Diffusion Equations and Geometric Inequalities

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Abstract. Let $\theta = (\theta_0, \theta_1)$ be a fixed vector in $\mathbf{R}^2$ with strictly positive components and suppose $\sigma_0, \sigma_1 > 0$. Set $\sigma_\theta = \theta_0\sigma_0 + \theta_1\sigma_1$ and, if $x_0, x_1 \in \mathbf{R}^n$, set $x_\theta = \theta_0 x_0 + \theta_1 x_1$. Moreover, for any $j \in \{0, 1, \theta\}$, let $c_j : \mathbf{R}^n \to \mathbf{R}$ be a continuous, bounded function and denote by $p_{\sigma_j, c_j}(t, x, y)$ the fundamental solution of the diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\sigma_j^2}{2}\triangle v - \frac{1}{\sigma_j^2} c_j(x) v, t > 0, x \in \mathbf{R}^n.$$

If

$$\frac{1}{\sigma_\theta} c_\theta(x_\theta) \leq \frac{\theta_0}{\sigma_0} c_0(x_0) + \frac{\theta_1}{\sigma_1} c_1(x_1),\ \ x_0, x_1 \in \mathbf{R}^n$$

then by applying the Girsanov transformation theorem of Wiener measure it is proved that

$$\sigma_\theta^n p_{\sigma_\theta, c_\theta}(t, x_\theta, y_\theta) \geq \left\{\sigma_0^n p_{\sigma_0, c_0}(t, x_0, y_0)\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}} \left\{\sigma_1^n p_{\sigma_1, c_1}(t, x_1, y_1)\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}}$$

for all $x_0, x_1, y_0, y_1 \in \mathbf{R}^n$ and $t > 0$. Finally, in the last section, another proof of this inequality is given more in line with earlier investigations in this field.

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1. Introduction

Suppose c is a real-valued function defined on a bounded convex domain K in $\mathbf{R}^n$ and let H_c be the operator $-\frac{1}{2}\triangle + c(x)$ in K equipped with the Dirichlet boundary condition zero. Here, as usual, $\triangle$ denotes Laplace operator. The function c is called potential function. During the past twenty years we have encountered several inequalities in diffusion theory which motivate either convex potential functions (Brascamp and Lieb [6] , [7]) or so-called $-\frac{1}{2}-$concave potential functions (Borell [3] , [4] , [5]). Here recall that a function f is said to be $-\frac{1}{2}-$concave if f is either zero everywhere or strictly positive everywhere and such that the function $f^{-\frac{1}{2}}$ is concave. The purpose of this paper is twofold. First, we want to point out a new method in this context and second, we want to give a more unified approach than has been done earlier. To begin with, however, we will tell more about the background.

Let $p_{\sigma,c}(t,x,y)$ denote the fundamental solution of the diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\sigma^2}{2}\triangle v - \frac{1}{\sigma^2}c(x)v, t > 0, x \in K$$

with the Dirichlet boundary condition zero on $t > 0$, $x \in \partial K$. Here σ is a positive parameter. For short, we write $p_{1,c}(t,x,y) = p_c(t,x,y)$. By applying log-concavity of Gaussian measures in $\mathbf{R}^n$, Brascamp and Lieb ([6] , [7]) proved that the fundamental solution $p_c(t,x,y)$ is a log-concave function of (x,y) for fixed $t > 0$, if the potential function c is convex. Here recall that a function f is said to be log-concave if f is non-negative and the function $\ln f$ is concave with values in $\{-\infty\} \cup \mathbf{R}$. From the above, Brascamp and Lieb among other things concluded that the ground state wave function of the Hamiltonian H_c is log-concave for a convex potential function c (the result is put into historical perspective by Kawohl [12]).

Let $g_c(x,y)$ be the Green function of the operator H_c so that

$$g_c(x,y) = \int_0^\infty p_c(t,x,y)dt.$$

A classical theorem by Gabriel says that the harmonic ball

$$\{x \in B; g_0(x,y) > r\}$$

is convex for fixed $y \in K$ and $r > 0$ (see e.g. Hörmander [11]). Stated otherwise, the Green function $g_0(x,y)$ is a quasi-concave function of x if y is fixed.

In [3] we applied the Gabriel line of reasoning to a Green function $g_c(x,y)$ corresponding to a $-\frac{1}{2}-$concave potential function c and obtained that $g_c(x,y)$ is a quasi-concave function of (x,y) if $n=2$ and that the function $g_c(x,y)^{\frac{1}{2-n}}$ is a convex function of (x,y) if $n \geq 3$. For short, given $n \geq 2$, we here say that the Green function g_c is $\frac{1}{2-n}-$convex. Later, by combining Brunn-Minkowski theory and the Feynman-Kac formula, we proved in [5] that the function $s \ln \{s^n p_c(s^2, x, y)\}$ is a concave function of $(s,x,y) \in \,]0,\infty[\, \times K \times K$, if the potential function is $-\frac{1}{2}-$concave. Interestingly enough, from this result, given $n \geq 2$, the $\frac{1}{n-2}-$convexity of the Green function g_c for a $-\frac{1}{2}-$concave potential function c follows by very simple means [5].

Now consider a situation where the potential function eventually depends on the parameter σ as well as on the position x in $\mathbf{R}^n$. Let $0 < \alpha \leq \beta$. If $c_\sigma(x) = c(x,\sigma)$ and the function

$$\frac{c(x,\sigma)}{\sigma}, x \in K, \alpha \leq \sigma \leq \beta$$

is convex, then Theorem 3.2 below implies that the function

$$\sigma \ln \{\sigma^n p_{c_\sigma,\sigma}(t,x,y)\}, (\sigma,x,y) \in [\alpha,\beta] \times K \times K$$

is concave for fixed $t > 0$. From this the above quoted results by Brascamp and Lieb and the author follow at once. Theorem 3.2 is the main contribution of this paper. Its proof is based on the Girsanov transformation theorem of Wiener measure and ideas from the theory of stochastic optimal control. In particular, we obtain a Brownian motion proof of the classical Brunn-Minkowski inequality. As far as we know, this approach to inequalities of the Brunn-Minkowski type is new, although very similar arguments appear in connection with the Merton portfolio problem in the theory of finance (see e.g. Fleming and Soner [10], p. 204).

Finally, in the last section we give another proof of Theorem 3.2 more in line with the papers [6], [7] and [5].

2. The Hamilton – Jacobi – Bellman equation

Suppose σ is a positive parameter and consider the diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\sigma^2}{2} \triangle v - \frac{1}{\sigma^2} c(x) v, t > 0, x \in \mathbf{R}^n$$

with the initial condition

$$v(0,x) = f(x), x \in \mathbf{R}^n$$

where $f(x) > 0$ for any $x \in \mathbf{R}^n$. The substitutions

$$V = -\sigma^2 \ln v$$

and

$$F = -\sigma^2 \ln f$$

reduce the above Cauchy problem to the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \mid \nabla V \mid^2 - c(x) = \frac{\sigma^2}{2} \triangle V, t > 0, x \in \mathbf{R}^n * (2.1) \qquad (1)$$

with the initial condition

$$V(0,x) = F(x), x \in \mathbf{R}^n.$$

To begin with in this section, we assume that c and F are infinitely many times differentiable with bounded derivatives of all orders ≥ 0. Our subsequent reasoning follows the Fleming and Soner book on stochastic optimal control [10] (especially pp. 257-258).

