On the connection between probabilistic and topological stability of weak Feller chains I

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Abstract

We consider a weak Feller Markov chain with transition kernel $P(x,\cdot)$ on a locally compact separable metric space $(X,\mathcal{B})$. Let $L(x,A) = P_x\{\tau_A < \infty\} \forall A \in \mathcal{B}$. We look at the connection between probabilistic and topological stability when the function $L(x,\cdot)$ admits a continuous component and at least one reachable point $x^* \in X$ exists.

Key words: Feller chains, continuous component, stability

0 Introduction

The purpose of this paper is to develop a set of convergence results for Feller chains on a topological space X; since in the Feller situation with a reachable point, ψ-irreducibility is equivalent to the so-called T-property [2], we need to look at the "Doeblin" decomposition of the state space X where the weaker condition of the existence of a continuous component of $L(x,\cdot)$ is also necessary and sufficient ([2], [3], [4]).

The first section describes the probabilistic and toplogical backgound; the results on the convergence of the weak Feller chains are stated in the second one, and the proofs are presented in Section 3. The work was motivated by [1]; in the Feller case it seems quite natural to look at the *weak* convergence to stationary distribution rather than at the *total variation* one.

1 Some preliminaries and background

1.1 Probabilistic concepts of stability

Let us consider a Markov chain $\{\Phi_n\}_0^\infty$ living on a locally compact separable space $(X,\mathcal{B})$, ($\mathcal{B} - \sigma$-algebra of all open subsets of X), and let τ_A be a stopping time defined $\forall A \in \mathcal{B}$ by $\tau_A = \inf\{n \geq 1, \Phi_n \in A\}$. For any $x \in A$, $A \in \mathcal{B}$ let us define functions $L(x, A)$ and $Q(x, A)$ by

$$\begin{aligned} L(x, A) = P_x\{\tau_A < \infty\} = P_x\{\Phi \text{ enters } A\}; \\ Q(x, A) = P_x\{\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \{\Phi_k \in A\}\} = \\ = P_x\{\Phi \in A \text{ i.o. }\}. \end{aligned}$$

Definition 1.1. A set $A \in \mathcal{B}$ is called "inessential" if $Q(x, A) = 0 \quad \forall x \in X$; otherwise, it is called "essential". If it cannot be written as a countable union of inessential sets, A is regarded as properly essential.

Definition 1.2. Let ψ be non-trivial σ-finite measure on $(X, \mathcal{B})$ such that $L(x, A) > 0 \quad \forall x \in X$ whenever $\psi(A) > 0$, $A \in \mathcal{B}$. Then the chain $\{\Phi_n\}$ is ψ-irreducible, and ψ is called an irreducibility measure.

Probabilistic concepts of stability 1: Harris recurrence

Definition 1.3. Suppose that the chain $\{\Phi_n\}_0^{\infty}$ is ψ-irreducible, and $Q(x, A) \equiv 1 \quad \forall A \in \mathcal{B}$ such that $\psi(A) > 0$. Then the Markov chain $\{\Phi_n\}_0^{\infty}$ is called Harris recurrent ([2], [3], [4]). A subset $B \subset X$ is called "absorbing" if $P(x, B) = 1 \quad \forall x \in B$. If B is absorbing, then the chain $\{\Phi_n\}_0^{\infty}$ may be restricted to B, and B is called a "Harris" if the restricted chain is Harris recurrent.

Definition 1.4. A σ-finite measure π on $\mathcal{B}$ with the property

$$\pi(A) = \pi P(A) \triangleq \int P(x, A)\pi(dx) \quad \forall A \in \mathcal{B}$$

will be called "invariant".

It is shown in ([2], [3] and [4]) that if $\{\Phi_n\}_0^{\infty}$ is Harris recurrent, then an essentially unique invariant measure π exists.

Probabilistic concept of stability 2: positive Harris recurrence

Definition 1.5. Suppose that $\{\Phi_n\}_0^{\infty}$ is Harris recurrent, and π is finite. Then $\{\Phi_n\}_0^{\infty}$ is called positive Harris recurent

1.2 Topological concepts of stability

Topological concept of stability 1: non-evanescence:

Definition 1.6. A Markov chain $\{\Phi_n\}_0^{\infty}$ will be called non-evanescent if $P_x\{\Phi \to \infty\} = 0 \quad \forall x \in X$.

