SOME OPTIMIZATION PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The problems studied in this paper are to find the minimum of $\int_0^1 |q(t)|\,dt$ or $\operatorname{ess\,sup}_{t\in[0,1]} |q(t)|$, where q appears in one of the coefficients of a second order differential equation whose solution satisfies certain boundary conditions. The infimum of the integral is in general not attained for any L^1-function q, but by a special transformation a related control problem is obtained where the infimum is attained and is the same as in the original problem. The new problem is solved by a detailed analysis of the necessary conditions for optimality. The other problem is solved in a similar way using a maximum principle for minimax problems.

1. INTRODUCTION

In this paper we study the differential equation

$$y'' - 2by' + (c - q(t))y = 0, \qquad t \in [0,1], \tag{1.1}$$

with boundary conditions

$$y(0) = a, \quad y'(0) = 1, \quad y(1) = 0, \tag{1.2}$$

and we consider the problem of minimizing

$$\int_0^1 |q(t)|\,dt \tag{1.3}$$

or

$$\operatorname*{ess\,sup}_{t\in[0,1]} |q(t)|\,. \tag{1.4}$$

This problem generalizes or is related to optimization problems that have been studied before. A special case (with $a = b = 0$) was studied by Borg [2] in connection with stability questions. In the papers [5], [6], and [8] optimization problems for the equation (1.1) with $b = c = 0$ were solved. There the integral (1.3) has a given value, and the problem is to maximize or minimize $y(1)$. These investigations were motivated by some previous results by Essén [3]. A differential equation of the form (1.1) (with $c = 0$) was used in [7] to describe the concentration of a substance being eliminated by enzymes in the liver, and some optimization problems concerning the flux (related to y') were considered.

If y is replaced by $e^{bt}y$, then we obtain a differential equation of the form (1.1) with $b = 0$ and c replaced by $c - b^2$. In (1.2) the value of $y'(0)$ is changed to $1 - ab$. It is therefore no restriction to let $b = 0$, and we will use the following formulation:

$$y'' + (c - q(t))y = 0, \qquad t \in [0,1], \tag{1.5}$$

$$y(0) = \alpha, \quad y'(0) = \beta, \quad y(1) = 0, \tag{1.6}$$

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where $\alpha = 0$ and $\beta = 1$, or $\alpha = 1$ and β arbitrary ($\beta = \frac{1}{a} - b$ in terms of the original parameters). First we consider (1.3).

2. Reformulation of the problem

The infimum of (1.3) is in general not attained for any L^1-function q. Following [5] we will first transform the problem into an equivalent one with bounded controls. Then, after a compactification and convexification, we obtain a problem whose infimum is attained, and it turns out that this gives the solution to the original problem.

First, let us write (1.5)–(1.6) as a first order system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= q(t)y_1 - cy_1, \end{aligned} \tag{2.1}$$

with boundary conditions

$$y_1(0) = \alpha, \quad y_2(0) = \beta, \quad y_1(1) = 0. \tag{2.2}$$

Here, $q \in L^1(0,1)$, and the solution of (2.1) is in the Carathéodory sense, i.e., y_1 and y_2 are absolutely continuous functions that satisfy (2.1) almost everywhere. Assume that (2.1) and (2.2) are satisfied for some q. Define

$$\varphi(\tau) = \int_0^\tau (1 + |q(s)|)\, ds, \quad \tau \in [0,1], \qquad t_1 = \varphi(1).$$

Then φ is absolutely continuous, strictly increasing, and maps $[0,1]$ onto $[0,t_1]$. Its inverse ψ is also absolutely continuous, and we define

$$x_i(t) = y_i(\psi(t)),\ i = 1,\, 2, \qquad u(t) = \frac{q(\psi(t))}{1 + |q(\psi(t))|}, \quad t \in [0, t_1].$$

Using properties of absolutely continuous functions (see in particular [10, I.4.43]) we see that x_1, x_2, and u satisfy

$$\begin{aligned} x_1' &= (1 - |u|)x_2, \\ x_2' &= ux_1 - c(1 - |u|)x_1, \end{aligned} \tag{2.3}$$

$$x_1(0) = \alpha, \quad x_2(0) = \beta, \quad x_1(t_1) = 0, \tag{2.4}$$

where

$$|u(t)| < 1, \quad t \in [0, t_1], \tag{2.5}$$

and t_1 satisfies

$$t_1 - 1 = \int_0^{t_1} |u(t)|\, dt. \tag{2.6}$$

That u is measurable follows from the fact that an absolutely continuous function maps measurable sets onto measurable sets. Conversely, if x_1 and x_2 satisfy (2.3) and (2.4) for some t_1 and some measurable function u satisfying (2.5) and (2.6), we define

$$\psi(t) = \int_0^t (1 - |u(s)|)\, ds, \quad t \in [0, t_1],$$

which is absolutely continuous, strictly increasing, and maps $[0, t_1]$ onto $[0, 1]$. With φ as the inverse of ψ, we let

$$y_i(\tau) = x_i(\varphi(\tau)),\ i = 1,\, 2, \qquad q(\tau) = \frac{u(\varphi(\tau))}{1 - |u(\varphi(\tau))|}, \quad \tau \in [0, 1].$$

Then y_1 and y_2 satsify (2.1) and (2.2). In any case

$$\int_0^1 |q(\tau)|\,d\tau = \int_0^{t_1} |u(t)|\,dt = t_1 - 1, \tag{2.7}$$

so we have transformed the problem to the time optimal problem for (2.3)–(2.4) with constraints (2.5)–(2.6).

Because of the strict inequality in (2.5), the infimum of t_1 is in general not attained. We therefore replace (2.5) by the constraint $|u(t)| \leq 1$. This does not change $\inf t_1$, which we will demonstrate later (at the end of Section 5) when the solution of the extended problem has been obtained. To handle the constraint (2.6) we introduce a state variable x_3 satisfying

$$x_3' = 1 - |u(t)|, \quad x_3(0) = 0, \quad x_3(t_1) = 1.$$

Because of a lack of convexity in u we still cannot be sure that the infimum is attained. Therefore we consider the relaxed problem (see [1, IV.3]) instead. Thus we will consider the following problem: Find $\min t_1$ subject to

$$\begin{aligned}
x_1' &= x_2 u_1, && (2.8)\\
x_2' &= x_1 u_2 - c x_1 u_1, && (2.9)\\
x_3' &= u_1, && (2.10)\\
x_1(0) = \alpha, \quad x_2(0) &= \beta, \quad x_3(0) = 0, && (2.11)\\
x_1(t_1) = 0, \quad & x_3(t_1) = 1, && (2.12)\\
u_1(\cdot),\, u_2(\cdot) &\text{ measurable}, && (2.13)\\
(u_1(t), u_2(t)) \in \Omega = \{(u_1, u_2) &\in \mathbb{R}^2 : 0 \leq u_1 \leq 1 - |u_2|\},\ t \in [0, t_1]. && (2.14)
\end{aligned}$$

According to an existence theorem such as [1, III.5.1] this problem has a solution (referred to as an optimal solution of (2.8)–(2.14)). We shall see that any solution is such that $u_1(t) = 1 - |u_2(t)|$ a.e. and is therefore a solution of the original problem (with $u = u_2$).

Remark. To be on the safe side we should convince ourselves that the set of solutions to (2.8)–(2.14) is not empty. This could easily be done now, but will follow from the next section where we construct explicit solutions that are candidates for the optimal solution.

3. Analysis of the necessary conditions

Let $(x, u) = (x_1, x_2, x_3, u_1, u_2)$. Assume that (x^*, u^*) on $[0, t_1^*]$ is an optimal solution of (2.8)–(2.14). Let $\eta = (\eta_1, \eta_2, \eta_3)$ and introduce the Hamiltonian function

$$H(x, u, \eta) = \eta_1 x_2 u_1 + \eta_2 (x_1 u_2 - c x_1 u_1) + \eta_3 u_1.$$

From the theory of necessary conditions for optimality (see, e.g., [1, Corollary V.3.1]) we know that there exist absolutely continuous functions $\eta_1(\cdot)$, $\eta_2(\cdot)$, $\eta_3(\cdot)$ on $[0, t_1^*]$ and a constant λ_0 such that

$$\eta_1' = -\frac{\partial H}{\partial x_1} = -\eta_2 u_2^* + c\eta_2 u_1^*, \tag{3.1}$$

$$\eta_2' = -\frac{\partial H}{\partial x_2} = -\eta_1 u_1^*, \tag{3.2}$$

$$\eta_3 = \text{const.}, \tag{3.3}$$

$$\eta_2(t_1^*) = 0, \tag{3.4}$$

$$H(x^*(t), u^*(t), \eta(t)) = \max_{u\in\Omega} H(x^*(t), u, \eta(t)) = -\lambda_0 \geq 0 \text{ a.e. on } [0, t_1^*], \tag{3.5}$$

$$(\lambda_0, \eta_1(t), \eta_2(t), \eta_3) \neq (0,0,0,0) \text{ for all } t. \tag{3.6}$$

By changing u_1^* and u_2^* on a set of measure zero, if necessary, we may assume that (3.5) holds for all $t \in [0, t_1^*]$. In the following we omit the stars on x^*, u^*, and t_1^*.

