On the stationary Povzner equation in $\mathbb{R}^n$.

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Abstract. The stationary Povzner equation is considered in a bounded and strictly convex domain of $\mathbb{R}^n$. Existence theorems are established for a class of collision kernels in the case of hard forces and for diffuse reflection boundary conditions. Generalizations with respect to the collision kernel and boundary conditions are discussed.

1 Introduction.

We consider the stationary Povzner equation for a strictly convex bounded, piecewise C^1 domain Ω in $\mathbb{R}^n$,

$$v \cdot \nabla_x f(x,v) = Q(f,f)(x,v), \; x \in \Omega, \; v \in \mathbb{R}^n. \tag{1.1}$$

The details of the collision operator $Q(f,f)$ will be introduced below. The boundary conditions are of diffuse reflection type

$$f(x,v) = M(x,v) \int_{v' \cdot n(x) < 0} \mid v' \cdot n(x) \mid f(x,v') dv', \tag{1.2}$$
$$x \in \partial\Omega, \quad v \cdot n(x) > 0.$$

Here $n(x)$ denotes the inward normal to the boundary. We shall restrict the discussion to the $\mathbb{R}^3$ case with no loss in generality methodwise. M is a given normalized half-space Maxwellian

$$M(x,v) = \frac{1}{2\pi T^2(x)} e^{-\frac{|v|^2}{2T(x)}},$$

such that $\frac{1}{T}$ is a measurable function on $\partial\Omega$, uniformly bounded away from 0 and ∞.
Let us first recall that in the case of the time-dependent Povzner equation

$$(f_t + v \cdot \nabla_x f)(t,x,v) = Q(f,f)(t,x,v), \quad t \in \mathbb{R}_+, \quad x \in \Omega, \quad v \in \mathbb{R}^3,$$

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the Cauchy problem has been studied by a number of authors. Existence and uniqueness results for global solutions in the whole space (i.e. $\Omega = I\!R^3$) have been given by Morgenstern [28] and Povzner [31]. Existence results for the Povzner equation in a bounded domain with periodic boundary conditions have been proven by Lachowicz [23]. The existence of non-negative L^1 solutions to the Povzner equation in a bounded domain with a general type of boundary conditions (including the maxwellian diffuse reflection case) has been established by Broman [8].

In this paper we concentrate on solutions to the stationary Povzner problem. Stationary solutions are of interest as candidates for the time asymptotics of evolutionary solutions. They also appear naturally in the resolution of boundary layer problems, when studying hydrodynamical limits of time dependent solutions. However, they cannot be obtained by the techniques so far used in the time-dependent case, since for the latter natural bounds on mass, energy and entropy provide the initial mathematical framework, whereas in the stationary case only bounds on mass, energy and entropy flows through the boundary are easily available. Instead the crucial steps in our proofs are based on estimates involving the entropy dissipation term. That requires a more delicate approximation approach than for time-dependent and earlier stationary kinetic problems.

Similar difficulties are faced when dealing with the stationary Boltzmann equation. In the slab case mathematical results on boundary value problems with large given indata are presented in a measure setting in [1], [11] and in an L^1 setting in [3], [4]. For the nonlinear Boltzmann equation the long time behaviour under constant temperature, diffuse reflection boundary conditions was treated in [5]. In all these results except [11] and [4], the control of the mass was obtained by introducing truncations in the collision operator for small velocities. In this paper, we do not introduce such truncations, but control the mass with the energy and the entropy production term.

We should also point out that a number of results are known concerning the non-linear stationary Boltzmann equation close to equilibrium, and solutions of the corresponding linearized equation. There more general approaches can be utilized. So e.g. in an $I\!R^n$ setting the solvability of boundary value problems for the Boltzmann equation in situations close to equilibrium is studied in [18], [19] and [20] in bounded domains, and for exterior regions in [34]. Stationary problems in small domains for the non-linear Boltzmann equation are studied in [21], [30]. The unique solvability of internal stationary problems for the Boltzmann equation at large Knudsen numbers

is established in [27]. Existence and uniqueness of stationary solutions of the linearized Boltzmann equation in a bounded domain is discussed in [26], and for the linear Boltzmann equation uniqueness in [32], [33], and existence in [12] and others. A classification of well-posed boundary value problems for the linearized Boltzmann equation is given in [16]. For discrete velocity models, in particular the Broadwell model, there are a number of stationary results in two dimensions, among them [6], [7], [13], [14], [15].
Let us next recall a few well known facts from the kinetic theory of neutral gases, and some details about the collision term in the Povzner equation. The gas is modelled as a density at position x and velocity v, and represented by a nonnegative function $f(x, v)$. In a number of different kinetic equations the evolution of f is driven by collisions, and the rate of change is defined through the collision term Q. In the Boltzmann equation, one of the assumptions in the derivation of the Boltzmann collision operator is that only pair collisions are significant and that each separate collision between two molecules occurs at one point in space. Povzner [31] proposed a modified Boltzmann collision operator, considering a 'smearing' process for the pair collisions. This modified Povzner collision operator looks as follows,

$$Q(f,f)(x,v) = \int_{\Omega\times I\!R^3} (f' f'_* - f f_*) B(x-y, v-v_*) dy dv_*,$$

where B is the collision kernel and $f = f(x,v)$, $f_* = f(y,v_*)$, $f' = f(x,v')$, $f'_* = f(y,v'_*)$. Here the post-collisional velocities v' and v'_* are linear functions of the pre-collisional velocities according to

$$v' = (I - a(x-y))v + a(y-x)v_*, v'_* = a(x-y)v + (I - a(y-x))v_*,$$

where a is a 3×3 matrix and I the 3×3 identity matrix. These last relations imply the conservation of momentum $v' + v'_* = v + v_*$. They also imply that simultaneously interchanging x with y and v with v_* gives an exchange between v' and v'_*. The conservation of energy $v^{'2} + v_*^{'2} = v^2 + v_*^2$ yields that $a(\zeta) = a(-\zeta)$ and $a^*(\zeta) = a(\zeta)$. In this paper we consider $a(\zeta) = \frac{\zeta^t . \zeta}{|\zeta|^2}$, so that

$$\begin{aligned} v' &= v - (v - v_*) \cdot \frac{x-y}{\mid x-y \mid} \frac{x-y}{\mid x-y \mid}, \\ v'_* &= v_* + (v - v_*) \cdot \frac{x-y}{\mid x-y \mid} \frac{x-y}{\mid x-y \mid}. \end{aligned} \tag{1.3}$$

That implies in particular that head-on collisions (when $x - y$ and $v_* - v$ are parallel) exchange (x, v), (y, v_*) into (x, v_*), (y, v).

The Povzner equation was first introduced for purely mathematical reasons and usually ignored by the physicists. However, when considering the Grad limit of a system of N interacting 'soft spheres', Cercignani [10] obtained a hierarchy of equations factorized by a Povzner-like equation. Lachowicz and Pulvirenti [24] considered a system of N spheres colliding at a stochastic distance. They proved that when N tends to infinity, the one-particle distribution function converges to a local Maxwellian with density, velocity and temperature satisfying the Euler equations. At an intermediate step the Povzner equation appears.

