On wave equations with supercritical nonlinearities

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Abstract

We prove that the solution operators $\mathcal{E}_t(\phi, \psi)$ for the nonlinear wave equations with supercritical nonlinearities are not Lipschitz mappings from a subset of the finite-energy space $(\dot{H}^1 \cap L_{\rho+1}) \times L_2$ to $\dot{H}^s_{q'}$ for $t \neq 0$, and $0 \leq s \leq 1, (n+1)/(1/2 - 1/q') = 1$. This is in contrast to the subcritical case, where the corresponding operators are Lipschitz mappings ([3], [6]). Here $\mathcal{E}_t(\phi, \psi) = u(\cdot, t)$, where u is a solution of

$$\begin{cases} \partial_t^2 u - \Delta_x u + m^2 u + |u|^{\rho-1} u = 0,\ t > 0,\ x \in \mathbf{R}^n, \\ u|_{t=0}(x) = \phi(x), \\ \partial_t u|_{t=0}(x) = \psi(x). \end{cases}$$

where $n \geq 4, m \geq 0$ and $\rho > \rho^* = (n+2)/(n-2)$ in the supercritical case.

1 Introduction

We will in this paper discuss properties of the semilinear semilinear hyperbolic equation

$$\partial_t^2 u - \Delta u + m^2 u + |u|^{\rho-1} u = 0,\ t > 0,\ x \in \mathbf{R}^n,\ u|_0 = \phi, \partial_t u|_0 = \psi, \qquad (1)$$

where $m \geq 0, \rho > 1$ (we will add more assumptions below) and the data ϕ, ψ belong to $(H^1 \cap L_{\rho+1}) \times L_2 = X_e$, the energy space for the nonlinear equation. If $m = 0$, the space H^1 is replaced by the corresponding homogeneous space $\dot{H}^1$. If $m > 0$, we denote the equation (1) by NLKG (the nonlinear Klein-Gordon equation), and if $m = 0$, we denote it by NLWE (the nonlinear wave equation). Notice that for any solution u the energy

$$E(u) = \frac{1}{2} \int (|\partial_x u|^2 + m^2 |u|^2 + |\partial_t u|^2) dx + \frac{1}{\rho + 1} \int |u|^{\rho+1} dx$$

is uniformly bounded for such data by the energy of $u(0)$. We will investigate the solution operator of (1) for small values of $t > 0$, in which case the behavior in time of the kernel of the solution operator as a map between L_p-spaces on Besov-spaces is the same for the NLWE and NLKG. In order to have a less complicated exposition, we will treat the case of the NLWE, i.e. the case $m = 0$ in (1). The results, with obvious modifications, work also for the NLKG, however.

Above $\dot{H}_p^s$ denotes the homogeneous Sobolev space of order s, based on L_p, where for $p = 2$ we usually drop the reference to the L_2-space, with norm

$$\|u\|_{\dot{H}_p^s} = \left(\int |\mathcal{F}^{-1}(\mathcal{F}u(\xi)|\xi|^s)(x)|^p dx \right)^{\frac{1}{p}},$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbf{R}^n$. Correspondingly, we define the H_p^s space as $\dot{H}_p^s \cap L_p$ with norm

$$\|u\|_{H_p^s} = \|u\|_{L_p} + \|u\|_{\dot{H}_p^s}.$$

We will investigate mappings

$$(\dot{H}_p^s \cap L_{\rho+1}) \times \dot{H}_p^{s-1} \ni (\phi, \psi) \mapsto \mathcal{E}_t(\phi, \psi) = u(t) \in \dot{H}_{p'}^{s'} \tag{2}$$

where u is a solution of (1) with data ϕ, ψ.

We know that for data in X_e the solution operator $\mathcal{E}_t$ of (1) is bounded on X_e, but from the above result we will derive the main theorem of this paper:

Theorem 1. *Let $n \geq 4$. If ρ is an odd integer, and $\rho > \frac{n+2}{n-2}$, then for t in a neighborhood of $t = 0$ the solution operator $\mathcal{E}_t$ of the (NLWE) is not a Lipschitz mapping at 0 from $X_e \cap (\dot{H}_q^s \times L_2)$ to $\dot{H}_{q'}^s$ where $(n+1)(\frac{1}{2} - \frac{1}{q'}) = 1$, q and q' are dual exponents and $0 \leq s \leq 1$.*

Notice that X_e is a subset of $\dot{H}_{q'}^s \times L_2$ if $s = \frac{1}{n+1}$. This means that $\mathcal{E}_t$ is not Lipschitz from $X_e \cap (\dot{H}_q^s \times L_2)$ to $\dot{H}_2^1$. The result in Theorem 1 holds also for $s \geq -\frac{n}{n+1}$, in which case $L_2 \subseteq \dot{H}_{q'}^s$. In the theorem, the same conclusion holds if we map to $\dot{H}_{q'}^{s'}$ where $0 \leq s' \leq s$, provided the space dimension is high enough ($n > 8$ will do).

The theorem holds also for the nonlinear Klein-Gordon equation as remarked above; that is, with an added mass term $m^2 u$, where the energy space is exactly $X_e \cap (L_2 \times L_2)$. As a contrast to the results of Theorem 1, in the subcritical case the corresponding map is Lipschitz, which e.g. as a consequence has the uniqueness of weak solutions in that case (cf. [6], [3] where a simple proof of the Lipschitz properties of the solution operator at $t = 0$ from X_e to $L_{q'}$ is given).

In the linear case the map is Lipschitz on the energy space; that is the case $s = -\frac{n}{n+1}$.

As far as the authors know, our theorem is one of the few results that distinguishes the sub- and supercritical cases for the NLWE. The critical case $\rho = \frac{n+2}{n-2}$ has properties that resembles those in the subcritical case (cf. [7], [8], [15], [16], [17]).

The result of the present paper is based on a construction used by one of the authors ([11]) in his thesis to prove that there is no improvement in the smoothing properties of the solution operator in the supercritical case, compared to the subcritical or linear cases. We have also been encouraged by a number of our collegues to have the results of [11] more generally available.

The proofs of the main results will be carried out in some detail in the following sections. For the convenience of the reader we have tried to give as complete and self-contained an exposition as is reasonably possible. We have, however, for the technical background on real interpolation and Besov spaces to refer the reader to [2] and [4].

2 Basic lemmas and concepts

The proof of the main result is composed of a local existence result for smooth solutions of (1) (Lemma 3), a decay result for smooth small solutions of (1) (Lemma 4) and a nonlinear interpolation result due to Peetre [14]. The nonlinear interpolation in our context will, however, be rather inspired by than directly using Peetre's result.

Below we formulate these lemmas, in the next section we discuss certain aspects of real interpolation and (truncated) Besov spaces. We then supply a proof of Lemma 4, and we then proceed to the proof of the main theorem in the last section. We give some auxiliary constructions in an appendix.

Next, we shall prove an $(H^s_p, H^{s'}_{p'})$-version of the following well-known H^s-existence result (for a proof see [5] and [9]):

Lemma 1 (Local existence of smooth solutions (H^s-version)). *Let $\phi \in H^s$, $\psi \in H^{s-1}$, where $s > \dfrac{n}{2}$. Then there exists a positive number $\tilde{T} = \tilde{T}(\|\phi\|_{H^s}, \|\psi\|_{H^{s-1}})$ and a constant $C > 0$ with the properties:*

There is a unique solution

$$u \in C^2([0, \tilde{T}), L_2)$$

of (1) such that

1. $\dfrac{d^i}{dt^t}u(t) \in C([0,\tilde{T}), H^{s-i})$ *for* $i = 0,1,2$

2. $\lim_{t\uparrow\tilde{T}} \|u(t)\|_{H^{s-i}} = +\infty$ *if* $\tilde{T} < \infty$.