Suppose $t > 0$ is fixed and let P be Wiener measure on the Banach space Ω of all continuous functions ω of $[0,t]$ into $\mathbf{R}^n$ with $\omega(0) = 0$. If $B(\omega) = \omega = (\omega_1(s), ..., \omega_n(s))_{0 \leq s \leq t}$, $\omega \in \Omega$, then B is a normalized Brownian motion in $\mathbf{R}^n$ relative to the probability measure P, that is, B is a centred Gaussian process in $\mathbf{R}^n$ relative to the probability measure P with

$$E^P\left[B_i(s_0)B_j(s_1)\right] = \begin{cases} 0, \; i \neq j \\ \min(s_0, s_1), \; i = j. \end{cases}$$

By setting

$$B_x^\sigma(s) = x + \sigma B(s), s \geq 0$$

the Feynman-Kac formula yields

$$v(t,x) = E^P\left[e^{-\frac{1}{\sigma^2}\left\{F(B_x^\sigma(t)) + \int_0^t c(B_x^\sigma(s))ds\right\}} \right]$$

and the assumptions on c and F imply that

$$\inf_{0 \leq s \leq t, x \in \mathbf{R}^n} v(s,x) > 0 * (2.2) \qquad (2)$$

and

$$\sup_{0\le s\le t, x\in\mathbf{R}^n} \mid \nabla v(s,x) \mid < \infty. * (2.3) \tag{3}$$

Let $u(s)$, $0 \le s \le t$, be a bounded, progressively measurable process and set

$$h(s) = h_u(s) = \int_0^s u(\lambda)d\lambda, 0 \le s \le t$$

and

$$dQ(\omega) = e^{-\frac{1}{2\sigma^2}\int_0^t |u(s)|^2 ds - \frac{1}{\sigma}\int_0^t u(s)d\omega(s)} dP(\omega).$$

Then, by the Girsanov theorem (see e.g. Nualart [15]),

$$\int_\Omega \varphi(\omega + \frac{1}{\sigma}h)dQ(\omega) = \int_\Omega \varphi(\omega)dP(\omega)$$

for any positive measurable function φ on Ω and it follows that

$$v(t,x) = E^P\left[e^{-\frac{1}{\sigma^2}\left\{F(B_x^\sigma(t)) + \int_0^t c(B_x^\sigma(s))ds\right\}}\right] =$$

$$E^Q\left[e^{-\frac{1}{\sigma^2}\left\{F(B_x^\sigma(t)+h(t)) + \int_0^t c(B_x^\sigma(s)+h(s))ds\right\}}\right] =$$

$$E^P\left[e^{-\frac{1}{\sigma^2}\left\{F(B_x^\sigma(t)+h(t)) + \int_0^t c(B_x^\sigma(s)+h(s))ds\right\}} e^{-\frac{1}{2\sigma^2}\int_0^t |u(s)|^2 ds - \frac{1}{\sigma}\int_0^t u(s)d\omega(s)}\right].$$

For short, we write

$$X(s) = X_u(s) = B_x^\sigma(s) + h_u(s), 0 \le s \le t$$

so that

$$v(t,x) =$$

$$E^P\left[e^{-\frac{1}{\sigma^2}\left\{F(X(t)) + \int_0^t c(X(s))ds\right\}} e^{-\frac{1}{2\sigma^2}\int_0^t |u(s)|^2 ds - \frac{1}{\sigma}\int_0^t u(s)d\omega(s)}\right]$$

and the Jensen inequality yields

$$\ln v(t,x) \ge -\frac{1}{\sigma^2}E^P\left[Y_u(t)\right] * (2.4) \tag{4}$$

where

$$Y_u(t) = F(X(t)) + \int_0^t (c(X(s)) + \frac{1}{2} \mid u(s) \mid^2)ds + \sigma \int_0^t u(s)d\omega(s).$$

Note that

$$E^P\left[\int_0^t u(s)d\omega(s)\right] = 0.$$

If we choose u in an appropriate way, it turns out that the random variable $Y_u(t)$ is constant with probability one, which implies that equality occurs in (2.4) for this choice of u. To find such a process u, first define

$$U(s,x) = -\nabla V(t-s,x), 0 \le s \le t.$$

¿From the assumptions on c and F we conclude that the function $U(s,x)$, $0 \le s \le t$, $x \in \mathbf{R}^n$, is bounded and continuous and, moreover, the equations (2.2) and (2.3) imply that there exists a constant $C > 0$ such that

$$\mid U(s,x) - U(s,y) \mid \le C \mid x - y \mid, 0 \le s \le t, x, y \in \mathbf{R}^n.$$

Therefore the stochastic differential equation

$$dX(s) = U(s, X(s))ds + \sigma d\omega(s), 0 \le s \le t$$

with the initial condition $X(0) = x$ possesses a unique solution. We set $u_0(s) = U(s, X(s))$, $0 \le s \le t$, and have $X(s) = x + \sigma\omega(s) + h_{u_0}(s) = B_x^\sigma(s) + h_{u_0}(s)$, $0 \le s \le t$. Moreover, we claim that the random variable $Y_{u_0}(t)$ is constant with probability one. To prove this claim we introduce the process

$$\xi(s) = V(t-s, X(s)) + \int_0^s (c(X(\lambda)) + \frac{1}{2} \mid u_0(\lambda) \mid^2) ds + \sigma \int_0^s u_0(\lambda) d\omega(\lambda)$$

defined for all $0 \le s \le t$ and have, recalling the Itô lemma,

$$d\xi(s) = -V_t(t-s, X(s)ds + \nabla V(t-s, X(s)) \cdot (u_0(s)ds + \sigma d\omega(s)) +$$

$$\frac{\sigma^2}{2}\triangle V(t-s, X(s))ds + (c(X(s)) + \frac{1}{2} \mid u_0(s) \mid^2)ds + \sigma u_0(s) d\omega(s).$$

Moreover, since the function $V(t,x)$ satisfies the Hamilton-Jacobi-Bellman equation (2.1), $d\xi(s) = 0$, and we conclude that ξ is constant with probability one. In particular, $\xi(t) = Y_{u_0}(t)$ is constant with probability one, which was to be proved.

¿From the above,

$$v(t,x) = \exp(-\inf_{u \in \mathcal{U}(t)} (J(t,x,u))$$

where

$$J(t,x,u) = \frac{1}{\sigma^2} E^P\left[Y(t,u)\right]$$

and where $\mathcal{U}(t)$ denotes the class of all bounded, progressively measurable processes $u(s)$, $0 \leq s \leq t$.

In the following, let

$$v^F_{\sigma,c}(t,x) = E^P\left[e^{-\frac{1}{\sigma^2}\left\{F(B^\sigma_x(t))+\int_0^t c(B^\sigma_x(s)))ds\right\}}\right]$$

and

$$J^F_{\sigma,c}(t,x,u) =$$

$$\frac{1}{\sigma^2} E^P\left[F(B^\sigma_x(t)+h_u(t)) + \int_0^t (c(B^\sigma_x(s)) + h_u(s)) + \frac{1}{2} \mid u(s) \mid^2) ds\right]$$

for all continuous and bounded functions F and c in $\mathbf{R}^n$. Then, from the above, it is simple to conclude that

$$v^F_{\sigma,c}(t,x) = \exp(-\inf_{u \in \mathcal{U}(t)} (J^F_{\sigma,c}(t,x,u)). * (2.5) \tag{5}$$

Below we will also make use of the short-hand notation

$$v^{A,F}_{\sigma,c}(t,x) = E^P\left[1_A(B^\sigma_x(t))e^{-\frac{1}{\sigma^2}\left\{F(B^\sigma_x(t))+\int_0^t c(B^\sigma_x(s)))ds\right\}}\right]$$

for any Borel set A in $\mathbf{R}^n$.

