STABILITY, ANALYTICITY, AND ALMOST BEST APPROXIMATION IN MAXIMUM-NORM FOR PARABOLIC FINITE ELEMENT EQUATIONS

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Dedicated to the memory of Fritz John

Abstract.

We consider semidiscrete solutions in quasi-uniform finite element spaces of order $O(h^r)$ of the initial-boundary value problem with Neumann boundary conditions for a second order parabolic differential equation with time independent coefficients in a bounded domain in R^N. We show that the semigroup on L_∞, defined by the semidiscrete solution of the homogeneous equation, is bounded and analytic uniformly in h. We also show that the semidiscrete solution of the inhomogeneous equation is bounded in the space-time L_∞-norm, modulo a logarithmic factor for $r = 2$, and we give a corresponding almost best approximation property.

1. Introduction.

Let Ω be a bounded domain in R^N with a sufficiently smooth boundary, where the space dimension $N \geq 2$ is arbitrary. Consider the parabolic initial–boundary value problem of finding a real-valued function $u = u(x,t)$ such that

$$\text{(1.1.a)} \quad u_t + Au \equiv \frac{\partial u}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + a_0(x)u = f, \quad \text{for } x \in \Omega,\ t > 0,$$

with the homogeneous natural conormal boundary condition

$$\text{(1.1.b)} \qquad \partial_{n_A} u(x,t) \equiv \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x,t) \sum_{j=1}^{N} a_{ij}(x) n_j(x) = 0, \quad \text{for } x \in \partial\Omega,\ t > 0,$$

and the initial condition

$$\text{(1.1.c)} \qquad u(x,0) = u_0(x), \quad \text{for } x \in \Omega.$$

Here, the second order partial differential operator A is uniformly elliptic and, only for brevity of the analysis below, assumed to be symmetric. It is also assumed that

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$a_0(x) \geq \underline{a}_0 > 0$ so that, in particular, the corresponding bilinear form $A(\cdot,\cdot)$ is coercive in the Sobolev space $W_2^1(\Omega) = H^1(\Omega)$. The coefficients $a_{ij}(x)$ and $a_0(x)$ are assumed to be sufficiently smooth and independent of t, again only for brevity.

We shall analyze semidiscrete continuous in time finite element approximations of (1.1). Thus let S_h, $0 < h < \frac{1}{2}$, denote finite element spaces. For the purposes of this introduction, we may think of them as at least continuous (thus conforming) piecewise polynomials of degree $r-1 \geq 1$ on partitions of Ω which fit the boundary exactly. No boundary conditions are imposed on the functions in S_h, since the conormal derivative condition (1.1.b) is a natural boundary condition. The S_h are defined on globally quasi-uniform partitions of Ω with the diameter of each element uniformly comparable to h. We seek a semidiscrete approximation to the solution of (1.1) as $u_h : \mathcal{C}^1[0,\infty) \to S_h$ such that

$$(1.2)\quad (u_{h,t},\chi) + A(u_h,\chi) = (f,\chi), \quad \forall \chi \in S_h, \quad t > 0, \quad \text{with } u_h(0) = u_{0h} \in S_h,$$

where (v,w) denotes the $L_2(\Omega)$ inner product $\int_\Omega v(x)w(x)dx$.

Let us begin with the case $f \equiv 0$ in (1.1) and (1.2). In semigroup notation, then

$$u(t) = E(t)u_0 = e^{-At}u_0 \quad \text{and} \quad u_h(t) = E_h(t)u_{0h} = e^{-A_h t}u_{0h},$$

where $A_h : S_h \to S_h$ is defined by $(A_h v,\chi) = A(v,\chi)$, $\forall \chi \in S_h$. It follows from the Hopf maximum-principle, see Protter and Weinberger [20, Theorem 8, p. 176], that $\|E(t)v\|_{L_\infty} \leq \|v\|_{L_\infty}$, for $v \in \mathcal{C}(\Omega)$, say. Such a maximum principle is not valid for the semidiscrete problem, but we shall show the semidiscrete stability result

$$(1.3)\qquad \|E_h(t)v_h\|_{L_\infty} \leq C\|v_h\|_{L_\infty}, \quad \text{for } v_h \in S_h.$$

Similarly, the smoothing estimate $\|E'(t)v\|_{L_\infty} \leq Ct^{-1}\|v\|_{L_\infty}$ holds for the continuous problem, and we shall establish the corresponding semidiscrete result

$$(1.4)\qquad \|E_h'(t)v_h\|_{L_\infty} \leq C(t+h^2)^{-1}\|v_h\|_{L_\infty}, \quad \text{for } v_h \in S_h.$$

For the inhomogeneous equation, we shall show the space-time stability estimate

$$(1.5)\qquad \|u_h\|_{L_\infty(Q_T)} \leq C\|u_{0h}\|_{L_\infty} + C\ell_{h,r}\|u\|_{L_\infty(Q_T)}, \quad \text{where } Q_T = \Omega \times [0,T],$$

with C independent of T. Here $\ell_{h,r}$ is a logarithmic factor which enters only if $r = 2$, $\ell_{h,r} = \ell_h := \log(1/h)$ if $r = 2$, $\ell_{h,r} = 1$ otherwise, and it will be seen in Section 7 that this factor is necessary when $r = 2$. We shall also see in Corollary 2.1 below that this result implies an almost best approximation property in the space-time maximum-norm. We remark that if $u_{0h} = P_h u$, with P_h the L_2-projection onto S_h, which is bounded in maximum-norm, then the first term on the right in (1.5) may be absorbed into the second. In this case we may think of u_h as a "parabolic projection" $\mathcal{P}_h u$ of u, and (1.5) may be expressed by saying that this projection $\mathcal{P}_h$ is bounded in $L_\infty(Q_T)$ with a logarithmic factor in the bound when $r = 2$.

The three results (1.3), (1.4), and (1.5) constitute our main results in this investigation.

We shall next comment on relations between these results. By a classical theorem of Hille, see e.g. Pazy [19, Thm. 2.5.2], (1.3) and (1.4) show that $E_h(t)$ is an analytic semigroup, uniformly in h, and lead to a resolvent estimate on an appropriate sector in the complex plane,

$$\|(z - A_h)^{-1} v_h\|_{L_\infty} \le M|z|^{-1}\|v_h\|_{L_\infty}, \quad \tfrac{1}{2}\pi - c \le |\arg z| \le \pi. \tag{1.6}$$

Conversely, this by itself leads to (1.3) and almost to (1.4) ($C(t+h^2)^{-1}$ would be replaced by Ct^{-1}). However, although for $r \ge 3$ the space-time stability result (1.5) implies the semigroup stability estimate (1.3), not even the combination of (1.3) and (1.4) will in general imply (1.5). (As an analogue, consider the case of $L_2(\Omega)$. Then (1.3) and (1.4) hold, but (1.5) is not true in $L_2(Q_T)$ or $L_\infty(0,T;L_2(\Omega))$, cf. Babuška and Osborn [2, p. 58] and Section 7 below.)

We remark that results such as the above for the time-continuous semigroup are useful in analyzing fully discrete approximations, cf. [30, Chapter 8] and Palencia [18].

We shall now briefly discuss previous work on maximum-norm estimates for finite element methods for parabolic problems; a general reference on such methods is [30]. We shall not consider the well–developed theory in L_2 or W_2^1 but mention the basic paper of Wheeler [34], which also contains some maximum norm estimates.

For some general results on maximum–norm error estimates in parabolic problems, we refer to Fujii [10], which used a maximum principle that is valid for acute–angle piecewise linear triangular elements and lumped mass matrices, Wheeler [35] for optimal order results in one space dimension, Bramble et al. [4, Sections 4 and 5] for certain rather special space dimensionally independent results, and Dobrowolski [7], [8], and Nitsche [15] for a variety of almost optimal order results.

More germane to our present investigation, [23] used weighted norm techniques to show the stability and smoothing estimates (1.3) and (1.4) for $N = r = 2$ (but with a factor ℓ_h). Also using weighted norm techniques, Nitsche and Wheeler [17] proved the space-time stabililty result (1.5) for $N = 2, 3$, and $r \ge 4$. Rannacher [21] showed the stability estimates (1.3) and (1.5) on a convex polygon in the plane, with C replaced by $C\ell_h^{3/2}$, and Chen [5] improved this to $C\ell_h$. (The techniques of the present paper do not immediately apply to a convex polygon.)

In [32] the problem was approached in the "reverse" manner of first proving the resolvent estimate (1.6) and then using that and the extra ingredient of maximum–norm stability of the elliptic projection to arrive at, essentially, the space-time stability result (1.5). (As remarked following (1.6), some extra such ingredient is necessary to show (1.5).) The investigation in [32] was restricted to one space dimension, and a further investigation so restricted was given in [31]. It does, in $N = 1$, what [23] did in $N = 2$. (The technique used was easier for $N = 2$ than for $N = 1$.) Still in the case of one space dimension, Crouzeix, Larsson, and Thomée [6] further extended the resolvent estimates of [32].

Using a simple and elegant technique, Palencia [18] has recently derived dimensionally independent resolvent estimates relying on the dimensionally independent elliptic results in [25]. Regrettably, his results are not quite sharp in the difficult transition region when $|z|$ goes from $o(h^{-2})$ to $O(h^{-2})$, and, as a consequence, his results for the stability of $E_h(t)$ are as follows: There exist three constants C, c_1, and c_2, such that $\|E_h(t)\|_{L_\infty} \le C$ for $0 \le t \le c_1 h^2$ and for $c_2 h^2 \ell_{h,r} \ell_h^2 \le t$. Hence, as far as stability is concerned, we are "merely" filling a very small gap in the time domain. The techniques of [18] similarly apply to the smoothing estimate (1.4). Resolvent estimates for A_h are clearly related to finite element approximation of singularly perturbed problems of elliptic–to–elliptic type, $-\varepsilon^2\Delta u + u = f$, cf. [26], and [32]. The same transition region as in [18] presented the major difficulties in [26] and [32]. To conclude this overview of previous results, we again cite the thesis of Chen [5]. He proves interior maximum–norm error estimates on unit-size domains for $N = 1, 2, 3, 4$, and 5. In a general broad outline his techniques are similar to ours, but crucial differences in detail limit his investigation to $N \leq 5$, while we would admit general space dimension in the corresponding situation.

The space dimension–independent proofs in the present investigation were inspired by the dimensionally independent proofs in the elliptic case in [25], and they are natural parabolic analogues of them. Essential new ingredients in carrying through our program of translation from the elliptic to the parabolic case are a "strong superapproximation" result and consequent local L_2–based error estimates for (1.2), basically given in [29] and used also in [5].

The study of maximum-norm estimates for finite element approximations to parabolic problems was preceded by analogous work in the theory for finite difference methods. A fundamental contribution in this regard is the celebrated paper by Fritz John [14].

2. Basic assumptions and notation, statements of the main results, and some preliminaries.

With $0 < h < \frac{1}{2}$ a parameter, let $\tau_j^h, j = 1, \cdots, J_h$, be disjoint open sets, "elements", which form a partition of Ω, i.e., $\bar\Omega = \cup_{j=1}^{J_h} \bar\tau_j^h$. For each such partition, let $S_h = S_h(\Omega) \subset W_\infty^1(\Omega)$ be a finite-dimensional space. Here, and in the rest of this paper, $W_p^l(D)$, with $1 \leq p \leq \infty, l = 0, 1, 2, \cdots$, and D a spatial set, will denote the standard Sobolev spaces and $\|\cdot\|_{W_p^l(D)}$ and $|\cdot|_{W_p^l(D)}$ their norms and semi-norms (containing only the derivatives of order l), respectively. Similarly, we use $W_p^{l,m}(Q)$ when Q is a space-time set; then l is the order of derivatives in space and m that of derivatives in time. When needed we shall also use the piecewise norms

$$\|v\|^{(h)}_{W_p^l(D)} = \Big(\sum_{\tau_j^h \cap D \neq \emptyset} \|v\|^p_{W_p^l(\tau_j^h \cap D)} \Big)^{1/p}.$$

We shall now formulate certain properties of standard finite element spaces in the conforming h-method setting as basic assumptions. We thus assume that there exist two positive constants K and k, and also two integers r and $\bar r, r \geq 2$, all independent of h, such that the properties A.0 - A.5 below hold. Our first property reflects the globally quasi-uniform nature of our partitions and that the boundaries of elements are "not too irregular".