Topological concept of stability 2: boundedness in probability

Definition 1.7. A Markov chain $\{\Phi_n\}$ will be called "bounded in probability" if $\forall x \in X$, $\forall \epsilon > 0 \exists K \subset X$, K-compact such that

$$\liminf_{n \to \infty} P_x\{\Phi_n \in K\} \geq 1 - \epsilon.$$

1.3 Continuous component, T- and Feller chains

Definition 1.8. A kernel T is called a continuous component of a function $K : (X, \mathcal{B}) \to \mathbb{R}_+$ if

(i) For $A \in \mathcal{B}$ the function $T(\cdot, A)$ is lower semi-continuous;

(ii) For all $x \in X$ and $A \in \mathcal{B}$, the measure $T(x, \cdot)$ satisfies $K(x, A) \geq T(x, A)$.

The continuous component T is called "non-trival" at x if $T(x, X) > 0$.

Definition 1.9. A chain will be called a T-chain if, for some a, the Markov transition function $K_a \triangleq \sum_{i=1}^{\infty} a(i) P^i$, where a is a probability on Z^+, admits a continuous component T which is non-trivial for all $x \in X$.

Definition 1.10. A chain will be called "lsc-chain" if the function $L(x, \cdot)$ admits a continuous component which is non-trivial $\forall x \in X$.

Definition 1.11. A chain will be called (i) weak Feller if $Pf \in C(X) \quad \forall f \in C^0(X)$, (ii) strong Feller if $Pf \in C(X)$ for every bounded and measurable f on X, and (iii) e-chain if $\forall f \in C^0(X)$, the sequence of $\{P^k f\}_{X=0}^{\infty}$ is equicontinuous on compact sets.

1.4 Links between topological and probabilistic stability

Theorem 1.1 (The Doeblin decomposition (Th. 6.1 in [4])). Suppose that X is a topological space and $\mathcal{B}$ is σ-field containing all open sets. If (a) X is T_1 (every singleton is closed) and has a countable basis for the topology; (b) $L(x, \cdot)$ has a continuous component T which is non-trivial at every recurrent point of X, then (i) X can be decomposed into

$$X = \sum_{i \in I} H_i + E$$

where I is countable, each H_i is a maximal Harris set, and E is not properly essential. The set R of recurrent points is contained in the Harris part $\sum_{i \in I} H_i$ of the space. The sets $R \cap H_i$ are both open and closed in the relative topology of R and hence card (I) is at most the number of topological components of R.

(ii) If (b) is replaced by (c) (L has an equivalent component), then each H_i is also topologically closed, and E can be further represented as $E = E' + E''$, where E' is open and $L(x, \sum H_i) > 0 \quad \forall x \in E'$, while E'' is both stochastically and topologicall closed (or empty).

Remark 1.1. *It was also shown by R. Tweedie in [4] that for a 'Doeblin decomposition' it is* also necessary *to have some topology in which (a) and (b) hold true.*

1.5 Recurrent points and chains with some topolgical solidarity properties

Definition 1.12 (Open set irreducibility). (i) A point $x \in X$ is called "reachable" if for every open $O \ni x$, $O \in \mathcal{B}$ and for any $y \in X$

$$\sum_{n=1}^{\infty} P^n(y, 0) > 0.$$

(ii) A chain $\{\Phi_n\}_0^\infty$ is called "open set irreducible" if every point $x \in X$ is reachable.

Definition 1.13. A point $x^* \in X$ is called topologically Harris "recurrent" (a point of sure return) if , respectively, $Q(x^*, O_{x^*}) = 1$, $(L(x^*, O_{x^*}) = 1)$ for every open neighbourhood $O_{x^*} \ni x^*$.

Definition 1.14. A point $x^* \in X$ is called "positive" if

$$\limsup_{n\to\infty} P^n(x^*, O_{x^*}) > 0$$

for every open neighbourhood $O_{x^*} \ni x^*$.

Theorem 1.2 ([2] Theorem 18.0.2). Suppose that $\mathbf{\Phi}$ is a chain on a topological space for which a reachable state $x^* \in X$ exists.

1. If the chain is a T-chain, then the following are equivalent:

   (a) $\mathbf{\Phi}$ is positive Harris;

   (b) $\mathbf{\Phi}$ is bounded in probability;

   (c) $\mathbf{\Phi}$ is non-evanescent and x^* is positive.