3. Application to diffusion equations

In what follows, $\theta = (\theta_0, \theta_1)$ denotes a fixed vector in $\mathbf{R}^2$ with strictly positive components. If $x_0, x_1 \in \mathbf{R}^n$, let

$$x_\theta = \theta_0 x_0 + \theta_1 x_1$$

and, if A_0 and A_1 denote subsets of $\mathbf{R}^n$, let

$$A_\theta = \left\{x_\theta; x_0 \in A_0 and x_1 \in A_1\right\}.$$

Suppose first that $\sigma_0, \sigma_1 > 0$ and let D_i, $i = 0, 1$, be subdomains of $\mathbf{R}^n$. Below we will often consider functions $\varphi_j : D_j \to \mathbf{R}$, $j = 0, 1, \theta$, which satisfy the inequality

$$\frac{1}{\sigma_\theta}\varphi_\theta(x_\theta) \leq \frac{\theta_0}{\sigma_0}\varphi_0(x_0) + \frac{\theta_1}{\sigma_1}\varphi_1(x_1), x_0 \in D_0, x_1 \in D_1.$$

Note that this inequality is true in the following situations:

Case 1: $\sigma_0 = \sigma_1$ and

$$\varphi_\theta(x_\theta) \leq \theta_0\varphi_0(x_0) + \theta_1\varphi_1(x_1), x_0 \in D_0, x_1 \in D_1.$$

Case 2: $\varphi_j(x) = \psi_j^2(x)$, $j = 0, 1, \theta$, where the ψ_j are non-negative and

$$\psi_\theta(x_\theta) \leq \theta_0\psi_0(x_0) + \theta_1\psi_1(x_1), x_0 \in D_0, x_1 \in D_1.$$

Case 3: $\varphi_j(x) = \frac{\sigma_j^4}{\psi_j^2(x)}$, $j = 0, 1, \theta$, where the ψ_j are positive and

$$\psi_\theta(x_\theta) \geq \theta_0\psi_0(x_0) + \theta_1\psi_1(x_1), x_0 \in D_0, x_1 \in D_1.$$

In connection with the last two cases it is useful to know that the function

$$\psi_\alpha(\lambda, \sigma) = \frac{\lambda^{\alpha+1}}{\sigma^\alpha}, \lambda \geq 0,\ \sigma > 03.1 \qquad (6)$$

is convex and positively homogeneous of degree one for $\alpha = 1$ and $\alpha = 2$, respectively.

THEOREM 3.1. *Let $\sigma_0, \sigma_1 > 0$ and suppose $c_j(x), F_j(x)$ $j = 0, 1, \theta$, are bounded, continuous functions defined for all $x \in \mathbf{R}^n$ such that*

$$\frac{1}{\sigma_\theta}c_\theta(x_\theta) \leq \frac{\theta_0}{\sigma_0}c_0(x_0) + \frac{\theta_1}{\sigma_1}c_1(x_1)$$

and

$$\frac{1}{\sigma_\theta}F_\theta(x_\theta) \le \frac{\theta_0}{\sigma_0}F_0(x_0) + \frac{\theta_1}{\sigma_1}F_1(x_1)$$

for all $x_0, x_1 \in \mathbf{R}^n$.

Then

$$v^{F_\theta}_{\sigma_\theta,c_\theta}(t,x_\theta) \ge \left\{v^{F_0}_{\sigma_0,c_0}(t,x_0)\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}} \left\{v^{F_1}_{\sigma_1,c_1}(t,x_1)\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}} 3.2 \tag{7}$$

for all $x_0, x_1 \in \mathbf{R}^n$ *and* $t \ge 0$.

Moreover,

$$v^{A_\theta,F_\theta}_{\sigma_\theta,c_\theta}(t,x_\theta) \ge \left\{v^{A_0,F_0}_{\sigma_0,c_0}(t,x_0)\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}} \left\{v^{A_1,F_1}_{\sigma_1,c_1}(t,x_1)\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}} 3.3 \tag{8}$$

for all $x_0, x_1 \in \mathbf{R}^n$, $t \ge 0$ *and Borel sets* A_0 *and* A_1 *in* $\mathbf{R}^n$.

Proof. Let $u_0, u_1 \in \mathcal{U}(t)$ and define

$$u_\theta(s) = \theta_0 u_0(s) + \theta_1 u_1(s), 0 \le s \le t.$$

Then

$$B^{\sigma_\theta}_{x_\theta}(s) + h_{u_\theta}(s) = \theta_0(B^{\sigma_0}_{x_0}(s)) + h_{u_0}(s)) + \theta_1(B^{\sigma_1}_{x_1}(s)) + h_{u_1}(s))$$

for all $0 \le s \le t$ and every fixed $\omega = B(\omega)$. Moreover, since the function ψ_1 defined by equation (3.1) is convex and positively homogeneous of degree one,

$$\frac{1}{\sigma_\theta}\left[F_\theta(B^{\sigma_\theta}_{x_\theta}(t) + h_{u_\theta}(t)) + \int_0^t (c_\theta(B^{\sigma_\theta}_{x_\theta}(s) + h_{u_\theta}(s)) + \frac{1}{2} \mid u_\theta(s) \mid^2)ds\right] \le$$

$$\frac{\theta_0}{\sigma_0}\left[F_0(B^{\sigma_0}_{x_0}(t) + h_{u_0}(t)) + \int_0^t (c_0(B^{\sigma_0}_{x_0}(s) + h_{u_0}(s)) + \frac{1}{2} \mid u_0(s) \mid^2)ds\right] +$$
$$\frac{\theta_1}{\sigma_1}\left[F_1(B^{\sigma_1}_{x_1}(t) + h_{u_1}(t)) + \int_0^t (c_1(B^{\sigma_1}_{x_1}(s) + h_{u_1}(s)) + \frac{1}{2} \mid u_1(s) \mid^2)ds\right]$$

and, hence,

$$\frac{1}{\sigma_\theta}E^P\left[F_\theta(B^{\sigma_\theta}_{x_\theta}(t) + h_{u_\theta}(t)) + \int_0^t (c(B^{\sigma_\theta}_{x_\theta}(s) + h_{u_\theta}(s)) + \frac{1}{2} \mid u_\theta(s) \mid^2)ds\right] \le$$

$$\frac{\theta_0}{\sigma_0}E^P\left[F_0(B^{\sigma_0}_{x_0}(t)+h_{u_0}(t))+\int_0^t(c(B^{\sigma_0}_{x_0}(s)+h_{u_0}(s))+\frac{1}{2}\mid u_0(s)\mid^2)ds\right]+$$
$$\frac{\theta_1}{\sigma_1}E^P\left[F_1(B^{\sigma_1}_{x_1}(t)+h_{u_1}(t))+\int_0^t(c(B^{\sigma_1}_{x_1}(s)+h_{u_1}(s))+\frac{1}{2}\mid u_1(s)\mid^2)ds\right]$$

that is,

$$\sigma_\theta J^F_{\sigma_\theta,c_\theta}(t,x_\theta,u_\theta)\le\theta_0\sigma_0 J^F_{\sigma_0,c_0}(t,x_0,u_0)+\theta_1\sigma_1 J^F_{\sigma_1,c_1}(t,x_1,u_1).$$

¿From this the inequality (3.2) is an immediate consequence of the representation formula (2.5) written in the form

$$-\sigma\ln v^F_{\sigma,c}(t,x)=\inf_{u\in\mathcal{U}(t)}\sigma J(t,x,u).$$

To prove the inequality (3.3) there is no loss of generality to assume that A_0 and A_1 are non-empty and compact. If $A\subseteq\mathbf{R}^n$ is non-empty and compact and $\varepsilon>0$, let

$$d(x,A)=\min\left\{\mid x-y\mid;y\in\mathbf{R}^n\right\},x\in\mathbf{R}^n,$$
$$A^\varepsilon=\{x\in\mathbf{R}^n;d(x,A)\le\varepsilon\}$$

and

$$\varphi^\varepsilon_A(x)=\min(\varepsilon,d(x,A)),x\in\mathbf{R}^n.$$

Then, if we define

$$\tilde{\varphi}_\varepsilon=\sigma_\theta\min(\frac{\theta_0}{\sigma_0},\frac{\theta_1}{\sigma_1})\varphi^1_{(A_\theta)^{\varepsilon(\theta_0+\theta_1)}}$$

it follows that

$$\frac{1}{\sigma_\theta}\tilde{\varphi}_\varepsilon(x_\theta)\le\frac{\theta_0}{\sigma_0}\varphi^\varepsilon_{A_0}(x_0)+\frac{\theta_1}{\sigma_1}\varphi^\varepsilon_{A_1}(x_1)$$

for all $x_0,x_1\in\mathbf{R}^n$, and the inequality (3.2) gives

$$v^{m\tilde{\varphi}_\varepsilon+F_\theta}_{\sigma_\theta,c_\theta}(t,x_\theta)\ge$$
$$\left\{v^{m\varphi^\varepsilon_{A_0}+F_0}_{\sigma_0,c_0}(t,x_0)\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}}\left\{v^{m\varphi^\varepsilon_{A_1}+F_1}_{\sigma_1,c_1}(t,x_1)\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}}$$

for all $x_0,x_1\in\mathbf{R}^n$, $t\ge0$ and $m\in\mathbf{N}_+$. The inequality (3.3) now follows for all non-empty and compact A_0 and A_1 by first letting $m\to\infty$ and then letting $\varepsilon\to0$. This completes the proof of Theorem 3.1.