A.0. Quasiuniformity and Trace.

(i) Each τ_j^h contains a ball of radius kh and is contained in a ball of radius Kh.

(ii) For $0 < h < \frac{1}{2}$,

$$\int_{\partial \tau_j^h} |\nabla v| \, d\sigma \leq K \big(h^{-1} |v|_{W_1^1(\tau_j^h)} + |v|_{W_1^2(\tau_j^h)} \big), \quad \forall v \in W_1^2(\tau_j^h), \ j = 1, \cdots, J_h.$$

For $D \subset \Omega, S_h(D)$ will denote the restriction of S_h to D. Our next assumption is a standard inverse property.

A.1. Inverse property.

Let $\chi \in S_h(D)$ where D is any union of closures of elements. Then

$$\|\chi\|^{(h)}_{W^l_p(D)} \leq Kh^{-(l-s)-N(\frac{1}{q}-\frac{1}{p})}\|\chi\|^{(h)}_{W^s_q(D)}, \quad \text{for } 0 \leq s \leq l \leq 2, 1 \leq q \leq p \leq \infty.$$

The next two assumptions are concerned with local approximation properties of the finite element spaces. The integer $r, r \geq 2$, will denote the highest order of approximation possible in L_p-norms for smooth functions. (In many but not all cases $r-1$ then corresponds to the maximal total polynomial degree of shape-functions included on all elements, sometimes nonlinearly mapped from a master element.)

It is convenient to introduce a special notation signifying functions which have "compact support in the interior of $D \subset \Omega$ modulo $\partial\Omega$". Thus let $\bar{D}^\partial = IntD \cup (\bar{D} \cap \partial\Omega)$. Then supp $v \subset \bar{D}^\partial$ means that v vanishes in a neighborhood of every point of ∂D that is not common with $\partial\Omega$. We shall write $S^0_h(\bar{D}^\partial)$ for the functions in S_h with support in $\bar{D}^\partial$.

Now let D_0 be a subset of Ω and, for $d > 0$, set $D_{jd} = \{x \in \Omega; \text{dist } (x, D_0) \leq jd\}, \quad j = 1, 2, 3, \ldots.$

A.2. Local approximation.

(i) There exists a linear operator $I_h : W^1_1(\Omega) \to S_h(\Omega)$ such that the following holds. If $d \geq kh$, then

$$\|I_h v - v\|^{(h)}_{W^s_p(D_d)} \leq Kh^{l-s}\|v\|_{W^l_p(D_{2d})}, \quad \text{for } 0 \leq s \leq l \leq r, 1 \leq p \leq \infty.$$

Furthermore, if supp $v \subset \bar{D}^\partial_d$, then $I_h v \in S^0_h(\bar{D}^\partial_{2d})$. Also, if $v|_{D_d} \in S_h(D_d)$, then $I_h v = v$ on D_0, and the bound above may be replaced by $Kh^{l-s}\|v\|_{W^l_p(D_{2d}\setminus D_0)}$.

(ii) $1 \in S_h(\Omega)$.

If the function to be approximated is of a certain special form, we have an assumption known as superapproximation.

A.3. Superapproximation.

Let $d \geq kh$ and $\omega \in \mathcal{C}^{\bar{r}}(D_{2d})$ with supp $\omega \subset \bar{D}^\partial_{2d}$. Then, for any $\psi \in S_h(D_{3d})$, there exists $\eta \in S^0_h(\bar{D}^\partial_{3d})$ such that

$$\|\omega\psi - \eta\|_{W^l_2(D_{3d})} \leq Kh\|\omega\|_{W^{\bar{r}}_\infty(D_{2d})}\|\psi\|_{W^l_2(D_{3d})}, \quad l = 0, 1.$$

Furthermore, if $\omega \equiv 1$ on D_d, then $\eta = \psi$ on D_0, and the last factor may be replaced by $\|\psi\|_{W^l_2(D_{3d}\setminus D_0)}$.

We next assume the existence of a certain convenient "delta-function".

A.4. Regularized delta-function.

Let $x_0 \in \bar{\Omega} \cap \bar{\tau}^h_j$. There exists a function $\tilde{\delta}_{x_0} \in \mathcal{C}^3$ with support in τ^h_j such that

$$\chi(x_0) = \int_{\tau^h_j} \chi\tilde{\delta}_{x_0}\, dx, \quad \forall\chi \in S_h,$$

and

$$\|\tilde{\delta}_{x_0}\|_{W^l_p} \leq Kh^{-l-N(1-\frac{1}{p})}, \quad \text{for } 1 \leq p \leq \infty,\ l = 0, 1, 2, 3.$$

We finally make an assumption about scaling. Let $B_R(x_0)$ denote the $N-$dimensional open ball of radius R centered at x_0.

A.5. Scaling.

Let $x_0 \in \bar{\Omega}$ and $R \geq kh$. The linear transformation $y = (x - x_0)/R$ takes $\Omega_R(x_0) = B_R(x_0) \cap \Omega$ into a new domain $\hat{\Omega}_1$ and $S_h(\Omega_R(x_0))$ into a new function space $\hat{S}_{h/R}(\hat{\Omega}_1)$. Then $\hat{S}_{h/R}(\hat{\Omega}_1)$ satisfies A.0 - A.4 with h replaced by h/R. The constants occurring remain unchanged.

Let us remark on our simplifying assumption that the partitions are exact, i.e., $\bar{\Omega} = \cup_j \bar{\tau}_j^h$, which is not necessarily satisfied in practice. In two dimensions, one may accomplish this by allowing "pieshaped" elements at the boundary. In three or more dimensions this does not work in general due to wedge-shaped pieces carrying different "polynomial" values from elements whose prolongations overlap in these wedges. However, in principle, one could extend the basic domain Ω to $\Omega_{ch^2} = \{x \in \Omega; \text{dist } (x, \Omega) \leq ch^2\}$, partition that domain while making sure that the meshdomain contains Ω (by taking c large enough), and then restrict to Ω. There are other exact methods, see Bernardi [3] for an exact triangulation of the unit ball in R^3. We also refer to Babuška et al. [1] for articles and further references to practice in finite element partitions. In our opinion the simplifying assumption is, for natural boundary conditions, akin to disregarding numerical quadrature.

A brief discussion of properties A.0 - A.5 can be found in [27, Appendix]. We remark that these properties are not necessarily independent; for instance it follows as in [27] that A.4 is a consequence of other properties.

We are now in a position to state our main results. Recall that we assume that A is uniformly elliptic, that the boundary $\partial\Omega$ and the coefficients of A are sufficiently smooth, and that a_0 is positive.

Theorem 2.1. *Suppose that A.0-A.5 hold. Then there exists a constant C independent of h and t such that*

$$\|E_h(t)\|_{L_\infty} + (t + h^2)\|E_h'(t)\|_{L_\infty} \leq C, \quad \textit{for } t \geq 0.$$

It may be seen from the proofs that both $E_h(t)$ and $E_h'(t)$ decay exponentially for t large.

Theorem 2.2. *Suppose that A.0-A.5 hold. Then there exists a constant C independent of h and T such that if u_h and u satisfy (1.2) and (1.1), then*

$$\|u_h\|_{L_\infty(Q_T)} \leq C\|u_{0h}\|_{L_\infty} + C\ell_{h,r}\|u\|_{L_\infty(Q_T)}, \quad \textit{for } T \geq 0.$$

A consequence of Theorem 2.2 is that the solution of (1.2) is an "almost best approximation" of the solution of (1.1).

Corollary 2.1. *Under the assumptions of Theorem 2.2, we have*

$$\|u_h - u\|_{L_\infty(Q_T)} \leq C\|u_{0h} - u_0\|_{L_\infty} + C\ell_{h,r} \min_{\chi \in \mathcal{C}([0,T];S_h)} \|u - \chi\|_{L_\infty(Q_T)}, \quad \textit{for } T \geq 0.$$

Proof. Let $\chi \in \mathcal{C}^1([0,T]; S_h)$. Then $u_h - \chi$ is the solution of (1.2) with f such that the exact solution of (1.1) (in weak form) is $u - \chi$, and we conclude from Theorem 2.2 that

$$\|u_h - \chi\|_{L_\infty(Q_T)} \leq C\|u_{0h} - \chi\|_{L_\infty} + C\ell_{h,r}\|u - \chi\|_{L_\infty(Q_T)}.$$

Writing $u_h - u = (u_h - \chi) - (u - \chi)$ and using the triangle inequality completes the proof since $\mathcal{C}^1[0,T]$ is dense in $\mathcal{C}[0,T]$. □

In the remainder of this section we shall introduce two facts that will be used repeatedly later. The first concerns the Green's function in the continuous problem (1.1).

Lemma 2.1. *Assume that the boundary $\partial\Omega$ and the coefficients of A are sufficiently smooth. Then, for any integer l_0 and multi-integer l, there exist constants C and $c > 0$ such that for the Green's function $G(x,t;y,s), t > s, x, y \in \Omega$, we have*

$$|D_t^{l_0} D_x^l G(x,t;y,s)| \le C((t-s)^{1/2} + |x-y|)^{-(N+2l_0+|l|)} e^{-c\frac{|x-y|^2}{t-s}}.$$

A proof can be found in Èĭdel'man and Ivasišen [9].

A second basic ingredient of our proofs is the detailed behavior of the L_2–projection $P_h : L_1(\Omega) \to S_h(\Omega)$.

Lemma 2.2. *(i) With $\tilde{\delta}_{x_0}$ given in A.4 and $\tilde{\delta}_{x_0,h} = P_h\tilde{\delta}_{x_0}$, we have*

$$(P_h v)(x_0) = \int_\Omega v(y)\tilde{\delta}_{x_0,h}(y)\,dy, \quad \textit{for } x_0 \in \Omega.$$

(ii) There exist constants C and $c > 0$ such that

$$|\tilde{\delta}_{x_0,h}(y)| \le Ch^{-N}e^{-c|x_0-y|/h}, \quad \forall x_0, y \in \Omega.$$

(iii) There exists a constant C such that

$$\|P_h\|_{L_p} \le C, \quad \textit{for } 1 \le p \le \infty.$$

Such standard results can be found in [33, cf. Lemma 7.2 in particular].

3. Proofs of Theorems 2.1 and 2.2, Part 1.

In this section we shall reduce the proofs to certain technical estimates for approximate Green's functions which will then be proven in Section 4. We begin with the case of bounded T, taking $T \le 1$ for definiteness; the case $T > 1$ will be treated at the end of this section.

Starting with the stability estimate of Theorem 2.1, let x_0 be any point in Ω. This point will be fixed throughout this and the next sections and often suppressed in the notation. Let $x_0 \in \bar{\tau}_0 = \bar{\tau}_{j_0}^h$, let $\tilde{\delta} = \tilde{\delta}_{x_0}$ be the regularized delta-function of Assumption A.4, and let $\tilde{\delta}_h = \tilde{\delta}_{x_0,h} = P_h\tilde{\delta}$. With $\Gamma_h = \Gamma_{x_0,h}(x,t) \in S_h$ given by

$$\Gamma_{h,t} + A_h\Gamma_h = 0, \quad \text{for } t > 0, \quad \text{with } \Gamma_h(0) = \tilde{\delta}_h, \tag{3.1}$$

we then have

$$(E_h(t)u_{0h})(x_0) = (\Gamma_h(t), u_{0h}).$$

Further, letting $\Gamma = \Gamma_{x_0}(x,t)$ be given by

$$\Gamma_t + A\Gamma = 0, \quad \text{for } t > 0, \quad \text{with } \Gamma(0) = \tilde{\delta}, \tag{3.2}$$

we may write, with $F = F_{x_0}(x,t) = \Gamma_h - \Gamma$,

$$(E_h(t)u_{0h})(x_0) = (F(t), u_{0h}) + (\Gamma(t), u_{0h}).$$

Here $|(\Gamma(t), u_{0h})| \le \|u_{0h}\|_{L_\infty}\|\Gamma(t)\|_{L_1}$ and, since $\Gamma(x,t) = \int_{\tau_0} G(x,y;t,0)\tilde{\delta}(y)\,dy$, the Green's function estimates of Lemma 2.1 and A.4 show that $\|\Gamma(t)\|_{L_1}$ is bounded (in fact, by $C\|\tilde{\delta}\|_{L_1}$). In order to secure the stability estimate, it hence remains to bound $\|F(t)\|_{L_1}$. But $F(t) = P_h\tilde{\delta} - \tilde{\delta} + \int_0^t F_s(s)ds$ and, by the stability of P_h in L_1 (Lemma 2.2), a bound would follow if we can show that

$$\|F_t\|_{L_1(Q_T)} \le C, \quad \text{for } 0 \le T \le 1. \tag{3.3}$$

This will be accomplished in the next section.