Let us conclude this introduction by detailing our results and methods of proofs. Later in this section a central existence theorem is stated for the stationary Povzner equation in the case of maxwellian diffuse reflection boundary conditions for the kernel $B = 1$. It is then established that the stationary Povzner problem is equivalent to another kinetic problem with collision frequency equal to unity via a transform of the *space* variables and involving the *mass*, here called the sm-transform. The transform was first introduced into radiative transfer and boundary layer studies, later in the mid 1950ies by M. Krook [22] into gas kinetic for the BGK equation, and recently used by C. Cercignani [11] for measure solutions to the Boltzmann equation in a slab. The second section is devoted to a crucial construction of approximated solutions to the transformed problem with a modified asymmetric collision operator in the case of maxwellian diffuse reflection boundary conditions. The asymmetry introduced in the collision operator allows monotonicity arguments which lead to uniqueness of the approximate solutions. Moreover, we prove pointwise bounds from below and from above of the mass flow through the boundary by taking into account the diffuse reflection type of the boundary conditions. In the third section the symmetry of the collision operator is reintroduced. Weak compactness in $L^1(\Omega \times I\!\!R^3)$ is obtained by controlling the approximate solutions inside $\Omega \times I\!\!R^3$ by their values at the outgoing boundary. There the transformed situation is being utilized, enabling a pointwise boundedness of the collision frequency. In the last section the passage to the limit for a solution of the transformed problem is performed. The mass of the transformed approximations is controlled by using the bounds of the energy and the entropy dissipation term. The technique of obtaining compactness directly from the entropy dissipation term without involving the entropy property was to our knowledge first introduced in [3] for a slab problem, and is here extended to a higher dimensional context. Towards the end of Section 4, generalizations from $B = 1$ to hard forces are also discussed in the main result of the paper, Theorem 4.8. With minor changes, but to the price of a few additional arguments the proof can be carried through for the original equation without introducing the sm-transform (cf [4] for such an alternative proof). The approach of this paper can also be used for given indata problems (cf [4]) in both the hard and the soft force cases.

Definition 1.1 *f is a weak solution to the stationary Povzner equations (1.1-2) if f belongs to $L^1(\Omega \times I\!\!R^3)$ and for any test function $\varphi \in C^1(\bar{\Omega} \times I\!\!R^3)$ vanishing on $\partial\Omega^- \cup \partial\Omega^0$,*

$$\int_{\Omega \times I\!\!R^3} (fv \cdot \nabla_x \varphi + Q(f,f)\varphi) dx dv$$
$$+ \int_{\partial\Omega^+} v \cdot n(x) M(x,v) \varphi(x,v) \Big(\int_{v' \cdot n(x) < 0} \mid v' \cdot n(x) \mid f(x,v') dv' \Big) dx dv = 0.$$

Here v' and v'_* are given by (1.3) and

$$\partial\Omega^- := \{(x,v) \in \partial\Omega \times I\!\!R^3; v \cdot n(x) < 0\},$$
$$\partial\Omega^0 := \{(x,v) \in \partial\Omega \times I\!\!R^3; v \cdot n(x) = 0\},$$
$$\partial\Omega^+ := \{(x,v) \in \partial\Omega \times I\!\!R^3; v \cdot n(x) > 0\}.$$

Let us next state a key result of this paper.

Theorem 1.2 *There is a weak solution f in $L^1(\Omega \times I\!\!R^3)$ with given total mass k to the stationary Povzner equations (1.1-2) when $B \equiv 1$.*

Extensions to more general pseudo-maxwellian and hard forces are given in Theorem 4.8.
Remark. The theorem holds with an analogous proof in $I\!\!R^n$, $n \geq 2$. It is obvious from the proof that given indata problems can also be treated by the method of this paper. In that case some of the more elaborate arguments from the present diffuse reflection case are not needed due to the a priori control of ingoing entropy flux.

Lemma 1.3 *(i) If f is a weak solution in $L^1_+(\Omega \times I\!\!R^3)$ to (1.1-2), then*

$$F(X,v) := f\Big(\frac{X}{\int f(x,v) dx dv}, v\Big)$$

is a weak solution to the sm-transformed problem

$$v \cdot \nabla_X F = \frac{1}{\int F(X,v) dX dv} \int F(X,v') F(Y,v'_*) dY dv_* - F,$$

$$
\begin{aligned}
&(\frac{X}{(\int F(X,v)dXdv)^{\frac{1}{4}}},v)\in\Omega\times I\!\!R^3, && (1.4)\\
&F(X,v)=M(\frac{X}{(\int F(X,v)dXdv)^{\frac{1}{4}}},v)\\
&\int_{w\cdot n(\frac{X}{(\int F(X,v)dXdv)^{\frac{1}{4}}})<0}\mid w\cdot n(\frac{X}{(\int F(X,v)dXdv)^{\frac{1}{4}}})\mid F(X,w)dw,\\
&(\frac{X}{(\int F(X,v)dXdv)^{\frac{1}{4}}},v)\in\partial\Omega^+. && (1.5)
\end{aligned}
$$

(ii) Reciprocally, if there is a positive real number k such that there is a weak solution F_k in $L^1_+(k\Omega\times I\!\!R^3)$ to the sm-transformed problem

$$
\begin{aligned}
&v\cdot\nabla_X F_k=\frac{1}{\int F_k(X,v)dXdv}\int F_k(X,v')F_k(Y,v'_*)dYdv_*-F_k,\\
&(\frac{X}{k},v)\in\Omega\times I\!\!R^3, && (1.6)\\
&F_k(X,v)=M(\frac{X}{k},v)\int_{w\cdot n(\frac{X}{k})<0}\mid w\cdot n(\frac{X}{k})\mid F_k(X,w)dw,\\
&(\frac{X}{k},v)\in\partial\Omega^+, && (1.7)
\end{aligned}
$$

then there is a weak solution $f\in L^1_+(\Omega\times I\!\!R^3)$ to (1..1-2), with $k=\int f(x,v)dxdv$.

<u>Proof of Lemma 1.3.</u>
(i). If f is a weak solution to (1.1-2) belonging to $L^1_+(\Omega\times I\!\!R^3)$, then $F(X,v):=f(\frac{X}{\int f(x,v)dxdv},v)$ is a weak solution to

$$
\begin{aligned}
&v\cdot\nabla_X F=\frac{1}{(\int f(x,v)dxdv)^4}\int F(X,v')F(Y,v'_*)dYdv_*-F,\\
&(\frac{X}{\int f(x,v)dxdv},v)\in\Omega\times I\!\!R^3.
\end{aligned}
$$

Then

$$
\int F(X,v)dXdv=(\int f(x,v)dxdv)^4
$$

implies (1.4). Finally, (1.5) is straightforward.
(ii). Reciprocally, if for some positive number k, there is a solution $F_k\in$

$L^1_+(k\Omega \times I\!R^3)$ to (1.6), let us define the $L^1_+(\Omega \times I\!R^3)$ function f by

$$f(x,v) = \frac{k^4}{\int F_k(X,v)dXdv} F_k(kx,v).$$

The function f satisfies

$$v \cdot \nabla_x f = \int f(x,v')f(y,v'_*)dydv_* - kf, \quad (x,v) \in \Omega \times I\!R^3,$$

and

$$k = \int f(x,v)dxdv,$$

which implies (1.1). Finally, (1.2) follows from the definition of f and (1.7).