EXAMPLE 3.1. Suppose $\theta = (\theta_0, \theta_1)$ is a vector in $\mathbf{R}^2$ with strictly positive components such that $\theta_0 + \theta_1 = 1$ and let f_j, $j = 0, 1, \theta$, be non-negative continuous functions in $\mathbf{R}^n$ which satisfy the inequality

$$f_\theta(x_\theta) \geq f_0^{\theta_0}(x_0) f_1^{\theta_1}(x_1)$$

for all $x_0, x_1 \in \mathbf{R}^n$. The Prékopa inequality says that

$$\int_{A_\theta} f_\theta(x)dx \geq \left\{ \int_{A_0} f_0(x)dx \right\}^{\theta_0} \left\{ \int_{A_1} f_1(x)dx \right\}^{\theta_1}$$

(Prékopa [16], [17]). In the special case $f_j = 1$, $j = 0, 1, \theta$, this inequality reads

$$m_n(A_\theta) \geq m_n^{\theta_0}(A_0) m_n^{\theta_1}(A_1)$$

where m_n denotes Lebesgue measure in $\mathbf{R}^n$. Since $m_n(\alpha A) = \alpha^n m_n(A)$, $\alpha \geq 0$, the Prékopa inequality thus implies the classical Brunn-Minkowski inequality

$$m_n^{\frac{1}{n}}(A_0 + A_1) \geq m_n^{\frac{1}{n}}(A_0) + m_n^{\frac{1}{n}}(A_1)$$

valid for all non-empty Borel sets A_0 and A_1 in $\mathbf{R}^n$. Conversely, the classical Brunn-Minkowski inequality implies the Prékopa inequality ([16], [17]).

The Prékopa inequality is an immediate consequence of Theorem 3.1. To see this there is no loss of generality to assume that

$$0 < \inf f_j \leq \sup f_j < \infty, j = 0, 1, \theta.$$

Furthermore, let $\sigma > 0$ and set

$$F_j = -\sigma^2 \ln f_j, j = 0, 1, \theta.$$

Then, if $\sigma_0 = \sigma_1 = \sigma$, $c_0 = c_1 = c_\theta = 0$, $x_0 = x_1 = 0$ and $t = 1$ in Theorem 3.1, the inequality (3.3) says that

$$\int_{A_\theta} f_\theta(x) e^{-\frac{|x|^2}{2\sigma^2}} dx \geq \left\{ \int_{A_0} f_0(x) e^{-\frac{|x|^2}{2\sigma^2}} dx \right\}^{\theta_0} \left\{ \int_{A_1} f_1(x) e^{-\frac{|x|^2}{2\sigma^2}} dx \right\}^{\theta_1}$$

and in the limit as $\sigma \to \infty$ we obtain the Prékopa inequality.

There is a complement to the Prékopa inequality for Gaussian measures which we would like to point out here. Put

$$\mu_\sigma(A) = \int_A e^{-\frac{|x|^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}^n}$$

for any Borel set A in $\mathbf{R}^n$ and $\sigma > 0$. If we choose $c_j = F_j = 0$, $j = 0, 1, \theta$ and $x_0 = x_1 = 0$ and $t = 1$ in Theorem 3.1, the inequality (3.3) implies that

$$\mu_{\sigma_\theta}(A_\theta) \geq \{\mu_{\sigma_0}(A_0)\}^{\frac{\theta_0 \sigma_0}{\sigma_\theta}} \{\mu_{\sigma_1}(A_1)\}^{\frac{\theta_1 \sigma_1}{\sigma_\theta}}$$

for all Borel sets A_0 and A_1 in $\mathbf{R}^n$. Here θ may be any vector in $\mathbf{R}^n$ with strictly positive components.

It is well known that the Prékopa inequality implies log-concavity of Wiener measure (Borell [1]) as well as various log-concavity properties of solutions to the classical diffusion equation in $\mathbf{R}^n$ with a convex potential function (Brascamp and Lieb [6, 7]) (cf *Case 1* above). The approach in this section based on transformation of Wiener measure is sometimes more direct.

EXAMPLE 3.2. Consider the Cauchy problem

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 \triangle v - \frac{1}{\sigma^2} c(x) v \\ v(0, x) = \exp(-\frac{F(x)}{\sigma^2}) \end{cases}$$

with c and F both convex. Then, in view of Theorem 3.1, we rediscover a result by Brascamp and Lieb stating that the function $v(t, x) = v^F_{\sigma,c}(t, x)$ is a log-concave function of x for fixed $t > 0$. Incidentally, let us note that the Hopf-Cole substitution

$$\mathbf{v}(t, x) = -\sigma^2 \nabla \ln v(t, x) = \nabla V_\sigma(t, x)$$

reduces the above Cauchy problem to the Burgers equation

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = \frac{1}{2}\sigma^2 \triangle \mathbf{v} + \nabla c(x) \\ \mathbf{v}(0, x) = \nabla F(x). \end{cases}$$

Therefore, if c and F are convex, the velocity field $\mathbf{v}(t, x)$ is the gradient of a convex function of x for every fixed $t > 0$.

Under the stronger assumptions that the functions c and F are non-negative with $\sqrt{c}$ and $\sqrt{F}$ both convex (cf *Case 2* above), the function

$$\frac{1}{\sigma} V_\sigma(t, x), x \in \mathbf{R}^n, \sigma > 0$$

is convex.

Now let D be a region in $\mathbf{R}^n$ and suppose the functions $c, F : D \to \mathbf{R}$ are continuous and bounded from below. We define the function $\tilde{c}$ on $\mathbf{R}^n$ equal to c in D and equal to ∞ off D. Similarly, we define the function $\tilde{F}$ on $\mathbf{R}^n$ equal to F in D and equal to ∞ off D and set

$$v_{\sigma,c}^{F}(t,x) = E^P\left[e^{-\frac{1}{\sigma^2}\left\{\tilde{F}(B_x^\sigma(t))+\int_0^t \tilde{c}(B_x^\sigma(s)))ds\right\}}\right].$$

Moreover, set

$$v_{\sigma,c}^{A,F}(t,x) = E^P\left[1_A(B_x^\sigma(t))e^{-\frac{1}{\sigma^2}\left\{\tilde{F}(B_x^\sigma(t))+\int_0^t \tilde{c}(B_x^\sigma(s)))ds\right\}}\right]$$

for any Borel set A contained in D. If $p_{\sigma,c}(t,x,y)$ denotes the fundamental solution of the diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\sigma^2}{2}\triangle v - \frac{1}{\sigma^2}c(x)v, t > 0, x \in D$$

with the Dirichlet boundary condition zero on $t > 0$, $x \in \partial D$,

$$v_{\sigma,c}^{A,F}(t,x) = \int_D 1_A(x)e^{-\frac{1}{\sigma^2}F(x)}p_{\sigma,c}(t,x,y)dy, t > 0, x \in D.$$

Theorem 3.1 now implies the following result, the proof of which is excluded here.