We now turn to the smoothing estimate of Theorem 2.1. Writing

$$t(E_h'(t)u_{0h})(x_0) = (tF_t(t), u_{0h}) + (t\Gamma_t(t), u_{0h}),$$

and again using the Green's function estimates, and that $tF_t(t) = \int_0^t (sF_s)_s\, ds$, the proof is reduced to showing that (since $(sF_s)_s = F_s + sF_{ss}$ and assuming (3.3))

$$\|tF_{tt}\|_{L_1(Q_T)} \le C, \quad \text{for } 0 \le T \le 1.$$

Again this will be accomplished in the next section.

For the refinement of the smoothing estimate when $t \le h^2$, note that $E_h'(t) = -A_hE_h(t)$. Since $\|A_h\|_{L_\infty} \le Ch^{-2}$ (cf, e.g., [23, Lemma 1.3]), $\|E_h'(t)\|_{L_\infty} \le Ch^{-2}$.

We next consider the space-time stability estimate of Theorem 2.2 for $T \le 1$. Assuming now also (3.3), it suffices to consider the case of $u_h(0) = P_hu_0$. We have

$$\begin{aligned}\frac{d}{dt}(u_h(t), \Gamma_h(T-t)) &= (u_{h,t}(t), \Gamma_h(T-t)) + A(u_h(t), \Gamma_h(T-t))\\ &= (f(t), \Gamma_h(T-t)) = (u_t(t), \Gamma_h(T-t)) + A(u(t), \Gamma_h(T-t)).\end{aligned}$$

Integrating, integrating by parts in the first term on the right, and using that $u_h(T)(x_0) = (u_h(T), \tilde{\delta}_h)$ and $u_h(0) = P_hu_0$ together with (3.2),

$$u_h(T)(x_0) = (u(T), \tilde{\delta}_h) + \int_0^T (u(t), F_t(T-t))\, dt + \int_0^T A(u(t), F(T-t))\, dt.$$

Since $\|\tilde{\delta}_h\|_{L_1} \le C$ and using also (3.3), the first two terms are bounded as stated, and it remains to consider the third term which we denote I. To bound it we integrate by parts over each element to obtain

$$I = \int_0^T \sum_{l=1}^{J_h} \Big(\int_{\tau_l^h} u(t)AF(T-t)\, dx + \int_{\partial\tau_l^h} u(t)\partial_{n_A}F(T-t)\, d\sigma \Big)\, dt.$$

Using Assumption A.0 (ii), we then have

$$|I| \le C\|u\|_{L_\infty(Q_T)}\big(\|F\|^{(h)}_{W_1^{2,0}(Q_T)} + h^{-1}\|F\|_{W_1^{1,0}(Q_T)}\big).$$

With I_h as in A.2, we write $F = (\Gamma_h - I_h\Gamma) + (I_h\Gamma - \Gamma)$ and obtain by A.1

$$\|F\|^{(h)}_{W_1^{2,0}(Q_T)} \le Ch^{-1}\|F\|_{W_1^{1,0}(Q_T)} + C|[I_h\Gamma - \Gamma]|_{Q_T},$$

where we have used the notation

$$|[X]|_Q = \|X\|^{(h)}_{W_1^{2,0}(Q)} + h^{-1}\|X\|_{W_1^{1,0}(Q)}$$

for a space-time set Q.

Lemma 3.1. *We have* $|[I_h\Gamma - \Gamma]|_{Q_T} \le C\ell_{h,r}$, *for* $0 \le T \le 1$.

Proof. With $\rho(x,t) = \max(|x-x_0|, t^{1/2})$ the parabolic distance between (x,t) and $(x_0, 0)$, let $Q^c = \{(x,t) \in Q_T : \rho(x,t) \le ch\}$. By Cauchy-Schwarz's inequality, A.2, a standard parabolic energy estimate, and A.4,

$$(3.4) \qquad |[I_h\Gamma - \Gamma]|_{Q^{2c}} \le (2ch)^{1+N/2}\|\Gamma\|_{W_2^{2,0}(Q_T)} \le Ch^{1+N/2}\|\tilde{\delta}\|_{W_2^1(\Omega)} \le C.$$

With c sufficiently large we have on $Q_T \backslash Q^c$, for any derivative D_x^α of order $\le r$, that $|D^\alpha\Gamma(x,t)| \le \rho(x,t)^{-N-r}$, see Lemma 2.1. Hence by A.2, and changing variables as $y = |x-x_0|t^{-1/2}$ so that $\rho(x,t) = t^{1/2}\max(1,y) \ge \frac{1}{2}t^{1/2}(1+y)$,

$$\begin{aligned} |[I_h\Gamma - \Gamma]|_{Q_T\backslash Q^{2c}} &\le Ch^{r-2}\|\rho^{-N-r}\|_{L_1(Q_T\backslash Q^c)} \\ &\le Ch^{r-2}\int_0^{c^2h^2} t^{-r/2}\int_{y\ge cht^{-1/2}} (1+y)^{-N-r}\,dy\,dt + Ch^{r-2}\int_{c^2h^2}^T t^{-r/2}\,dt \\ &\le Ch^{r-2}\int_0^{c^2h^2} h^{-r}\,dt + C\ell_{h,r} \le C\ell_{h,r}. \end{aligned}$$

Together with (3.4) this proves the lemma. □

Assuming (3.3), we have thus obtained

$$|u_h(T)(x_0)| \le C\|u\|_{L_\infty(Q_T)}(h^{-1}\|F\|_{W_1^{1,0}(Q_T)} + \ell_{h,r}).$$

Hence Theorem 2.2 would follow, for $0 \le T \le 1$, from

$$h^{-1}\|F\|_{W_1^{1,0}(Q_T)} \le C\ell_{h,r}, \quad \text{for } 0 \le T \le 1.$$

Again, this will be established in the next section.

Summarizing the development above, in order to complete the proof in the case $0 \le T \le 1$, it remains to prove the following proposition which will be done in Section 4 below.

Proposition 3.1. *There exists a constant* C *such that, for* $F = \Gamma_h - \Gamma$ *given by (3.1) and (3.2),*

$$\|F_t\|_{L_1(Q_T)} + \|tF_{tt}\|_{L_1(Q_T)} + h^{-1}\ell_{h,r}^{-1}\|F\|_{W_1^{1,0}(Q_T)} \le C, \quad \textit{for } 0 \le T \le 1.$$

We shall finally consider the case $T > 1$. In doing so we shall use the Ritz projection $R_h : L_\infty(\Omega) \cup H^1(\Omega) \to S_h(\Omega)$ defined by

$$A(R_hv - v, \chi) = 0, \quad \forall \chi \in S_h.$$

From [22, Remark following Theorem 2.2] (cf. [25] for the case of Dirichlet boundary conditions), we have

$$(3.5) \qquad \|R_h\|_{L_\infty} \le C\ell_{h,r}.$$

We begin with the following lemma which shows Theorem 2.1 for $T > 1$.

Lemma 3.2. *Let $q \geq 0$. We have, with λ_1 the smallest eigenvalue of A and $m > N/2$,*

$$\|A_h^q E_h(t) v_h\|_{L_\infty} \leq C t^{-q-m} e^{-\lambda_1 t/2} \|v_h\|_{L_\infty}, \quad \textit{for } v_h \in S_h, \quad t > 0.$$

Proof. We first show that

$$\|\chi\|_{L_\infty} \leq C \|A_h^m \chi\|_{L_2}, \quad \forall \chi \in S_h, \quad \text{for } m > N/2. \tag{3.6}$$

For this we quote the following result from [4, Lemma 4.1] (valid with (3.5)),

$$\|A_h^{-1} \chi\|_{L_{p_{j+1}}} \leq C \|\chi\|_{L_{p_j}}, \quad \forall \chi \in S_h, \quad \text{if } \frac{1}{p_j} - \frac{1}{p_{j+1}} < \frac{1}{N},$$

from which (3.6) follows by choosing an increasing sequence of p_j with $1/p_j - 1/p_{j+1} < 1/N, p_1 = 2, p_m = \infty$, and adding the corresponding results. Applying (3.6) with $\chi = A_h^q E_h(t) v_h$, we find, with $\{\lambda_j^h\}_{j=1}^{n_h}$ the increasing eigenvalues of A_h (since, as is well known, $\lambda_1^h \geq \lambda_1$),

$$\begin{aligned} \|A_h^q E_h(t) v_h\|_{L_\infty} &\leq C \|A_h^{q+m} E_h(t) v_h\|_{L_2} \\ &= C \max_j \left((\lambda_j^h)^{q+m} e^{-\lambda_j^h t} \right) \|v_h\|_{L_2} \leq C t^{-q-m} e^{-\lambda_1 t/2} \|v_h\|_{L_\infty}. \end{aligned} \qquad \square$$

We next show Theorem 2.2 for $T > 1$, assuming it holds for $T = 1$.

Lemma 3.3. *Assume that Theorem 2.2 holds for $T = 1$. Then it holds for any $T > 1$, with C independent of T.*

Proof. By the previous lemma it suffices to consider the case $u_{0h} = P_h u_0$. We need to show that

$$\|u_h(T)\|_{L_\infty} \leq C \ell_{h,r} \|u\|_{L_\infty(Q_T)}, \quad \text{for } T > 1.$$

Assume $T > 1$. Let $\varphi \in C^\infty(-\infty, 0]$ with $\varphi(t) = 1$ in $(-\frac{1}{2}, 0]$, and $\varphi(t) = 0$ for $t < -1$, and let $\varphi_T(t) = \varphi(t - T)$. We set $v = u\varphi_T$ and $w = u - v$ and find that

$$\begin{aligned} v_t + Av &= f_1 = f\varphi_T + u\varphi_T', \quad \text{for } t \geq 0, \qquad \text{with } v(0) = 0, \\ w_t + Aw &= f_2 = f(1 - \varphi_T) - u\varphi_T', \quad \text{for } t \geq 0, \qquad \text{with } w(0) = u_0. \end{aligned}$$

We denote by v_h and w_h the corresponding semidiscrete solutions and note that $u_h = v_h + w_h$. Since Theorem 2.2 is assumed to hold for $T = 1$ and v and v_h vanish for $t \leq T - 1$, we may apply it to the time interval $(T - 1, T)$ to obtain

$$\|v_h(T)\|_{L_\infty} \leq C \ell_{h,r} \|v\|_{L_\infty(Q_T)} \leq C \ell_{h,r} \|u\|_{L_\infty(Q_T)}.$$

By Duhamel's principle, we have

$$w_h(T) = E_h(T) P_h u_0 + \int_0^T E_h(T - t) P_h f_2(t)\, dt. \tag{3.7}$$

Here $f_2(t) = 0$ for $t > T - \frac{1}{2}$, and, using that $f = u_t + Au$ and $P_h Au = A_h R_h u$, we find

$$P_h f_2(t) = (1-\varphi_T)(P_h u_t + A_h R_h u) - \varphi_T' P_h u = (1-\varphi_T) A_h R_h u + \frac{d}{dt}((1-\varphi_T) P_h u),$$

so that, after integrating by parts in the last term and using (3.7),

$$w_h(T) = \int_0^{T-\frac{1}{2}} (1-\varphi_T) A_h E_h(T-t)(R_h - P_h) u(t)\, dt.$$

Hence, using Lemma 3.2, (3.5), and Lemma 2.2 (iii),

$$\|w_h(T)\|_{L_\infty} \le C(\|R_h\|_{L_\infty} + \|P_h\|_{L_\infty}) \|u\|_{L_\infty(Q_T)} \le C\ell_{h,r} \|u\|_{L_\infty(Q_T)}.$$

Together these estimates show the lemma. □

4. Proofs of Theorems 2.1 and 2.2, Part 2.

In this section we shall prove Proposition 3.1.