2 Approximate solutions to the transformed problem.

Let $k > 0$, $r > 0$ and $j \in I\!N$ be given. The aim of this section is to construct solutions $f^{j,r}$ to the following approximate problem

$$\begin{aligned}
&v \cdot \nabla_x f^{j,r} = \frac{1}{\int f^{j,r} dydv_*} \int \chi^r(x-y,v,v_*)\chi^j(x-y,v,v_*) \\
&fracf^{j,r}1 + \frac{f^{j,r}}{j}(x,v')\frac{f^{j,r}}{1+\frac{f^{j,r}}{j}}(y,v'_*)dydv_* - f^{j,r}\frac{\int \chi^r\chi^j f^{j,r}(y,v_*)dydv_*}{\int f^{j,r}dydv_*}, \\
&(\frac{x}{k},v) \in \Omega \times I\!R^3, \\
&f^{j,r}(x,v) = M(\frac{x}{k},v)\frac{\beta^{j,r}(x)}{\int_{\partial(k\Omega)}\beta^{j,r}(x)dx}, \quad (\frac{x}{k},v) \in \partial\Omega^+,
\end{aligned} \tag{2.1}$$

where,

$$\beta^{j,r}(x) = \int_{w \cdot n(\frac{\cdot}{k})<0} \mid w \cdot n(\frac{\cdot}{k}) \mid (f^{j,r}(.,w)dw * \varphi^n_x)(x).$$

Here, φ^n_x is a normalized density with support in a neighbourhood of x, defined as follows. Divide $\partial\Omega$ into 2^n subdomains S^n_j each with area $\mid S^n_j \mid = 2^{-n} \mid \partial\Omega \mid, j = 1,...,2^n$, so that for fixed x the maximal diameter

D_n of $S_j^n, j = 1, ..., 2^n$ tends to zero when $n \to \infty$. Denote the characteristic function of S_j^n by $\chi_{S_j^n}$ and define

$$\varphi_x^n(y) = \frac{2^n}{\mid \partial\Omega \mid}\chi_{S_j^n}(x)\chi_{S_j^n}(y), \quad x \in S_j^n.$$

The functions χ^r and χ^j are C_0^∞, invariant under the transformations $(v, v_*) \to (v', v'_*)$, $(v, v_*) \to (v_*, v)$. They are defined from a function ψ on $I\!R$. Take $\frac{1}{m} \ll r$, $\psi(s) = 0$ for $s < -\frac{1}{m}$, $\psi(s) = 1$ for $s > \frac{1}{m}$, $\psi(s)$ increasing from 0 to 1 in the interval $-\frac{1}{m} < s < \frac{1}{m}$, $\psi \in C^\infty$. Define

$$\chi^r(x - y, v, v_*) = \psi^r(v)\psi^r(v_*)\psi^r(v')\psi^r(v'_*)$$
$$\psi(1 - \frac{2}{m} - \mid \frac{v - v_*}{\mid v - v_* \mid} \cdot \frac{x - y}{\mid x - y \mid} \mid)\psi(\mid \frac{v - v_*}{\mid v - v_* \mid} \cdot \frac{x - y}{\mid x - y \mid} \mid - \frac{2}{m}),$$

with $\psi^r(v) = \psi(\mid v \mid -r)$. Analogously define

$$\chi^j(x - y, v, v_*) = \psi^j(v)\psi^j(v_*)\psi^j(v')\psi^j(v'_*),$$

where $\psi^j(v) = \psi(j - \mid v \mid^2)$ and $m = 2$.
In the paper $\mid \Omega \mid$ and $\mid \partial\Omega \mid$ denote volume in $I\!R^3$ of Ω, respectively area of $\partial\Omega$, and D is the diameter of Ω. Define $s^\pm$ by

$$s^+(y, v) := inf\{s > 0;\ (y - sv, v) \in \partial\Omega^+\}, \quad (y, v) \in \bar{\Omega} \times I\!R^3,$$

and

$$s^-(y, v) := inf\{s > 0;\ (y + sv, v) \in \partial\Omega^-\}, \quad (y, v) \in \bar{\Omega} \times I\!R^3.$$

Denote by

$$\nu(f)(x, v) := \frac{\int_{k\Omega \times I\!R^3} \chi^r\chi^j f(y, v_*)dydv_*}{\int_{k\Omega \times I\!R^3} f(y, v_*)dydv_*}.$$

The following geometric property of $\partial\Omega$ will be used in the sequel. For $x \in \partial\Omega$, $\omega \in I\!R^3$ with $\mid \omega \mid= 1$, $\omega \cdot n(x) < 0$, set $y = x - s^+(x, \omega)\omega$. Assume that the Jacobian $\mid \frac{D\omega}{Dy} \mid$ is uniformly bounded away from zero and infinity as a function of x, $y \in \partial\Omega$. Since the truncation χ^r will be removed only at the very end of the proof in Section 4, we shall skip the index r in $\chi^r = \chi$,

$f^{j,r} = f^j$, $\beta^{j,r} = \beta^j$, $\nu_j^r = \nu_j$ and elsewhere. Let $0 < \alpha \leq 1$ be given. Let K be the closed and convex subset of $L^1(k\Omega \times I\!\!R^3) \times L^1(\partial k\Omega)$ defined by

$$K = \{f \in L^1(k\Omega \times I\!\!R^3);\ 0 \leq f(x,v) \leq e^j\}$$
$$\times \{\rho \in L^1(\partial k\Omega);\ 0 \leq \rho(x), \int_{\partial k\Omega} \rho(x)dx = 1\}.$$

The boundary integral (2.1) is related to the total inflow condition $\int_{\partial k\Omega} \rho(x)dx = 1$ in the definition of K.
For $\frac{1}{m} > \delta > 0$ let μ_δ be a mollifier in v with support in $\mid v \mid \leq \delta$. Denote by

$$Q^+(F,f)(x,v) := \frac{1}{\int f dy dv_*} \int \chi\chi^j \frac{F}{1+\frac{F}{j}}(x,v') \frac{f}{1+\frac{f}{j}}(y,v'_*) dy dv_*.$$

Let T be the map defined on K by $T(f,\rho) = (F,\sigma)$, where F is the solution to

$$\alpha F + v \cdot \bigtriangledown F = Q^+(F,f) * \mu_\delta - F \frac{\int_{k\Omega \times I\!\!R^3} \chi\chi^j f(y,v_*) dy dv_*}{\int f(y,v_*) dy dv_*},$$
$$(\frac{x}{k}, v) \in \Omega \times I\!\!R^3,$$
$$F(x,v) = M(\frac{x}{k}, v)(\rho * \varphi_x^n)(x), \quad (\frac{x}{k}, v) \in \partial\Omega^+, \tag{2.2}$$

and

$$\sigma(x) = \frac{\int_{w \cdot n(\frac{x}{k})<0} \mid w \cdot n(\frac{x}{k}) \mid F(x,w) dw}{\int_{\partial(k\Omega)^-} \mid w \cdot n(\frac{x}{k}) \mid F(x,w) dw dx}.$$

Below we shall state the results with respect to k, but for easy reading only carry out the proofs for $k = 1$.