As in [5] we write H as

$$H = s_1 u_1 + s_2 u_2, \tag{3.7}$$

where

$$s_1 = \eta_1 x_2 - c\eta_2 x_1 + \eta_3, \qquad s_2 = \eta_2 x_1,$$

and we will work in the s-plane, $s = (s_1, s_2)$.

We see that the differential equations (3.1)–(3.2) for η_1 and η_2 are the same as (2.8)–(2.9) if η_1 is replaced by x_2 and η_2 by $-x_1$. From the conditions (2.12) and (3.4) it then follows that there exists a constant δ such that

$$\eta_1(t) = \delta x_2(t), \qquad \eta_2(t) = -\delta x_1(t).$$

Thus

$$s_1 = \delta x_2^2 + c\delta x_1^2 + \eta_3, \qquad s_2 = -\delta x_1^2. \tag{3.8}$$

First, let us show that $\lambda_0 \neq 0$. Assume the contrary. It then follows from (2.14), (3.5), and (3.7) that $s_2(t) = 0$, $s_1(t)u_1(t) = 0$, and $s_1(t) \leq 0$ for all t. Since x_1 cannot be identically 0, we must have $\delta = 0$ according to (3.8). Then $s_1(t) = \eta_3 \leq 0$. By (3.6) we must have $\eta_3 < 0$. But then $u_1(t) = 0$ for all t, which is impossible since $\int_0^{t_1} u_1(t)\,dt = 1$ [see (2.10)–(2.12)]. Therefore we must have $\lambda_0 < 0$, and we may assume that $\lambda_0 = -1$.

Thus the maximum principle (3.5) gives

$$s_1(t)u_1(t) + s_2(t)u_2(t) = \max_{u\in\Omega}[s_1(t)u_1 + s_2(t)u_2] = 1 \tag{3.9}$$

for all $t \in [0, t_1]$. This makes it possible to express $u_1(t)$ and $u_2(t)$ in terms of $s_1(t)$ and $s_2(t)$. Indeed, if we let $s = (s_1, s_2)$ and

$$\begin{aligned}
D_1 &= \{s : s_1 > |s_2|\}, \\
D_2 &= \{s : s_2 > 0 \text{ and } s_2 > s_1\}, \\
D_3 &= \{s : s_2 < 0 \text{ and } s_2 < -s_1\}, \\
L_2 &= \{s : s_2 = s_1 > 0\}, \\
L_3 &= \{s : s_2 = -s_1 < 0\},
\end{aligned}$$

then it follows from (3.9) that

$$\begin{aligned} u_1(t) &= 1, \quad u_2(t) = 0, \quad \text{if } s(t) \in D_1, \\ u_1(t) &= 0, \quad u_2(t) = 1, \quad \text{if } s(t) \in D_2, \\ u_1(t) &= 0, \quad u_2(t) = -1, \quad \text{if } s(t) \in D_3, \\ u_1(t) + u_2(t) &= 1, \quad 0 \le u_1(t) \le 1, \quad \text{if } s(t) \in L_2, \\ u_1(t) - u_2(t) &= 1, \quad 0 \le u_1(t) \le 1, \quad \text{if } s(t) \in L_3. \end{aligned}$$

The location of $s(t)$ in the s-plane is illustrated in Fig. 1. To see how $s(t)$ moves we compute the derivatives, using (2.8)–(2.9) and (3.8):

$$s_1' = Su_2, \tag{3.10}$$

$$s_2' = -Su_1, \tag{3.11}$$

where

$$S = 2\delta x_1 x_2. \tag{3.12}$$

Thus, the sign of S determines how $s(t)$ moves. This is indicated in Fig. 1.

s_2 L_2

$u_1 + u_2 = 1$

$S > 0$ $S < 0$

D_2 $u_1 = 0$ $u_2 = 1$

$S > 0$ D_1 $u_1 = 1$ $u_2 = 0$

s_1

D_3 $u_1 = 0$ $u_2 = -1$

$S < 0$

$S < 0$ $S > 0$

$u_1 - u_2 = 1$

L_3

FIGURE 1. Possible movements in the s-plane.

We also need

$$S' = 2\delta \frac{d}{dt}(x_1 x_2) = 2(s_1 - \eta_3 + 2cs_2)u_1 - 2s_2 u_2.$$

In particular,

$$S' = -2 \quad \text{in } D_2 \cup D_3, \tag{3.13}$$

$$S' = 2(1 - \eta_3) + 4cs_2 \quad \text{in } D_1. \tag{3.14}$$

By the same type of arguments as in [5] we find that the set of t-values for which s belongs to L_2 or L_3 is empty or consists of a finite number of intervals. Let us assume that $s(t) \in L_2$ on an interval $[t', t'']$ of positive length. On this interval, $S = (u_1 + u_2)S = (s_1 - s_2)' = 0$ by (3.10)–(3.11). Since $s_2 = -\delta x_1^2 = 1$, (3.12) implies that $x_2 = 0$ on $[t', t'']$. Then (2.9) gives $u_2 = cu_1$, and since $u_1 + u_2 = 1$,

$$u_1 = \frac{1}{1+c}, \qquad u_2 = \frac{c}{1+c}, \tag{3.15}$$

and we must have $c \geq 0$. In the same way, if $s(t) \in L_3$ on $[t', t'']$, then we must have $c \leq 0$.

Since $s_2(t_1) = -\delta x_1^2(t_1) = 0$, $s(t) \in D_1$ in the last stage. We may start in D_2, but once $s(t)$ has left D_2 it cannot come back, because if it did, S would be negative and decreasing [see (3.13)], and $s(t)$ could not leave D_2 again.

There are two main cases, $c > 0$ and $c < 0$, and also the special case $c = 0$, and we shall analyse them separately. We want to find all solutions of (2.8)–(2.14), (3.1)–(3.6). For some values of the parameters there will be more than one solution. Then we must compare the different candidates to find out which one is optimal.

4. The case $c > 0$

In this section we assume that $c > 0$. Let

$$k = \sqrt{c}\,.$$

Since $s_2(t_1) = 0$, there is a maximal interval $[\tau_1, t_1]$ such that $s(t) \in D_1$ for all $t \in (\tau_1, t_1]$. On this interval we have $u_1 = 1$, $u_2 = 0$, and [by (2.8)–(2.9)]

$$x_1' = x_2, \tag{4.1}$$

$$x_2' = -k^2 x_1. \tag{4.2}$$

Using $x_1(t_1) = 0$, we find that the solution is

$$x_1(t) = -x_2(t_1)\frac{1}{k}\sin k(t_1 - t), \tag{4.3}$$

$$x_2(t) = x_2(t_1)\cos k(t_1 - t), \tag{4.4}$$

so that

$$s_2(t) = -\frac{\delta x_2^2(t_1)}{k^2}\sin^2 k(t_1 - t). \tag{4.5}$$

We must have $x_2(t_1) \neq 0$ [otherwise x_1 and x_2 would be indentically 0, contradicting (2.11)].

We now consider a number of cases classified after their behaviour in the s-plane.

Case 1. It is possible that s belongs to D_1 for all $t \in [0, t_1]$ except for some isolated points where s may lie on L_2 or L_3 . In this case conditions (2.10)–(2.12) for x_3 give

$$\int_0^{t_1} u_1(t)\,dt = t_1 = 1.$$

The solution (4.3)–(4.4) holds for all $t \in [0, t_1]$. The initial conditions (2.11) give

$$-\frac{x_2(1)}{k}\sin k = \alpha, \qquad x_2(1)\cos k = \beta.$$

Therefore this case may occur only if

$$\sin k = 0, \quad \text{if } \alpha = 0,$$

$$\beta = -k\cot k, \quad \text{if } \alpha = 1.$$

If this case does not occur, then $\tau_1 > 0$ and $|s_2(\tau_1)| = 1$. Let $\ell = t_1 - \tau_1$. Then (4.5) shows that we must have $0 < k\ell \leq \frac{\pi}{2}$. We also define

$$\lambda = k\cot k\ell, \tag{4.6}$$

so that $\frac{x_2}{x_1}(\tau_1) = -\lambda$.

Consider first the case $s_2(\tau_1) = 1$. Before τ_1 we may have a D_2-stage; we call this a solution of type D_2D_1 following the notation in [5]. It is also possible [if $S(\tau_1) = 0$] for s to oscillate in D_1 between (1,1) on L_2 and (1,0) a number of times, and each time s reaches L_2 it may stay there for a while before entering D_1. We obtain the following possibilities:

$$D_2D_1,\ D_2L_2(D_1L_2)^{n-1}D_1\ (n\geq 1),\ L_2(D_1L_2)^{n-1}D_1\ (n\geq 1),$$
$$D_1L_2(D_1L_2)^{n-1}D_1\ (n\geq 1).$$

By τ_1, τ_2, τ_3, ... we denote the t-values (in decreasing order) when we enter or leave L_2; some of them may coincide.