We shall decompose Q_T into "parabolic annuli". For this, let $d_j = 2^{-j}$, j integer, and let $Q_j = \{(x,t) \in Q_T; d_j \le \rho(x,t) \le 2d_j\}$, with $\rho(x,t) = \max(|x - x_0|, t^{1/2})$ again denoting the parabolic distance to $(x_0, 0)$, and similarly $\Omega_j = \{x \in \Omega; d_j \le |x - x_0| \le 2d_j\}$. Then, for some fixed J_0 (depending on Ω and the fact that $T \le 1$), and any $J_* > J_0$,

$$Q_T = \Big(\bigcup_{j=J_0}^{J_*} Q_j \Big) \cup Q_*, \quad \text{where } Q_* = \{(x,t) \in Q_T : \rho(x,t) \le d_{J_*}\}.$$

We shall refer to Q_* as the "innermost" set, and ultimately we shall choose J_* such that $d_{J_*} \approx C_* h$ with C_* sufficiently large. Note that then $J_* \approx \log(1/h)$. Constants C will, as usual, change freely but will be independent of C_*. We shall write $\sum_{*,j}$ when the innermost set is included and $\sum_j$ when it is not.

In this proof, almost all norms occuring will be L_2-based norms. We shall therefore write $\|v\|_D$ and $|||v|||_Q$ for L_2-norms over space and space-time sets, respectively, and $\|v\|_{k,D}$ and $|||v|||_{k,Q}$ when up to k spatial derivatives are included. Time-derivatives will always be displayed explicitly.

Denoting the sum of norms in Proposition 3.1 by $\mathcal{M}$, we have by Cauchy-Schwarz's inequality

$$\mathcal{M} \le \sum_{*,j} (2d_j)^{1+N/2} (|||F_t|||_{Q_j} + |||tF_{tt}|||_{Q_j} + h^{-1}\ell_{h,r}^{-1} |||F|||_{1,Q_j}).$$

Here, since $t \le 4d_j^2$ on Q_j, the part of $\mathcal{M}$ over Q_* is bounded by

$$\mathcal{I} = C(C_* h)^{1+N/2} \big[|||\Gamma_t|||_{Q_T} + |||\Gamma_{h,t}|||_{Q_T} + C_*^2 h^2 (|||\Gamma_{tt}|||_{Q_T} + |||\Gamma_{h,tt}|||_{Q_T}) + h^{-1}\ell_{h,r}^{-1} (|||\Gamma|||_{1,Q_T} + |||\Gamma_h|||_{1,Q_T}) \big].$$

The remaining terms are bounded by $Cd_j^{1+N/2} K_j$ where

$$K_j = |||F_t|||_{Q_j} + d_j^2 |||F_{tt}|||_{Q_j} + \mu_j |||F|||_{1,Q_j}, \quad \text{with } \mu_j = h^{-1}\ell_{h,r}^{-1} + d_j^{-1}; \tag{4.1}$$

here the term $d_j^{-1}|||F|||_{1,Q_j}$ has simply been added in K_j for the purpose of a later "kick-back" argument. It follows that

$$\mathcal{M} \le \mathcal{I} + C\mathcal{K}, \quad \text{where } \mathcal{K} = \sum_j d_j^{1+N/2} K_j. \tag{4.2}$$

We begin by estimating the first term on the right and show that

$$\mathcal{I} \le C C_*^{3+N/2}. \tag{4.3}$$

This follows by standard energy arguments and using A.1 and A.4. For example, we have by (3.1)

$$\|\Gamma_{h,tt}\|^2 + \tfrac{1}{2}\frac{d}{dt}A(\Gamma_{h,t}, \Gamma_{h,t}) = 0,$$

so that, by inverse estimates, A.1, and by A.4,

$$|||\Gamma_{h,tt}|||_{Q_T} \le A^{1/2}(A_h\tilde{\delta}_h, A_h\tilde{\delta}_h) \le Ch^{-3}\|\tilde{\delta}_h\| \le Ch^{-3}\|\delta_h\| \le Ch^{-3-N/2},$$

and thus $h^{3+N/2}|||\Gamma_{h,tt}|||_{Q_T} \le C$.

To treat the terms involved in K_j we shall use local energy-based estimates for functions $e = z_h - z$ satisfying

$$(e_t, \chi) + A(e, \chi) = 0, \quad \forall \chi \in S_h,\ t > 0. \tag{4.4}$$

We set $Q_j' = Q_{j-1} \cup Q_j \cup Q_{j+1}$ and correspondingly for Ω_j'. The following proposition will be proven in Section 6. For readability, we shall use mnemonic acronyms; thus Initial terms, approXimation terms, and Higher order slush terms. We shall apply this with $e = F$ and F_t, respectively. In either case, $z(0) = \tilde{\delta} = 0$ or $A\tilde{\delta} = 0$ on $\Omega_j' = Q_j' \cap \{t = 0\}$, while $z_{0h} = P_h\tilde{\delta}$ or $A_hP_h\tilde{\delta}$ does not necessarily vanish on Ω_j'.

Proposition 4.1. *For any $q > 0$ there exists C such that the following holds. Let $e = z_h - z$ satisfy (4.4) with $z(0) = 0, z_h(0) = z_{0h}$ on Ω_j'. Then*

$$|||e_t|||_{Q_j} + d_j^{-1}|||e|||_{1,Q_j} \le C\big(I_j(z_{0h}) + X_j(\zeta) + H_j(e) + d_j^{-2}|||e|||_{Q_j'}\big), \tag{4.5}$$

where, with $\zeta = I_hz - z$,

$$\begin{aligned} I_j(z_{0h}) &= \|z_{0h}\|_{1,\Omega_j'} + d_j^{-1}\|z_{0h}\|_{\Omega_j'}, \\ X_j(\zeta) &= d_j|||\zeta_t|||_{1,Q_j'} + |||\zeta_t|||_{Q_j'} + d_j^{-1}|||\zeta|||_{1,Q_j'} + d_j^{-2}|||\zeta|||_{Q_j'}, \\ H_j(e) &= (h/d_j)^q(|||e_t|||_{Q_j'} + d_j^{-1}|||e|||_{1,Q_j'}). \end{aligned}$$

We now apply Proposition 4.1 to estimate the terms in K_j involved in $\mathcal{K}$ as defined in (4.2) and (4.1). We first note that, when treating the term $d_j^2|||F_{tt}|||_{Q_j}$ (i.e., $e = F_t$), the last term in (4.5) becomes $|||F_t|||_{Q_j'}$ which already occurs in K_j, apart from Q_j being replaced by Q_j'. Taking intermediate sets between Q_j and Q_j', we may then estimate

$$K_j \le C(\widehat{I_j} + \widehat{X_j} + \widehat{H_j} + \mu_j d_j^{-1}|||F|||_{Q_j'}), \tag{4.6}$$

where $\widehat{I}_j$, $\widehat{X}_j$, $\widehat{H}_j$ now contain μ_j in front of some terms.

We shall next in turn estimate the first three terms on the right. When doing so, we shall also be more explicit about what they contain. We begin with the initial terms $\widehat{I}_j$ and show that there exist C and $c > 0$ such that

$$\begin{aligned}\widehat{I}_j &= d_j^2\|F_t(0)\|_{1,\Omega_j'} + d_j\|F_t(0)\|_{\Omega_j'} + d_j\mu_j\|F(0)\|_{1,\Omega_j'} + \mu_j\|F(0)\|_{\Omega_j'}\\ &\le Ch^{-1-N/2}e^{-cd_j/h}.\end{aligned}$$

Let us consider only $\|F_t(0)\|_{1,\Omega_j'}$ which happens to be the hardest term to treat. Since $F_t(0) = A_h\tilde{\delta}_h - A\tilde{\delta} = A_h\tilde{\delta}_h$ on Ω_j' (if C_* is large enough), we obtain, using an inverse estimate (with a slight abuse of notation),

$$\|F_t(0)\|_{1,\Omega_j'} \le Ch^{-1}\|A_h\tilde{\delta}_h\|_{\Omega_j'} = Ch^{-1}\sup(A_h\tilde{\delta}_h, v),$$

where the supremum is taken over v with supp $v \subset \Omega_j'$ and $\|v\|_{\Omega_j'} = 1$. For each such v

$$(A_h\tilde{\delta}_h, v) = (A_h\tilde{\delta}_h, P_hv) = A(\tilde{\delta}_h, P_hv) = A(P_h\tilde{\delta}, P_hv).$$

Considering separately the contributions of the last term from Ω_j'' (which we define as $(\Omega_j')' = \Omega_{j-2}\cup\Omega_{j-1}\cup\Omega_j\cup\Omega_{j+1}\cup\Omega_{j+2}$) and $\mathcal{C}\Omega_j''$, we obtain by inverse estimates and decay properties of the L_2-projection, see Lemma 2.2, and A.4,

$$\begin{aligned}|(A_h\tilde{\delta}_h, v)| &\le Ch^{-2}\big(\|P_h\tilde{\delta}\|_{\Omega_j''}\,\|P_hv\| + \|P_h\tilde{\delta}\|_{L_1(\Omega)}\,\|P_hv\|_{L_\infty(\mathcal{C}\Omega_j'')}\big)\\ &\le Ch^{-2-N}d_j^{N/2}e^{-cd_j/h}.\end{aligned}$$

Thus

$$d_j^2\|F_t(0)\|_{1,\Omega_j'} \le Ch^{-3-N}d_j^{N/2+2}e^{-cd_j/h} \le Ch^{-1-N/2}e^{-cd_j/h}.$$

The remaining terms are treated analogously. The contribution to $\mathcal{K}$ of the $\widehat{I}_j$ is hence bounded by

$$\sum_j d_j^{1+N/2}\widehat{I}_j \le C\sum_j (d_j/h)^{1+N/2}e^{-cd_j/h} \le C. \tag{4.7}$$

We shall next consider the approximation terms in (4.6) and show

$$\widehat{X}_j \le C(\ell_{h,r}^{-1}h^{r-2}d_j^{1-N/2-r} + h^{r-1}d_j^{-N/2-r}).$$

$\widehat{X}_j$ consists of several terms, of which we shall exhibit but two. We first consider the "top order" term. By approximation theory and the Green's function estimates of Lemma 2.1,

$$d_j^3|||(I_h\Gamma - \Gamma)_{tt}|||_{1,Q_j'} \le Cd_j^3h^{r-1}|||\Gamma_{tt}|||_{r,Q_j''} \le Ch^{r-1}d_j^{-N/2-r}.$$

Next consider

$$\mu_j|||I_h\Gamma - \Gamma|||_{1,Q_j'} \le C\mu_jh^{r-1}|||\Gamma|||_{r,Q_j''} \le C\mu_jh^{r-1}d_j^{1-N/2-r},$$

which is bounded as desired since $\mu_j = h^{-1}\ell_{h,r}^{-1} + d_j^{-1}$. The rest of the terms appearing in $\widehat{X_j}$ are treated in a similar fashion. The contribution to $\mathcal{K}$ of the $\widehat{X_j}$ is thus bounded by (since C_* will eventually be fixed independently of h)

$$\sum_j d_j^{1+N/2}\widehat{X_j} \le C\ell_{h,r}^{-1}\sum_j (h/d_j)^{r-2} + C\sum_j (h/d_j)^{r-1} \le C. \tag{4.8}$$

Note that for $r=2$ the first sum on the right is compensated for by the factor ℓ_h^{-1}.

We next bound $\widehat{H_j}$. Replacing Q_j' by the full set Q_T, we have

$$\widehat{H_j} \le C(h/d_j)^q\big[d_j^2|||F_{tt}|||_{Q_T} + d_j|||F_t|||_{1,Q_T} + \mu_j(d_j|||F_t|||_{Q_T} + |||F|||_{1,Q_T})\big].$$

Writing $|F| \le |\Gamma_h| + |\Gamma|$, etc., and estimating the terms individually by standard energy arguments (cf. the proof of (4.3)), we have

$$h^3|||F_{tt}|||_{Q_T} + h^2|||F_t|||_{1,Q_T} + h|||F_t|||_{Q_T} + |||F|||_{1,Q_T} \le Ch^{-N/2},$$

so that taking $q = 4 + N/2$, since $h/d_j \le C$,

$$\widehat{H_j} \le C(h/d_j)^q\big[d_j^2h^{-3} + d_jh^{-2} + \mu_j(d_jh^{-1}+1)\big]h^{-N/2} \le Chd_j^{-2-N/2}.$$

The contribution to $\mathcal{K}$ of the $\widehat{H_j}$ is thus bounded by

$$\sum_j d_j^{1+N/2}\widehat{H_j} \le C\sum_j h/d_j \le C. \tag{4.9}$$

From (4.6), (4.7), (4.8), and (4.9) we hence obtain

$$\mathcal{K} = \sum_j d_j^{1+N/2}K_j \le C + C\sum_j d_j^{N/2}\mu_j|||F|||_{Q_j'}.$$

In the next lemma we estimate $|||F|||_{Q_j'}$ by a duality argument.