Lemma 2.1 *There is a positive lower bound c_l for $\int F(x,v) dx dv$, with c_l independent of $0 < \alpha \leq 1$, $0 < \delta < \frac{1}{m}$, and of $(f,\rho) \in K$.*

<u>Proof of Lemma 2.1.</u> It follows from the exponential form of (2.2) and the boundedness from above of ν by 1 that

$$F(x,v) \geq M(x - s^+(x,v)v, v)\rho * \varphi_x^n(x - s^+(x,v)v) e^{-(1+\alpha)s^+(x,v)}.$$

And so for, say, $1 \leq \mid v \mid \leq 2$ and for some c independent of $(f,\rho) \in K$,

$$F(x,v) \geq c\rho * \varphi_x^n(x - s^+(x,v)v),$$

$$\int_{\Omega\times I\!R^3} F(x,v)dxdv \geq c\int_A v\cdot n(x)s^-(x,v)\rho * \varphi_x^n(x)dxdv,$$

with

$$A = \{(x,v);\ x\in\partial\Omega,\ 1\leq\mid v\mid\leq 2,\ v\cdot n(x)>0\}.$$

It follows that for some $c_l>0$,

$$\int_{\Omega\times I\!R^3} F(x,v)dxdv \geq c_l\int_{\partial\Omega}\rho(y)dy = c_l. \qquad \Box$$

Lemma 2.2

$$\sup_{\partial k\Omega}\sigma(x)\leq c_{jn},$$

with c_{jn} independent of α and of $(f,\rho)\in K$ when

$$\int_{k\Omega\times I\!R^3} f(y,v_*)dydv_* \geq c_l.$$

<u>Proof of Lemma 2.2.</u> Obviously in (2.2),

$$\sup_{\partial\Omega}\rho*\varphi_x^n \leq \sup_{n,x}\varphi_x^n < \infty,$$

and the gain term $Q^+(F,f)*\mu_\delta$ is bounded by

$$c_j = \frac{4\pi}{3}\mid\Omega\mid(j+1)^{\frac{3}{2}}\frac{j^2}{c_l}.$$

So from the exponential form of (2.2), for $r-\frac{2}{m}\leq\mid v\mid\leq\sqrt{j+1}$,

$$\begin{aligned}
&F(x,v) = \\
&M(x-s^+(x,v)v,v)\rho*\varphi_x^n(x-s^+(x,v)v)e^{-\alpha s^+(x,v)-\int_{-s^+(x,v)}^0\nu(f)(x+sv,v)ds} \\
&+\int_{-s^+(x,v)}^0 e^{\alpha s-\int_s^0\nu(f)(x+\tau v,v)d\tau}Q^+(F,f)*\mu_\delta(x+sv,v)ds \\
&\leq \sup_{y\in\partial\Omega}M(y,v)\sup_{n,y,z}\varphi_y^n(z)+\frac{c_jD}{r-\frac{2}{m}}\psi^{j+1}(v).
\end{aligned}$$

In fact this estimate holds for all of $\Omega\times I\!R^3$. Also an estimate downward in the exponential form of outgoing F by ingoing ones gives

$$\int_{\partial\Omega^-}\mid v\cdot n(x)\mid F(x,v)dxdv \geq c_r\int_{\partial\Omega}\rho(y)dy = c_r>0.$$

And so

$$\sigma(x) \leq c_{jn},$$

for some c_{jn} independent of α and of $(f,\rho) \in K$ with $\int f(y,v_*)dydv_* \geq c_l$. $\square$

It follows that T maps

$$K_{jn} = \{(f,\rho) \in K; \quad \int f dydv_* \geq c_l, \quad \rho \leq c_{jn}\}$$

into itself. For $(f,\rho) \in K_{jn}$, one solution F is obtained as the strong L^1 limit of the nonnegative monotone sequence (F_m), bounded from above, defined by $F_0 = 0$ and

$$\begin{aligned}
&\alpha F_{m+1} + v \cdot \bigtriangledown F_{m+1} = Q^+(F^m, f) * \mu_\delta \\
&-F_{m+1} \frac{\int \chi\chi^j f(y,v_*)dydv_*}{\int f(y,v_*)dydv_*}, \quad (x,v) \in \Omega \times I\!R^3, \\
&F_{m+1}(x,v) = M(x,v)(\rho * \varphi_x^n)(x), \quad (x,v) \in \partial\Omega^+.
\end{aligned}$$

Moreover, F is unique since if there were another solution G, then

$$\begin{aligned}
&\alpha(F-G) + v \cdot \bigtriangledown(F-G) = (Q^+(F,f) - Q^+(G,f)) * \mu_\delta \\
&-(F-G)\frac{\int \chi\chi^j f(y,v_*)dydv_*}{\int f(y,v_*)dydv_*}, \quad (x,v) \in \Omega \times I\!R^3, \\
&(F-G)(x,v) = 0, \quad (x,v) \in \partial\Omega^+.
\end{aligned} \tag{2.3}$$

Multiplying (2.3) by $sgn(F-G)$ and integrating it over $\Omega \times I\!R^3$ leads to

$$\alpha \int_{\Omega \times I\!R^3} \mid F - G \mid dxdv \leq 0,$$

which implies that $F = G$.