3. *Let* $\mathcal{K} = 1 + \|\phi\|_{H^s_p} + \|\psi\|_{H^{s-1}}$. *Then*

$$\tilde{T} \geq C\mathcal{K}^{-(\rho-1)}.$$

Let $E(t)v$ denote the solution of LWE/LKG with data $u|_{t=0} = v$, $\partial_t u|_{t=0} = 0$. We have the following standard estimate, see [3], [12].

Lemma 2 ($(H^s_p, H^{s'}_{p'})$**-decay estimates).** *Let* $s, s' \in \mathbf{R}$ *and* $1 < p \leq 2 \leq p' < \infty$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$, *and set* $\delta = \frac{1}{2} - \frac{1}{p'}$. *Assume that*

$$0 \leq 1 - (n+1)\delta + s - s'.$$

Then there exists a constant $C > 0$ *such that*

$$\|E(t)v\|_{H^{s'}_{p'}} \leq C \begin{cases} t^{1-2n\delta+s-s'}\|v\|_{H^s_p}, & 0 < t < 1, \\ (1+t)^{-(n-1)\delta}(1+tm)^{-\delta}\|v\|_{H^s_p}, & 1 \leq t \end{cases}$$

We can now give the local $H^s_p, H^{s'}_{p'}$-existence result:

Lemma 3 (Local existence of smooth solutions $(H^s_p, H^{s'}_{p'})$**-version).** *Let* $\phi \in H^s_p, \psi \in H^{s-1}_p, s' > n/p'$, *and let* $1/p + 1/p' = 1$ *and* s, s' *restricted as in Lemma 2. Then there exists a positive number* $\tilde{T} = \tilde{T}(\|\phi\|_{H^s_p}, \|\psi\|_{H^{s-1}_p})$ *and a constant* C *with the properties.*

There is a unique solution

$$u \in C^2([0,\tilde{T}), H_{p'})$$

of (1) such that

1. $\dfrac{d^i}{dt^i}u(t) \in C([0,\tilde{T}), H^{s'-i}_{p'})$ *for* $i = 0,1,2$

2. $\lim_{t\uparrow\tilde{T}} \|u(t)\|_{H^{s'-i}_{p'}} = +\infty$ *if* $\tilde{T} < \infty$.

3. *Let* $\mathcal{K} = 1 + \|\phi\|_{H^s_p} + \|\psi\|_{H^{s-1}_p}$. *Then*

   (i) $\tilde{T} > T_0 = C\mathcal{K}^{\frac{\rho-1}{1-(n-1)\delta}}$

   (ii) $\|u(t)\|_{H^{s'}_{p'}} \leq \mathcal{K}$ *for* $t \in [0, \frac{T_0}{2}]$.

The proof is the same as in the H^s-case, using now Lemma 2 to obtain

$$\|u(t)\|_{H^{s'}_{p'}} \leq \|u_0(t)\|_{H^{s'}_{p'}} + C\int_0^t (t-\tau)^{-(n-1)\delta}\|f(u)\|_{H^s_p} d\tau$$

where u_0 is a solution of the linear equation with data ϕ, ψ (and has $\mathcal{K}$=constant, actually, then $\tilde{T} = \infty$), where we can estimate

$$\|f(u) - f(v)\|_{H^s_p} \leq C(\|u\|_{H^{s'}_{p'}} + \|v\|_{H^{s'}_{p'}} + 1)^{\rho-1}\|u - v\|_{H^{s'}_{p'}}.$$

An application of the fixed-point theorem now completes the proof of Lemma 3.

Lemma 4 (Decay result for smooth solutions). *Let $p' \geq 2, s' \geq 0$ and set $\delta = \frac{1}{2} - \frac{1}{p'}$. Assume that $n \geq 4$ and $\rho > \rho^* = \frac{n+2}{n-2}$. Then for smooth, small and compactly supported data there exists a global smooth solution $u(t)$ of (1) and constant $C = C(u, s') > 1$ such that*

$$\frac{1}{C}(1+t)^{-(n-1)\delta}(1+tm)^{-\delta} \leq \|u(t)\|_{\dot{H}^{s'}_{p'}} \leq C(1+t)^{-(n-1)\delta}(1+tm)^{-\delta},$$

for all $t > 0$.

The proof of the next result is implicit in the arguments in Section 5.

Proposition 1 (Mapping results for the solution operator of NLWE). *Let $0 < T \in \mathbf{R}$ and $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and set $\delta = \frac{1}{2} - \frac{1}{p'}$. Moreover let $s \geq 1$ and $s' \geq 0$. Assume that $n \geq 4$ and that ρ is an odd integer $> \rho^* = \frac{n+2}{n-2}$. Then there exists data $(\phi, \psi) \in (H^s_p \cap L_{\rho+1}) \times H^{s-1}_p$ such that for $0 < t \leq T$, $\mathcal{E}_t(\phi, \psi)$ is not bounded in $\dot{H}^{s'}_{p'}$, provided that*

$$\begin{cases} p'((n-1)\delta - \frac{2}{\rho-1} - s') + p(\frac{2}{\rho-1} + s) < 0 \\ (n-1)\delta < 1 \end{cases}$$

3 Besov spaces and real interpolation

In this section we will shortly remind us of the basic definitions and properties of real interpolation and Besov spaces. We will then introduce a scale of semi-norms on Besov space which we, for the sake of simplicity, call truncated Besovspaces. These truncated Besov-spaces will come up naturally in our problems, as shown in Section 5 below. The basic references are [1], [2] and [4], to which we refer the reader for additional information.

Let $C_1 \subset C_0$ be a Banach space couple. Then the K-functional $K(t,\phi;C_0,C_1)$ is defined by

$$K(t,\phi;C_0,C_1) = \inf\{\|\phi_0\|_{C_0} + t\|\phi_1\|_{C_1}; \phi = \phi_0 + \phi_1, \phi_i \in C_i\},$$

where $\phi \in C_1$. We define $C_{\theta,q} = (C_0,C_1)_{\theta,q}$ as the completion of C_1 in the norm

$$\|\phi\|_{C_{\theta,q}} = (\int_0^\infty (t^{-\theta}K(t,\phi;C_0,C_1))^q \frac{dt}{t})^{1/q}.$$

If $C_1 = H_p^{s_1}, C_0 = H_p^{s_0}, s_1 > s_0$, then $C_{\theta,q}$ defines the Besov space $B_p^{s,q}$, where $s = (1-\theta)s_0 + \theta s_1$. By the definition, the family of Besov spaces have a number of natural convexity and inclusion properties, for which we refer the reader to the already given basic references, and in particular to [2].

We will need a set of truncated integrals of K-functionals, defined by Banach couples based on H_p^s-spaces (including H^s-spaces). The properties of these will, in many respects, be parallel to those of the ordinary Besov spaces.

Let $t^* < 1$, and define the $C_{\theta,q}(t^*)$-norm (seminorm) by

$$\|\phi\|_{C_{\theta,q}(t^*)} = \left(\int_{t^*}^\infty (t^{-\theta}K(t,\phi;C_0,C_1))^q \frac{dt}{t}\right)^{1/q}.$$

In case $C_{\theta,q} = B_p^{s,q}$, we use the obvious notation $B_p^{s,q}(t^*)$. In our applications $\phi = \phi(\tau)$ will depend on a parameter τ, and $t^* = t^*(\tau) \to 0$ as $\tau \to 0$.