Case 2 D_2D_1. On $[0,\tau_1]$ we have $u_1=0$, $u_2=1$, $x_1=\alpha$, and $x_2(t)=\beta+\alpha t$. The following equations must hold:

$$x_2(\tau_1)=\beta+\alpha\tau_1=-\lambda\alpha,$$
$$\int_0^{t_1}u_1(t)\,dt=\ell=1.$$

Then $\lambda=k\cot k$, $\alpha=1$, and $\tau_1=-\beta-k\cot k$ [see (4.6)]. Since $\tau_1>0$ we have the conditions

$$\beta<-k\cot k,\quad 0<k\leq\frac{\pi}{2}. \tag{4.7}$$

Also, $t_1=\tau_1+\ell=\tau_1+1$, so that the value of the objective functional (2.7) is

$$t_1-1=-\beta-k\cot k.$$

Thus a solution of the type D_2D_1 may be optimal only if (4.7) is satisfied. Conversely, as soon as (4.7) is satisfied, a solution (optimal or not) of this type can be constructed. A similar remark can be made for all the other cases.

In the cases involving L_2 we must have $k\ell=\frac{\pi}{2}$, because $S(\tau_1)=-s_2'(\tau_1)=0$. Also, $S(\tau_j)=0$ and $x_2(\tau_j)=0$ for all points τ_j [for the last statement use (3.12) combined with $s_2=1$]. On a typical D_1-interval $[\tau_{2i+1},\tau_{2i}]$ from L_2 back to L_2 we can solve the differential equations with the conditions $x_2(\tau_{2i+1})=0$ and $s_2(\tau_{2i+1})=1$ to obtain $s_2(t)=-\delta x_1^2(t)=\cos^2 k(t-\tau_{2i+1})$. Thus $k(\tau_{2i}-\tau_{2i+1})=\pi$, or $\tau_{2i}-\tau_{2i+1}=2\ell=\frac{\pi}{k}$.

Case 3a $D_2L_2(D_1L_2)^{n-1}D_1$. On the first interval $[0,\tau_{2n}]$ (the D_2-stage) we have as in Case 2, $x_2(t)=\beta+\alpha t$, so that $x_2(\tau_{2n})=\beta+\alpha\tau_{2n}=0$. Thus $\alpha=1$, and $\tau_{2n}=-\beta$. If T_{L_2} is the sum of the lengths of all L_2-intervals $[\tau_{2i},\tau_{2i-1}]$, we obtain from (3.15)

$$\int_0^{t_1}u_1(t)\,dt=(n-1)\frac{\pi}{k}+T_{L_2}\frac{1}{1+k^2}+\frac{\pi}{2k}=(n-\tfrac{1}{2})\frac{\pi}{k}+\frac{T_{L_2}}{1+k^2}=1,$$
$$t_1=\tau_{2n}+(n-1)\frac{\pi}{k}+T_{L_2}+\frac{\pi}{2k}=(n-\tfrac{1}{2})\frac{\pi}{k}-\beta+T_{L_2},$$
$$t_1-1=-\beta+T_{L_2}\frac{k^2}{1+k^2}=-\beta+k[k-(n-\tfrac{1}{2})\pi]. \tag{4.8}$$

The conditions for this case are $\tau_{2n}>0$ and $T_{L_2}\geq 0$, i.e., $\beta<0$ and $k\geq(n-\frac{1}{2})\pi$. If $n=1$ and $k=\frac{\pi}{2}$, this case is included in Case 2, so we consider only $k>\frac{\pi}{2}$ here. However, since the right-hand side of (4.8) is decreasing in n, we need only consider the largest n such that $(n-\frac{1}{2})\pi\leq k$. For a fixed n the restrictions on k are

$$\frac{\pi}{2}<k<\frac{3\pi}{2}\quad\text{if } n=1,$$
$$(n-\tfrac{1}{2})\pi\leq k<(n+\tfrac{1}{2})\pi\quad\text{if } n>1.$$

Case 3b $L_2(D_1L_2)^{n-1}D_1$. The analysis is the same as in Case 3a with $\beta = 0$, except that $T_{L_2} > 0$ ($T_{L_2} = 0$ is included in Case 1). Thus the conditions for this case are

$$\beta = 0, \quad (n - \tfrac{1}{2})\pi < k \le (n + \tfrac{1}{2})\pi.$$

Case 4 $D_1L_2(D_1L_2)^{n-1}D_1$. On the first interval $[0, \tau_{2n}]$ (the D_1-stage) (4.1)–(4.2) and (2.11) give

$$x_1(t) = \alpha \cos kt + \frac{\beta}{k} \sin kt,$$
$$x_2(t) = \beta \cos kt - \alpha k \sin kt.$$

From $x_2(\tau_{2n}) = 0$ we get

$$\beta \cos k\tau_{2n} = \alpha k \sin k\tau_{2n}.$$

If $\alpha = 0$, then $k\tau_{2n} = \frac{\pi}{2}$ (since $0 < k\tau_{2n} < \pi$). If $\alpha = 1$, then $k \tan k\tau_{2n} = \beta$, or

$$k\tau_{2n} = \frac{\pi}{2} - \arctan \frac{k}{\beta}.$$

As in Case 3 we must have

$$\int_0^{t_1} u_1(t)\, dt = \tau_{2n} + (n - \tfrac{1}{2})\frac{\pi}{k} + T_{L_2} \frac{1}{1 + k^2} = 1,$$
$$t_1 = \tau_{2n} + (n - \tfrac{1}{2})\frac{\pi}{k} + T_{L_2},$$
$$t_1 - 1 = T_{L_2} \frac{k^2}{1 + k^2} = k(k - n\pi + \arctan \frac{k}{\beta}). \tag{4.9}$$

The only further condition is $T_{L_2} > 0$ ($T_{L_2} = 0$ is covered by Case 1), i.e., $k > n\pi - \arctan \frac{k}{\beta}$, but we need only consider the largest n for which this is true. For a fixed n the restriction on k and β is

$$n\pi < k + \arctan \frac{k}{\beta} \le (n + 1)\pi. \tag{4.10}$$

Next we consider the case $s_2(\tau_1) = -1$ in the same manner. Since $c > 0$, there cannot be any L_3-interval according to Section 3. The possible forms are

$$D_3(D_1D_3)^n D_1 \ (n \ge 0), \quad D_1D_3(D_1D_3)^{n-1}D_1 \ (n \ge 1).$$

If $S(\tau_1) = 0$, then s cannot belong to D_3 just before τ_1 (in D_3, $s_1'' = -S' = 2$), and we must have Case 1. Thus $S(\tau_1) < 0$, and $k\ell < \frac{\pi}{2}$. Consider a typical D_3-interval $[\tau_{2i}, \tau_{2i-1}]$ from L_3 back to L_3. Since $s_1'' = 2$, $s_1(\tau_{2i}) = s_1(\tau_{2i-1}) = 1$, it follows that $s_1'(\tau_{2i} + 0) = -s_1'(\tau_{2i-1} - 0)$, i.e., $S(\tau_{2i}) = -S(\tau_{2i-1})$, and also that $\tau_{2i-1} - \tau_{2i} = S(\tau_{2i})$. Furthermore, x_1 is constant on the interval, and (3.12) implies that

$$\frac{x_2}{x_1}(\tau_{2i}) = -\frac{x_2}{x_1}(\tau_{2i-1}).$$

Consider a typical D_1-interval $[\tau_{2i+1}, \tau_{2i}]$ from L_3 back to L_3. There s_2 satisfies [see (3.11) and (3.14)]

$$s_2'' + 4k^2 s_2 = 2(\eta_3 - 1),$$
$$s_2(\tau_{2i+1}) = s_2(\tau_{2i}) = -1,$$

the solution of which satisfies $s_2'(\tau_{2i+1}+0) = -s_2'(\tau_{2i}-0)$. Thus $S(\tau_{2i+1}) = -S(\tau_{2i})$, and (3.12) implies that

$$\frac{x_2}{x_1}(\tau_{2i+1}) = -\frac{x_2}{x_1}(\tau_{2i}).$$

Since $s_2(t) = -\frac{1}{\sin^2 k\ell}\sin^2 k(t_1-t)$ on $[\tau_1, t_1]$, it follows that $S(\tau_1) = -s_2'(\tau_1+0) = -2k\cot k\ell = -2\lambda$; see (4.6). Also $\frac{x_2}{x_1}(\tau_1) = -\lambda$. Thus

$$\frac{x_2}{x_1}(\tau_j) = \lambda(-1)^j, \quad S(\tau_j) = 2\lambda(-1)^j. \tag{4.11}$$

Then $s_2(t) = -\frac{1}{\sin^2 k\ell}\sin^2 k(t+\ell-\tau_{2i})$ on $[\tau_{2i+1}, \tau_{2i}]$, which shows that $\tau_{2i} - \tau_{2i+1} = 2\ell$. Thus all D_3-intervals $[\tau_{2i}, \tau_{2i-1}]$ have length 2λ, and all D_1-intervals $[\tau_{2i+1}, \tau_{2i}]$ have length 2ℓ. The sum of the lengths of all D_1-intervals (including the first and the last one) is equal to $\int_0^{t_1} u_1(t)\,dt = 1$, and $t_1 - 1$ is equal to the sum of the lengths of all D_3-intervals (including the first one).