Let us prove that T is continuous for the strong topology of L^1. If (f_l, ρ_l) converges to (f,ρ) in $L^1(\Omega \times I\!R^3) \times L^1(\partial\Omega)$, denote by $(F_l, \sigma_l) = T(f_l, \rho_l)$. It is enough to prove that there is a subsequence of (F_l, ρ_l) converging to $(F,\sigma) = T(f,\rho)$, because of the uniqueness of the solution to (2.2). But $\int \chi\chi^j f_l dydv_*$ and $\int f_l dydv_*$ converge to $\int \chi\chi^j f dydv_*$ respectively $\int f dydv_*$. There is a subsequence, still denoted (f_l) such that decreasingly $G_l := \sup_{m \geq l} f_m$, and increasingly $g_l := \inf_{m\geq l} f_m$ converge to f in L^1. Let (S_l) and (s_l) be the sequences of solutions to

$$\alpha S_l + v\cdot\nabla_x S_l = \Big(\frac{1}{\int g_l dydv_*}\int \chi\chi^j \frac{S_l}{1+\frac{S_l}{j}}(x,v')\frac{G_l}{1+\frac{G_l}{j}}(y,v'_*)dydv_*\Big)*\mu_\delta$$
$$-S_l\frac{\int\chi\chi^j g_l(y,v_*)dydv_*}{\int G_l(y,v_*)dydv_*},\quad (x,v)\in\Omega\times I\!R^3,$$
$$S_l(x,v) = M(x,v)(\rho*\varphi^n_x)(x),\quad (x,v)\in\partial\Omega^+,$$

and

$$\alpha s_l + v\cdot\nabla_x s_l = \Big(\frac{1}{\int G_l dydv_*}\int \chi\chi^j \frac{s_l}{1+\frac{s_l}{j}}(x,v')\frac{g_l}{1+\frac{g_l}{j}}(y,v'_*)dydv_*\Big)*\mu_\delta$$
$$-s_l\frac{\int\chi\chi^j G_l(y,v_*)dydv_*}{\int g_l(y,v_*)dydv_*},\quad (x,v)\in\Omega\times I\!R^3,$$
$$s_l(x,v) = M(x,v)(\rho*\varphi^n_x)(x),\quad (x,v)\in\partial\Omega^+.$$

(S_l) is a non-increasing sequence. Indeed, $S_l = \lim_{m\to+\infty} S_l^m$, with $S_l^0 = 0$ and S_l^{m+1} solution to

$$\alpha S_l^{m+1} + v\cdot\nabla_x S_l^{m+1}$$
$$= \Big(\frac{1}{\int g_l dydv_*}\int \chi\chi^j \frac{S_l^m}{1+\frac{S_l^m}{j}}(x,v')\frac{G_l}{1+\frac{G_l}{j}}(y,v'_*)dydv_*\Big)*\mu_\delta$$
$$-S_l^{m+1}\frac{\int\chi\chi^j g_l(y,v_*)dydv_*}{\int G_l(y,v_*)dydv_*},\quad (x,v)\in\Omega\times I\!R^3,$$
$$S_l^{m+1}(x,v) = M(x,v)(\rho*\varphi^n_x)(x),\quad (x,v)\in\partial\Omega^+. \tag{2.4}$$

¿From (2.4) in exponential form it is easy to see that $(S_l^m)_l$ is nonincreasing in l for any m. Analogously, (s_l) is a non-decreasing sequence. Moreover, it can be proved from the iterates that $s_l \leq F_l \leq S_l$. Then (S_l) decreasingly converges in L^1 to some S and (s_l) increasingly converges in L^1 to some s, which are solutions to

$$\alpha S + v\cdot\nabla_x S = \Big(\frac{1}{\int f dydv_*}\int \chi\chi^j \frac{S}{1+\frac{S}{j}}(x,v')\frac{f}{1+\frac{f}{j}}(y,v'_*)dydv_*\Big)*\mu_\delta$$
$$-S\frac{\int\chi\chi^j f(y,v_*)dydv_*}{\int f(y,v_*)dydv_*},\quad (x,v)\in\Omega\times I\!R^3,$$
$$S(x,v) = M(x,v)(\rho*\varphi^n_x)(x),\quad (x,v)\in\partial\Omega^+, \tag{2.5}$$

and

$$\alpha s+v\cdot\nabla_x s=\Big(\frac{1}{\int f dydv_*}\int\chi\chi^j\frac{s}{1+\frac{s}{j}}(x,v')\frac{f}{1+\frac{f}{j}}(y,v'_*)dydv_*\Big)*\mu_\delta$$

$$-s\frac{\int\chi\chi^j f(y,v_*)dydv_*}{\int f(y,v_*)dydv_*},\quad (x,v)\in\Omega\times I\!\!R^3,$$

$$s(x,v)=M(x,v)(\rho*\varphi^n_x)(x),\quad (x,v)\in\partial\Omega^+. \tag{2.6}$$

By the uniqueness of the solution to the systems (2.5) and (2.6), $S=s=F$. It then follows that (F_l) converges to F in L^1. Also, for $x\in\partial\Omega$

$$\int_{w\cdot n(x)<0}\mid w\cdot n(x)\mid F_l(x,w)dw=$$

$$\int_{w\cdot n(x)<0}\mid w\cdot n(x)\mid M(x-s^+(x,w)w,w)$$

$$(\rho_l*\varphi^n_x)(x-s^+(x,w)w)e^{-\alpha s^+(x,w)-\int_{-s^+(x,w)}^0\nu(f_l)(x+\tau w,w)d\tau}dw$$

$$+\int_{w\cdot n(x)<0}\mid w\cdot n(x)\mid\Big(\int_{-s^+(x,w)}^0 e^{\alpha s-\int_s^0\nu(f_l)(x+\tau w,w)d\tau}Q^+(F_l,f_l)*\mu_\delta(x+sw,w)ds\Big)dw,$$

converges in $L^1(\partial\Omega)$ to

$$\int_{w\cdot n(x)<0}\mid w\cdot n(x)\mid F(x,w)dw=$$

$$\int_{w\cdot n(x)<0}\mid w\cdot n(x)\mid M(x-s^+(x,w)w,w)$$

$$(\rho*\varphi^n_x)(x-s^+(x,w)w)e^{-\alpha s^+(x,w)-\int_{-s^+(x,w)}^0\nu(f)(x+\tau w,w)d\tau}dw$$

$$\int_{w\cdot n(x)<0}\mid w\cdot n(x)\mid\Big(\int_{-s^+(x,w)}^0 e^{\alpha s-\int_s^0\nu(f)(x+\tau w,w)d\tau}Q^+(F,f)*\mu_\delta(x+sw,w)ds\Big)dw.$$

Hence (σ_l) converges in $L^1(\partial\Omega)$ to σ, which ends the proof of the continuity of T.

The compactness of T is a consequence of the following argument. Let (F_m,σ_m) and (f_m,ρ_m) be sequences in $L^1(\Omega\times I\!\!R^3)\times L^1(\partial\Omega)$ with (f_m,ρ_m) bounded and $(F_m,\sigma_m):=T(f_m,\rho_m)$. The sequences $(\frac{F_m}{1+\frac{F_m}{j}})$ as well as $(v\cdot\nabla_x\frac{F_m}{1+\frac{F_m}{j}})$ are uniformly bounded in L^∞, hence weakly compact in L^1.