THEOREM 3.2. *Let $\sigma_0, \sigma_1 > 0$ and let D_i, $i = 0, 1$, be subdomains of $\mathbf{R}^n$. Furthermore, suppose $c_j, F_j : D_j \to \mathbf{R}$, $j = 0, 1, \theta$, are continuous functions which are bounded from below and such that*

$$\frac{1}{\sigma_\theta}c_\theta(x_\theta) \le \frac{\theta_0}{\sigma_0}c_0(x_0) + \frac{\theta_1}{\sigma_1}c_1(x_1)$$

and

$$\frac{1}{\sigma_\theta}F_\theta(x_\theta) \le \frac{\theta_0}{\sigma_0}F_0(x_0) + \frac{\theta_1}{\sigma_1}F_1(x_1)$$

for all $x_0 \in D_0$ and $x_1 \in D_1$.

Then

$$v^{F_\theta}_{\sigma_\theta,c_\theta}(t,x_\theta) \geq \left\{v^{F_0}_{\sigma_0,c_0}(t,x_0)\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}} \left\{v^{F_1}_{\sigma_1,c_1}(t,x_1)\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}} \text{ 3.4} \tag{9}$$

for all $x_0 \in D_0$, $x_1 \in D_1$ *and* $t \geq 0$.

Moreover,

$$v^{A_\theta,F_\theta}_{\sigma_\theta,c_\theta}(t,x_\theta) \geq \left\{v^{A_0,F_0}_{\sigma_0,c_0}(t,x_0)\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}} \left\{v^{A_1,F_1}_{\sigma_1,c_1}(t,x_1)\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}} \text{ 3.5} \tag{10}$$

for all $x_0 \in D_0$, $x_1 \in D_1$, $t \geq 0$ *and Borel sets* $A_i \subseteq D_i$, $i = 0, 1$.

The following example draws the attention to a certain construction method of $-\frac{1}{2}$−concave functions, which is immediate from the Brascamp and Lieb papers ([6], [7]). Furthermore, we will point out that Theorem 3.2 here yields an alternative to the Brascamp and Lieb approach. Below, we let $v^F_{1,c} = v^F_c$ and $v^0_{1,c} = v_c$.

EXAMPLE 3.3. Suppose $\theta = (\theta_0, \theta_1)$ is a vector in $\mathbf{R}^2$ with strictly positive components such that $\theta_0 + \theta_1 = 1$ and suppose D_i, $i = 0, 1$ are bounded domains in $\mathbf{R}^n$. Furthermore, let $c_j(x) : D_j \to \mathbf{R}$, $j = 0, 1, \theta$, be continuous functions which are bounded from below and such that

$$c_\theta(x_\theta) \leq \theta_0 c_0(x_0) + \theta_1 c_1(x_1)$$

for all $x_0 \in D_0$ and $x_1 \in D_1$. Then, by Brascamp and Lieb ([6], [7])

$$v_{c_\theta}(t,x_\theta) \geq \{v_{c_0}(t,x_0)\}^{\theta_0} \{v_{c_1}(t,x_1)\}^{\theta_1}$$

for all $x_0 \in D_0$ and $x_1 \in D_1$. Alternatively, this inequality follows from Theorem 3.2. The Prékopa inequality now gives that

$$\int_{D_\theta} v_{c_\theta}(t,x)dx \geq \left\{\int_{D_0} v_{c_0}(t,x)dx\right\}^{\theta_0} \left\{\int_{D_1} v_{c_1}(t,x)dx\right\}^{\theta_1}.$$

Moreover, since the limit

$$\lambda_{c_j}(D_j) = -\lim_{t\to\infty} \frac{1}{t} \ln \int_{D_j} v_{c_j}(t,x)dx$$

is equal to the smallest eigenvalue of the operator $-\frac{1}{2}\triangle + c_j(x)$ in D_j with the Dirichlet boundary condition zero, Brascamp and Lieb ([6], [7]) concluded that

$$\lambda_{c_\theta}(D_\theta) \le \theta_0 \lambda_{c_0}(D_0) + \theta_1 \lambda_{c_1}(D_1).$$

For a zero potential in D, $\lambda_0(\alpha D) = \alpha^{-2}\lambda_0(D)$, $\alpha > 0$, and it follows that

$$\lambda_0^{-\frac{1}{2}}(D_\theta) \ge \theta_0 \lambda_0^{-\frac{1}{2}}(D_0) + \theta_1 \lambda_0^{-\frac{1}{2}}(D_1)$$

if $c_j = 0$, $j = 0, 1, \theta$.

¿From the above it is possible to construct $-\frac{1}{2}-$concave functions as follows. Suppose K is a bounded convex domain in $\mathbf{R}^n$ and let $F = \{0\} \times \mathbf{R}^{n-1}$. We define $R = \{x \in \mathbf{R}^n; (x+F) \cap K \ne \emptyset\}$. Furthermore, let H_0^x be the negative $(n-1)-$dimensional Laplace operator in $(x+F)\cap K$ equipped with the Dirichlet boundary condition zero on the relative boundary of $(x+F)\cap K$ viewed as a subset of $x+F$. If $\lambda(x)$ denotes the smallest eigenvalue of H_0^x for $x \in R$, then the function $\lambda(x)$, $x \in R$ is $-\frac{1}{2}-$concave.

COROLLARY 3.1. *Let $\sigma_0, \sigma_1 > 0$ and let D_i, $i = 0, 1$, be subdomains of $\mathbf{R}^n$. Furthermore, suppose $c_j : D_j \to \mathbf{R}$, $j = 0, 1, \theta$, are continuous functions which are bounded from below and such that*

$$\frac{1}{\sigma_\theta} c_\theta(x_\theta) \le \frac{\theta_0}{\sigma_0} c_0(x_0) + \frac{\theta_1}{\sigma_1} c_1(x_1)$$

for all $x_0 \in D_0$ and $x_1 \in D_1$.

Then

$$p_{\sigma_\theta, c_\theta}(t, x_\theta, y_\theta) \prod_{k=1}^{n} a_\theta^{(k)} \ge$$

$$\left\{ p_{\sigma_0, c_0}(t, x_0, y_0) \prod_{k=1}^{n} a_0^{(k)} \right\}^{\frac{\theta_0 \sigma_0}{\sigma_\theta}} \left\{ p_{\sigma_1, c_1}(t, x_1, y_1) \prod_{k=1}^{n} a_1^{(k)} \right\}^{\frac{\theta_1 \sigma_1}{\sigma_\theta}}$$

for all $x_0, y_0 \in D_0$, $x_1, y_1 \in D_1$ and $t > 0$ and all vectors $a_0 = (a_0^{(1)}, ..., a_0^{(n)})$ and $a_1 = (a_1^{(1)}, ..., a_1^{(n)})$ with non-negative components or, stated otherwise,

$$\sigma_\theta^n p_{\sigma_\theta, c_\theta}(t, x_\theta, y_\theta) \ge \{\sigma_0^n p_{\sigma_0, c_0}(t, x_0, y_0)\}^{\frac{\theta_0 \sigma_0}{\sigma_\theta}} \{\sigma_1^n p_{\sigma_1, c_1}(t, x_1, y_1)\}^{\frac{\theta_0 \sigma_1}{\sigma_\theta}}$$

for all $x_0, y_0 \in D_0$, $x_1, y_1 \in D_1$ *and* $t > 0$.

Corollary 3.1 is an immediate consequence of Theorem 3.2 and the following standard lemma in Brunn-Minkowski theory.