Case 5 $D_3(D_1D_3)^nD_1$. On the first interval $[0, \tau_{2n+1}]$ we have $u_1 = 0$, $u_2 = -1$, $x_1 = \alpha$, and $x_2(t) = \beta - \alpha t$. From condition (4.11) at τ_{2n+1} we see that $\alpha = 1$, and

$$\beta - \tau_{2n+1} = -\lambda, \quad \tau_{2n+1} = \beta + \lambda.$$

Furthermore,

$$2n\ell + \ell = 1, \quad \ell = \frac{1}{2n+1},$$

and

$$t_1 - 1 = \tau_{2n+1} + 2n\lambda = \beta + (2n+1)k\cot\frac{k}{2n+1}. \tag{4.12}$$

The conditions $0 < \tau_{2n+1} \le 2\lambda$ give $-\lambda < \beta \le \lambda$, i.e.,

$$-k\cot\frac{k}{2n+1} < \beta \le k\cot\frac{k}{2n+1}, \quad 0 < \frac{k}{2n+1} < \frac{\pi}{2}. \tag{4.13}$$

Since the right-hand side of (4.12) is increasing in n, we need only consider the smallest n for which (4.13) holds. That is, for a fixed n we consider those k and β that satisfy

$$-k\cot k < \beta \le k\cot k, \quad 0 < k < \frac{\pi}{2}, \quad \text{if } n = 0,$$

$$-k\cot\frac{k}{2n+1} < \beta \le -k\cot\frac{k}{2n-1} \quad \text{or}$$

$$k\cot\frac{k}{2n-1} < \beta \le k\cot\frac{k}{2n+1}, \quad \text{if } k < (n-\tfrac{1}{2})\pi,\ n \ge 1,$$

$$-k\cot\frac{k}{2n+1} < \beta \le k\cot\frac{k}{2n+1}, \quad \text{if } (n-\tfrac{1}{2})\pi \le k < (n+\tfrac{1}{2})\pi,\ n \ge 1.$$

Case 6 $D_1D_3(D_1D_3)^{n-1}D_1$. On the first interval $[0, \tau_{2n}]$, x_1 and x_2 satisfy (4.1)–(4.2). Using (4.11) the solution can be written

$$x_1(t) = \frac{x_1(\tau_{2n})\sin k(\ell + t - \tau_{2n})}{\sin k\ell},$$

$$x_2(t) = \frac{kx_1(\tau_{2n})\cos k(\ell + t - \tau_{2n})}{\sin k\ell}.$$

We must have $0 < \tau_{2n} \le 2\ell$, so that

$$-\frac{\pi}{2} < -k\ell \le k(\ell - \tau_{2n}) < k\ell < \frac{\pi}{2}. \tag{4.14}$$

Consider $t = 0$. If $\alpha = 0$, then $\sin k(\ell - \tau_{2n}) = 0$, and $\tau_{2n} = \ell$ by (4.14). If $\alpha = 1$, then

$$\beta = \frac{x_2(0)}{x_1(0)} = k \cot k(\ell - \tau_{2n}).$$

By (4.14), $\beta \neq 0$, and

$$k(\ell - \tau_{2n}) = \arctan \frac{k}{\beta}.$$

We must also have

$$\tau_{2n} + (2n-1)\ell = 1.$$

If $\alpha = 0$, then $\tau_{2n} = \ell = \frac{1}{2n}$. If $\alpha = 1$, then

$$k(2n\ell - 1) = \arctan \frac{k}{\beta}, \quad \ell = \frac{1}{2n}\big(1 + \frac{1}{k} \arctan \frac{k}{\beta}\big).$$

By (4.14) the conditions on k and β are

$$-\frac{k}{2n+1} \le \arctan \frac{k}{\beta} < \frac{k}{2n-1}, \tag{4.15}$$

$$k + \arctan \frac{k}{\beta} < n\pi. \tag{4.16}$$

Finally,

$$t_1 - 1 = 2n\lambda = 2nk \cot \big[\frac{1}{2n}(k + \arctan \frac{k}{\beta})\big].$$

Again it is enough to consider the smallest n such that (4.15) and (4.16) hold. For each n we therefore only consider k and β that satisfy (4.15) and

$$(n-1)\pi \le k + \arctan \frac{k}{\beta} < n\pi.$$

So far we have found six different possible forms for the optimal solution. Let us summarize the results by listing the conditions for each case and the corresponding value of $t_1 - 1$.

(1)
$$\beta = -k \cot k \quad \text{if } \alpha = 1, \quad \sin k = 0 \quad \text{if } \alpha = 0,$$
$$t_1 - 1 = 0.$$

(2)
$$\beta < -k \cot k, \quad 0 < k < \frac{\pi}{2},$$
$$t_1 - 1 = -\beta - k \cot k = F_0(k, \beta).$$

(3)
$$\frac{\pi}{2} < k < \frac{3\pi}{2}, \quad n = 1,$$
$$(n - \tfrac{1}{2})\pi \le k < (n + \tfrac{1}{2})\pi, \quad n > 1,$$
$$\beta \le 0,$$
$$t_1 - 1 = -\beta + k[k - (n - \tfrac{1}{2})\pi] = F_n(k, \beta) \quad (n \ge 1).$$

$$n\pi < k + \arctan\frac{k}{\beta} \le (n+1)\pi, \quad \beta \ne 0, \tag{4}$$

$$t_1 - 1 = k\big(k - n\pi + \arctan\frac{k}{\beta}\big) = G_n(k,\beta) \quad (n \ge 1),$$

$$\big[\arctan\frac{k}{\beta} = 0 \text{ if } \alpha = 0\big].$$

$$-k\cot k < \beta \le k\cot k, \quad 0 < k < \frac{\pi}{2},\ n = 0, \tag{5}$$

$$-k\cot\frac{k}{2n+1} < \beta \le -k\cot\frac{k}{2n-1} \quad \text{or}$$

$$k\cot\frac{k}{2n-1} < \beta \le k\cot\frac{k}{2n+1}, \quad 0 < k < (n - \tfrac{1}{2})\pi,\ n \ge 1,$$

$$-k\cot\frac{k}{2n+1} < \beta \le k\cot\frac{k}{2n+1}, \quad (n-\tfrac{1}{2})\pi \le k < (n+\tfrac{1}{2})\pi,\ n \ge 1.$$

$$t_1 - 1 = \beta + (2n+1)k\cot\frac{k}{2n+1} = H_n(k,\beta) \quad (n \ge 0).$$

$$(n-1)\pi \le k + \arctan\frac{k}{\beta} < n\pi, \quad \beta \ne 0, \tag{6}$$

$$\beta > k\cot\frac{k}{2n-1} > 0 \quad \text{or} \quad \beta \le -k\cot\frac{k}{2n+1} < 0,$$

$$t_1 - 1 = 2nk\cot\big[\frac{1}{2n}(k + \arctan\frac{k}{\beta})\big] = J_n(k,\beta) \quad (n \ge 1).$$

Consider the region defined by the inequalities in (6); see Fig. 2.

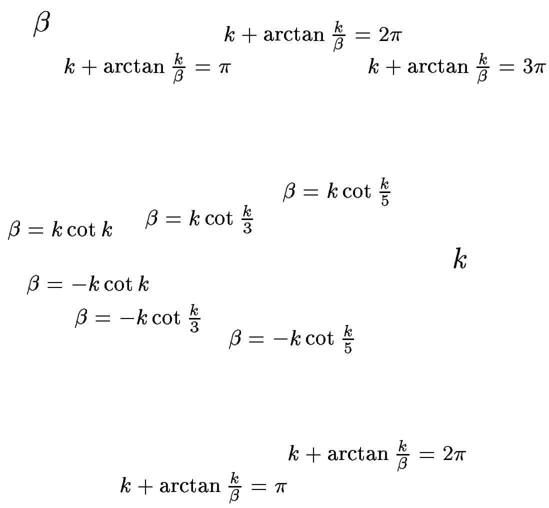

FIGURE 2. The regions defined by (6).