Also
$(\int \chi\chi^j \frac{f_m}{1+\frac{f_m}{j}}(y,v'_*)\mu_\delta(\bar{v}-v)dydv'_*)$ is compact in L^1, and so the sequence

$$Q^+(F_m,f_m)*\mu_\delta(x,\bar{v}) = \frac{1}{\int f_m dydv_*}\int \frac{F_m}{1+\frac{F_m}{j}}(x,v')$$
$$(\int \chi\chi^j \frac{f_m}{1+\frac{f_m}{j}}(y,v'_*)\mu_\delta(\bar{v}-v)dydv'_*)dv'$$

is compact in $L^1(\Omega\times I\!R^3)$ by the averaging lemma (cf [17]). And so the exponential form of (2.2) for F_m together with the compactness in $L^1(\Omega\times I\!R^3)$ of

$$M(x-s^+(x,v)v)(\rho_m*\varphi^n_x)(x-s^+(x,v)v)$$
$$e^{-\alpha s^+(x,v)-\int_{-s^+(x,v)}^0 \nu(f_m)(x+\tau v,v)d\tau}$$

due to the boundedness of (ρ_m) and the convolution with φ^n_x, implies that (F_m) is compact in $L^1(\Omega\times I\!R^3)$. Moreover,

$$\int_{w\cdot n(x)<0} \mid w\cdot n(x)\mid M(x-s^+(x,w)w,w)$$
$$(\rho_m*\varphi^n_x)(x-s^+(x,w)w)e^{-\alpha s^+(x,w)-\int_{-s^+(x,w)}^0 \nu(f_m)(x+\tau v,v)d\tau}dw$$

is compact in $L^1(\partial\Omega)$. Together with the compactness of $(Q^+(F_m,f_m)*\mu_\delta)$ in $L^1(\Omega\times I\!R^3)$, this makes (σ_m) compact in $L^1(\partial\Omega)$.

T is thus continuous and compact from the closed and convex subset K_{jn} of $L^1(\Omega\times I\!R^3)\times L^1(\partial\Omega)$ into a bounded subset of K_{jn}, so by Schauder's fixed point theorem it has a fixed point (F^α,σ^α), solution to (2.2) with $f=f^\alpha,\rho=\sigma^\alpha$. Arguing similarly to the preceeding analysis of T, we can also by compactness pass to the limit when α tends to zero. That limit F^δ is then a solution to

$$v\cdot\nabla F^\delta = \frac{1}{\int F^\delta dydv_*}(\int \chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x,v')\frac{F^\delta}{1+\frac{F^\delta}{j}}(y,v'_*)dydv_*)*\mu_\delta$$
$$-F^\delta\frac{\int\chi\chi^j F^\delta(y,v_*)dydv_*}{\int F^\delta(y,v_*)dydv_*},\quad (x,v)\in\Omega\times I\!R^3,$$
$$F^\delta(x,v) = M(x,v)(\frac{\int_{w\cdot n(.)<0}\mid w\cdot n(.)\mid F^\delta(.,w)dw}{\int_{\partial\Omega^-}\mid w\cdot n(y)\mid F^\delta(y,w)dwdy})*\varphi^n_x(x),$$
$$(x,v)\in\partial\Omega^+. \quad (2.7)$$

In order to remove the μ_δ convolution, we shall next prove that the family $(F^\delta)_{\delta>0}$ is strongly compact in L^1. Denote by

$$q^\delta(x,v) := \int \chi\chi^j \frac{F^\delta}{1+\frac{F^\delta}{j}}(x,v')\frac{F^\delta}{1+\frac{F^\delta}{j}}(y,v'_*)dydv_*.$$

To prove the compactness of the family $(Q^+(F^\delta,F^\delta))$, and again using the previous compactness argument for $Q^+(F_m,f_m)*\mu_\delta$, it remains to show that

$$\int_{\Omega\times I\!R^3} \mid Q^+(F^\delta,F^\delta)*\mu_\delta - Q^+(F^\delta,F^\delta)\mid dxdv \to 0 \tag{2.8}$$

when $\delta\to 0$ (cf [25]). Similarly to the earlier analysis of T, the compactness of $(Q^+(F^\delta,F^\delta)_{\delta>0}$ implies the compactness of $(F^\delta)_{\delta>0}$.

The proof of (2.8) comes back to proving the strong L^1 translational equicontinuity in the v variable of (q^δ). Use the Hilbert-Carleman parametrization

$$q^\delta(x,v) = \int (\int_{-s^+(x,\frac{v'-v}{|v'-v|})}^{s^-(x,\frac{v'-v}{|v'-v|})} s^2 \frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid},v')ds)$$
$$(\int_{E_{v,v'}} \frac{1}{\mid v'-v\mid^2}\chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x,v'_*)dv'_*)dv',$$

where $E_{v,v'}$ is the plane containing v and orthogonal to $v-v'$. We split $q^\delta(x,v+h)-q^\delta(x,v)$ into $A^{\delta,h}(x,v)+B^{\delta,h}(x,v)$, where

$$A^{\delta,h}(x,v) := \int (\int s^2 \frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid},v')ds)$$
$$(\int_{E_{v+h,v'}} \frac{dv'_*}{\mid v'-v-h\mid^2}\chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x,v'_*) - \int_{E_{v,v'}} \frac{dv'_*}{\mid v'-v\mid^2}\chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x,v'_*))dv',$$

and

$$B^{\delta,h}(x,v) :=$$
$$\int(\int s^2\int[\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v-h}{\mid v'-v-h\mid},v') - \frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid},v')]ds)$$
$$(\int_{E_{v+h,v'}} \frac{dv'_*}{\mid v'-v-h\mid^2}\chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x,v'_*))dv'.$$

First

$$\| A^{\delta,h} \|_{L^2}^2 \leq c \int_0^1 d\theta \int dx dv \mid D_v C^\delta(x,v) \mid^2,$$

where

$$C^\delta(x, v+\theta h) := \int_{E_{v,v'}} \frac{dv'_*}{\mid v'-v \mid^2} \chi\chi^j \frac{F^\delta}{1+\frac{F^\delta}{j}}(x, v'_*)\}(x, v+\theta h)\cdot h.$$

Hence by [35],

$$\| A^{\delta,h} \|_{L^2}^2 \leq c \mid h \mid^2 \int_{\Omega\times|v|\leq\sqrt{j+1}} (\frac{F^\delta}{1+\frac{F^\delta}{j}})^2(x,v) dx dv,$$

which tends to zero when h tends to zero. This proves the translational equicontinuity of the $A^{\delta,h}$-term.
The $B^{\delta,h}$ estimate is connected to averaging. We give a direct proof. For h small

$$\begin{aligned} \mid B^{\delta,h}(x,v) \mid \leq c \int \mid \int s^2 [\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v-h}{\mid v'-v-h \mid}, v') \\ -\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v \mid}, v')] ds \mid dv'. \end{aligned} \tag{2.9}$$