Lemma 4.2. *With $m_{ij} = (\min(d_i/d_j, d_j/d_i))^{1+N/2}$, we have*

$$|||F|||_{Q_j'} \le Ch^2d_j^{-1-N/2} + C\sum_{*,i} m_{ij}(h^2|||F_t|||_{Q_i} + h|||F|||_{1,Q_i}).$$

Proof. Writing $[v,w]$ for the L_2-space-time inner product over Q_T, we have

$$|||F|||_{Q_j'} = \sup\{[F,v]; \text{supp } v \subset Q_j',\ |||v|||_{Q_j'} = 1\}.$$

For each fixed such v, let w solve the dual problem

$$-w_t + Aw = v, \quad \text{with } w(T) = 0.$$

Integrating by parts and using the corresponding notation $A[\cdot,\cdot]$, we then have

$$[F,v] = (F(0),w(0)) + [F_t,w] + A[F,w]. \tag{4.10}$$

We consider the terms in order.

We have for any $\chi_0 \in S_h$ (and C_* large enough)

$$(F(0), w(0)) = (P_h\tilde{\delta} - \tilde{\delta}, w(0) - \chi_0)$$
$$= (P_h\tilde{\delta}, w(0) - \chi_0)_{\Omega_j''} + (P_h\tilde{\delta} - \tilde{\delta}, w(0) - \chi_0)_{\mathcal{C}\Omega_j''} \equiv I_1 + I_2.$$

Choosing $\chi_0 = I_h w(0)$, using the exponential decay properties of P_h, and the standard a priori energy estimate $\|w(0)\|_1 \leq C|||v|||_{Q_T} = C$,

$$|I_1| \leq C d_j^{N/2} h^{-N} e^{-cd_j/h} h \|w(0)\|_1 \leq C d_j^{N/2} h^{-N+1} e^{-cd_j/h}.$$

Since P_h is stable in L_1, $|I_2| \leq Ch^2 \|w(0)\|_{W^2_\infty(D_j)}$, where D_j is a set containing $\mathcal{C}\Omega_j''$ but whose parabolic distance to Ω_j' is greater than Cd_j. By the Green's function estimates, $\|w(0)\|_{W^2_\infty(D_j)} \leq C d_j^{-1-N/2}$, and thus

$$|(F(0), w(0))| \leq C d_j^{N/2} h^{-N+1} e^{-cd_j/h} + Ch^2 d_j^{-1-N/2} \leq Ch^2 d_j^{-1-N/2},$$

which bounds the first term in (4.10) as desired.

We now consider the remaining two terms on the right of (4.10). We have

$$[F_t, w] + A[F, w] = [F_t, w - I_h w] + A[F, w - I_h w] =$$
$$\leq C \sum_{*,i} (|||F_t|||_{Q_i} |||w - I_h w|||_{Q_i} + |||F|||_{1,Q_i} |||w - I_h w|||_{1,Q_i})$$
$$\leq C \sum_{*,i} (h^2 |||F_t|||_{Q_i} + h|||F|||_{1,Q_i}) |||w|||_{2,Q_i'}.$$

Recalling the definition of w, the Green's function estimates give

$$|||w|||_{2,Q_i'} \leq C(\min(d_i/d_j, d_j/d_i))^{1+N/2} = Cm_{ij},$$

for $|i - j| \geq 2$; this is also true when $|i - j| \leq 1$ by a standard a priori energy estimate, namely, $|||w|||_{2,Q_T} \leq C|||v|||_{Q_T} = C$. This proves the lemma. □

The contribution to $\mathcal{K}$ is then

$$\sum_j d_j^{N/2} \mu_j |||F|||_{Q_j'} \leq C \sum_j h d_j^{-1} + C \sum_j d_j^{N/2} \mu_j \sum_{*,i} m_{ij} (h^2 |||F_t|||_{Q_i} + h|||F|||_{1,Q_i}).$$

Clearly the first term is bounded and, after changing the order of summation and using elementary geometric sums, the second is bounded by

$$C \sum_{*,i} (|||F_t|||_{Q_i} h^2 + |||F|||_{1,Q_i} h) \Big(\sum_{j \leq i} + \sum_{j > i} \Big) d_j^{N/2} \mu_j m_{ij}$$
$$\leq C \sum_{*,i} (|||F_t|||_{Q_i} h^2 + |||F|||_{1,Q_i} h) d_i^{1+N/2} \mu_i d_i^{-1}$$
$$\leq C C_*^{N/2} + C \sum_i d_i^{1+N/2} (|||F_t|||_{Q_i} + \mu_i |||F|||_{1,Q_i})(h/d_i)$$
$$\leq C C_*^{N/2} + C \sum_i d_i^{1+N/2} K_i (h/d_i).$$

We hence find that

$$\mathcal{K} \leq C(1 + C_*^{N/2}) + C C_*^{-1} \mathcal{K}.$$

With C_* large enough this shows that $\mathcal{K}$ is bounded, and, together with (4.2) and (4.3), it shows that $\mathcal{M}$ is bounded. The proof of Proposition 3.1 is now complete. □

5. Strong superapproximation.

We shall now state and show the strong superapproximation result that we will need in our proof of Proposition 4.1 in Section 6.

Let $D_0 \subset \Omega$ and $D_d = \{x \in \Omega; \mathrm{dist}(x, D_0) \le d\}$. Let $\omega = \omega_d(x)$ be a smooth function depending on the parameter d with $\mathrm{supp}\,\omega \subset \bar{D}_0^\partial$ and with $|D^\alpha \omega_d| \le Cd^{-|\alpha|}$, $|\alpha| = 0, \dots, \bar{r}$. By the scaling assumption A.5 and the superapproximation assumption A.3 (applied with $d \approx 1$), if $\psi \in S_h$, the function $\omega_d\psi$ can be approximated by a $\chi \in S_h^0(\bar{D}_d^\partial)$ so that

$$(5.1) \quad d\|\omega_d\psi - \chi\|_{D_d} \le Ch\|\psi\|_{D_d}, \quad d^2\|\omega_d\psi - \chi\|_{1,D_d} \le Ch(d|\psi|_{1,D_d} + \|\psi\|_{D_d}).$$

The following theorem shows that a local elliptic projection of $\omega_d\psi$ may be approximated in $S_h^0(\bar{D}_d^\partial)$ to order $O(h)$ in $W_2^1(D_d)$-norm with an error bound requiring only the $L_2(D_d)$-norm of ψ, which is sharper than in (5.1). For unit-size interior domains such a result was shown in [29]. For $\mathcal{D} \subset \Omega$ we define $R_h^{\mathcal{D}} : W_2^1(\mathcal{D}) \to S_h(\mathcal{D})$ by

$$(5.2) \qquad A^{\mathcal{D}}(R_h^{\mathcal{D}} v - v, \eta) = 0, \quad \forall \eta \in S_h(\mathcal{D}),$$

where in $A^{\mathcal{D}}(\cdot,\cdot)$ the integration is extended only over $\mathcal{D}$.

Theorem 5.1. *There exist C and c independent of h and d such that, if $d \ge ch$, then for any $\psi \in S_h$ there exists $\chi \in S_h^0(\bar{D}_d^\partial)$ such that*

$$(5.3) \qquad d^2\|R_h^{D_d}(\omega_d\psi) - \chi\|_{1,D_d} + d\|\omega_d\psi - \chi\|_{D_d} \le Ch\|\psi\|_{D_d}.$$

We remark that, in terms of the global Ritz projection on Ω, we similarly have

$$d^2\|R_h(\omega_d\psi) - \chi\|_{1,\Omega} \le Ch\|\psi\|_{D_d},$$

as may be seen from the proof below.

Proof. A major point of the theorem is that the constant C does not depend on the particular shape or size of the set D_0. Following directly the proof in [29] for interior domains, the constants would, in particular, depend on the constant in the W_2^2 regularity estimate for the elliptic problem on D_d, on a bound in the extension inequality from $W_2^2(D_d)$ to $W_2^2(\Omega)$, and on the constant in Poincaré's inequality for D_d. Even though we shall only apply the theorem with D_0 equal to the intersection of Ω with a ball or an annulus with center in Ω, we see no way of directly controlling such constants; indeed, we do not even know if W_2^2 regularity holds on every such set.

In order to be able to control the constants we shall instead proceed in three steps as follows. In Step 1 we consider unit size sets, $d = 1$, of the following two types

(i) *D_0 and D_d are concentric balls with $D_d \subset \Omega$,*
(ii) *the boundary of $\partial\Omega$ is straight and D_0 and D_d are the intersections of Ω with two concentric balls centered on $\partial\Omega$.*

In Step 2 we then consider general sets of type (i) and also $\mathcal{C}^2$-diffeomorphic images of sets of type (ii) and apply an argument using a partition of unity to treat general sets of unit size. Finally, in Step 3, the case $0 < d < 1$ will be reduced to the case $d = 1$ by a scaling argument.

Step 1. With $d = 1$ we write $\omega = \omega_1$. For the purpose of the later scaling argument in Step 3, we introduce

$$A_\mu(v, w) = A_0(v, w) + \mu(a_0 v, w), \quad \text{with } 0 < \mu \le 1,$$

where A_0 is the principal part of $A = A_1$. Let $R^{D_1}_{\mu,h}$ denote the elliptic projection defined as in (5.2) by the form $A^{D_1}_\mu(\cdot,\cdot)$. We have supp $\omega \subset \bar{D}^\partial_0$ and introduce another cut-off function $\widetilde{\omega}$ such that $\widetilde{\omega} \equiv 1$ on $D_{0.7}$ with supp $\widetilde{\omega} \subset \bar{D}^\partial_{0.8}$. For given $\psi \in S_h$, and with I_h the local approximant described in A.2 of Section 2, we set $\chi = I_h(\widetilde{\omega} R^{D_1}_{\mu,h}(\omega\psi))$; we may assume supp $\chi \subset \bar{D}^\partial_{0.8}$. We shall show that, with C independent of μ and h,

$$(5.4) \qquad \|R^{D_1}_{\mu,h}(\omega\psi) - \chi\|_{1,D_1} + \|\omega\psi - \chi\|_{D_1} \le Ch\|\psi\|_{D_{0.3}}.$$

In particular, with $\mu = 1$, this shows (5.3) in the case considered. It also implies

$$(5.5) \qquad |A^{D_1}_\mu(\nu, \omega\psi - \chi)| \le Ch\Big(\|\nabla\nu\|_{D_1} + \mu\|\nu\|_{D_1}\Big)\|\psi\|_{D_{0.3}}, \quad \forall \nu \in S_h.$$

For later purposes we note that, conversely, if (5.5) holds, then $A^{D_1}_\mu(\nu, \omega\psi - \chi) = A^{D_1}_\mu(\nu, R^{D_1}_{\mu,h}(\omega\psi) - \chi)$, so that taking $\nu = R^{D_1}_{\mu,h}(\omega\psi) - \chi$ shows that (5.4) holds for $\mu = 1$.

In order to show (5.4) we shall need some preliminaries. We recall that $0 < \underline{a}_0 \le a_0 \le \overline{a}_0$ in $\bar{\Omega}$ and let α be the ellipticity constant of A_0 and $\kappa = \max_{ij} \|a_{ij}\|_{C^1(\bar{\Omega})}$.