Let us show that Case 6 is always better than Case 5 in this region. If (k,β) is a point there with $\beta > 0$, then

$$k\cot\frac{k}{2m-1} < \beta \le k\cot\frac{k}{2m+1} \qquad \text{for some } m \ge n.$$

Since $\frac{k}{2n} < \frac{1}{2n}(k + \arctan\frac{k}{\beta}) < \frac{\pi}{2}$,

$$\begin{aligned} J_n(k,\beta) < 2nk\cot\frac{k}{2n} &< \beta + (2n+1)k\cot\frac{k}{2n+1} \\ &= H_n(k,\beta) \le H_m(k,\beta). \end{aligned}$$

If $\beta < 0$, then $\beta \le -k\cot\frac{k}{2n+1}$, or $\arctan\frac{k}{\beta} \ge -\frac{k}{2n+1}$, so that $k + \arctan\frac{k}{\beta} \ge \frac{2nk}{2n+1}$, and

$$-k\cot\frac{k}{2m+3} < \beta \le -k\cot\frac{k}{2m+1} \qquad \text{for some } m \ge n,$$

$$\begin{aligned} J_n(k,\beta) \le 2nk\cot\frac{k}{2n+1} &\le \beta + (2n+1)k\cot\frac{k}{2n+1} \\ &= H_n(k,\beta) \le H_{m+1}(k,\beta). \end{aligned}$$

Let us compare Case 6 and Case 4 (with $n-1$ instead of n, $n \ge 2$) in this region. Let $v = k + \arctan\frac{k}{\beta}$, so that $(n-1)\pi < v < n\pi$. Then

$$\begin{aligned} J_n(k,\beta) - G_{n-1}(k,\beta) &= 2nk\cot\frac{v}{2n} - k[v-(n-1)\pi] \\ &= k[2n\cot\frac{v}{2n} - v + (n-1)\pi]. \end{aligned}$$

Let v_n be the unique solution of the equation

$$2n\cot\frac{v}{2n} = v - (n-1)\pi, \qquad (n-1)\pi < v < n\pi.$$

Then

$$\begin{aligned} G_{n-1}(k,\beta) < J_n(k,\beta) &\quad \text{if } (n-1)\pi < k + \arctan\frac{k}{\beta} < v_n, \\ G_{n-1}(k,\beta) = J_n(k,\beta) &\quad \text{if } k + \arctan\frac{k}{\beta} = v_n, \\ J_n(k,\beta) < G_{n-1}(k,\beta) &\quad \text{if } v_n < k + \arctan\frac{k}{\beta} < n\pi. \end{aligned}$$

If $\beta < 0$ and $k + \arctan\frac{k}{\beta} > \pi$, then

$$\begin{aligned} G_n(k,\beta) - F_{n+1}(k,\beta) &= \beta + k(\arctan\frac{k}{\beta} + \frac{\pi}{2}) \\ &= k(\arctan\frac{k}{\beta} + \frac{\beta}{k} + \frac{\pi}{2}) < 0, \end{aligned}$$

so that $G_n < F_{n+1}$, and also $G_n < F_n$.

In the region

$$n\pi < k + \arctan\frac{k}{\beta} < (n+1)\pi,$$

$$0 < \beta \le k\cot\frac{k}{2n+1},$$

we want to compare H_n and G_n. The equation

$$H_n(k,\beta) - G_n(k,\beta) = \beta + (2n+1)k\cot\frac{k}{2n+1} - k(k - n\pi + \arctan\frac{k}{\beta}) = 0$$

defines a curve that divides the region into two parts. In the left part $G_n < H_n$, and in the right part $H_n < G_n$. If instead $-k\cot\frac{k}{2n+3} < \beta < 0$, we compare H_{n+1} and G_n. In the same way we consider the regions where Cases (2) and (6), (3) and (6), and (3) and (5) overlap. The result is illustrated in Fig. 3 where each region is labelled with the name of the function that yields the smallest value of $t_1 - 1$.

β

J_1 G_1 J_2 G_2 J_3 G_3 J_4

H_0 H_1 H_2 H_3 k

F_1

F_0

J_1 G_1 J_2 G_2 J_3

FIGURE 3. Regions for different forms of the optimal solution.

5. THE CASE $c \leq 0$

First we assume that $c < 0$. Let

$$\kappa = \sqrt{-c}\,.$$

In this case we obtain for $t \in [\tau_1, t_1]$

$$\begin{aligned} x_1(t) &= -x_2(t_1)\frac{1}{\kappa}\sinh\kappa(t_1 - t), \\ x_2(t) &= x_2(t_1)\cosh\kappa(t_1 - t). \end{aligned}$$

These formulae, like many others in this section, may be obtained from the corresponding formulae in the previous section by replacing k by $i\kappa$.

This time too $s_2(t)$ may belong to D_1 a.e., but only if $\beta = -\kappa\coth\kappa$ (see the argument in Case 1). If this is not the case, then $\tau_1 > 0$, and $|s_2(\tau_1)| = 1$. As in (4.6) we obtain

$$\frac{x_2}{x_1}(\tau_1) = -\lambda = -\kappa\coth\kappa\ell, \qquad \ell = t_1 - \tau_1.$$

If $s_2(\tau_1) = 1$, then the only possibility is a solution of the form D_2D_1 (since we cannot stay on L_2 when $c < 0$). As in Case 2, $\ell = 1$,

$$\frac{x_2}{x_1}(\tau_1) = \beta + \tau_1 = -\lambda = -\kappa\coth\kappa,$$

and

$$t_1 - 1 = -\beta - \kappa\coth\kappa = \tilde{F}_0(\kappa, \beta), \tag{5.1}$$

where

$$\beta < -\kappa\coth\kappa.$$

If $s_2(\tau_1) = -1$, we may have the cases

$$D_3(D_1D_3)^n D_1 \ (n \geq 0), \quad D_1D_3(D_1D_3)^{n-1}D_1 \ (n \geq 1)$$

as before. We still have the relations (4.11).

Case $\tilde{5}$ $D_3(D_1D_3)^nD_1$. Instead of (4.12) and (4.13) we get

$$t_1 - 1 = \beta + (2n+1)\kappa \coth \frac{\kappa}{2n+1} = \tilde{H}_n(\kappa,\beta), \tag{5.2}$$
$$-\kappa \coth \frac{\kappa}{2n+1} < \beta \leq \kappa \coth \frac{\kappa}{2n+1}.$$

Since $\tilde{H}_n(\kappa,\beta)$ is increasing in n, we need only consider

$$-\kappa \coth \kappa < \beta \leq \kappa \coth \kappa \quad \text{if } n = 0,$$
$$-\kappa \coth \frac{\kappa}{2n+1} < \beta \leq -\kappa \coth \frac{\kappa}{2n-1} \quad \text{or}$$
$$\kappa \coth \frac{\kappa}{2n-1} < \beta \leq \kappa \coth \frac{\kappa}{2n+1} \quad \text{if } n \geq 1.$$

Case $\tilde{6}$ $D_1D_3(D_1D_3)^{n-1}D_1$. On $[0,\tau_{2n}]$ we have as in Case 6

$$\frac{x_2}{x_1}(t) = \kappa \coth \kappa(\ell + t - \tau_{2n}).$$

If $\alpha = 0$, then $\tau_{2n} = \ell$, and if $\alpha = 1$, then

$$\beta = \kappa \coth \kappa(\ell - \tau_{2n}).$$

Furthermore, $\tau_{2n} + (2n-1)\ell = 1$, so that $\tau_{2n} = \ell = \frac{1}{2n}$ if $\alpha = 0$, and

$$\ell = \frac{1}{2n}\big(1 + \frac{1}{\kappa}\operatorname{arctanh}\frac{\kappa}{\beta}\big) \quad \text{if } \alpha = 1.$$

Also,

$$t_1 - 1 = 2n\lambda = 2n\kappa \coth \kappa\ell$$
$$= 2n\kappa \coth\big[\frac{1}{2n}(\kappa + \operatorname{arctanh}\frac{\kappa}{\beta})\big] = \tilde{J}_n(\kappa,\beta). \tag{5.3}$$

The conditions $0 < \tau_{2n} \leq 2\ell$ imply $\frac{1}{2n+1} \leq \ell < \frac{1}{2n-1}$, so that

$$-\tanh\frac{\kappa}{2n+1} \leq \frac{\kappa}{\beta} < \tanh\frac{\kappa}{2n-1},$$

i.e.,

$$\beta > \kappa \coth \frac{\kappa}{2n-1} \quad \text{or} \quad \beta \leq -\kappa \coth \frac{\kappa}{2n+1}.$$

But if this is satisfied for some n, it is satisfied for $n = 1$, and since $\tilde{J}_n(\kappa,\beta)$ is increasing in n, we need only consider the case $n = 1$.