Write the difference within absolute values in (2.9),

$$\int s^2 [\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v-h}{\mid v'-v-h \mid}, v') - \frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v \mid}, v')] ds,$$

with each term $\frac{F^\delta}{1+\frac{F^\delta}{j}}$ in mild form. Then there first appears a difference of boundary terms whose integral tends to zero when h tends to zero, essentially because of the convolution with $\varphi^n_\cdot$. There also appears a difference of gain terms along characteristics. There finally appear differences for the loss terms and for $(1+\frac{F^\delta}{j})^{-2}$ which can be treated analogously to the differences in the gain term, only simpler. The gain term contribution to $B^{\delta,h}(x,v)$ is a v' integral of an expression of the type

$$\int_{-s^+(x,\frac{v'-v}{|v'-v|})}^{s^-(x,\frac{v'-v}{|v'-v|})} s^2 [\int_{-s^+(x+s\frac{v'-v-h}{|v'-v-h|},v')}^{0} ds_1 \int \chi\chi^j \frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v-h}{\mid v'-v-h \mid}+s_1 v', v'_{1h})$$

$$\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_1,v'_{1h*})dy_1dw_1$$

$$-\int_{-s^+(x+s\frac{v'-v}{|v'-v|},v')}^0 ds_1\int\chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid}+s_1v',v'_1)$$

$$\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_1,v'_{1*})dy_1dw_1]ds, \tag{2.10}$$

where

$$v'_{1h*}:=v'_*(x+s\frac{v'-v-h}{\mid v'-v-h\mid}+s_1v',y_1,v',w_1),$$

$$v'_{1h}:=v'(x+s\frac{v'-v-h}{\mid v'-v-h\mid}+s_1v',y_1,v',w_1),$$

$$v'_{1*}:=v'_*(x+s\frac{v'-v}{\mid v'-v\mid}+s_1v',y_1,v',w_1),\quad v'_1:=v'(x+s\frac{v'-v}{\mid v'-v\mid}+s_1v',y_1,v',w_1).$$

We split (2.10) into a difference in the $\chi\chi^j$ terms which tends to zero when $h\to 0$, and the sum of

$$\int_{-s^+(x,\frac{v'-v}{|v'-v|})}^{s^-(x,\frac{v'-v}{|v'-v|})}s^2\int_{-s^+(x+s\frac{v'-v-h}{|v'-v-h|},v')}^0\int\chi\chi^j[\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v-h}{\mid v'-v-h\mid}+s_1v',v'_{1h})$$

$$-\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid}+s_1v',v'_1)]\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_1,v'_{1h*})dy_1dw_1dsds_1, \tag{2.11}$$

$$\int_{-s^+(x,\frac{v'-v}{|v'-v|})}^{s^-(x,\frac{v'-v}{|v'-v|})}s^2\int_{-s^+(x+s\frac{v'-v-h}{|v'-v-h|},v')}^0\int\chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid}+s_1v',v'_1)$$

$$(\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_1,v'_{1h*})-\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_1,v'_{1*}))dy_1dw_1dsds_1, \tag{2.12}$$

and

$$\int_{-s^+(x,\frac{v'-v}{|v'-v|})}^{s^-(x,\frac{v'-v}{|v'-v|})}s^2[\int_{-s^+(x+s\frac{v'-v-h}{|v'-v-h|},v')}^{-s^+(x+s\frac{v'-v}{|v'-v|},v')}\int\chi\chi^j\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid}+s_1v',v'_1)$$

$$\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_1,v'_{1*})dy_1dw_1dsds_1. \tag{2.13}$$

The contribution of the integral in v' of (2.13) to the integral over $\Omega\times I\!\!R^3$ of $B^{\delta,h}$ tends to zero when h tends to zero, because the volume of integration tends to zero with h. As for the two differences in brackets in (2.11) and (2.12), we repeat the procedure of expressing them in mild form. Terms like

$$\int_{-s^+(x,\frac{v'-v}{|v'-v|})}^{s^-(x,\frac{v'-v}{|v'-v|})} s^2[\int_{-s^+(x+s\frac{v'-v-h}{|v'-v-h|},v')}^0 \int \chi\chi^j$$
$$(\int_{-s^+(x+s\frac{v'-v-h}{|v'-v-h|}+s_1v',v'_{1h*})}^0 ds_2\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v-h}{\mid v'-v-h\mid}+s_1v'+s_2v'_{1h*},v'_{2h})$$
$$-\int_{-s^+(x+s\frac{v'-v}{|v'-v|}+s_1v',v'_{1*})}^0 ds_2\frac{F^\delta}{1+\frac{F^\delta}{j}}(x+s\frac{v'-v}{\mid v'-v\mid}+s_1v'+s_2v'_{1*},v'_2))$$
$$\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_1,v'_{1h})\frac{F^\delta}{1+\frac{F^\delta}{j}}(y_2,v'_{2h*})dy_1dw_1dy_2dw_2dsds_1$$

will appear. Sets where $\frac{v'-v-h}{|v'-v-h|}$, v', and v'_{1h*} are close to parallel can be made as small as desired, to make the corresponding integrals as small as desired because of the uniform boundedness of $\frac{F^\delta}{1+\frac{F^\delta}{j}}$. Except for the sets where $\frac{v'-v-h}{|v'-v-h|}$, v' and v'_{1h*} are close to parallel, we can perform a change of variables with uniformly bounded Jacobians

$$(s,s_1,s_2)\to X:=x+s\frac{v'-v-h}{\mid v'-v-h\mid}+s_1v'+s_2v'_{1h*},$$

as well as a similar change of variables in the term without h. We end up with a difference of volume integrals integrated over volumes differing by $c\mid h\mid$ and integrals containing the difference of the Jacobians. Each such term tends to zero when h tends to zero.

We can now pass to the limit in (2.7), when δ tends to zero. The limit $F^{j,n}$ is a solution to

$$v\cdot\bigtriangledown F^{j,n}=\frac{1}{\int F^{j,n}dydv_*}\int\chi\chi^j\frac{F^{j,n}}{1+\frac{F^{j,n}}{j}}(x,v')\frac{F^{j,n}}{1+\frac{F^{j,n}}{j}}(y,v'_*)dydv_*$$

$$-F^{j,n}\frac{\int \chi\chi^j F^{j,n}(y,v_*)dydv_*}{\int F^{j,n}(y,v_*)dydv_*}, \quad (x,v)\in\Omega\times I\!R^3,$$

$$F^{j,n}(x,v)=M(x,v)\Big(\frac{\int_{w\cdot n(.)<0}\mid w\cdot n(.)\mid F^{j,n}(.,w)dw}{\int_{\partial\Omega^-}\mid w\cdot n(y)\mid F^{j,n}(y,w)dwdy}\Big)*\varphi^n_x(x),$$

$$(x,v)\in\partial\Omega^+. \tag{2.14}$$

The aim of the present section to construct a solution to the approximate problem (2.1) has then been achieved.

Lemma 2.3 *Let $f = f^{j,r,n}$ denote a solution to the approximate problem (2.14), and for $x\in\partial k\Omega$ set*

$$\rho(x):=\int_{v\cdot n(\frac{x}{k})<0}\mid v\cdot n(\frac{x}{k})\mid f(x,v)dv, \quad \sigma(x):=\frac{\rho(x)}{\int_{\partial k\Omega}\rho(x)dx}.$$

Then

$$\sigma(x)\geq c_1>0, \quad x\in\partial k\Omega,$$

with c_1 only depending on $\partial\Omega$ and M but not on j,r,n.