We will need some simple inclusions which follow (as in the case $t^* = 0$) by convexity:

$$B_p^{s,q}(t^*) \supset B_p^{s,2}(t^*),\ q > 2.$$

Next, the Besov spaces can be described in terms of integrals of moduli of continuity. A similar result holds for the truncated Besov-norms $B_p^{s,q}(t^*)$

Lemma 5. *Assume $0 \le s \le s_0 \le s'$, where $s < s'$ are non-integers. Let $\theta \in (0,1)$ be defined by $s_0 = (1-\theta)s + \theta s'$. Moreover, let $[s_0]$ denote the integer part of s_0. Let r be an integer ≥ 1 and let $\omega_p^{(r)}(t,f)$ denote the r-th modulus of continuity of f in L_p, i.e.*

$$\omega_p^{(r)}(t,f) = \sup_{|h|\le t} \|\sum_{k=0}^r \binom{r}{k}(-1)^{r-k} f(\cdot + kh)\|_{L_p}.$$

Then there is a constant $C > 0$ such that

$$\left(\int_{t^*}^\infty (t^{-\theta}K(t,f;\dot{H}_p^s,\dot{H}_p^{s'}))^q \frac{dt}{t}\right)^{\frac{1}{q}} \ge$$
$$\ge C\left(\int_{t^*}^\infty (t^{-s_0+[s_0]} \sum_{|\alpha|=[s_0]} \omega_p^{(r)}(t,D^\alpha f))^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

for $2 < p = q < \infty$ *and* $0 < t^* < 1$.

Let us remind us of the following well-known interpolation result: Let H_p^s denote the (usual) Sobolev spaces, and $\dot{H}_p^s$ the corresponding homogeneous spaces. Then for $1 \le p \le \infty$ and $s \le s'$

$$(\dot{H}_p^s, \dot{H}_p^{s'})_{\theta,q} = \dot{B}_p^{(1-\theta)s+\theta s'=s^*,q}$$

where $\dot{B}_p^{s,q}$ has the intrinsic norm (among many)

$$\|f\|_{\dot{B}_p^{s,q}} \simeq \sum_{|\alpha|=[s]} \left(\int_0^\infty (t^{-s+[s]}\omega_p^{(r)}(t, D^\alpha f))^q \frac{dt}{t} \right)^{\frac{1}{q}} \tag{3}$$

where $[s] = \sup\{m \in \mathbf{Z} : m \le s\}$ and $r \ge 1$. Here $\dot{B}_p^{s,q}$ denotes the standard homogeneous Besov space.

Thus our lemma establishes an estimate of the form (3) also for our truncated Besov spaces.

In order to prove our lemma, we need some estimates of the K-functional. We will use the following estimates, assuming that $0 \le s < s' \le 1$ (the modification for general s', s will be obvious). We let $p > 2$. The following estimate and formula are well known.

$$K(t, f, L_p, \dot{H}_p^m) \ge C\omega_p^{(m)}(t^{1/m}, f),\ m \in \mathbf{Z}_+. \tag{4}$$

(See [1], p.339)

$$\begin{aligned} K(t, f; \dot{H}_p^s, \dot{H}_p^{s'}) \simeq & \left(\int_0^{t^{1/\lambda}} (\sigma^{-\theta_0} K(\sigma, f; L_p, \dot{H}_p^2))^p \frac{d\sigma}{\sigma} \right)^{1/p} + \\ & + t\left(\int_{t^{1/\lambda}}^\infty (\sigma^{-\theta_1} K(\sigma, f; L_p, \dot{H}_p^2))^p \frac{d\sigma}{\sigma} \right)^{1/p} \end{aligned} \tag{5}$$

where $\theta_0 = s/2$, $\theta_1 = s'/2$ and $\lambda = \theta_1 - \theta_0 > 0$. This follows from Holmstedt's iteration formula (see [2], p. 52), since $\dot{H}_p^s = (L_p, \dot{H}_p^2)_{\theta_0,p}$ and $\dot{H}_p^{s'} = (L_p, \dot{H}_p^2)_{\theta_1,p}$.

Now (4) and (5) give

$$\begin{aligned} K(t, f; \dot{H}_p^s, \dot{H}_p^{s'}) \ge C\Bigg\{ & \left(\int_0^{t^{1/\lambda}} (\sigma^{-\theta_0}\omega_p^{(2)}(\sqrt{\sigma}, f))^p \frac{d\sigma}{\sigma} \right)^{1/p} \\ & + t\left(\int_{t^{1/\lambda}}^\infty (\sigma^{-\theta_1}\omega_p^{(2)}(\sqrt{\sigma}, f))^p \frac{d\sigma}{\sigma} \right)^{1/p} \Bigg\}. \end{aligned}$$

The truncated Besov space norm on $B_p^{s_0,p}(t^*)$ is given by

$$\left(\int_{t^*}^{\infty}(t^{-\theta}K(t,f;\dot{H}_p^s,\dot{H}_p^{s'}))^p\frac{dt}{t}\right)^{1/p},$$

where $\theta=\frac{s_0-s}{s'-s}$. So, let us first compute

$$\left(\int_{t^*}^{\infty}t^{(1-\theta)p}\left(\int_{t^{1/\lambda}}^{\infty}(\sigma^{-\theta_1}\omega_p^{(2)}(\sqrt{\sigma},f))^p\frac{d\sigma}{\sigma}\right)\frac{dt}{t}\right)^{1/p} \tag{6}$$

In order to do this, we will use Marchaud's inequality (see [1] p.333).

Let $0<k<r,\ k,r\in\mathbf{Z}_+$. Then

$$2^{k-r}\omega_p^{(r)}(t,f)\leq\omega_p^{(k)}(t,f)\leq Ct^k\int_t^{\infty}\frac{\omega_p^{(r)}(u,f)}{u^k}\frac{du}{u},t>0 \tag{7}$$

Let $k=1,r=2$ and change variables $u\to\sqrt{\sigma}$ in the right-hand side integral in (7). Then

$$\omega_p^{(1)}(t,f)\leq Ct\int_t^{\infty}\frac{\omega_p^{(2)}(u,f)}{u}\frac{du}{u}=C\frac{t}{2}\int_{t^2}^{\infty}\frac{\omega_p^{(2)}(\sigma^{1/2},f)}{\sigma^{1/2}}\frac{d\sigma}{\sigma} \tag{8}$$

Let us first estimate (6) for $p=1$ and $p=\infty$, while keeping $\omega_p^{(2)}$ fixed, and then obtain a general estimate by an ordinary interpolation

$$\int_{t^{1/\lambda}}^{\infty}\sigma^{-\theta_1}\omega_p^{(2)}(\sigma^{1/2},f)\frac{d\sigma}{\sigma}=\int_{t^{1/\lambda}}^{\infty}\sigma^{1/2-\theta_1}\frac{\omega_p^{(2)}(\sigma^{1/2},f)}{\sigma^{1/2}}\frac{d\sigma}{\sigma}$$

$$\geq C(t^{1/\lambda})^{-1/2}t^{(\frac{1}{2}-\theta_1)/\lambda}\omega_p^{(1)}(t^{1/2\lambda},f)=$$

$$=Ct^{-(1/\lambda)\theta_1}\omega_p^{(1)}(t^{1/2\lambda},f)\geq C\frac{1}{2}t^{-\frac{\theta_1}{\lambda}}\omega_p^{(2)}(t^{1/2\lambda},f)$$

where we used (8) and (7), and the fact that $1/2-\theta_1\geq 0$.