   If any of these equivalent conditions hold and if the chain is aperiodic, then for each initial state $x \in X$,

   $$\|P^k(x, \cdot) - \pi\| \to 0 \quad \text{as } k \to \infty.$$

2. If the chain is an e-chain, then the following are equivalent:

   (a) There exists a unique invariant probability π and for every initial condition $x \in X$ and each bounded continuous function $f \in C(X)$,

   $$\lim_{k\to\infty} \bar{P}_k(x, f) = \pi(f)$$
   $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\Phi_i) = \pi(f) \quad \text{in probability;}$$

(b) $\mathbf{\Phi}$ is bounded in probability on average;

(c) $\mathbf{\Phi}$ is non-evanescent and x^* is positive;

If any of these equivalent conditions hold and if the reachable state is "aperiodic", then for each initial state $x \in X$,

$$P^k(x, \cdot) \xrightarrow{\text{w}} \pi \quad \text{as } k \to \infty.$$

It seems quite natural to consider the asymptotic behavior of weak Feller chains, "lying somewhere between" T- and e-chains. As preparation for that, we state the following lemma:

Lemma 1.1. Suppose that the chain $\{\Phi_n\}_0^\infty$ is weak Feller and the reachable point x^* exists. Then the following are equivalent:

1. Φ is a T-chian
2. Φ is a ψ-irreducible with $\psi = T(x^*, \cdot)$.

Proof. The proof is based on Lemma 6.1.4 and Th. 6.2.9 in [2]. □

2 Main results

2.1 Chains on locally compact and separable (metric) space X

Preparing for our first result, we look at the following lemma.

Lemma 2.1 ([2]. Lemma 18.4.1). If $\mathbf{\Phi}$ is a Feller chain and if a reachable state x^* exists, then for any pre-compact neighborhood O containing x^*,

$$\{\mathbf{\Phi} \to \infty\} = \{\mathbf{\Phi} \in O \text{ i.o.}\}^c \quad \text{a.s. } [P_*]$$

Proof. Since $L(x, O)$ is a lower semicontinuous function (ii) of x by Proposition 6.1.1 in [2], and since by reachability it is strictly positive everywhere, it follows that $L(x, O)$ is bounded from below on compact subsets of X.

Letting $\{O_n\}$ denote a sequence of pre-compact open subsets of X with $O_n \uparrow X$, it follows that $O_n \rightsquigarrow O^{*)}$ for each n, and hence by Theorem 9.1.3. in [2], we have

$$\{\mathbf{\Phi} \in O_n \text{i.o.}\} \subseteq \{\mathbf{\Phi} \in O \text{ i.o. }\} \quad \text{a.s. } [P_*].$$

This immediately implies that

$$\{\mathbf{\Phi} \to \infty\}^c = \cup_{n \geq 1}\{\mathbf{\Phi} \in O_n \text{i.o.}\} \subseteq \{\mathbf{\Phi} \in O \text{i.o.}\} \quad \text{a.s.} [P_*],$$

and since it is obvious that $\{\mathbf{\Phi} \to \infty\} \subseteq \{\mathbf{\Phi} \in O \text{ i.o.}\}^c$, this proves the lemma. □

$^{*)}O_n \rightsquigarrow O$ see [2], Chapter 4.

Theorem 2.1. Suppose that the assumptions a) and b) of Th. 1.1 hold true for a weak open set irreducible Feller chain. Then the following conditions are equivalent:

1. $\{\Phi_n\}_{n=0}^{\infty}$ is non-evanescent;
2. $\{\Phi_n\}_n^{\infty}\}_{n=0}^{\infty}$ is Harris recurrent.

Remark 2.1. *It is easily seen that $\{\Phi_n\}$ becomes a T-chain by Lemma 1.1.*

Theorem 2.2. Suppose that the assumptions of Th. 2.1 hold true. Then the following are equivalent:

1. Φ is positive Harris;
2. Φ is bounded in probability and $\bar{P}_k(x,\cdot) \stackrel{\mathrm{w}}{\to} \pi$ for any initial condition $x \in X$;
3. Φ is non-evanescent and x^* is positive;

If any of these equivalent conditions hold and if the chain is aperiodic, then for each initial state $x \in X$,

$$\|P^k(x,\cdot) - \pi\| \to 0 \quad \text{as} \quad k \to \infty,$$

and Φ is an e-chain.