If $\kappa \coth \frac{\kappa}{2n-1} < \beta \leq \kappa \coth \frac{\kappa}{2n+1}$ $(n \geq 1)$, then

$$\tilde{J}_1(\kappa,\beta) \leq 2\kappa \coth \frac{(n+1)\kappa}{2n+1} < \tilde{H}_n(\kappa,\beta),$$

and if $-\kappa \coth \frac{\kappa}{2n+1} < \beta \leq -\kappa \coth \frac{\kappa}{2n-1}$ $(n \geq 2)$, then

$$\tilde{J}_1(\kappa,\beta) \leq 2\kappa \coth \frac{(n-1)\kappa}{2n-1} < 2\kappa \coth \frac{\kappa}{2n+1} < \tilde{H}_n(\kappa,\beta).$$

For $-\kappa \coth \frac{\kappa}{3} < \beta < -\kappa \coth \kappa$ we have

$$\tilde{F}_0(\kappa,\beta) = -\beta - \kappa \coth \kappa < \beta + 3\kappa \coth \frac{\kappa}{3} = \tilde{H}_1(\kappa,\beta).$$

Finally, we must compare $\tilde{J}_1$ and $\tilde{F}_0$ for $\beta \le -\kappa \coth \frac{\kappa}{3}$. The equation

$$\tilde{J}_1(\kappa,\beta) - \tilde{F}_0(\kappa,\beta) = 2\kappa \coth \big[\frac{1}{2}(\kappa + \operatorname{arctanh} \frac{\kappa}{\beta})\big] + \beta + \kappa \coth \kappa = 0$$

determines a curve above which $\tilde{F}_0 < \tilde{J}_1$, and below which $\tilde{J}_1 < \tilde{F}_0$.

The solution in the case $c = 0$ can be obtained if we let $\kappa \to 0$ (or $k \to 0$ in the previous section).

Now we know the solution for each (κ, β). The result is illustrated in Fig. 4. The minimum values of $t_1 - 1$ are given by (5.1), (5.2), and (5.3) with $n = 0$ or $n = 1$.

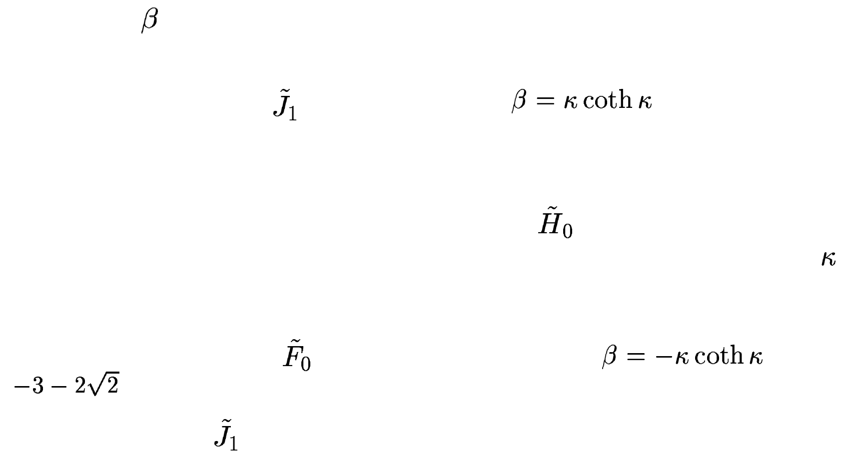

FIGURE 4. Regions for different forms of the optimal solution.

Remark. In Section 2 we stated that $\inf t_1$ is not changed when passing from the constraint $|u(t)| < 1$ to $|u(t)| \le 1$. We now know that the optimal controls of the latter problem are such that $u(t) = 0$ on an interval $(\tau_1, t_1]$. Let u be such an optimal control. Define for $0 < \varepsilon < 1$,

$$u_\varepsilon(t) = \begin{cases} (1-\varepsilon)u(t) & \text{on } [0, \tau_1), \\ u_0 & \text{on } (\tau_1, t_{1,\varepsilon}], \end{cases}$$

where u_0 is a constant, and

$$\begin{aligned} \int_0^{t_{1,\varepsilon}} |u_\varepsilon(t)|\, dt &= (1-\varepsilon)\int_0^{\tau_1} |u(t)|\, dt + |u_0|(t_{1,\varepsilon} - \tau_1) \\ &= (1-\varepsilon)(t_1 - 1) + |u_0|(t_{1,\varepsilon} - \tau_1) = t_{1,\varepsilon} - 1, \end{aligned}$$

i.e.,

$$(1 - |u_0|)(t_{1,\varepsilon} - \tau_1) = t_1 - \tau_1 - \varepsilon(t_1 - 1). \tag{5.4}$$

The corresponding solution $(x_{1,\varepsilon}, x_{2,\varepsilon})$ of the differential equations differs from (x_1, x_2) at τ_1 by $O(\varepsilon)$ (as $\varepsilon \to 0$). On $[\tau_1, t_{1,\varepsilon}]$ we can write down the solution explicitly. The condition $x_{1,\varepsilon}(t_{1,\varepsilon}) = 0$ gives an equation for u_0 [$t_{1,\varepsilon}$ only appears in the expression (5.4)]. The solution is $u_0 = O(\varepsilon)$, and then (5.4) gives $t_{1,\varepsilon} = t_1 + O(\varepsilon)$. This proves the statement about $\inf t_1$.

6. Two minimax problems

Instead of the L^1-norm in (1.3) we now consider the L^∞-norm (1.4). The corresponding optimization problem (a minimax problem) is solved below (Problem 1). In [2] a problem of this type was studied with the additional constraint $\int_0^1 q(t)\,dt = 0$. We are going to solve this problem too (Problem 2).

Problem 1. This time there is no transformation of the problem, but we prefer to use standard notation, so we consider the following problem: Minimize

$$C(u) = \operatorname*{ess\,sup}_{t\in[0,1]} |u(t)| \tag{6.1}$$

subject to

$$x_1' = x_2, \tag{6.2}$$

$$x_2' = ux_1 - cx_1, \tag{6.3}$$

$$x_1(0) = \alpha, \quad x_2(0) = \beta, \tag{6.4}$$

$$x_1(1) = 0. \tag{6.5}$$

There exists an optimal control – see [9, p. 262]. Let $(x^*, u^*) = (x_1^*, x_2^*, u^*)$ be an optimal solution of (6.2)–(6.5). We can apply [4, Theorem 2.2] to obtain necessary conditions for optimality. Let

$$C^* = \min_{u\in L^\infty(0,1)} C(u) = \operatorname*{ess\,sup}_{t\in[0,1]} |u^*(t)|\,.$$

As before we define

$$k = \sqrt{c} \qquad \text{if } c > 0,$$
$$\kappa = \sqrt{-c} \qquad \text{if } c \le 0.$$

If $C^* = 0$, then $u^*(t) = 0$ a.e., and as in Section 4, Case 1, and the corresponding case in Section 5 we must have

$$\begin{aligned}
&c > 0 \quad \text{and} \quad \sin k = 0, \text{ if } \alpha = 0,\\
&\beta = -k\cot k, \text{ if } c > 0,\ \alpha = 1,\\
&\beta = -\kappa\coth\kappa, \text{ if } c < 0,\ \alpha = 1,\\
&\beta = -1, \text{ if } c = 0,\ \alpha = 1.
\end{aligned}$$

Now, assume that $C^* > 0$. Then condition (vi$'$) in [4, p. 402] is satisfied because

$$\operatorname*{ess\,sup}_{t\in[0,1]} |u^*(t) - \varepsilon u^*(t)| = (1-\varepsilon)C^* < C^*, \quad \text{if } 0 < \varepsilon \le \bar\varepsilon < 1.$$

Introduce

$$H(x, u, \eta) = \eta_1 x_2 + \eta_2(ux_1 - cx_1). \tag{6.6}$$

Then Theorem 2.2 in [4] tells us that there are absolutely continuous functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ such that $(\eta_1(t), \eta_2(t)) \ne (0,0)$ for all t,

$$\eta_1' = -\frac{\partial H}{\partial x_1} = -\eta_2 u^* + c\eta_2, \tag{6.7}$$

$$\eta_2' = -\frac{\partial H}{\partial x_2} = -\eta_1, \tag{6.8}$$

$$\eta_2(1) = 0, \tag{6.9}$$

and the following maximum principle is satisfied:

$$H(x^*(t), u^*(t), \eta(t)) = \max_{u \in \Omega} H(x^*(t), u, \eta(t)), \tag{6.10}$$

where

$$\Omega = \{u : |u| \leq C^*\}.$$

Thus, if

$$s(t) = \eta_2(t) x_1^*(t),$$

then

$$u^*(t) = C^* \operatorname{sign} s(t), \quad \text{if } s(t) \neq 0. \tag{6.11}$$

Let us omit the stars on x^* and u^* in the following. We find from (6.2)–(6.3), (6.5), and (6.7)–(6.9) that there exists a constant $\delta \neq 0$ such that

$$\eta_1(t) = \delta x_2(t), \qquad \eta_2(t) = -\delta x_1(t). \tag{6.12}$$

Thus, $s(t) = -\delta x_1^2(t)$, which shows that s can only have isolated zeroes and cannot change sign. Thus [by (6.11)] $u(t) = C^*$ a.e. or $u(t) = -C^*$ a.e.