Thus (6) is (for $p=1$) estimated from below by

$$\int_{t^*}^{\infty}t^{1-\theta-\frac{\theta_1}{\lambda}}\omega_p^{(2)}(t^{\frac{1}{2\lambda}},f)\frac{dt}{t}\geq C\int_{(t^*)^{1/2\lambda}}^{\infty}t^{-(2\lambda\theta+2\theta_1-2\lambda)}\omega_p^{(2)}(t,f)\frac{dt}{t}$$

Since $\lambda = \theta_1 - \theta_0$, $\theta_1 = s'/2$, $\theta_0 = s/2$ and $\theta = \frac{s_0 - s}{s' - s}$, we get finally a lower bound of (6) in the case of $p = 1$ ($\lambda \in (0, 1/2)$ and $t^* < 1$)

$$C \int_{t^*}^{\infty} t^{-s_0} \omega_p^{(2)}(t, f) \frac{dt}{t}.$$

In case $p = \infty$, we estimate (6) as follows:

$$\sup_{t \geq t^*} [t^{-\theta} t \sup_{u \geq t^{1/\lambda}} (u^{-\theta_1} \omega_p^{(2)}(u^{1/2}, f))] \geq$$

$$\geq \sup_{t \geq t^*} (t^{1-\theta} t^{-\theta_1/\lambda} \omega_p^{(2)}(t^{1/2\lambda}, f)) \geq$$

$$\geq C \sup_{t \geq (t^*)^{1/2\lambda}} (t^{2\lambda - 2\lambda\theta - 2\theta_1} \omega_p^{(2)}(t, f)) \geq \sup_{t \geq t^*} (t^{-s_0} \omega_p^{(2)}(t, f))$$

and so the interpolation gives a lower bound of (6) for general p, $1 \leq p \leq \infty$, as

$$\left(\int_{t^*}^{\infty} (t^{-s_0} \omega_p^{(2)}(t, f))^p \frac{dt}{t} \right)^{1/p} \tag{9}$$

and hence (9) is also a lower bound for the semi-norm in $B_p^{s_0, p}(t^*)$. In addition, using (7) and Hardy's inequality, we may (as in the case $t^* = 0$) replace $\omega_p^{(2)}$ in (9) with $\omega_p^{(1)}$, and the lemma is proved.

4 Proof of Lemma 4

Our proof differs slightly from that of Kumlin [11], which is modelled on a method due to Klainerman/Ponce [12], but differs substantially from an earlier proof of von Wahl [18].

We first prove the existence of smooth global solutions of the (NLWE) for small and smooth data. As a consequence we obtain estimates for the rate of decay of such solutions in $\dot{H}_p^s$-norms as $t \to \infty$. We then estimate the decay of the contribution to the solution by the nonlinear term, and show that for the values of ρ we consider, this contribution decays faster than the solution of the linear wave equation. Thus the maximal rate of decay of the nonlinear and linear wave equations are the same in the supercritical case. As the asymptotic behaviour of the solution of the linear wave equation is well known, we are in position to prove the decay estimate in Lemma 4 thus completing the proof of Lemma 4.

Let u and u_0 be solutions of the nonlinear and linear wave equations, respectively, with the same initial data. Assume that this data belong to $\dot{H}^{\bar{s}} \times \dot{H}^{\bar{s}-1}$, $\bar{s}$ large,

$\bar{s} \in \mathbf{N}$, with small norm in this space, which will be made precise below. Then u and u_0 exist locally in t, see Lemma 1, belong to $\dot{H}^{\bar{s}}$ and

$$u(t) = u_0(t) - \int_0^t E(t-\tau) f(u(\tau)) d\tau, \tag{10}$$

where $E(t)\phi$ denotes the solution of the linear wave equation with data $v = 0, \partial_t v = \phi$. From the energy estimates for the linear wave equation we get

$$\|u(t)\|_{\dot{H}^{\bar{s}}} \leq \|u_0(t)\|_{\dot{H}^{\bar{s}}} + \int_0^t \|f(u(\tau))\|_{\dot{H}^{\bar{s}-1}} d\tau.$$

Hence the Hölder inequality together with the Gagliardo-Nirenberg inequality applied to $f(u) = |u|^{\rho-1}u$ with $\rho > \frac{n+2}{n-2}$ an odd integer yields

$$\|f(u)\|_{\dot{H}^{\bar{s}-1}} \leq \|u\|_{L_r}^{\rho-\theta} \|u\|_{\dot{H}^{\bar{s}}}^{\theta},$$

where

$$\theta \equiv \frac{\bar{s}-1}{\bar{s}},\ r \equiv 2\frac{\rho-\theta}{1-\theta}. \tag{11}$$

With $\|u\|_{\dot{H}^{\bar{s}}}^{\theta} \leq 1 + \|u\|_{\dot{H}^{\bar{s}}}$ we obtain

$$\|u(t)\|_{\dot{H}^{\bar{s}}} \leq \|u_0(t)\|_{\dot{H}^{\bar{s}}} + C\int_0^t \|u(\tau)\|_{L_r}^{\rho-\theta} d\tau + C\int_0^t \|u(\tau)\|_{L_r}^{\rho-\theta} \|u(\tau)\|_{\dot{H}^{\bar{s}}} d\tau$$

and hence by Gronwall's inequality,

$$\|u(t)\|_{\dot{H}^{\bar{s}}} \leq C(\|u_0(t)\|_{\dot{H}^{\bar{s}}} + CI_\theta(t)) \exp(CI_\theta(t))$$

where

$$I_\theta(t) \equiv \int_0^t \|u(\tau)\|_{L_r}^{\rho-\theta} d\tau. \tag{12}$$

Thus

$$\sup_t I_\theta(t) < \infty \tag{13}$$

will imply that

$$\sup_t \|u(t)\|_{\dot{H}^{\bar{s}}} < \infty \tag{14}$$

since

$$\sup_t \|u_0(t)\|_{\dot{H}^{\bar{s}}} < C(\|u_0(0)\|_{\dot{H}^{\bar{s}}}, \|\partial_t u_0(0)\|_{\dot{H}^{\bar{s}-1}}) < \infty,$$

where $C(0,0) = 0$ and C is a continuous function at the origin. See Lemma 1. This proves, provided (13) is satisfied, the existence of global smooth solutions of the NLWE in the case of small, smooth data. Notice that we have essentially only used L_2-estimates so far. In order to prove (13) and at the same time prove the L_p-decay estimates, we will turn to the use of the $(H_p^s, H_{p'}^{s'})$-estimates in Lemma 2.

¿From lemma 1 we conclude that

$$\|u(t)\|_{H_{p'}^{s'}} \le C\|u_0(t)\|_{H_{p'}^{s'}} + \int_0^t (t-\tau)^{-(n-1)\delta}(1+m(t-\tau))^{-\delta}\|f(u(\tau))\|_{H_p^s} d\tau.$$

Here we assume that

$$(n+1)\delta \le 1 + s - s', \quad (n-1)\delta < 1. \tag{15}$$

The norms are non-homogeneous, in contrast to the discussion above, Let $[s] \equiv \max\{k \in \mathbf{N} : s \ge k\}$. A straightforward but tedious calculation with the Hölder and Gagliardo-Nirenberg inequalities yields

$$\|f(u)\|_{\dot{H}_p^s} \le \|f(u)\|_{\dot{H}_p^{[s]}} + \|f(u)\|_{\dot{H}_p^{[s]+1}} \le C(\|u\|^{\eta}_{\dot{H}^{\hat{s}_1}} + \|u\|^{\eta}_{\dot{H}^{\hat{s}_2}})\|u\|^{\rho-\eta}_{L_{\hat{r}}}$$

where

$$\hat{s}_1 \equiv \frac{[s]}{\eta} < \hat{s}_2 \equiv \frac{[s]+1}{\eta} \le \bar{s},\ \hat{r} = \frac{\rho-\eta}{\frac{1}{p} - \eta\frac{1}{2}}, \quad \eta \in (0,1). \tag{16}$$

Here we assume

$$\frac{1}{\hat{r}} \ge \frac{1}{p'} - \frac{s'}{n} > 0, \quad \hat{r} \ge p', \tag{17}$$

which implies that $H_{p'}^{s'} \subset L_{\hat{r}}$. We conclude that

$$\|f(u)\|_{\dot{H}_p^s} \le C(\|u\|^{\eta}_{\dot{H}^{\hat{s}_1}} + \|u\|^{\eta}_{\dot{H}^{\hat{s}_2}})\|u\|^{\rho-\eta}_{H_{p'}^{s'}}.$$