Lemma 5.1. *Let $\mathcal{D} \subset \Omega$ be any convex domain or the image under a $\mathcal{C}^2$-diffeomorphism of a convex domain. Then there exists a unique solution $V \in W^2_2(\mathcal{D})$ of the problem*

$$(5.6) \qquad A^{\mathcal{D}}_\mu(V, \zeta) = (v, \zeta), \quad \forall\, \zeta \in W^1_2(\mathcal{D}).$$

Further, there exists a constant M which depends only on $\alpha, \kappa, \underline{a}_0, \overline{a}_0$, meas$(\mathcal{D})$, diam$(\mathcal{D})$, the Lipschitz constants in a suitable atlas for $\partial\mathcal{D}$, k and K of A.2, and the $\mathcal{C}^2$ diffeomorphism, such that

$$(5.7) \qquad \|\nabla(R^{\mathcal{D}}_{\mu,h}V - V)\|_{\mathcal{D}} + \mu^{1/2}\|R^{\mathcal{D}}_{\mu,h}V - V\|_{\mathcal{D}} \le Mh\|v\|_{\mathcal{D}}.$$

Proof. We assume first that $\mathcal{D}$ is convex. The existence of a unique $V \in W^1_2(\mathcal{D})$ satisfying (5.6) follows since A_μ is coercive, and, following Grisvard [12] with minor modifications, we shall show that $V \in W^2_2(\mathcal{D})$ and

$$(5.8) \qquad |V|_{2,\mathcal{D}} + \|\nabla V\|_{\mathcal{D}} + \mu\|V\|_{\mathcal{D}} \le M\|v\|_{\mathcal{D}}.$$

Setting $\zeta = V$ in (5.6), we find

$$\|\nabla V\|^2_{\mathcal{D}} + \mu\|V\|^2_{\mathcal{D}} \le MA^{\mathcal{D}}_\mu(V, V) = M(v, V) \le \tfrac{1}{2}\mu\|V\|^2_{\mathcal{D}} + \tfrac{1}{2}M^2\mu^{-1}\|v\|^2_{\mathcal{D}},$$

which shows $\mu\|V\|_{\mathcal{D}} \le M\|v\|_{\mathcal{D}}$. Next, taking $\zeta = 1$ in (5.6), $(v - \mu a_0 V, 1) = 0$, so that, with $\overline{V}$ the mean-value of V over $\mathcal{D}$,

$$A_0(V,V) = (v - \mu a_0 V, V - \overline{V}). \tag{5.9}$$

From Gilbarg and Trudinger [11, (7.45), p. 164] we quote the Poincaré inequality

$$\|V - \overline{V}\|_{\mathcal{D}} \le \Big(\frac{\omega_N}{\operatorname{meas}\mathcal{D}}\Big)^{1-1/N} (\operatorname{diam}\mathcal{D})^N \|\nabla V\|_{\mathcal{D}}, \tag{5.10}$$

valid for any convex domain with $\omega_N = 2\pi^{N/2}/(N\Gamma(N/2))$. Thus from (5.9), with a new M, $\|\nabla V\|_{\mathcal{D}} \le M(\|v\|_{\mathcal{D}} + \mu\|V\|_{\mathcal{D}})$, so that also $\|\nabla V\|_{\mathcal{D}}$ is bounded as claimed in (5.8).

We next turn to $|V|_{2,\mathcal{D}}$. Approximating $\mathcal{D}$ by $\mathcal{C}^2$-domains and using a limiting argument as in [12, proof of Theorem 3.2.1.3], it suffices to consider convex $\mathcal{C}^2$-domains $\mathcal{D}$. From [12, equation (3,1,3,9)] we have

$$\alpha^2 |V|^2_{2,\mathcal{D}} \le 2\|A_0 V\|^2_{\mathcal{D}} + 4N^2\kappa^4\alpha^{-2}\|\nabla V\|^2_{\mathcal{D}},$$

and, since $\|A_0 V\|_{\mathcal{D}} \le \|A_\mu V\|_{\mathcal{D}} + \mu\overline{a}_0\|V\|_{\mathcal{D}}$, this completes the proof of (5.8) for $\mathcal{D}$ convex. The extension to smooth diffeomorphic mappings of such domains is obvious.

We now consider (5.7). Since $R^{\mathcal{D}}_{\mu,h}V$ is the best approximation to V in the energy norm over $\mathcal{D}$ associated with $A^{\mathcal{D}}_\mu$, and since $\mu \le 1$, we have

$$\|\nabla(R^{\mathcal{D}}_{\mu,h}V - V)\|_{\mathcal{D}} + \mu^{1/2}\|R^{\mathcal{D}}_{\mu,h}V - V\|_{\mathcal{D}} \le M\|V - \chi\|_{1,\mathcal{D}}, \quad \forall \chi \in S_h(\mathcal{D}).$$

Since the constant functions belong to $S_h(\mathcal{D})$ (A.2 (ii)),

$$\min_{\chi\in S_h(\mathcal{D})} \|V - \chi\|_{1,\mathcal{D}} = \min_{\chi\in S_h(\mathcal{D})} \|V - \overline{V} - \chi\|_{1,\mathcal{D}},$$

where again $\overline{V}$ denotes the mean-value of V over $\mathcal{D}$. Extending $V - \overline{V}$ continuously in W^2_2 from $\mathcal{D}$ to Ω (this is where the Lipschitz nature of $\partial\mathcal{D}$ enters, cf. Stein [28, Theorem 5, p. 181]), this is bounded using A.2, Poincaré's inequality (see (5.10)), and (5.8), by

$$Mh\|V - \overline{V}\|_{2,\mathcal{D}} \le Mh\Big(|V|_{2,\mathcal{D}} + \|\nabla V\|_{\mathcal{D}} + \|V - \overline{V}\|_{\mathcal{D}}\Big) \le Mh\|v\|_{\mathcal{D}}.$$

This shows the lemma in the convex case.

For the diffeomorphically mapped case, one only has to take care to let the constant $\overline{V}$ be the mean-value of the corresponding function V when mapped to the convex domain. □

Note that Lemma 5.1 applies to domains of the type considered in Step 1, since their measures, diameters, and Lipschitz constants are appropriately uniformly bounded.

Our next lemma contains some auxiliary estimates for $R^{D_1}_{\mu,h}$.

Lemma 5.2. *Let $\widetilde{\psi} = R^{D_1}_{\mu,h}(\omega\psi)$. We then have, with C independent of μ and h,*

$$\|\widetilde{\psi} - \omega\psi\|_{D_1} \leq Ch\|\psi\|_{D_{0.3}} \tag{5.11}$$

and

$$\|\widetilde{\psi}\|_{1,D_1\setminus D_{0.5}} \leq C\|\widetilde{\psi}\|_{D_1\setminus D_{0.3}}. \tag{5.12}$$

Proof. The inequality (5.11) follows using a duality argument and superapproximation. In fact, for v with support in D_1 and $\|v\|_{D_1} = 1$, let V be as in Lemma 5.1. We have for suitable choice of $\eta \in S_h(D_1)$

$$\begin{aligned}(\omega\psi - \widetilde{\psi}, v) &= A^{D_1}_\mu(\omega\psi - R^{D_1}_{\mu,h}(\omega\psi), V - R^{D_1}_{\mu,h}V) = A^{D_1}_\mu(\omega\psi - \eta, V - R^{D_1}_{\mu,h}V)\\ &\leq C\|\omega\psi - \eta\|_{1,D_1} h\|v\|_{D_1} \leq Ch^2\|\psi\|_{1,D_{0.2}} \leq Ch\|\psi\|_{D_{0.3}}.\end{aligned}$$

Here we have used in turn (5.7), the superapproximation property A.3 (for the suitable choice of η), and an inverse property.

In order to prove (5.12), consider $\eta \in S_h(D_1)$ with $\eta \equiv 0$ on $\bar{D}_0 \supseteq \operatorname{supp}\omega$. Then

$$A^{D_1}_\mu(\widetilde{\psi}, \eta) = A^{D_1}_\mu(\omega\psi, \eta) = 0, \tag{5.13}$$

i.e., $\widetilde{\psi}$ is "discrete A_μ–harmonic" on $D_1\setminus D_0$. Inequalities of the type (5.12) were first proved for $\widetilde{\psi}$ satisfying (5.13) on interior subdomains in [16] and up to boundaries in [24, Lemma 4.4]. For completeness we include a proof.

Let $\varphi = 1$ on $D_1 \setminus D_{0.5}$ with supp $\varphi \subset \bar{D}_1 \setminus D_{0.4}$. A straightforward calculation using (5.13) yields

$$\begin{aligned}\|\varphi\nabla\widetilde{\psi}\|^2_{D_1} &\leq C\int_{D_1}\Big(\sum_{i,j=1}^N \varphi^2 a_{ij}\frac{\partial\tilde{\psi}}{\partial x_i}\frac{\partial\tilde{\psi}}{\partial x_j} + \mu a_0\varphi^2\tilde{\psi}^2\Big)\,dx\\ &= C(A^{D_1}_\mu(\widetilde{\psi}, \varphi^2\widetilde{\psi}) + W) = C(A^{D_1}_\mu(\widetilde{\psi}, \varphi^2\widetilde{\psi} - \eta) + W),\end{aligned} \tag{5.14}$$

for all $\eta \in S_h(D)$ with $\eta = 0$ on $D_{0.4}$ ($\supseteq D_0$). Here by superapproximation with η suitable

$$|A^{D_1}_\mu(\tilde{\psi}, \varphi^2\tilde{\psi} - \eta)| \leq Ch\|\tilde{\psi}\|^2_{1,D_1\setminus D_{0.3}},$$

and, for the commutator W, we have

$$\begin{aligned}|W| &= \Big|\int_{D_{0.5}\setminus D_{0.4}} \sum_{i,j=1}^N a_{ij}\frac{\partial\tilde{\psi}}{\partial x_i}(2\varphi\frac{\partial\varphi}{\partial x_j})\tilde{\psi}\,dx\Big|\\ &\leq C\|\varphi\nabla\widetilde{\psi}\|_{D_1}\|\widetilde{\psi}\|_{D_{0.5}\setminus D_{0.4}} \leq \epsilon\|\varphi\nabla\widetilde{\psi}\|^2_{D_1} + C_\epsilon\|\widetilde{\psi}\|^2_{D_{0.5}\setminus D_{0.4}}.\end{aligned}$$

Using the above and a kick-back argument in (5.14), we find

$$\|\tilde{\psi}\|^2_{1,D_1\setminus D_{0.5}} \leq \|\varphi\nabla\widetilde{\psi}\|^2_{D_1} + \|\varphi\tilde{\psi}\|^2_{D_1} \leq C\big(h\|\tilde{\psi}\|^2_{1,D_1\setminus D_{0.3}} + \|\widetilde{\psi}\|^2_{D_1\setminus D_{0.3}}\big).$$

Iterating this argument with an obvious change in sets we obtain

$$\|\widetilde{\psi}\|^2_{1,D_1\setminus D_{0.6}} \le C\big(h^2\|\widetilde{\psi}\|^2_{1,D_1\setminus D_{0.3}} + \|\widetilde{\psi}\|^2_{D_1\setminus D_{0.3}}\big),$$

and an inverse inequality completes the proof of (5.12). □

We are now in a position to prove (5.4) and complete Step 1. Recalling that $\chi = I_h(\tilde{\omega}\tilde{\psi})$, we write

$$\widetilde{\psi} - \chi = \big[\widetilde{\omega}\widetilde{\psi} - I_h(\widetilde{\omega}\widetilde{\psi})\big] + \big[(1-\widetilde{\omega})\widetilde{\psi}\big] = I_1 + I_2.$$

Since $\tilde{\omega} = 1$ on $D_{0.7}$, by A.2, $I_h(\tilde{\omega}\tilde{\psi}) = \tilde{\psi}$ on $D_{0.6}$. Then from A.2 and (5.12)

$$\|\widetilde{\psi} - \chi\|_{1,D_1} \le \|I_1\|_{1,D_1} + \|I_2\|_{1,D_1} \le C\|\widetilde{\psi}\|_{1,D_1\setminus D_{0.5}} \le C\|\widetilde{\psi}\|_{D_1\setminus D_{0.3}}.$$

Since supp $\omega \subset \bar{D}_0^{\partial}$, we have, using this and (5.11),

$$\|\widetilde{\psi} - \chi\|_{1,D_1} \le C\|\widetilde{\psi} - \omega\psi\|_{D_1} \le Ch\|\psi\|_{D_{0.3}}. \tag{5.15}$$

Furthermore, using (5.11) and (5.15),

$$\|\omega\psi - \chi\|_{D_1} \le \|\omega\psi - \widetilde{\psi}\|_{D_1} + \|\widetilde{\psi} - \chi\|_{D_1} \le Ch\|\psi\|_{D_{0.3}}. \tag{5.16}$$

Taken together (5.15) and (5.16) prove (5.4) and hence complete the proof of Theorem 5.1 in the case $d = 1$ and (i), (ii).

