappa_1(w,\eta_0')$ and in $\kappa(w,\eta_0')$, so it only remains to show that $\kappa_1(w,\eta_0')$ and $\kappa(w,\eta_0')$ are decreasing functions of η_0'. This, however, is immediate upon recalling that $\kappa_1(w,\eta_0')$ is the number of single-site connected components of $\omega(w,\eta_0')$ that intersect the "neighbourhood" $\{u \in \mathbf{Z}^d : d(u,w) \leq 1\}$ of w, and noting that $\kappa(w,\eta_0')$ is the total number of connected components of $\omega(w,\eta_0')$ intersecting this neighbourhood. $\Box$

Proof of Theorem 2.4: By Theorem 2.3, we only need to show the monotonicity statement which is implicit in Theorem 2.4, i.e. that if $\lambda_1 \leq \lambda_2$, then the existence of more than one Gibbs measure at intensity λ_1 implies the same thing at intensity λ_2. So suppose that we have more than one Gibbs measure at intensity λ_1. By Proposition 2.5 and Lemma 6.5, this implies that

$$\nu_A^{\lambda_1,\gamma}(X(v) = A) - \nu_A^{\lambda_1,\gamma}(X(v) = B) > 0,$$

i.e. that

$$\lim_{i\to\infty} \left(\nu_{S_i,A}^{\lambda_1,\gamma}(X(v) = A) - \nu_{S_i,A}^{\lambda_1,\gamma}(X(v) = B)\right) > 0 \tag{28}$$

(the limit exists by Proposition 2.5). Set $p_1 = \frac{\lambda_1}{1+\lambda_1}$ and $p_2 = \frac{\lambda_2}{1+\lambda_2}$. By Lemma 5.5, (28) is the same as having

$$\lim_{i\to\infty} \mu_{S,1}^{p_1,\gamma}(Y(v) = 1,\, v \leftrightarrow \mathbf{Z}^d \setminus S) > 0.$$

By Lemma 6.6, this implies

$$\lim_{i\to\infty} \mu_{S,1}^{p_2,\gamma}(Y(v) = 1,\, v \leftrightarrow \mathbf{Z}^d \setminus S) > 0,$$

so that another application of Lemma 5.5 yields

$$\lim_{i\to\infty} \left(\nu_{S_i,A}^{\lambda_2,\gamma}(X(v) = A) - \nu_{S_i,A}^{\lambda_2,\gamma}(X(v) = B)\right) > 0.$$

Hence

$$\nu_A^{\lambda_2,\gamma}(X(v) = A) > \nu_A^{\lambda_2,\gamma}(X(v) = B)$$

so by Lemma 6.5 again we have more than one Gibbs measure at intensity λ_2. $\Box$

7 A multitype generalization

A natural generalization of the VLWR model is to allow three or more different types of particles, rather than just the two types A and B. Let $q \geq 2$ be an integer, let as usual $G = (V, E)$ be a finite graph, and let $A_1, \ldots, A_q$ represent q different types of particles. Call a configuration $\xi \in \{A_1, \ldots, A_q, 0\}^V$ feasible if for no $i, j \in \{1, \ldots, q\}$ with $i \neq j$ and no $v, w \in V$ with $d(v, w) \leq 2$ we have $\xi(v) = A_i$ and $\xi(w) = A_j$. Let $n_{A_i}(\xi)$ denote the number of vertices $v \in V$ for which $\xi(v) = A_i$, and define $n_*(\xi)$ as in Section 2.

Definition 7.1: *The multitype VLWR measure $\nu_G^{q,\lambda,\gamma}$ on $\{A_1, \ldots, A_q, 0\}^V$ with parameters $q \in \{2, 3, \ldots\}$, $\lambda \geq 0$ and $\gamma \geq 0$ is the probability measure which, to each $\xi \in \{A_1, \ldots, A_q, 0\}^V$, assigns probability*

$$\nu_G^{q,\lambda,\gamma}(\xi) = \begin{cases} \frac{1}{Z_G^{q,\lambda,\gamma}} \lambda^{\sum_{i=1}^q n_{A_i}(\xi)} \gamma^{n_*(\xi)} & \text{if } \xi \text{ is feasible} \\ 0 & \text{otherwise.} \end{cases}$$