STANDARD LEMMA. *Suppose* $\Psi : [0,\infty[\times [0,\infty[\to [0,\infty[$ *is a continuous, positively homogeneous function of degree one, increasing in each variable separately, and such that* $\Psi(\xi,\eta) = 0$, *if* $\xi = 0$ *or* $\eta = 0$. *Moreover, let* $\Omega_0, \Omega_1 \subseteq \mathbf{R}^n$ *be open and suppose* $\varphi_j : \Omega_j \to [0,\infty[$, $j = 0, 1, \theta$, *are continuous functions.*

The following assertions are equivalent:

(i)

$$\int_{A_\theta} \varphi_\theta(x)dx \geq \Psi(\int_{A_0} \varphi_0(x)dx, \int_{A_1} \varphi_1(x)dx)$$

for all Borel sets $A_i \subseteq \Omega_i$, $i = 0, 1$;

(ii)

$$\varphi_\theta(x_\theta) \prod_{k=1}^{n} a_\theta^{(k)} \geq \Psi(\varphi_0(x_0) \prod_{k=1}^{n} a_0^{(k)}, \varphi_1(x_1) \prod_{k=1}^{n} a_1^{(k)})$$

for all $x_0 \in \Omega_0$, $x_1 \in \Omega_1$, *and all vectors* $a_0 = (a_0^{(1)}, ..., a_0^{(n)})$ *and* $a_1 = (a_1^{(1)}, ..., a_1^{(n)})$ *with non-negative components.*

In particular, if $\sigma_0, \sigma_1 > 0$, *the following assertions are equivalent:*

$(i)'$

$$\int_{A_\theta} \varphi_\theta(x)dx \geq \left\{\int_{A_0} \varphi_0(x)dx\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}} \left\{\int_{A_1} \varphi_1(x)dx\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}}$$

for all open $A_i \subseteq \Omega_i$, $i = 0, 1$;

$(ii)'$

$$\sigma_\theta^n \varphi_\theta(x_\theta) \geq \{\sigma_0^n \varphi_0(x_0)\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}} \{\sigma_1^n \varphi_1(x_1)\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}}$$

for all $x_0 \in \Omega_0,\ x_1 \in \Omega_1$.

The equivalence of (i) and (ii) in the Standard Lemma is proved in [2] and [5]. In view of this result, the equivalence of $(i)'$ and $(ii)'$ above is a consequence of the following lemma.

LEMMA 3.1. *Let* $\sigma_0, \sigma_1 > 0$ *and let* ξ, η *and* ς *be non-negative real numbers such that*

$$\sigma_\theta^n \xi \geq \{\sigma_0^n \eta\}^{\frac{\theta_0 \sigma_0}{\sigma_\theta}} \{\sigma_1^n \varsigma\}^{\frac{\theta_1 \sigma_1}{\sigma_\theta}} . * (3.6) \tag{11}$$

Then

$$\xi \prod_{k=1}^{n} a_\theta^{(k)} \geq \left\{ \eta \prod_{k=1}^{n} a_0^{(k)} \right\}^{\frac{\theta_0 \sigma_0}{\sigma_\theta}} \left\{ \varsigma \prod_{k=1}^{n} a_1^{(k)} \right\}^{\frac{\theta_1 \sigma_1}{\sigma_\theta}}$$

for all vectors $a_0 = (a_0^{(1)}, ..., a_0^{(n)})$ *and* $a_1 = (a_1^{(1)}, ..., a_1^{(n)})$ *with non-negative components.*

Proof. The function

$$\sigma \ln \frac{\lambda}{\sigma}, \sigma, \lambda > 0 * (3.7) \tag{12}$$

is concave and positively homogeneous of degree one. Therefore, if $a_0 = (a_0^{(1)}, ..., a_0^{(n)})$ and $a_1 = (a_1^{(1)}, ..., a_1^{(n)})$ are vectors with non-negative components,

$$\frac{a_\theta^{(k)}}{\sigma_\theta} \geq \left\{ \frac{a_0^{(k)}}{\sigma_0} \right\}^{\frac{\theta_0 \sigma_0}{\sigma_\theta}} \left\{ \frac{a_1^{(k)}}{\sigma_1} \right\}^{\frac{\theta_1 \sigma_1}{\sigma_\theta}} \quad k = 1, ..., n.$$

By multiplying all these n inequalities and the inequality in (3.6), Lemma 3.1 follows at once.

Again let D be a subdomain of $\mathbf{R}^n$ and suppose c is a continuous potential function defined in D which is bounded from below. The solution of the diffusion equation

$$\frac{\partial w}{\partial t} = \frac{1}{2} \triangle w - c(x) w, t > 0, x \in D$$

with the initial condition

$$w(0,x) = 1_A(x), x \in D$$

and with the Dirichlet boundary condition zero on $t>0$, $x \in \partial D$, is denoted by $w_c^A(t,x)$. Clearly,

$$w_c^A(t,x) = v_{1,c}^{A,0}(t,x).$$

Moreover, if

$$q = \sigma^4 c$$

we have

$$v_{\sigma,q}^{A,0}(t,x) = E^P\left[1_A(B_x^\sigma(t))e^{-\frac{1}{\sigma^2}\int_0^t \tilde{q}(B_x^\sigma(s)))ds}\right] =$$

$$E^P\left[1_A(B_x^\sigma(t))e^{-\sigma^2\int_0^t \tilde{c}(B_x^\sigma(s)))ds}\right]$$

and since the stochastic processes $(\sigma B(s))_{s\geq 0}$ and $(B(\sigma^2 s))_{s\geq 0}$ are equivalent,

$$v_{\sigma,q}^{A,0}(t,x) = E^P\left[1_A(B_x^1(\sigma^2 t))e^{-\sigma^2\int_0^t \tilde{c}(B_x^1(\sigma^2 s)))ds}\right] =$$

$$E^P\left[1_A(B_x^1(\sigma^2 t))e^{-\int_0^{\sigma^2 t} \tilde{c}(B_x^1(s)))ds}\right].$$

Accordingly,

$$w_c^A(\sigma^2 t,x) = v_{\sigma,q}^{A,0}(t,x)$$

and writing $p_{1,c}(t,x,y) = p_c(t,x,y)$ we have

$$p_c(\sigma^2 t,x,y) = p_{\sigma,q}(t,x,y).$$

COROLLARY 3.2. *Let D_i, $i=0,1$, be subdomains of $\mathbf{R}^n$. Furthermore, suppose $c_j : D_j \to [0,\infty[$, $j=0,1,\theta$, are continuous functions such that*

$$c_\theta^{-\frac{1}{2}}(x_\theta) \geq \theta_0 c_0^{-\frac{1}{2}}(x_0) + \theta_1 c_1^{-\frac{1}{2}}(x_1)$$

for all $x_0 \in D_0$, $x_1 \in D_1$.

Then

$$w_{c_\theta}^{A_\theta}(s_\theta^2, x_\theta) \geq \left\{w_{c_0}^{A_0}(s_0^2,x_0)\right\}^{\frac{\theta_0 s_0}{s_\theta}} \left\{w_{c_1}^{A_1}(s_1^2,x_1)\right\}^{\frac{\theta_1 s_1}{s_\theta}} * (3.8) \qquad (13)$$

for all $x_0 \in D_0$, $x_1 \in D_1$, $s_0, s_1 > 0$ *and all Borel sets* $A_i \subseteq D_i$, $i = 0, 1$.

Moreover,

$$s_\theta^n p_{c_\theta}(s_\theta^2, x_\theta, y_\theta) \geq \left\{ s_0^n p_{c_0}(s_0^2, x_0, y_0) \right\}^{\frac{\theta_0 s_0}{s_\theta}} \left\{ s_1^n p_{c_1}(s_1^2, x_1, y_1) \right\}^{\frac{\theta_1 s_1}{s_\theta}} * (3.9) \quad (14)$$

for all $x_0, x_1, y_0, y_1 \in \mathbf{R}^n$ and $s_0, s_1 > 0$.

Here $0^{-\frac{1}{2}}$ shall be interpreted as ∞.

A slightly weaker result than Corollary 3.2 is obtained in [5].