When $u(t) = \pm C^*$, the solution of (6.2)–(6.4) gives

$$x_1(t) = \alpha \cos k_\pm t + \frac{\beta}{k_\pm} \sin k_\pm t, \tag{6.13}$$

where

$$k_\pm = \sqrt{c \mp C^*}.$$

If $c \mp C^* < 0$, we replace $k_\pm$ by $i\kappa_\pm$, where

$$\kappa_\pm = \sqrt{\pm C^* - c},$$

and use the corresponding hyperbolic functions in (6.13). From (6.5) and (6.13) we obtain

$$\alpha \cos k_\pm + \frac{\beta}{k_\pm} \sin k_\pm = 0, \tag{6.14}$$

or

$$\alpha \cosh \kappa_\pm + \frac{\beta}{\kappa_\pm} \sinh \kappa_\pm = 0. \tag{6.15}$$

In the case $\alpha = 0$, only (6.14) is possible, and we get $\sin k_\pm = 0$, $k_\pm = n\pi$ $(n = 1, 2, \dots)$, so that $C^* = c - n^2\pi^2$ and $C^* = n^2\pi^2 - c$ in the two cases. In the first case we must have $c > n^2\pi^2$, and we should choose n as large as possible. In the second case we must choose the smallest $n \geq 1$ such that $n^2\pi^2 > c$. Thus

$$C^* = \begin{cases} \pi^2 - c & \text{for } c < \pi^2, \\ \min\left(c - n^2\pi^2, (n+1)^2\pi^2 - c\right) & \text{for } n^2\pi^2 \leq c < (n+1)^2\pi^2. \end{cases}$$

In the case $\alpha = 1$, we get from (6.14) or (6.15)

$$k_\pm \cot k_\pm = -\beta \qquad \text{or} \qquad \kappa_\pm \coth \kappa_\pm = -\beta.$$

Let for any β and $n = 1, 2, \dots$, and for $\beta > -1$ and $n = 0$, $k_n(\beta)$ be the solution of $k \cot k = -\beta$, $n\pi < k < (n+1)\pi$, and let for $\beta < -1$, $\kappa_0(\beta)$ be the solution of $\kappa \coth \kappa = -\beta$,

$\kappa > 0$. Let also $\kappa_0(-1) = 0$. Then we must have $k_\pm = k_n(\beta)$ for some $n \geq 1$, or $k_\pm = k_0(\beta)$ if $\beta > -1$, $k_\pm = \kappa_0(\beta)$ if $\beta \leq -1$. Thus, if $\beta > -1$, then

$$C^* = \begin{cases} k_0^2(\beta) - c & \text{for } c < k_0^2(\beta), \\ \min\bigl(c - k_n^2(\beta),\, k_{n+1}^2(\beta) - c\bigr) & \text{for } k_n^2(\beta) \leq c < k_{n+1}^2(\beta),\ n \geq 0, \end{cases}$$

and if $\beta \leq -1$, then

$$C^* = \begin{cases} -c - \kappa_0^2(\beta) & \text{for } c \leq -\kappa_0^2(\beta), \\ \min\bigl(\kappa_0^2(\beta) + c,\, k_1^2(\beta) - c\bigr) & \text{for } -\kappa_0^2(\beta) < c < k_1^2(\beta), \\ \min\bigl(c - k_n^2(\beta),\, k_{n+1}^2(\beta) - c\bigr) & \text{for } k_n^2(\beta) \leq c < k_{n+1}^2(\beta),\ n \geq 1. \end{cases}$$

Problem 2. Minimize (6.1) subject to (6.2)–(6.5) and $\int_0^1 u(t)\,dt = 0$. The solution turns out to be rather complicated, and therefore we restrict ourselves to the special case $\alpha = 0$.

The analysis follows the same lines as in the previous problem. The Hamiltonian (6.6) is modified by the addition of a term $\eta_3 u$, where η_3 is a constant. Then s is defined as

$$s = \eta_2 x_1 + \eta_3.$$

The equations (6.7)–(6.12) are still valid, but this time we have that $\eta_1(t)$, $\eta_2(t)$, and η_3 are not all zero.

If $\eta_3 = 0$, it would follow that s is of constant sign (except for isolated zeroes). But then $u(t) = C^* \operatorname{sign} s(t)$ cannot satisfy the condition $\int_0^1 u(t)\,dt = 0$ (we are assuming $C^* > 0$). Thus $\eta_3 \neq 0$. Also $\delta \neq 0$, since $\delta = 0$ would give $s = \eta_3 \neq 0$.

The set $\{t : s(t) \neq 0\}$ consists of finitely many disjoint open intervals (cf. Section 3), so that s may only be 0 on finitely many intervals. Assume that $s = 0$ on an interval of positive length. Then $s' = -2\delta x_1 x_2 = 0$ on this interval. We must have $x_2 = 0$ on the interval since $x_1^2 = \eta_3/\delta \neq 0$. It follows from the differential equation (6.3) for x_2 that $u(t) = c$ on any interval where $s = 0$.

As before we let τ_1 be the last zero of $s(t)$. Since $s(1) = \eta_3 \neq 0$, we have $\tau_1 < 1$. On $[\tau_1, 1]$, (6.2)–(6.3) and (6.5) give

$$\begin{aligned} x_1(t) &= -x_2(1)\frac{1}{k_1}\sin k_1(1-t), \\ x_2(t) &= x_2(1)\cos k_1(1-t), \end{aligned}$$

where $k_1 = k_+ = \sqrt{c - C^*}$ if $s > 0$, and $k_1 = k_- = \sqrt{c + C^*}$ if $s < 0$. Here k_1 may be complex, but if, for instance, $c - C^* < 0$, we replace k_+ by $i\kappa_+$, $\kappa_+ = \sqrt{C^* - c}$. We also get

$$s(t) = \eta_3 - \delta x_2^2(1)\frac{1}{k_1^2}\sin^2 k_1(1-t).$$

Put $\ell_1 = 1 - \tau_1$. Since $s(\tau_1) = 0$ and $s(t) \neq 0$ for $t \in (\tau_1, 1]$, we see that $k_1\ell_1 \leq \frac{\pi}{2}$, when k_1 is real and positive. We get

$$\frac{x_2}{x_1}(\tau_1) = -k_1 \cot k_1\ell_1 \leq 0.$$

If $k_1 = i\kappa_1$, we get instead

$$\frac{x_2}{x_1}(\tau_1) = -\kappa_1 \coth \kappa_1\ell_1 < 0,$$

and if $k_1 = 0$, $\frac{x_2}{x_1}(\tau_1) = -\frac{1}{\ell_1} < 0$. Thus $\frac{x_2}{x_1}(\tau_1) < 0$ except when $k_1\ell_1 = \frac{\pi}{2}$. Consider first $\frac{x_2}{x_1}(\tau_1) < 0$. Since $s' = -2\eta_3\frac{x_2}{x_1} \neq 0$ when $s = 0$, s must change sign at τ_1. Since $s(0) = s(1)$, there must be a τ_2, $0 < \tau_2 < \tau_1$, such that $s(\tau_2) = 0$ and $s \neq 0$ on (τ_2, τ_1). On (τ_2, τ_1), k_1

in the differential equations is replaced by k_2, where $k_2 = k_-$ if $k_1 = k_+$ and vice versa. We must have k_2 real and positive since if $k_2 = i\kappa_2$, we find for s on (τ_2, τ_1) that

$$s(t) = \eta_3 \left\{ \frac{1}{\kappa_2} \frac{x_2}{x_1}(\tau_1) \sinh 2\kappa_2(\tau_1 - t) - \left[1 + \frac{1}{\kappa_2^2}\big(\frac{x_2}{x_1}(\tau_1)\big)^2\right] \sinh^2 \kappa_2(\tau_1 - t) \right\},$$

and this cannot fulfil $s(\tau_2) = 0$. The same is true if $k_2 = 0$. Now, define ℓ_2 by

$$\frac{x_2}{x_1}(\tau_1) = -k_2 \tan k_2\ell_2, \quad 0 < k_2\ell_2 < \frac{\pi}{2}.$$

Then $s(t)$ can be written as

$$s(t) = -\frac{\eta_3}{\cos^2 k_2\ell_2} \sin k_2(\tau_1 - t) \sin k_2(t - \tau_1 + 2\ell_2), \quad \tau_2 \le t \le \tau_1.$$

This shows that $\tau_1 - \tau_2 = 2\ell_2$. We also see that $s'(\tau_2) = -s'(\tau_1)$, so that $\frac{x_2}{x_1}(\tau_2) = -\frac{x_2}{x_1}(\tau_1)$. Then s changes sign at τ_2, and in the stage before τ_2 we have the same differential equations as on $(\tau_1, 1]$. Using the condition $s'(\tau_2) = -s'(\tau_1)$, we see that at $t = \tau_2 - \ell_1$ we have the same situation as at $t = 1$ (i.e., $x_1 = 0$ and $s = \eta_3$), so either $\tau_2 - \ell_1 = 0$ or $s = 0$ at $\tau_3 = \tau_2 - 2\ell_1$, and $\frac{x_2}{x_1}(\tau_3) = -\frac{x_2}{x_1}(\tau_2)$. In general we may have solutions of the form

$$+\ -\ (+\,-)^{n-1}+\,, \qquad -\ +\ (-\,+)^{n-1}\ - \qquad (n \ge 1),$$

where the sequence of plus and minus signs indicates the signs of s.