By similar calculations it follows that

$$\|f(u)\|_{L_p} \le C\|u\|^{\eta}_{\dot{H}^1}\|u\|^{\rho-\eta}_{H_{p'}^{s'}}$$

provided

$$\frac{n(1+2(1-\eta)\delta)+2\eta}{n-2\delta n} \le \rho \le \eta + \frac{1}{p(\frac{1}{p'} - \frac{s'}{n})}[1 - \rho\eta\frac{n-2}{2n}]. \tag{18}$$

Note that if $(n-1)\delta < 1$ but close to 1, we have to let $n \ge 4$, in which case any odd integer $\rho > \frac{n+2}{n-2}$ will satisfy this inequality for η close to 1. If $n = 3$ this poses a restriction on δ since $\frac{n(1+2(1-\eta)\delta)+2\eta}{n-2\delta n}$ becomes arbitrarily large if $n=3$ and $(n-1)\delta$ approaches 1. The estimates above yield

$$\|f(u)\|_{H^s_p} \le C(\|u\|_{\dot H^{\bar s}}^{\eta(1-\lambda_1)}\|u\|_{\dot H^1}^{\eta\lambda_1} + \|u\|_{\dot H^{\bar s}}^{\eta(1-\lambda_2)}\|u\|_{\dot H^1}^{\eta\lambda_2} + \|u\|_{\dot H^1}^{\eta})\|u\|_{H^{s'}_{p'}}^{\rho-\eta},$$

where

$$\lambda_1 \equiv \frac{\bar s - \hat s_1}{\bar s - 1} > \lambda_2 \equiv \frac{\bar s - \hat s_2}{\bar s - 1} > 0,\ \hat s_1 > 1. \tag{19}$$

If we let $\epsilon_0 = \|u(0)\|_{\dot H^{s'}} + \|u(0)\|_{\dot H^1} \ll 1$, we obtain

$$\|u(t)\|_{H^{s'}_{p'}} \le \|u_0(t)\|_{H^{s'}_{p'}} + \epsilon \int_0^t (t-\tau)^{-(n-1)\delta}(1 + I_\theta^\kappa(\tau))\exp(CI_\theta(\tau))\|u(\tau)\|_{H^{s'}_{p'}}^{\rho-\eta} d\tau$$

where $\epsilon = \epsilon(\epsilon_0)$ is a non-decreasing function with $\epsilon(0) = 0$ and $\kappa > 0$. Now, set

$$m(t) \equiv \sup_{\tau \le t} \|u(\tau)\|_{H^{s'}_{p'}}(1+\tau)^{(n-1)\delta}$$

and define $m_0(t)$ in the same way in terms of u_0 instead of u. Moreover we assume that

$$\frac{1}{p'} - \frac{s'}{n} \le \frac{1}{r'}, \quad p' \le r \tag{20}$$

and

$$(\rho - 1)(n-1)\delta > 1 \tag{21}$$

which implies

$$I_\theta(t) \le Cm^{\rho-\theta}(t)$$

and we obtain

$$m(t) \le m_0(t) + \epsilon \int_0^t (t-\tau)^{-(n-1)\delta}(1+\tau)^{-(n-1)\delta(\rho-1)}(1 + m(\tau)^{\kappa'})\exp(Cm(\tau)^{\kappa''})d\tau,$$

for some $\kappa', \kappa'' > 0$. We conclude that for some $\mu > 0$

$$m(t) \le m_0(t) + \epsilon C(2 + m(t)^\mu)\exp(Cm(t)^\mu). \tag{22}$$

But now

$$m_0(t) \le \tilde\epsilon(\epsilon_0) \ll 1,$$

where $\tilde{\epsilon}(\epsilon_0)$ has the same properties as $\epsilon(\epsilon_0)$ above. See [11]. Hence since $m(0)$ is small, any $m(t)$ satisfying (22) will be bounded by a constant (depending on ϵ_0, κ' and κ''). Thus

$$\|u(t)\|_{H^{s'}_{p'}} \leq C(1+t)^{-(n-1)\delta}$$

and with (20) we have the same estimate for $\|u(t)\|_r$. Consequently we obtain

$$I_\theta(t) \leq C \int_0^t (1+\tau)^{-(\rho-\theta)(n-1)\delta} d\tau < C < \infty,$$

by our choice assuming that $(n-1)\delta < 1$ is close enough to 1.

This not only proves the boundedness of $\dot{H}^s$ for $1 \leq s \leq \bar{s}$, but also that for given s', p' satisfying (11), (15), (16), (17), (20), (21)

$$\|u(t)\|_{H^{s'}_{p'}} \leq C(1+t)^{-(n-1)\delta} \tag{23}$$

for $(n-1)\delta < 1$. In addition the proof gives that

$$\| \int_t^\infty E(t-\tau) f(u(\tau)) d\tau \|_{H^{s'}_{p'}} = o((1+t)^{-(n-1)\delta}), \tag{24}$$

with the same choice of parameters.

Now (23) implies the existence of a solution u_+ of the linear wave equation such that

$$u(t) = u_+(t) + \int_t^\infty E(t-\tau) f(u(\tau)) d\tau.$$

By well known asymptotic properties of solutions to the linear wave equation

$$\|u_+(t)\|_{\dot{H}^{s'}_{p'}} \geq C(1+t)^{-(n-1)\delta}, \quad C > 0.$$

See [11]. Thus by (24) the difference between u and u_+ behaves as $o((1+t)^{-(n-1)\delta})$ as $t \to \infty$ in $\dot{H}^{s'}_{p'}$, and we obtain

$$\|u(t)\|_{\dot{H}^{s'}_{p'}} \geq \|u_+(t)\|_{\dot{H}^{s'}_{p'}} - \|u(t) - u_+(t)\|_{\dot{H}^{s'}_{p'}} \geq C(1+t)^{-(n-1)\delta}. \tag{25}$$

This completes the proof of Lemma 2 for the particular choice of parameters s', p' above. Moreover since (23) and (24) are valid with s' replaced by s'' for $0 \leq s'' < s'$, due to the embedding $H^{s''}_{p'} \subset H^{s'}_{p'}$, (25) follows for $\|u(t)\|_{\dot{H}^{s'}_{p'}}$ replaced by $\|u(t)\|_{\dot{H}^{s''}_{p'}}$. Finally for $s''' > s'$ the corresponding estimate from below is concluded from the interpolation inequality $\|u(t)\|_{\dot{H}^{s'}_{p'}} \leq (\|u(t)\|_{\dot{H}^{s''}_{p'}})^\omega \cdot (\|u(t)\|_{\dot{H}^{s'''}_{p'}})^{1-\omega}$, where $s' = s''\omega + s'''(1-\omega)$, and (25). The proof is completed.

Let us remark that from the above construction it follows that the constant C in (25) is bounded from below by a constant times the $\dot{H}^s_p$-norm of the data.