Step 2. Let $\widehat{B}_0$ and $\widehat{B}_1$ be concentric balls in R^N with centers at the origin, and let $\widehat{D}_0 = \widehat{B} \cap \{y_N \ge 0\}, \widehat{D}_1 = \widehat{B}_1 \cap \{y_N \ge 0\}$. Let $\overline{x}$ be any point on $\partial\Omega$. Taking smaller balls, if necessary, there is a $\mathcal{C}^2$-diffeomorphism $\Phi : \widehat{B}_1 \to D_1 \subset \Omega$ in which the equatorial plane maps to a (piece of) $\partial\Omega$ containing $\overline{x} = \Phi(0)$. With $D_0 = \Phi(\widehat{D}_0)$ and $D_1 = \Phi(\widehat{D}_1)$ it is clear that the whole development in Step 1 holds for D_0 and D_1 but, of course, now with constants depending on Φ (and its inverse).

A general set $D_0 \subset \bar{D}_1^{\partial}$ can be covered by concentric balls, half-balls or maps of such as above, let us call them here $\mathcal{O}_{0,i}, \mathcal{O}_{1,i}, i = 1, \dots, I$, in such a way that $D_0 \subset \cup_{i=1}^I \mathcal{O}_{0,i}$ and $\cup_{i=1}^I \mathcal{O}_{1,i} \subset D_1$. Using a smooth partition of unity $\lambda_i, i = 1, \dots, I$, of D_0 subordinate to $\mathcal{O}_{1,i}$, and letting $\chi_i \in S_h^0(\bar{\mathcal{O}}_{1,i}^{\partial})$ denote what we have found above on $\mathcal{O}_{0,i}, \mathcal{O}_{1,i}$ for the functions $\lambda_i\omega\psi$, we set $\chi := \sum_{i=1}^I \chi_i$. Then

$$A_\mu^{D_1}(\nu, \omega\psi - \chi) = \sum_i A_\mu^{\mathcal{O}_{1,i}}(\nu, \lambda_i\omega\psi - \chi_i)$$

$$\le Ch\sum_i (\|\nabla\nu\|_{\mathcal{O}_{1,i}} + \mu\|\nu\|_{\mathcal{O}_{1,i}})\|\psi\|_{\mathcal{O}_{1,i}} \le Ch(\|\nabla\nu\|_{D_1} + \mu\|\nu\|_{D_1})\|\psi\|_{D_1},$$

and so (5.5) follows in general. The argument for the second term in (5.3) is analogous. It is not hard to see that the construction of the $\mathcal{O}_{0,i}, \mathcal{O}_{1,i}$, and the partitions of unity λ_i can be arranged so that the resulting constants are uniform for $\frac{1}{2} \le d \le 1$, say.

Step 3. We finally consider the case $d < 1$ which we shall reduce to the case $d \approx 1$ by a scaling argument. Using a suitable partition of unity (now on size d) as in Step 2, it suffices to consider the situation (i) or (ii) (or, (ii) mapped) now involving two domains D_0 and D_d of diameters d and $2d$, respectively, and centered at $\bar{x}$. Let $\widehat{D}_0$ and $\widehat{D}_1$ be the images under the scaling map $\widehat{x} - \bar{x} = (x - \bar{x})/d$. Then, setting $E_1 = \|\omega_d \psi - \chi\|$ and $E_2 = A(\nu, \omega_d \psi - \chi)$, we have for their scaled counterparts

$$\widehat{E}_1 = \|\widehat{\omega}_d \widehat{\psi} - \widehat{\chi}\| = d^{-N/2} E_1 \quad \text{and} \quad \widehat{E}_2 = \widehat{A}_{d^2}(\widehat{\nu}, \widehat{\omega}_d \widehat{\psi} - \widehat{\chi}) = d^{2-N} E_2.$$

Here $\widehat{A}_{d^2}(\nu, \omega) = \widehat{A}_0(\nu, \omega) + d^2(\widehat{a}_0 \nu, \omega)$, $\widehat{a}_0(\widehat{x}) = a_0(x)$ and $\widehat{a}_{ij}(\widehat{x}) = a_{ij}(x)$, and the integrations are taken over $\widehat{D}_1$. Note that the maximum norms of the coefficients and their derivatives are not increased under this transformation and that any finite number of derivatives of $\widehat{\omega}_d$ are bounded independent of d. Also, $\partial \hat{D}_1$ becomes smoother, so that the Lipschitz constants of $\partial \hat{D}_1$ involved in extension operators are controlled. Using the scaling hypothesis A.5 with $\widehat{h} = hd^{-1}$ and taking $\mu = d^2$, we may apply the results in (5.3) for unit sized sets to the transformed problem to obtain

$$|\widehat{E}_1| \leq C\widehat{h}\|\widehat{\psi}\|_{\widehat{D}_1} \quad \text{and} \quad |\widehat{E}_2| \leq C\widehat{h}\Big(\|\nabla \widehat{\nu}\|_{\widehat{D}_1} + d^2\|\widehat{\nu}\|_{\widehat{D}_1}\Big)\|\widehat{\psi}\|_{\widehat{D}_1}.$$

Transforming back to the original variables

$$|E_1| = d^{N/2}|\widehat{E}_1| \leq Chd^{-1}\|\psi\|_{D_d}$$

and

$$\begin{aligned} |E_2| = d^{N-2}|\widehat{E}_2| &\leq Ch\Big(d^{-2}\|\nabla \nu\|_{D_d} + d^{-1}\|\nu\|_{D_d}\Big)\|\psi\|_{D_d} \\ &\leq Chd^{-2}\|\nu\|_{1,D_d}\|\psi\|_{D_d}. \end{aligned}$$

After an argument as that giving (5.4) from (5.5), this completes the proof. □

6. Local energy error estimates in parabolic problems.

In this section we shall prove Proposition 4.1. As in the previous section the proof follows the ideas of [29], where interior unit-size domains were treated.

Lemma 6.1. *Let $D \subset \Omega, I = [\tau, T]$ with $\tau \geq 0$, and consider the space-time cylinder $Q = D \times I$. Let $D_d = \{x \in \Omega;\, dist\,(x, D) \leq d\}$, $I_{d^2} = [\max(0, \tau - d^2), T]$, and $Q_d = D_d \times I_{d^2}$. Let $e(t) = z_h(t) - z(t)$ with $z_h \in S_h$ satisfy*

(6.1) $\quad (e_t, \chi) + A(e, \chi) = 0,\ \forall \chi \in S_h^0(\bar{D}_{2d}^{\partial}),\ t > 0,\ z(0) = 0,\ z_h(0) = z_{0h}$ *in* D_d.

For any $q > 0$ there exist C and $c > 0$ such that if $d \geq ch$ then, with $\delta_d = 1$ if $d^2 > \tau, \delta_d = 0$ otherwise

$$\begin{aligned} |||e_t|||_Q + d^{-1}|||e|||_{1,Q} \leq C\big[&\delta_d(\|z_{0h}\|_{1,D_d} + d^{-1}\|z_{0h}\|_{D_d}) \\ &+ d|||z_t|||_{1,Q_d} + |||z_t|||_{Q_d} + d^{-1}|||z|||_{1,Q_d} + d^{-2}|||z|||_{Q_d} \\ &+ (h/d)^q(|||e_t|||_{Q_d} + d^{-1}|||e|||_{1,Q_d}) + d^{-2}|||e|||_{Q_d}\big]. \end{aligned}$$

Proof of Proposition 4.1. Since the sets Q_j in Proposition 4.1 are built up of the two cylinders $\Omega_j \times [0, d_j^2]$ and $\{x \in \Omega; |x - x_0| \le 2d_j\} \times [d_j^2, 4d_j^2]$, and since $z(0) = 0$ on Ω_j', (4.5) follows with $\zeta = -z$ from this lemma, with $\tau = 0$ for the first cylinder and $\tau = d_j$ for the second and with $d = d_j/2$ for both. Writing $z_h - z = (z_h - I_h z) + (I_h z - z)$ establishes it as stated. □

Proof of Lemma 6.1. Let now $\omega(x,t) = \omega_1(x)\omega_2(t)$ be a cut-off function such that $\omega_1(x) = 1$ on D, supp $\omega_1 \subset \bar{D}_d^\partial$, and $\|\omega_1^{(l)}\|_{L_\infty} \le Cd^{-l}$, for $l = 0, 1$, and such that $\omega_2(t) = 1$ on I, supp $\omega_2 \subset \bar{I}_{d^2}^\partial$ (in the natural sense), while $\|\omega_2^{(l)}\|_{L_\infty} \le Cd^{-2l}$ for $l = 0, 1$. For brevity of notation we define, for a function $g = g(x,t)$,

$$A_g(v,w) = \int_\Omega g^2 \Big(\sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + a_0 v w \Big)\, dx,$$

and

$$\|v\|_{g,1} = A_g(v,v)^{1/2}, \quad |||v|||_{g,1} = \Big(\int_0^T A_g(v,v)\, dt \Big)^{1/2}.$$

We shall first consider the case when $z \equiv 0$ in (6.1), i.e., z_h is "discrete parabolic". We have for any $\chi(t) \in S_h^0(\bar{D}_{2d}^\partial)$

$$\begin{aligned} \tfrac{1}{2}\frac{d}{dt}\|\omega z_h\|^2 + A_\omega(z_h, z_h) &= \big[(z_{h,t}, \omega^2 z_h - \chi) + A^{D_{2d}}(z_h, R_h^{D_{2d}}(\omega^2 z_h) - \chi) \big] \\ &\quad + (\omega\omega_t z_h, z_h) + \big[A_\omega(z_h, z_h) - A(z_h, \omega^2 z_h) \big] \equiv I_1 + I_2 + I_3. \end{aligned}$$

To treat I_1 we shall use the strong superapproximation result of Theorem 5.1, which allows us to choose $\chi(t)$ in $S_h^0(\bar{D}_{2d}^\partial)$ such that

$$|I_1| \le C\big(\|z_{h,t}\|_{D_{2d}} h d^{-1} \|z_h\|_{D_{2d}} + \|z_h\|_{1,D_{2d}} h d^{-2} \|z_h\|_{D_{2d}} \big).$$

Further $|I_2| \le Cd^{-2}\|z_h\|_{D_{2d}}^2$, and, for the commutator I_3,

$$|I_3| = \Big| \int_\Omega 2\omega \Big(\sum_{i,j=1}^N a_{ij} \frac{\partial z_h}{\partial x_i} \frac{\partial \omega}{\partial x_j} \Big) z_h\, dx \Big| \le Cd^{-1} \|z_h\|_{\omega,1} \|z_h\|_{D_{2d}}.$$

Using the arithmetic-geometric mean inequality, integrating in time and taking square roots, and noting that $\omega_2(0) = 0$ if $\tau > d^2$, we hence find

$$(6.2) \quad |||z_h|||_{\omega,1} \le C\big[\delta_d \|z_{0h}\|_{D_{2d}} + h(|||z_{h,t}|||_{Q_{2d}} + d^{-1}|||z_h|||_{1,Q_{2d}}) + d^{-1}|||z_h|||_{Q_{2d}} \big].$$

We remark that (6.2) can be proved without use of strong superapproximation.