The only remaining possibility is that $k_1\ell_1 = \frac{\pi}{2}$. Then $s'(\tau_1) = 0$, and s may be zero on an interval. If $s \ne 0$ on an interval with endpoint t' where $s(t') = 0$, then $s(t) = \eta_3 \sin^2 k_\pm(t - t')$ on that interval. If $s(1) = \eta_3 > 0$, then $k_+ = k_1$ and k_- are both real. If $\eta_3 < 0$, then k_+ may be imaginary, $k_+ = i\kappa_+$. Since $-\eta_3 \sinh^2 \kappa_+(t - t')$ has no other zero than t', s may be positive only on an interval $[0, t')$ in that case. But this cannot happen because $s(0) = s(1)$. Thus s must have the same sign as η_3 when it is not 0. To satisfy $\int_0^1 u(t)\,dt = 0$, $s(t)$ must be 0 on one or more intervals. There $u(t) = c$ (see above), and it follows from the definition of C^* that $C^* \ge |c|$. If $\eta_3 > 0$, then $k_1 = k_+ = \sqrt{c - C^*}$. But this is impossible since $c - C^* \le 0$. Thus $\eta_3 < 0$, and we may have a solution of the form $-\ 0\ (-\ 0)^{n-1} -$.

Let us investigate the three possible cases.

Case 1 $+\ -\ (+\,-)^{n-1}+$. In this case $k_1 = k_+$, $k_2 = k_-$. The length of each $-$-interval is $2\ell_-$, and the length of each $+$-interval is $2\ell_+$, except the first and the last one whose length is ℓ_+. We have the relation

$$-\frac{x_2}{x_1}(\tau_1) = k_+ \cot k_+\ell_+ = k_- \tan k_-\ell_-, \quad 0 < k_+\ell_+ < \frac{\pi}{2},\ 0 < k_-\ell_- < \frac{\pi}{2},$$

or

$$\kappa_+ \coth \kappa_+\ell_+ = k_- \tan k_-\ell_-.$$

The condition $\int_0^1 u(t)\,dt = 0$ implies that

$$\ell_+ + (n-1) \cdot 2\ell_+ + \ell_+ = \frac{1}{2}, \qquad n \cdot 2\ell_- = \frac{1}{2},$$

i.e.,

$$\ell_+ = \ell_- = \frac{1}{4n}, \quad n = 1, 2, \dots .$$

Consider the system

$$\begin{aligned} &y\cot\frac{y}{4n} = x\tan\frac{x}{4n}, \\ &x^2+y^2 = 2c, \\ &0<y<x<2n\pi, \end{aligned} \qquad \text{or} \qquad \begin{aligned} &y\coth\frac{y}{4n} = x\tan\frac{x}{4n}, \\ &x^2-y^2 = 2c, \\ &0<x<2n\pi,\ y\ge 0. \end{aligned}$$

The solution (when it exists) gives $k_- = x$, $k_+ = y$ or $\kappa_+ = y$, and then $C^* = k_-^2 - c$. It can be shown that x^2+y^2 is decreasing along the curve $y\cot\frac{y}{4n} = x\tan\frac{x}{4n}$, and x^2-y^2 is decreasing along the curve $y\coth\frac{y}{4n} = x\tan\frac{x}{4n}$. Therefore the solution is unique, and as Fig. 5 (in which the solid curves illustrate the case $n=2$) shows it exists when $c < n^2\pi^2$. The value $n\pi$ is the solution of $y\cot\frac{y}{4n} = x\tan\frac{x}{4n}$ when $y=x$. The figure also shows that we obtain the first form of the solution when $c > \alpha_n^2/2$, where α_n is the solution of $\tan\frac{x}{4n} = 4n$, and the second form when $c \le \alpha_n^2/2$. In the figure, $c_1 > \alpha_2^2/2$, $0 < c_2 < \alpha_2^2/2$, and $c_3 < 0$. Let us write $f_n(c) = k_-^2 - c$, and also $f_n(n^2\pi^2) = 0$. It is enough to consider the smallest n such that $c < n^2\pi^2$.

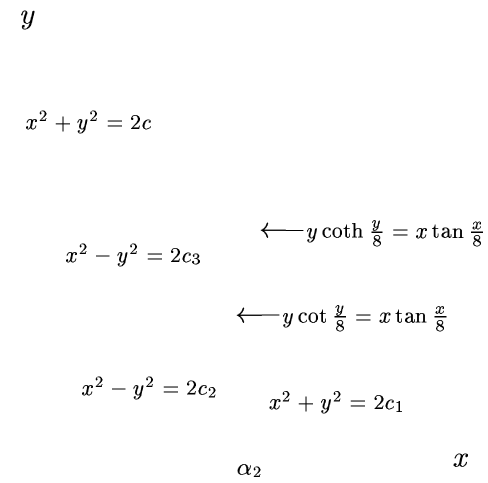

FIGURE 5. Curves for finding k_- and k_+.

Case 2 $-\ +\ (-\,+)^{n-1}\ -$. This is like the previous case with $+$ and $-$ interchanged, but this time k_+ and k_- are both real and positive. We have the equation

$$k_-\cot k_-\ell_- = k_+\tan k_+\ell_+, \quad 0 < k_-\ell_- < \frac{\pi}{2},\ 0 < k_+\ell_+ < \frac{\pi}{2},$$

and as before $\ell_+ = \ell_- = \frac{1}{4n}$. This time $k_+ = x$, $k_- = y$ is the solution of

$$\begin{aligned} &y\cot\frac{y}{4n} = x\tan\frac{x}{4n}, \\ &x^2+y^2 = 2c, \\ &0<x<y<2n\pi, \end{aligned}$$

and $C^* = k_-^2 - c = g_n(c)$. We see from Fig. 5 (the dashed curves) that the solution exists for $n^2\pi^2 < c < 2n^2\pi^2$. It is enough to consider the largest n that satisfies these inequalities. Note that $k_- < \sqrt{2c}$ implies $g_n(c) < c$.

Case 3 $-\ 0\ (-\ 0)^{n-1}\ -$. We have $k_-\ell_- = \frac{\pi}{2}$. If T_0 is the total length of the intervals where $s=0$, then

$$T_0 + 2n\ell_- = 1,$$

and

$$\int_0^1 u(t)\,dt = cT_0 - 2n\ell_- C^* = 0.$$

Thus

$$C^* = c(\frac{1}{2n\ell_-} - 1) = c(\frac{k_-}{n\pi} - 1),$$

and

$$k_- = \sqrt{c + C^*} = \sqrt{\frac{ck_-}{n\pi}},$$

so that $k_- = \frac{c}{n\pi}$. We must have $T_0 > 0$, which implies $c > n^2\pi^2$. The corresponding value of C^* is

$$C^* = c(\frac{c}{n^2\pi^2} - 1).$$

But we must also have $C^* \geq c$, which gives $c \geq 2n^2\pi^2$.

When $n \geq 3$, then $(n+1)^2 < 2n^2$, and Case 3 cannot be optimal since Case 2 gives a cost less than c. For $n = 2$, Case 3 may be considered for $8\pi^2 \leq c < 9\pi^2$, but then Case 1 is better. For $n = 1$, however, there is an interval $2\pi^2 \leq c < c'$ where Case 3 is better than Case 1. This fact shows that Satz IV.I in [2] is not correct. A numerical computation gives $c' \approx 21.81$.

Now we have found that the solution of the problem is

$$C^* = \begin{cases} f_1(c) \quad \text{if } c \leq \pi^2, \\ g_1(c) \quad \text{if } \pi^2 < c < 2\pi^2, \\ c(\frac{c}{\pi^2} - 1) \quad \text{if } 2\pi^2 \leq c \leq c', \\ f_2(c) \quad \text{if } c' < c \leq 4\pi^2, \\ \min(g_2(c), f_3(c)) \quad \text{if } 4\pi^2 < c < 8\pi^2, \\ f_3(c) \quad \text{if } 8\pi^2 \leq c \leq 9\pi^2, \\ \min(g_n(c), f_{n+1}(c)) \quad \text{if } n^2\pi^2 < c \leq (n+1)^2\pi^2,\ n \geq 3. \end{cases}$$

This is illustrated in Fig. 6.

C^*

c

FIGURE 6. The optimal value as a function of c.

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