Proof. Without loss of generality we may assume that the potential functions in Corollary 3.2 are strictly positive. Let $\sigma_0, \sigma_1 > 0$ and define

$$q_j = \sigma_j^4 c_j, \; j = 0, 1, \theta.$$

Then

$$\frac{1}{\sigma_\theta} q_\theta(x_\theta) \leq \frac{\theta_0}{\sigma_0} q_0(x_0) + \frac{\theta_1}{\sigma_1} q_1(x_1)$$

for all $x_0 \in D_0$, $x_1 \in D_1$ since the function ψ_2 defined by equation (3.1) is convex and positively homogeneous of degree one (cf *Case 3* above). The inequality (3.8) now follows at once from Theorem 3.2 and the inequality (3.9) follows from (3.8) and the Standard Lemma. This concludes our proof of Corollary 3.2.

EXAMPLE 3.4. Suppose K is a bounded, convex domain in $\mathbf{R}^n$ and let $c : K \to [0, +\infty[$ be a continuous function. Furthermore, let Y be a killed Brownian motion in K such that, for any starting point $y_0 \in K$ and $m \in \mathbf{N}_+$,

$$P\left[Y(t_1) \in A_1, ..., Y(t_m) \in A_m \mid Y(0) = y_0\right] =$$

$$\int_{A_1 \times ... \times A_m} \prod_{k=1}^{m} p_c(t_k - t_{k-1}, y_{k-1}, y_k) dy_1 ... dy_m$$

for all $0 = t_0 < t_1 < t_2 < ... < t_m$ and all Borel sets $A_i \subseteq K$ (for details, see e.g. Dynkin [8]). In our point of view the process Y is killed at the boundary of K.

Let A be a Borel set in K. If the process Y starts at the point $x \in K$, the expected time the process visits A until it is killed is given by

$$U_A(x) = \int_0^\infty \int_A p_c(t, x, y) dt dy.$$

The potential U_A need not be quasi-concave if A is convex (in the Newtonian case $n = 3$ and $c = 0$, the potential U_A cannot be quasi-concave for all $r > 2$ and $0 < \varepsilon < 1$, if $A = \{x; 0 \leq x_1 \leq 1, 0 \leq x_i \leq \varepsilon \min(x_1, 1 - x_1), i = 2, 3\}$ and $K = \{x; \mid x \mid < r\}$). The situation is different if we change time to log-time. Here τ is called log-time, if $t = e^\tau$ and t is usual time. If the process starts at the point $x \in K$, the expected log-time the process visits A until it is killed is given by

$$U_A^{\log}(x) = \int_{-\infty}^\infty \int_A p_c(e^\tau, x, y) d\tau dy =$$

$$2 \int_0^\infty \int_A p_c(s^2, x, y) \frac{1}{s} ds dy.$$

In what follows, suppose the function $c : K \to [0, \infty[$ is $-\frac{1}{2}$−concave and let $A \subseteq K$ be open and convex. Moreover, let $\theta = (\theta_0, \theta_1)$ be a vector in $\mathbf{R}^2$ with strictly positive components such that $\theta_0 + \theta_1 = 1$ and suppose $x_0, x_1, y_0, y_1 \in K$ and $s_0, s_1 > 0$. Finally, let $a_0 = (a_0^{(1)}, ..., a_0^{(n+1)})$ and $a_1 = (a_1^{(1)}, ..., a_1^{(n+1)})$ be vectors in $\mathbf{R}^{n+1}$with non-negative components. As the function in (3.7) is concave, Corollary 3.2 and the Standard Lemma now imply that

$$p_c(s_\theta^2, x_\theta, y_\theta) \frac{1}{s_\theta} \prod_{k=1}^{n+1} a_\theta^{(k)} \geq$$

$$\left\{ p_c(s_0^2, x_0, y_0) \frac{1}{s_0} \prod_{k=1}^{n+1} a_0^{(k)} \right\}^{\frac{\theta_0 s_0}{s_\theta}} \left\{ p_c(s_1^2, x_1, y_1) \frac{1}{s_1} \prod_{k=1}^{n+1} a_1^{(k)} \right\}^{\frac{\theta_1 s_1}{s_\theta}}$$

and, accordingly,

$$p_c(s_\theta^2, x_\theta, y_\theta) \frac{1}{s_\theta} \prod_{k=1}^{n+1} a_\theta^{(k)} \geq$$

$$\min \left\{ p_c(s_0^2, x_0, y_0) \frac{1}{s_0} \prod_{k=1}^{n+1} a_0^{(k)}, p_c(s_1^2, x_1, y_1) \frac{1}{s_1} \prod_{k=1}^{n+1} a_1^{(k)} \right\}.$$

By applying the Standard Lemma with $\Psi(\xi, \eta) = \min(\xi, \eta)$ it follows that

$$U_A^{\log}(x_\theta) \geq \min \left(U_A^{\log}(x_0), U_A^{\log}(x_1) \right)$$

for all $x_0, x_1 \in K$ and we conclude that the function $U_A^{\log}$ is quasi-concave.

Again, as in Example 3.4, suppose the potential function c is $-\frac{1}{2}-$concave and defined in a bounded convex domain K in $\mathbf{R}^n$. If

$$g_c(x,y) = \int_0^\infty p_c(t,x,y)dt$$

we remarked above that the potential

$$\int_A g_c(x,y)dy, x \in K$$

need not be quasi-concave even if $A \subseteq K$ is convex. However, this potential is quasi-concave if $A = K$. In fact, by applying the maximum principle of subharmonic functions we concluded in [4] that the function

$$\sqrt{\int_K g_c(x,y)dy}, x \in K$$

is concave. The special case $c = 0$ was settled independently and at the same time by Kawohl [13] using a similar method. Actually, we proved in [4] that the function

$$\left\{\int_B g_c(x,y)f^p(y)dy\right\}^{\frac{1}{2+p}}, x \in \mathbf{R}^n$$

is concave if $0 \leq p \leq 1$, $f \geq 0$ is concave and the potential function c is $-\frac{1}{2}-$concave. We think a Brownian motion approach to this property of the Green function $g_c(x,y)$ would be of great interest.

4. An alternative proof of Theorem 3.1

In view of the Standard Lemma, Theorem 3.1 and the following theorem are equivalent.

THEOREM 4.1. *Let* $\sigma_0, \sigma_1 > 0$ *and suppose* $c_j : \mathbf{R}^n \to \mathbf{R}$, $j = 0, 1, \theta$, *are bounded, continuous functions such that*

$$\frac{1}{\sigma_\theta}c_\theta(x_\theta) \leq \frac{\theta_0}{\sigma_0}c_0(x_0) + \frac{\theta_1}{\sigma_1}c_1(x_1)$$

for all $x_0, x_1 \in \mathbf{R}^n$. *Then*

$$\sigma_\theta^n p_{\sigma_\theta, c_\theta}(t, x_\theta, y_\theta) \geq \left\{\sigma_0^n p_{\sigma_0, c_0}(t, x_0, y_0)\right\}^{\frac{\theta_0 \sigma_0}{\sigma_\theta}} \left\{\sigma_1^n p_{\sigma_1, c_1}(t, x_1, y_1)\right\}^{\frac{\theta_1 \sigma_1}{\sigma_\theta}}$$

for all $x_0, x_1, y_0, y_1 \in \mathbf{R}^n$ *and* $t > 0$.

In this section we want to show that the Standard Lemma implies Theorem 4.1 without any use of the Girsanov theorem. To this end we first discuss a suitable representation of the fundamental solution $p_{\sigma,c}(t, x, y)$. In this discussion it is assumed that the potential function $c : \mathbf{R}^n \to \mathbf{R}$ is bounded and continuous.