5 Proof of Theorem 1

Let $A_0 = X_e^1 \cap H_q^s$ and $A_1 = X_e^1 \cap H_q^{s'}$, where $X_e^1 = \dot{H}_2^1 \cap L_{\rho+1}$ and let $B_0 = \dot{H}_{q'}^s$, $B_1 = \dot{H}^1 \cap \dot{H}_{q'}^{s'}$. Consider $\phi \in A_{\theta,q'} = (A_0, A_1)_{\theta,q'}$ for $0 < \theta < 1$, $1 \le q' < \infty$ being fixed for time being. For each $t > 0$ there are $\phi_0^t \in A_0, \phi_1^t \in A_1$ such that

$$\phi = \phi_0^t + \phi_1^t,$$
$$K(t, \phi; A_0, A_1) \sim \|\phi_0^t\|_{A_0} + t\|\phi_1^t\|_{A_1},$$

where K is the K-functional associated with the Banach space couple $A_1 \subset A_0$. We observe that from this and the fact that $A_{\theta,q'} \subseteq A_{\theta,\infty}$

$$\|\phi_1^t\|_{A_1} \le C\frac{1}{t}K(t, \phi; A_0, A_1) \le C\|\phi\|_{A_{\theta},q'}t^{\theta-1},$$

where C is independent of t and ϕ. Now let T denote the mapping $\phi \mapsto T\phi = \{u_\phi(\tau)\}_{\tau\in\mathbf{R}}$, where $u_\phi(\tau)$ denotes the solution of NLWE with data $(u, \partial_t u)|_0 = (\phi, 0)$ at time τ where τ is close to 0. We note that the statement of existence of a uniquely defined solution for finite energy data is part of the hypothesis of Theorem 1. More precisely we assume that T is uniformly Lipschitz at $\tau = 0$, i.e.

$$\|T\phi(\tau) - T\psi(\tau)\|_{B_0} \le C\|\phi - \psi\|_{A_0},$$

for $\tau \in (-\eta, \eta)$ some $\eta > 0$, and C independent of τ, ϕ and ψ. Moreover from the local existence result Lemma 3, and the corresponding L_p-estimates, we conclude that for $s' > \frac{n}{q'}$ and $\delta_{q'} = 1/2 - 1/q'$, we have that $u_{\phi_1^t}(\tau)$ exists as a $H_{q'}^{s'}$-valued function for

$$\tau \le C(\|\phi_1^t\|_{A_1}^{-(\rho-1)})^{\frac{1}{\mu}},\ \mu = 1 - (n-1)\delta_{q'}.$$

and in particular, for

$$\tau \le \tilde{C}(\|\phi\|_{A_{\theta,q'}}t^{\theta-1})^{\frac{-(\rho-1)}{\mu}} = \tilde{C}\|\phi\|_{A_{\theta,q'}}^{-\frac{\rho-1}{\mu}}t^{\frac{(\rho-1)(1-\theta)}{\mu}} = \tau^*(t), \tag{26}$$

where $\tilde{C}$ is a fixed numerical constant.

Furthermore Lemma 3 implies that

$$\|u_{\phi_1^t}(\tau)\|_{B_1} \le C\|\phi_1^t\|_{A_1}$$

for $|\tau| \le \frac{1}{2}\tau^*$.

Consider $K(t, u_\phi(\tau); B_0, B_1)$. We immediately observe that from the hypothesis of the theorem we have

$$K(t, u_\phi(\tau); B_0, B_1) \le \|u_\phi(\tau) - u_{\phi_1^t}(\tau)\|_{B_0} + t\|u_{\phi_1^t}(\tau)\|_{B_1}.$$

Here the right-hand side exists for $|\tau| \leq \frac{1}{2}\tau^*$. Hence for $|\tau| \leq \frac{1}{2}\tau^*(t)$, $t > 0$ we obtain

$$K(t, u_\phi(\tau); B_0, B_1) \leq CK(t, \phi; A_0, A_1).$$

Now set

$$\|u_\phi(\tau)\|_{B_{\theta,q'}(t^*)} = \left(\int_{t^*}^{\infty} (t^{-\theta} K(t, u_\phi(\tau); B_0, B_1))^{q'} \frac{dt}{t} \right)^{\frac{1}{q'}}, \tag{27}$$

where t^* denotes the inverse function of τ^* in (26), i.e.,

$$t^* = \hat{C} \|\phi\|_{A_{\theta,q'}}^{\frac{1}{1-\theta}} \tau^{\frac{\mu}{(\rho-1)(1-\theta)}}.$$

This yields

$$g_\phi(\tau^*) \equiv \sup_{2\tau \leq \tau^*} \|u(\tau)\|_{B_{\theta,q'}(t^*)} \leq C\|\phi\|_{A_{\theta,q'}} < \infty.$$

The contradiction that will eventually falsify the Lipschitz continuity of the solution operator at $\tau = 0$ follows by supplying a function $\phi \in A_{\theta,q'}$ such that

$$\sup_{2\tau \leq \tau^*} \|u_\phi(\tau)\|_{B_{\theta,q'}(t^*)} = \infty.$$

We next construct data ϕ based on scaling and dilation. Let $\Phi \in C_0^\infty$, supp $\Phi \subseteq \{|x| \leq 1\}$, and let us for a given sequence $R_j, R_j \downarrow 0$, and given $\tau \geq 1$ choose a sequence $y_j \in \mathbf{R}^n$ such that for some $\epsilon > 0$

(a) $\inf_{j \neq k} \{|y_j - y_k| - R_j - R_k\} \geq \tau$

(b) $\inf_{j \neq k} |y_j - y_k| > 2 + 2\frac{\tau}{R_j} + 2 + \frac{\epsilon}{R_j}$.

Let u_j be the solution of the NLWE with data

$$\phi_j = R_j^{-\frac{2}{\rho-1}} \Phi\left(\frac{x - y_i}{R_j}\right), \quad \psi_j = 0,$$

The finite speed of propagation for solutions of the NLWE implies with (a) that

$$\bigcup_{|t| \leq \tau} \text{supp } u_j(\cdot, t) \bigcap \bigcup_{|t| \leq \tau} \text{supp } u_k(\cdot, t) = \emptyset, \quad j \neq k.$$

and hence $u_\phi = \sum u_j$ is a solution of the NLWE for $|t| \leq \tau$, which by standard results is unique (see [10]). Let $\phi = \sum \phi_j$.

We have for $\bar{s} \geq 0$

$$\|\phi_j\|_{L_{\rho+1}} \sim R_j^{-\frac{2}{\rho-1}+\frac{n}{\rho+1}}$$
$$\|\phi_j\|_{\dot{H}_2^1} \sim R_j^{-\frac{2}{\rho-1}-1+\frac{n}{2}}$$
$$\|\phi_j\|_{H_p^{\bar{s}}} \sim R_j^{-\frac{2}{p-1}-\bar{s}+\frac{n}{p}}$$

and thus

$$\phi \in H_p^{\bar{s}} \cap \dot{H}_2^1 \cap L_{\rho+1}$$

if

$$\sum R_j^{\kappa} < \infty \tag{28}$$

for $\kappa = \min\big\{ -\frac{2}{\rho-1} + \frac{n}{\rho+1}, -\frac{2}{\rho-1} - 1 + \frac{n}{2}, -\frac{2}{\rho-1} - \bar{s} + \frac{n}{p}\big\}$.

Moreover, provided (28) is valid, there is a $C < \infty$ such that

$$\max(\|\phi\|_{L_{\rho+1}}, \|\phi\|_{\dot{H}_2^1}, \|\phi\|_{H_p^{\bar{s}}}) \leq C < \infty,$$

where C is independent of $\{y_j\}_{j=1}^{\infty}$.

We will prove that $u = u_\phi$ with our choice of the parameters, and the suitable choice of R_j's will be such that $g_\phi(\tau)$ is not uniformly bounded as $\tau \to 0^+$. This in turn will enable us to draw the conclusions stated in Theorem 1.

Let $s^* = (1-\theta)s + \theta s'$.

Now, by the same argument used to prove that $g_\phi(\tau)$ was bounded in the case of Lipschitz continuity in $\dot{H}_{q'}^s$, also proves that $u_j \in \dot{H}_{q'}^{s^*}$. Actually, this is an direct application of Lemma 3. In this case global existence in time is evident, and also Lipschitz continuity in $\dot{H}_{q'}^s$. In addition, by the (non-linear) energy inequality, $u_j \in \dot{H}_{q'}^{\epsilon}$ for some $\epsilon > 0$, and $2 < q'(< \rho+1)$. In particular, $u_j \in H_{q'}^{[s^*]}$, where we may assume that $1 \leq [s^*] < s^*$. If $[s^*] = s^*$, we should replace $[s^*]$ with $s^* - 1$, and the first order differences below with the second order differences. The main point is that the non-zero coupling constant and the supercritical value of ρ, allows for the conclusion that $u_j \in H_{q'}^{[s^*]}$ uniformly (in j). This will be crucial in our estimates from below of $\omega_{q'}^{(1)}(t, \partial^{[s^*]}u)$, and so of $g_\phi(\tau)$.