Taking $q = 2$ and identifying A_1 (resp. A_2) with A (resp. B) yields the ordinary VLWR model. Note, however, that taking $q = 3$ gives something entirely different from the trinary model considered in Section 4. The multitype VLWR model can, of course, be extended to $\mathbf{Z}^d$ in the usual way, and it turns out that much of the theory for the VLWR model obtained in previous sections can be extended in a straightforward manner to the multitype VLWR model. For $q \geq 3$ and $\gamma = q$, the results in Section 3 can be extended to yield an euqivalence between the multitype VLWR model and a kind of multitype beach model considered by Burton and Steif [5]. For $\gamma \geq 1$, we can also replace the volume-interaction factor by a $(q+1)$st particle type, and get multitype analogues of the results in Section 4.

The multitype VLWR model also admits a random-cluster representation as in Section 5; the factor $2^{k_2(\eta)}$ in (11) then has to be replaced by $q^{k_2(\eta)}$. The proof of Theorem 2.3 can then be extended in a completely straightforward manner to prove the following multitype generalization.

Theorem 7.2: *For fixed $d \geq 2$, $q \geq 2$ and $\gamma > 0$, the multitype VLWR model on $\mathbf{Z}^d$ with parameters q, λ and γ has a unique Gibbs measure if λ is taken to be sufficiently small. If instead λ is taken to be sufficiently large, then the model has more than one Gibbs measure.*

The extension of Theorem 2.4 to the multitype case is somewhat trickier, but can still be done: The proofs of Lemma 6.5 and Theorem 2.4 easily extend to show that for $d \geq 2$, $q \geq 2$ and $\gamma \geq q$, there exists a critical value $\lambda_c = \lambda_c(d, q, \gamma)$ such that

$$\nu_{A_i}^{q,\gamma,\lambda}(X(v) = A_i) - \nu_{A_i}^{q,\gamma,\lambda}(X(v) = A_j) \begin{cases} > 0 & \text{if } \lambda > \lambda_c \\ = 0 & \text{if } \lambda < \lambda_c, \end{cases} \tag{29}$$

for $j \neq i$, where $\nu_{A_i}^{q,\gamma,\lambda}$ is defined analogously to $\nu_A^{\gamma,\lambda}$. It then only remains to prove a multitype analogue of Lemma 6.5, i.e. to show that we have uniqueness of Gibbs measures if and only if

$$\nu_{A_i}^{q,\gamma,\lambda}(X(v) = A_i) = \nu_{A_i}^{q,\gamma,\lambda}(X(v) = A_j). \tag{30}$$

As before, the 'only if' direction is obvious, but the 'if' direction is not. The proof of Lemma 6.5 does not extend to the multitype case, because it makes essential use of stochastic domination with respect to the ordering $A < 0 < B$, and the state space $\{A_1, \ldots, A_q, 0\}$ does not admit any natural such ordering. Instead, we have to make use of the random-cluster representation in a careful adaption of a proof of a Potts model analogue of the equivalence asserted in (30). That proof is due to Aizenman *et al.* [1] and can also be found in [15]. Although somewhat tedious, this approach does work, and we obtain the following generalization of Theorem 2.4.

Theorem 7.3: *For fixed $d \geq 2$, $q \geq 2$ and $\gamma \geq q$, there exists a critical value $\lambda_c = \lambda_c(d, q, \gamma)$ such that for $\lambda < \lambda_c$, the multitype VLWR model on $\mathbf{Z}^d$ with parameters q, λ and γ has a unique Gibbs measure, whereas for $\lambda > \lambda_c$ the model has more then one Gibbs measure.*

The lack of a natural total ordering of $\{A_1, \ldots, A_q, 0\}$ makes parts of Proposition 2.5 impossible to extend to a multitype setting; there seems to be no sensible analogue of (5) for $q \geq 3$. On the other hand, the random-cluster approach shows that for $\gamma \geq q$, the limit $\nu_{A_j}^{q,\lambda,\gamma} = \lim_{i \to \infty} \nu_{S_i, A_j}^{q,\lambda,\gamma}$ is well defined, a result that we made implicit use of in writing down (29).

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