To begin with consider the Feynman-Kac formula

$$v_{\sigma,c}^F(t, x) = E^P \left[e^{-\frac{1}{\sigma^2}\left\{F(B_x^\sigma(t)) + \int_0^t c(B_x^\sigma(s))ds\right\}} \right]$$

so that

$$v_{\sigma,c}^F(t, x) = \int_{\mathbf{R}^n} e^{-\frac{F(y)}{\sigma^2}} p_{\sigma,c}(t, x, y) dy$$

with

$$p_{\sigma,c}(t, x, y) = p(\sigma^2 t, x, y) E \left[e^{-\frac{1}{\sigma^2}\int_0^t c(t, B_x^\sigma(s))ds} \mid B_x^\sigma(t) = y \right]$$

and

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}^n} e^{-\frac{|x-y|^2}{2t}}, t > 0, x, y \in \mathbf{R}^n.$$

Recall that

$$P\left[B_x^\sigma(s_1) \in dx_1, ..., B_x^\sigma(s_N) \in dx_N\right] =$$

$$\prod_{k=1}^N p(\sigma^2(s_k - s_{k-1}), x_k, x_{k-1}) dx_1 ... dx_N, 0 < s_1 < ... < s_N$$

where $s_0 = 0$ and $x_0 = x$. The Brownian bridge with standard deviation $\sigma > 0$ which starts at the point $x \in \mathbf{R}^n$ at time 0 and ends at the point $y \in \mathbf{R}^n$ at time $t > 0$ is denoted by

$$B_{x,y}^{\sigma,t} = (B_{x,y}^{\sigma,t}(s))_{0 \leq s \leq t}.$$

By definition, $B_{x,y}^{\sigma,t}(0) = x$ and $B_{x,y}^{\sigma,t}(t) = y$, and if

$$0 = s_0 < s_1 < ... < s_N < s_{N+1} = t4.1 \tag{15}$$

then

$$P\left[B^{\sigma,t}_{x,y}(s_1)\ \in dx_1, ..., B^{\sigma,t}_{x,y}(s_N)\ \in dx_N\right] =$$

$$\prod_{k=1}^{N+1} p(\sigma^2(s_k - s_{k-1}), x_k, x_{k-1}) \frac{dx_1...dx_N}{p(\sigma^2 t, x, y)}$$

where $x_0 = x$ and $x_{N+1} = y$. Therefore, if $s_1, ...s_N$ are as in (4.1) and $\varphi : \mathbf{R}^N \to \mathbf{R}$ is a bounded, continuous function,

$$E\left[\varphi(B^{\sigma,t}_{x,y}(s_1)\ , ..., B^{\sigma,t}_{x,y}(s_N))\ \right] =$$

$$E\left[\varphi(B^{\sigma}_x(s_1)\ , ..., B^{\sigma}_x(s_N))\ |\ B^{\sigma}_x(t) = y\right]$$

and it follows that

$$p_{\sigma,c}(t, x, y) = p(\sigma^2 t, x, y) E\left[e^{-\frac{1}{\sigma^2}\int_0^t c(B^{\sigma,t}_{x,y}(s))ds}\right].$$

Now let $N \in \mathbf{N}_+$ and set

$$\varepsilon = \varepsilon_N = \frac{t}{N+1},$$

$$s_k = s_{kN} = \varepsilon k, k = 0, ..., N+1,$$

and

$$A_N = E\left[e^{-\frac{\varepsilon}{\sigma^2}\sum_{k=1}^N c(B^{\sigma,t}_{x,y}(s_k))}\right].$$

Clearly,

$$\lim_{N\to\infty} A_N = E\left[e^{-\frac{1}{\sigma^2}\int_0^t c(B^{\sigma,t}_{x,y}(s))ds}\right].$$

Furthermore,

$$A_N = \int_{\mathbf{R}^{nN}} e^{-\frac{\varepsilon}{\sigma^2}\sum_{k=1}^N c(x_k)} \prod_{k=1}^{N+1} p(\sigma^2\varepsilon, x_k, x_{k-1}) \frac{dx_1...dx_N}{p(\sigma^2 t, x, y)}$$

where $x_0 = x$ and $x_{N+1} = y$. A rewriting gives

$$A_N =$$

$$\frac{1}{p(\sigma^2 t, x, y)} \int_{\mathbf{R}^{nN}} e^{-\frac{\varepsilon}{\sigma^2}\left\{\frac{1}{2}\sum_{k=1}^{N+1}|\frac{x_k - x_{k-1}}{\varepsilon}|^2 + \sum_{k=1}^N c(x_k)\right\}} \frac{dx_1...dx_N}{\sqrt{2\pi\varepsilon\sigma^2}^{n(N+1)}}$$

and we have

$$p_{\sigma,c}(t,x,y) =$$

$$\lim_{N\to\infty}\int_{\mathbf{R}^{nN}} e^{-\frac{\varepsilon}{\sigma^2}\left\{\frac{1}{2}\sum_{k=1}^{N+1}|\frac{x_k-x_{k-1}}{\varepsilon}|^2+\sum_{k=1}^{N}c(x_k)\right\}}\frac{dx_1...dx_N}{\sqrt{2\pi\varepsilon\sigma^2}^{n(N+1)}}$$

(cf Feynman [9]).

Proof of Theorem 4.1. If $x_0,...,x_{N+1}\in\mathbf{R}^n$, we will write

$$\mathbf{x} = (x_0 \mid ... \mid x_{N+1}).$$

Here $\mathbf{x}$ is considered a vector in $\mathbf{R}^{n(N+2)}$. Moreover, for any fixed $j=0,1,\theta$, we define

$$f_j(\mathbf{x}) =$$

$$e^{-\frac{\varepsilon}{\sigma^2}\left\{\frac{1}{2}\sum_{k=1}^{N+1}|\frac{x_k-x_{k-1}}{\varepsilon}|^2+\sum_{k=1}^{N}c_j(x_k)\right\}}\frac{1}{\sqrt{2\pi\varepsilon\sigma_j^2}^{n(N+1)}}$$

and conclude that

$$\sigma_\theta^{n(N+1)}f_\theta(\mathbf{x}_\theta)\geq\left\{\sigma_0^{n(N+1)}f_0(\mathbf{x}_0)\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}}\left\{\sigma_1^{n(N+1)}f_1(\mathbf{x}_1)\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}}$$

for all $\mathbf{x}_0,\mathbf{x}_1\in\mathbf{R}^{n(N+2)}$ since the function

$$\frac{\mid x\mid^2}{t}, x\in\mathbf{R}^n, t>0$$

is convex and positively homogeneous of degree one. Thus, in view of the Standard Lemma,

$$\sigma_\theta^n\int_{\mathbf{R}^{nN}}f_\theta(\xi_\theta\mid x_1...x_N\mid\eta_\theta)dx_1...dx_N\geq$$

$$\left\{\sigma_0^n\int_{\mathbf{R}^{nN}}f_0(\xi_0\mid x_1...x_N\mid\eta_0)dx_1...dx_N\right\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}}\times$$

$$\left\{\sigma_1^n\int_{\mathbf{R}^{nN}}f_1(\xi_1\mid x_1...x_N\mid\eta_1)dx_1...dx_N\right\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}}$$

for all $\xi_0,\xi_1,\eta_0,\eta_1\in\mathbf{R}^n$. By letting $N\to\infty$ we have

$$\sigma_\theta^n p_{\sigma_\theta,c_\theta}(t,\xi_\theta,\eta_\theta)\geq\{\sigma_0^n p_{\sigma_0,c_0}(t,\xi_0,\eta_0)\}^{\frac{\theta_0\sigma_0}{\sigma_\theta}}\{\sigma_1^n p_{\sigma_1,c_1}(t,\xi_1,\eta_1)\}^{\frac{\theta_1\sigma_1}{\sigma_\theta}}$$

for all $\xi_0, \xi_1, \eta_0, \eta_1 \in \mathbf{R}^n$, which proves Theorem 4.1.

Summing up, we claim that the idea to transform Wiener measure to obtain inequalities of the Brunn-Minkowski type has increased our understanding of this class of inequalities in measure theory as well as in diffusion theory although alternative methods are available. But still there are a variety of problems in connection with these inequalities, where, apparently, all known methods fail (see e.g. Ledoux and Talagrand $[14, p.456, Problem 1]$ and [5]).

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