Next, with this choice of s^*,

$$\begin{aligned}&\Big(\int_{t^*}^{\infty} (t^{-\theta} K(t, f; \dot{H}_{q'}^s, \dot{H}_{q'}^{s'}))^2 \frac{dt}{t}\Big)^{\frac{1}{q'}} \\ &\geq C \sum_{j=1}^{n} \Big(\int_{t^*}^{\infty} (t^{[s^*]-s^*} \omega_{q'}^{(1)}(t; \frac{\partial^{[s^*]}}{\partial x_j^{[s^*]}} f))^{q'} \frac{dt}{t}\Big)^{\frac{1}{q'}}\end{aligned} \tag{29}$$

uniformly for $0 < t^* \le 1$ by Lemma 5.

We note that

$$\left(\omega_{q'}^{(1)}\left(t; \sum_{j=1}^{N} \frac{\partial^{[s^*]}}{\partial x_1^{[s^*]}} u_j(\cdot,\tau)\right)\right)^{q'} \ge$$
$$\ge \sup_{|h|\le t} \int_{B(0;\min_{i\ne j}\{|y_i-y_j|-2-2\tau-2t\})} |\sum_{j=1}^{N} \frac{\partial^{[s^*]}}{\partial x_1^{[s^*]}} (u_j(x+h,\tau) - u_j(x,\tau))|^{q'} dx =$$
$$= \sup_{|h|\le t} \sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q'} \int_{B(0;\dots)} |\frac{\partial^{[s^*]} v}{\partial x_1^{[s^*]}} (\frac{x+h}{R_j}, \frac{\tau}{R_j}) - \frac{\partial^{[s^*]} v}{\partial x_1^{[s^*]}} (\frac{x}{R_j}, \frac{\tau}{R_j})|^{q'} dx =$$
$$= \sup_{|h|\le t} \sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q'+n} \times$$
$$\times \int_{B(0;\dots R_j)} |\frac{\partial^{[s^*]} v}{\partial x_1^{[s^*]}} (x+\frac{h}{R_j}, \frac{\tau}{R_j}) - \frac{\partial^{[s^*]} v}{\partial x_1^{[s^*]}} (x, \frac{\tau}{R_j})|^{q'} dx,$$

where v is the solution of NLWE with data $v|_0 = \Phi$ and $\partial_t v|_0 = 0$. Notice that we only need to consider the case $t \le 1$ in the following.

Now, set

$$\lambda(\eta,\tau,A) = \int_A |\partial_x^{[s^*]} v(x+\eta,\tau) - \partial_{x_1}^{[s^*]} v(x,\tau)|^{q'} dx$$

where $\eta \in \mathbf{R}^n, \tau \in \mathbf{R}_+$ and $A \subset \mathbf{R}^n$.

With this notation and the estimate above

$$\omega_{q'}^{(1)}(t; \sum \partial_{x_1}^{[s^*]} u_j(\cdot,\tau))^{q'} \ge$$
$$\ge \sup R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q+n} \lambda\left(\frac{h}{R_j}, \frac{\tau}{R_j}, B\right)$$

where $B = B(0, \min_{\substack{i\ne j \\ i,j\le N}} \{|y_i - y_j| - 2 - 2\frac{\tau}{R_j} - 2t\} R_j)$. Since $t \le 1$, we have

$$\lambda\left(\frac{h}{R_j}, \frac{\tau}{R_j}, B(\rho_j R_j)\right) \ge \lambda(\frac{h}{R_j}, \frac{\tau}{R_j}, B(0,\epsilon))$$

where $\rho_j = \inf_{i\ne j}\{|y_i - y_j| - 2 - 2\frac{\tau}{R_j} - 2t\}$, and hence $\rho_j R_j \ge \epsilon$ for $t \le 1$ by (b). Since $\partial_{x_i}^{[s^*]} u_j \in L_{q'}$ independent of our choice of y_j, we get that there is a $c_1 > 0$ independent of R_j such that

$$(1-c_1)\lambda(h,\tau,B(0,\epsilon/R_j)) \ge c_1 \lambda(h,\tau,\mathbf{R}^n \setminus B(0,\epsilon/R_j)).$$

for $\tau \le 1$, $|h| \le 1$.

Let $\kappa = -\{2/(\rho-1) + [s^*]\}q'$. Then

$$\omega_{q'}^{(1)}(t; \sum_{j=1}^{N} \partial_{x_1}^{[s^*]} u_j(\cdot,\tau))^{q'} \geq$$
$$\geq C \sum_{j=1}^{N} R_j^{-\kappa+n} \lambda(\frac{h}{R_j}, \frac{\tau}{R_j}, B(0,\epsilon)) \geq$$
$$\geq C \sum_{j=1}^{N} R_j^{-\kappa+n} \lambda(\frac{h}{R_j}, \frac{\tau}{R_j}, \mathbf{R}^n)$$

It remains to estimate $\lambda(\frac{h}{R_j}, \frac{\tau}{R_j}, \mathbf{R}^n)$ from below in terms of R_j, h and τ. To this effect we introduce the notation

$$\partial_h v(x,\tau) = v(x+h,\tau) - v(x,\tau).$$

We will in Appendix 1 below show how to construct data Φ such that

$$\|\partial_h \Phi\|_{\dot{H}_q^s} \geq C \min(1,h).$$

By the remark in the end of the proof of Lemma 3, we then obtain

$$\|\partial_h v(\cdot,\tau)\|_{\dot{H}_{q'}^{s'}} \geq C \min(1,|h|) \cdot \min(1,\tau)^{-(n-1)\delta_{q'}}, \quad \tau > 0$$

where $\delta_{q'} = \frac{1}{2} - \frac{1}{q'} > 0$. The calculations above yield

$$\lambda(\eta,\tau) \equiv \lambda(\eta,\tau,\mathbf{R}^n) \geq C(\min(1,|\eta|)\min(1,\tau)^{-(n-1)\delta_{q'}})^{q'}, \quad \tau > 0.$$

Using this estimate, we obtain

$$\int_{t^*}^{\infty} (t^{-s^*+[s^*]} \omega_{q'}^{(1)}(\sum_{j=1}^{N} \frac{\partial[s^*]}{\partial x_1^{[s^*]}} u_j(\cdot,\tau); t))^{q'} \frac{dt}{t} \geq$$
$$\geq C \int_{t^*}^{\infty} \sup_{|h|\leq t} \sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q'+n} \lambda\Big(\frac{h}{R_j}, \frac{\tau}{R_j}\Big) t^{\{-s^*+[s^*]\}q'} \frac{dt}{t} \tag{30}$$
$$\geq C \sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q'+n} \int_{t^*}^{\infty} \Big(\min(1,\frac{t}{R_j}) \min(1,\frac{\tau}{R_j})^{-(n-1)\delta_{q'}} t^{-s^*+[s^*]}\Big)^{q'} \frac{dt}{t}$$

¿From Lemma 3 it follows that

$$t^*(\tau) \sim \tau^{\frac{1}{(1-\theta)(\rho-1)}(1-(n-1)\delta_{q'})} \tag{31}$$

and hence for $\tau > 0$ small, $\tau \ll t^*(\tau)$ provided the exponent of τ is < 1.