Corollary 2.1. *Suppose that a) and c) of Th. 1.1 hold for a weak Feller chain with a reachable point x^*. Let $L(x^*, O(x)) > 0$ for every neighbourhood $O(x)$ of x and every $x \in X$. Then the assertions of Th. 2.1 and Th. 2.2 hold true.*

2.2 Chains on compact state space

Corollary 2.2. *Let $\{\Phi_n\}$ be a ψ-irreducible weak Feller chain on a compact T_1-space with a countable basis for the topology. If, in addition, the function $L(x,\cdot)$ has an non-trivial continuous component $\forall x \in X$, then $\{\Phi_n\}$ is positive Harris recurrent.*

Corollary 2.3. *Let $\{\Phi_n\}$ be a weak Feller chain with a reachable point x^* on a compact T_1-space with a countable basis for the topology. Suppose that the function $L(x,\cdot)$ has a component which is non-trivial and continuous at x^*. Then there exists a unique invariant probability π such that for some $\rho < 1$, $R < \infty$ and $m \geq 1$*

$$\|\frac{1}{m}\sum_{i=1}^{m} P^{k+1}(x,\cdot) - \pi\| \leq R\rho^k,\ x \in X,\ k \in Z_+.$$

3 Proofs

1) Proof of Th. 2.1. (i) $\Rightarrow$ (ii) Suppose that $\{\Phi_n\}_0^\infty$ is non-evanescent. Then by Lemma 18.4.1 in [2] for every pre-compact $O \ni x^*$

$$Q(x^*, 0) = P_{x^*}\{\Phi \in O \quad \text{i.o.}\} = 1,$$

which implies that every point of X is topologically Harris recurrent. Because of reachability, it follows that $L(x, (x^*)) > 0 \quad \forall x \in X$ and every neighbourhood O_{x^*} of x^*; the assumptions of the Corollary 6.4 in [4] are satisfied and Harris recurrence of the Markov chain $\{\Phi_n\}$ follows.

(ii) $\Rightarrow$ (i) Suppose that the chain $\{\Phi_n\}$ is Harris recurrent. By Lemma 1.1, Φ_n is a T-chain. The non-evanescense of $\{\Phi_n\}$ then follows by Th. 9.2.2 in [2]. $\square$

Proof of Th. 2.2. By the assumptions of the theorem, $\{\Phi_n\}$ is a ψ-irreducible T-chain, and the assertions of the theorem follow by Th. 18.0.2 in [2]. $\square$

Proof of Corollary 2.1. By Corollary 6.5 in [4], we can state recurrence of the point $x \in X$ if for some recurrent point x^* $L(x^*, O(x)) > 0$ for every neighbourhood $O(x)$ of x. If the chain $\{\Phi_n\}$ is non-evanescent, then by Lemma 18.4.2 [2], x^* is recurrent, and by the assumptions of the corollary, every point in X is recurrent. Then the assumptions of Th. 2.1 are satisfied. $\square$

Proof of Corollary 2.2. The proof of the corollary is based on the following result. Proposition 3.5 in [4]: (i) Suppose that X is compact and K_α has an everywhere-non-trivial continuous component T. If the chain is φ-irreducible for some φ, then it is positive recurrent.

(ii) Suppose that X is a compact T_1-space with a countable basis for the topology and L has an everywhere-non-trivial continuous component T. Then there is no stochastically closed set of non-recurrent points.*)

By Proposition 3.5 (ii), $\{\Phi_n\}$ has a reachable recurrent point x^* and by Lemma 1.1, $\{\Phi_n\}$ is a T-chain. Then, by Proposition 3.5 (i), $\{\Phi_n\}$ is positive Harris recurrent. $\square$

Proof of Corollary 2.3. By the assumptions of the corollary, there is a point x^* such that $L(x, O_{x^*})$ is bounded below on compact subsets of X for every neighbourhood O_{x^*} of x^*. Then $\inf_{x \in X} L(x, O_{x^*}) > 0$ for every neighbourhood O_{x^*} of x^*. Applying Th. 4.2 in [4], we obtain that the chain $\{\Phi_n\}$ is Harris recurrent, and hence, T-chain. Thus, the convergence result is an application of Th. 7.1 in [3]. $\square$

*)See [4] for the necessary definitions.

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