Similarly, again using strong superapproximation, and in this case the use of it is essential,

$$\begin{aligned} \|\omega^2 z_{h,t}\|^2 + \tfrac{1}{2}\frac{d}{dt} A_{\omega^2}(z_h, z_h) &= \big[(z_{h,t}, \omega^4 z_{h,t} - \chi) + A^{D_{2d}}(z_h, R_h^{D_{2d}}(\omega^4 z_{h,t}) - \chi) \big] \\ + \int_\Omega 2\omega^3 \omega_t \Big(\sum_{i,j=1}^N a_{ij} \frac{\partial z_h}{\partial x_i} \frac{\partial z_h}{\partial x_j} + a_0 z_h^2 \Big)\, dx &+ \int_\Omega 4\omega^3 \sum_{i,j=1}^N a_{ij} \frac{\partial z_h}{\partial x_i} \frac{\partial \omega}{\partial x_i} z_{h,t}\, dx \\ \le C\big(h d^{-1} \|z_{h,t}\|_{D_{2d}}^2 + h d^{-2} \|z_h\|_{1,D_{2d}} \|z_{h,t}\|_{D_{2d}} &+ d^{-2}\|z_h\|_{\omega,1}^2 \big) + \tfrac{1}{2}\|\omega^2 z_{h,t}\|^2, \end{aligned}$$

so that

$$\text{(6.3)} \qquad |||z_{h,t}|||_Q \le \delta_d \|z_{0h}\|_{1,D_{2d}} + Cd^{-1}|||z_h|||_{\omega,1} + C(h/d)^{1/2}(|||z_{h,t}|||_{Q_{2d}} + d^{-1}|||z_h|||_{1,Q_{2d}}).$$

Multiplying (6.2) by d^{-1} and adding a sufficiently small multiple of (6.3) so that the term $d^{-1}|||z_h|||_{\omega,1}$ may be kicked back and then, iterating with respect to the quantity $|||z_{h,t}|||_{Q_{2d}} + d^{-1}|||z_h|||_{1,Q_{2d}}$ (and changing notation for the Q_{jd} sets), we obtain

$$\text{(6.4)} \qquad |||z_{h,t}|||_Q + d^{-1}|||z_h|||_{1,Q} \le C\big[\delta_d(\|z_{0h}\|_{1,D_{2d}} + d^{-1}\|z_{0h}\|_{D_{2d}}) + (h/d)^q(|||z_{h,t}|||_{Q_{2d}} + d^{-1}|||z_h|||_{1,Q_{2d}}) + d^{-2}|||z_h|||_{Q_{2d}}\big].$$

This is Lemma 6.1 with $z \equiv 0$ (except for Q_d being Q_{2d}).

We now take another cut-off function $\tilde{\omega} \equiv 1$ on Q_{4d} with support in Q_{6d}, set $\tilde{z} = \tilde{\omega} z$, and let $\tilde{z}_h$ be given by (with $\tilde{e} = \tilde{z}_h - \tilde{z}$),

$$\text{(6.5)} \qquad (\tilde{e}_t, \chi) + A(\tilde{e}, \chi) = 0, \quad \forall \chi \in S_h,\ t > 0, \quad \text{with } \tilde{z}_h(0) = \tilde{z}(0) = 0.$$

Standard parabolic global energy arguments give, with $|||\cdot||| = |||\cdot|||_{Q_T}$,

$$|||\tilde{e}|||_1^2 \le C|||\tilde{e}_t|||\,|||\tilde{z}||| + C|||\tilde{z}|||_1^2, \quad \text{and} \quad |||\tilde{e}_t|||^2 \le |||\tilde{z}_t|||^2 + C|||\tilde{e}|||_1\,|||\tilde{z}_t|||_1,$$

from which we conclude, using the arithmetic-geometric mean inequality,

$$\text{(6.6)} \qquad \begin{aligned} |||\tilde{e}_t||| + d^{-1}|||\tilde{e}|||_1 &\le C(d|||\tilde{z}_t|||_1 + |||\tilde{z}_t||| + d^{-1}|||\tilde{z}|||_1 + d^{-2}|||\tilde{z}|||) \\ &\le C(d|||z_t|||_{1,Q_{6d}} + |||z_t|||_{Q_{6d}} + d^{-1}|||z|||_{1,Q_{6d}} + d^{-2}|||z|||_{Q_{6d}}) \equiv K_d(z). \end{aligned}$$

On Q_{6d} we write $e = z_h - z = Z_h + \tilde{e}$ where $Z_h = z_h - \tilde{z}_h \in S_h$ satisfies

$$(Z_{h,t}, \chi) + A(Z_h, \chi) = 0, \quad \forall \chi \in S_h^0(\bar{D}_{4d}^{\partial}).$$

Using (6.4) with $z_h = Z_h = e - \tilde{e}$, the triangle inequality, and (6.6), we have

$$\begin{aligned} |||Z_{h,t}|||_Q + d^{-1}|||Z_h|||_{1,Q} &\le C\big[\delta_d(\|z_{0h}\|_{1,D_{6d}} + d^{-1}\|z_{0h}\|_{D_{6d}}) \\ &\quad + (h/d)^q(|||Z_{h,t}|||_{Q_{6d}} + d^{-1}|||Z_h|||_{1,Q_{6d}}) + d^{-2}|||Z_h|||_{Q_{6d}}\big] \\ &\le C\big[\delta_d(\|z_{0h}\|_{1,D_{6d}} + d^{-1}\|z_{0h}\|_{D_{6d}}) + (h/d)^q(|||\tilde{e}_t||| + d^{-1}|||\tilde{e}|||_1) \\ &\quad + (h/d)^q(|||e_t|||_{Q_{6d}} + d^{-1}|||e|||_{1,Q_{6d}}) + d^{-2}(|||e|||_{Q_{6d}} + |||\tilde{e}|||)\big] \\ &\le C\big[\delta_d(\|z_{0h}\|_{1,D_{6d}} + d^{-1}\|z_{0h}\|_{D_{6d}}) + K_d(z) \\ &\quad + (h/d)^q(|||e_t|||_{Q_{6d}} + d^{-1}|||e|||_{1,Q_{6d}}) + d^{-2}|||e|||_{Q_{6d}} + d^{-2}|||\tilde{e}|||\big]. \end{aligned}$$

Since $e = Z_h + \tilde{e}$, and again in view of the triangle inequality and (6.6), it now remains only to estimate the last term $d^{-2}|||\tilde{e}|||$ above. Introducing the symmetric positive semidefinite operator $T_h = A_h^{-1}P_h$, we may write (6.5) as (cf., e.g., [30, p.30])

$$T_h\tilde{e}_t + \tilde{e} = (R_h - I)\tilde{z}, \quad t > 0, \quad \text{with } \tilde{e}(0) = 0.$$

Taking inner products with $\tilde{e}$ and integrating in time, we obtain

$$|||\tilde{e}||| \le |||(R_h - I)\tilde{z}||| \le Ch|||\tilde{z}|||_1 \le Cd|||z|||_{1,Q_{6d}},$$

which completes the proof of Lemma 6.1. □

7. Sharpness of the space-time stability estimate for $r = 2$.

In this final section we shall show that the logarithmic factor ℓ_h is needed in Theorem 2.2 for $r = 2$. We consider the plane case, $N = 2$, and shall base our analysis on an example given by Haverkamp [13] in the elliptic case.

We consider functions u which do not depend on t and their piecewise linear semidiscrete approximations $u_h(\cdot, t) \in S_h$ given by

$$(7.1) \qquad (u_{h,t}, \chi) + A(u_h - u, \chi) = 0, \quad \forall \chi \in S_h, \quad \text{for } t > 0.$$

For initial conditions we take (as an example) $u_h(0) = P_h u$; note that there is then no logarithmic factor present at initial time.

We shall show that there are positive c and T such that, for $u = u(x; h)$ suitably chosen,

$$(7.2) \qquad \|u_h(T)\|_{L_\infty} \geq c\ell_h \|u\|_{L_\infty}, \quad \text{with } \ell_h = \log(1/h),$$

and hence $\|u_h\|_{L_\infty(Q_T)} \geq c\ell_h \|u\|_{L_\infty(Q_T)}$, which would show our claim.

We shall first show that, as time increases, $u_h(t)$ approaches the Ritz projection $R_h u$. Indeed, from (7.1), where we may replace u inside $A(\cdot, \cdot)$ by $R_h u$, we have

$$u_{h,t} + A_h u_h = A_h R_h u, \quad \text{for } t > 0, \quad \text{with } u_h(0) = P_h u.$$

By Duhamel's principle

$$u_h(t) = E_h(t) P_h u + \int_0^t E_h(t-s)\, A_h R_h u \, ds = E_h(t) P_h u + (1 - E_h(t)) R_h u.$$

Thus $u_h(t) - R_h u = E_h(t)(P_h u - R_h u)$. By Lemma 3.2, $\|E_h(t)\|_{L_\infty} \leq C t^{-m} e^{-\lambda_1 t/2}$ so that

$$\|u_h(t) - R_h u\|_{L_\infty} \leq C t^{-m} e^{-\lambda_1 t/2} \|P_h u - R_h u\|_{L_\infty} \leq \tfrac{1}{2} \|P_h u - R_h u\|_{L_\infty},$$

if t is large enough. Hence, using the stability of P_h in the maximum-norm,

$$\|u_h(t)\|_{L_\infty} \geq \tfrac{1}{2} \|R_h u\|_{L_\infty} - C\|u\|_{L_\infty}, \quad \text{for } t \geq t_0 > 0.$$

To show (7.2) it therefore now suffices to exhibit a family of functions $u(x) = u(x; h)$ such that

$$(7.3) \qquad \|R_h u(\cdot; h)\|_{L_\infty} \geq c\ell_h \|u(\cdot; h)\|_{L_\infty};$$

this is where the elliptic example of [13] enters. Let $S = (0,1) \times (0,1)$, with the Laplace operator with homogeneous Dirichlet boundary conditions. Subdivide S into triangles via a four-directional mesh (x-direction, y-direction, and $\pm 45^o$ directions), and take piecewise linear functions which obey the essential Dirichlet boundary conditions. In [13] twice continuously differentiable functions $u = u(x; h)$ were constructed such that, with $\bar{x} = (\frac{1}{2}, \frac{1}{2})$,

$$(7.4) \qquad |(R_h^0 u)(\bar{x})| \geq \beta \ell_h \|u\|_{L_\infty}.$$

Here R_h^0 is the Ritz projection for the Laplacian with homogeneous Dirichlet conditions on ∂S, $\beta > 0$ is independent of h, and the fucntions u further satisfy $D^\alpha u|_{\partial S} = 0$, for $|\alpha| \le 2$, and

$$\|u\|_{W^1_\infty(S)} \le Ch^{-1}\|u\|_{L_\infty(S)}. \tag{7.5}$$

(In fact, the Ritz projection $R_h^0 u$ of the highly oscillatory function $u(x;h)$ is essentially the discrete Green's function associated with $\bar{x}$.)

It now remains to adapt this example to our coercive case with natural boundary conditions, with $\partial\Omega$ smooth. Let then Ω be a plane domain with smooth boundary and $\bar{S} \subset \Omega$. The triangulations of Ω are such that they reduce to those described above on S. We now extend the functions $u(x;h)$ by zero to Ω, they remain $\mathcal{C}^2$ functions, and take $A = -\Delta + 1$, with natural Neumann boundary conditions $\partial u/\partial n = 0$ on $\partial\Omega$. Let R_h be the Ritz projection with respect to A. Then, with the "discrete error" $\epsilon = R_h u - R_h^0 u$,

$$\int_\Omega \nabla\epsilon \cdot \nabla\chi \, dx = \int_\Omega (u - R_h u)\chi \, dx, \quad \forall \chi \in S_h^0(S). \tag{7.6}$$

From (7.4) we have

$$|R_h u(\bar{x})| \ge \beta \ell_h \|u\|_{L_\infty(S)} - |\epsilon(\bar{x})|. \tag{7.7}$$

Setting $S_0 = [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]$, and using in turn the discrete Sobolev inequality in two space dimensions (cf. [23, Lemma 1.1]), the interior estimates of Nitsche-Schatz [16] applied to (7.6), a standard duality estimate, and (7.5), we find that

$$\begin{aligned} |\epsilon(\bar{x})| &\le C\ell_h^{1/2}\|\epsilon\|_{1,S_0} \le C\ell_h^{1/2}(\|\epsilon\|_S + \|u - R_h u\|_S) \\ &\le Ch\ell_h^{1/2}(\|R_h u - u\|_{1,\Omega} + \|R_h^0 u - u\|_{1,S}) \le Ch\ell_h^{1/2}\|u\|_{W^1_\infty(S)} \le C\ell_h^{1/2}\|u\|_{L_\infty(S)}. \end{aligned}$$

Thus from (7.7)

$$|(R_h u)(\bar{x})| \ge (\beta\ell_h - C\ell_h^{1/2})\|u\|_{L_\infty(\Omega)} \ge \tfrac{1}{2}\beta\ell_h\|u\|_{L_\infty(\Omega)},$$

for h small enough. Hence (7.3) follows and our claim is shown.

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