For fixed N we chose τ so small such that

$$R_1 > R_2 > \ldots > R_N > 2t^*(\tau) \gg \tau > 0.$$

Then the right-hand side of (30) can be estimated from below by

$$C\sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q'+n}\int_{t^*}^{R_j}+\int_{R_j}^{\infty}\big(\min(1,\frac{t}{R_j})\min(1,\frac{\tau}{R_j})^{-(n-1)\delta_{q'}}t^{-s^*+[s^*]}\big)^{q'}\frac{dt}{t} =$$

$$= C\sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q'+n}\{\int_{t^*}^{R_j} R_j^{-q'}t^{(1-s^*+[s^*])q'-1}dt +$$

$$+\int_{R_j}^{\infty} t^{(-s^*+[s^*])q'-1}dt\}R_j^{(n-1)\delta_{q'}q'} \geq$$

$$\geq C\sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+[s^*]\}q'+n}\{R_j^{-q'}[R_j^{(1-s^*+[s^*])q'} - (t^*)^{(1-s^*+[s^*])q'}] +$$

$$R_j^{(-s^*+[s^*])q'}\}R_j^{(n-1)\delta_{q'}q'} \geq C\sum_{j=1}^{N} R_j^{-\{\frac{2}{\rho-1}+s^*\}q'+n+(n-1)\delta_{q'}q'}.$$

Here we have $C = C(\tau)$.

In view of (28) the data ϕ satisfies

$$\phi \equiv \sum \phi_j \in H_q^{s^*} \cap \dot{H}_2^1 \cap L_{\rho+1}, \quad \psi \equiv 0 \in H_q^{s^*-1} \cap L_2, \tag{32}$$

provided

$$\begin{cases} q\big(\frac{2}{\rho-1}+s^*\big) < n \\ 2\big(\frac{2}{\rho-1}+1\big) < n \end{cases} \tag{33}$$

for suitable choice of R_j. Note that the last inequality in (33) is equivalent to

$$\rho > \frac{n+2}{n-2}.$$

In order to estimate $g_\phi(\tau)$, we note that

$$g_\phi(\tau) \geq \sum_{j=1}^{N}\bigg(\int_{t^*(\tau)}^{1}(t^{[s^*]-s^*}\omega_{q'}^{(1)}(t,\frac{\partial^{[s^*]}}{\partial x_j^{[s^*]}}u_\phi(\tau)))^{q'}\frac{dt}{t}\bigg)^{\frac{1}{q'}},$$

which follows from (27) and (31). We find that (33) holds, and in the same time we can choose R_j so that the sum in the estimate from below of (30) tends to infinity with N, provided

$$\begin{cases} -q'\big(\frac{2}{\rho-1}+s^*\big)+q'(n-1)\delta_{p'} < \min\big(-q(\frac{2}{\rho-1}+s^*), -2(\frac{2}{\rho-1}+1)\big) \\ (n-1)\delta_{q'} < 1 \\ \rho > \frac{n+2}{n-2} \end{cases} \tag{34}$$

Can (34) be satisfied? Since $\delta_{q'}(n+1)=1$, the first inequality in (34) and in (33) are satisfied if

$$\frac{n-1}{2q}-\frac{2}{\rho-1} < s^* < \frac{n}{q}-\frac{2}{\rho-1}. \tag{35}$$

This is certainly satisfied if ρ is an odd integer larger than $\frac{n+2}{n-2}$. This also gives the value of θ, by

$$\theta = \frac{s^*-s}{s'-s}. \tag{36}$$

We can satisfy (31) with exponent < 1, provided q' satisfies $(n+1)\delta_{q'}=1$, if

$$(1-\theta)(\rho-1) > 1-(n-1)\delta_{q'} = \frac{2}{n+1}$$

and since $\rho-1$ is at least 2 and $(1-\theta)$ has a minimum value of $\frac{s'-s^*}{s'}$ for $0 \le s \le 1$, we get by using (36) and that $s' > \frac{n}{q'}$ that (31) holds with exponent < 1, for

$$s^* < s'(1-\frac{2}{(n+2)(\rho-1)})$$

which can easily be verified to be consistent with s^* satisfying (35) for $n \ge 4$, with $(n+1)\delta_{q'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$.

If s is not the same in the definition of B_0 and A_0, this reflects in the expression for θ, which poses restrictions on θ which for differences in the s-values close to 1 poses lower bounds on s and s', i.e., on the space dimension in Lemma 1.

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Appendix 1

We now give the construction of the data Φ used in our proof of the main theorem.

$\partial_h v$ satisfies the equation

$$\big(\frac{\partial}{\partial t}\big)^2\partial_h v - \Delta\partial_h v + \big(\sum_{j=0}^{\rho-1}(v(\cdot+h,\cdot))^j(v(\cdot,\cdot))^{\rho-1-j}\big)\partial_h v = 0$$

with initial data

$$\begin{cases} \partial_h v|_{t=0} = \partial_h\Phi \\ \frac{\partial}{\partial t}\partial_h v|_{t=0} = 0. \end{cases}$$

A crucial estimate is that of $\|\partial_h\Phi\|_{H_p^s}$ from below in terms of $|h|$, where $s \notin \mathbf{N}$. Choose $\Phi \in C_0^\infty(\mathbf{R}^n)$ such that

1) $\Phi(x) = \frac{1}{([s]+2)!}x_1^{[s]+2}, \quad x \in B(0,\frac{1}{2})$
2) supp $\Phi \subset B(0;1)$.

Hence we get $\frac{\partial^{[s]}}{\partial x^{[s]}{}_1}\Phi(x) = \frac{1}{2}x_1^2$ for $x \in B(0,\frac{1}{2})$ and in particular

$$\begin{aligned}\Delta_\eta\partial_h\frac{\partial^{[s]}}{\partial x_1^{[s]}}\Phi(x) &= \frac{1}{2}((x_1+h_1+\eta_1)^2 - (x_1+\eta_1)^2 - (x_1+h_1)^2 + x_1^2) = \\ &= h_1\eta_1\end{aligned}$$

for $x, x+h, x+\eta, x+h+\eta \in B(0,\frac{1}{2})$. Assume $h = (h_1, 0, \dots, 0)$ so that $|h_1| = |h|$. Consider the cases:

a) $|h|$ small, e.g. $|h| \le \frac{1}{8}$
b) $|h|$ large, e.g. $|h| \ge 9$

Case a: For $0 < t < \frac{1}{8}$ we have

$$(\omega_p(\partial_h\frac{\partial^{[s]}}{\partial x_1^{[s]}}\Phi(\cdot);t))^p \gtrsim |h|^p t^p$$

and for $8 < t$ we have

$$(\omega_p(\partial_h\frac{\partial^{[s]}}{\partial x_1^{[s]}}\Phi(\cdot);t))^p \gtrsim |h|^p.$$

Hence we have

$$(\omega_p(\partial_h\frac{\partial^{[s]}}{\partial x_1^{[s]}}\Phi(\cdot);t))^p \gtrsim |h|^p\min(1,t^p), \quad t > 0.$$

Case b: For $t > 0$ we have

$$(\omega_p(\partial_h \frac{\partial^{[s]}}{\partial x_1^{[s]}} \Phi(\cdot); t))^p \gtrsim \min(1, t^p), \quad t > 0.$$

We conclude that for a general h we obtain

$$\omega_p(\partial_h \frac{\partial^{[s]}}{\partial x_1^{[s]}} \Phi(\cdot); t) \gtrsim \min(1, |h|) \min(1, t), \quad t > 0.$$

This gives the estimate

$$\begin{aligned}(\|\partial_h \Phi\|_{\dot{H}_p^s})^p &= \int_0^\infty (t^{-s-[s]} \omega_p(\partial_h \frac{\partial^{[s]}}{\partial x_1^{[s]}} \Phi(\cdot, t))^p \frac{dt}{t} \\ &\gtrsim (\min(1, |h|))^p \int_0^\infty (t^{-s-[s]} \min(1, t))^p \frac{dt}{t} \geq C \min(1, |h|)^p.\end{aligned}$$