<u>Proof of Lemma 2.3.</u> The ingoing mass flow equals one. Take a closed connected part S of $\partial\Omega$ with $\mid S\mid=\frac{|\partial\Omega|}{4}$ and $\int_S\sigma*\varphi^n_x(x)dx\geq\frac{1}{4}$. Fix a ball $\mathcal{B}$ inside Ω and with centre x_0. For each $x_s\in\partial\Omega$, the line l through x_s and x_0 intersect $\partial\mathcal{B}$ in between at $x\in\partial\mathcal{B}$ with (antipodal) intersection $x_a\in\partial\mathcal{B}$ and $x_t\in\partial\Omega$. Choose a symmetric polar cap on $\partial\mathcal{B}$ with centre at x, such that its projection S in $\partial\Omega$ around x_s has area $\mid S\mid=\frac{1}{4}\mid\partial\Omega\mid$. Make a similar polar cap on $\partial\mathcal{B}$ around x_a with projection T to $\partial\Omega$ around x_t and area $\mid T\mid=\frac{1}{4}\mid\partial\Omega\mid$. Cover $\partial\Omega$ with N such domains $S_j, j=1,...,N$, so that the corresponding T_j's also cover $\partial\Omega$. Clearly, for at least one j,

$$\int_{S_j}\sigma*\varphi^n_x dx\geq\frac{1}{N}.$$

Also for $x\in T_j$, $v\cdot n(x)<0$,

$$f(x,v)\geq f(x-s^+(x,v)v,v)exp(-\int_{-s^+(x,v)}^0\nu(x+sv,v)ds),$$

and so for $x\in T_j$,

$$\rho(x)\geq\int_{A_{x_j}}\mid v\cdot n(x)\mid M(x-s^+(x,v)v,v)\sigma*\varphi^n_x(x-s^+(x,v)v)$$

$$exp(-\int_{-s^+(x,v)}^0\nu(x+sv,v)ds)dv,$$

with

$$A_{x_j} = \{v \in I\!R^3;\ x - s^+(x,v)v \in S_j,\ v \cdot n(x) < 0,\ 1 \leq \mid v \mid \leq 10\}.$$

Also

$$\min_j \inf_{x \in T_j} \inf_{v \in A_{x_j}} \mid \frac{v}{\mid v \mid} \cdot n(x) \mid = c_2 > 0,$$

$$\inf_{1 \leq |v| \leq 10, x \in \partial\Omega} M(x,v) = c_3 > 0,$$

$$\inf_{(x,v) \in \partial\Omega^-, 1 \leq |v| \leq 10} exp(-\int_{-s^+(x,v)}^0 \nu(x+sv,v)ds)$$
$$\geq \inf_{(x,v) \in \partial\Omega^-, 1 \leq |v| \leq 10} exp(-s^+(x,v)) = c_4 > 0.$$

Set $y = x - s^+(x,v)v$ and change variables

$$v = \mid v \mid \frac{v}{\mid v \mid} = t\omega \to (t,y), \quad dv = t^2 dt \mid \frac{D\omega}{Dy} \mid dy.$$

By hypothesis

$$\min_{1 \leq j \leq N} \inf_{x \in T_j} \inf_{v \in A_{x_j}} \mid \frac{D\omega}{Dy} \mid = c_5 > 0.$$

Hence for $x \in T_j$

$$\rho(x) \geq \int_{S_j} c_5 c_4 c_3 c_2 \sigma * \varphi_y^n(y) dy \geq \frac{c_5 c_4 c_3 c_2}{N} = c_6.$$

Here c_6 only depends on M and Ω. Using the lower bound c_6 for ρ on T_j (and n large) by a similar argument $\rho(x) \geq c_7 > 0$ for $x \in S_j$, with c_7 only depending on M and Ω. Finally using the geomerical fact that $x \in \partial\Omega \setminus (S_j \cup T_j)$ is "far" from either S_j or T_j, as well as the lower bounds c_6 and c_7 for ρ on S_j and T_j, gives in the same way a lower bound for ρ on all of $\partial\Omega$ independent of j, r and (large) n. By Green's formula for (2.14) the outflow is bounded by the inflow,

$$\int_{\partial\Omega} \rho(x) dx \leq \int \sigma * \varphi_x^n(x) dx = 1,$$

and the lemma follows. □

3 Reintroduction of the gain-loss symmetry.

In this section we shall remove the asymmetry between the gain and the loss terms by taking the limit $j \to \infty$. The smooth increase of ψ from zero to one in the interval $[-\frac{1}{m}, \frac{1}{m}]$ was needed in Section 2 for the Radon transform argument. That smoothness will also be removed from χ^r by keeping r fixed, but letting $m = j \to \infty$.

Lemma 3.1 *If F^j is a solution to (2.14), then for any $r > 0$,*

$$\int_{|v|\geq r} F^j(x,v)dxdv \leq c_8 e^{\frac{2kD}{r}}, \tag{3.1}$$

and

$$\int_{|v|\geq r} | v |^2 F^j(x,v)dxdv \leq c_9 e^{\frac{2kD}{r}}. \tag{3.2}$$

Proof of Lemma 3.1.
Multiplying (2.14) by 1 and $| v |^2$ respectively, and integrating over $\Omega \times I\!R^3$ implies that

$$\begin{aligned}\int_{\partial\Omega^-} | v \cdot n(x) | F^j(x,v)dxdv &\leq \int_{\partial\Omega^+} v \cdot n(x) F^j(x,v)dxdv \\ &= \int_{\partial\Omega} \sigma^j(x)dx = 1,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\int_{\partial\Omega^-} | v \cdot n(x) || v |^2 F^j(x,v)dxdv \\ &\leq \int_{\partial\Omega^+} v \cdot n(x) | v |^2 F^j(x,v)dxdv \\ &\qquad = c_{10} \int_{\partial\Omega} \sigma^j(x)dx = c_{10}.\end{aligned} \tag{3.4}$$

Also (2.14) implies that

$$\frac{d}{ds}(F^j(x+sv,v)e^{\int_0^s \nu_j(x+\tau v,v)d\tau}) \geq 0,$$

so that

$$\begin{aligned}F^j(x,v) &\leq F^j(x+s^-(x,v)v,v)e^{\int_0^{s^-(x,v)} \nu_j(x+\tau v,v)d\tau} \\ &\leq F^j(x+s^-(x,v)v,v)e^{\frac{D}{r}}, \quad | v |\geq r.\end{aligned}$$

Then

$$
\begin{aligned}
&\int_{\Omega\times\{v;|v|\geq r\}} F^j(x,v)dxdv \\
&= \int_{\partial\Omega^-,|v|\geq r}\int_{-s^+(x,v)}^0 F^j(y+sv,v)ds \mid v\cdot n(y)\mid dydv \\
&\leq e^{\frac{D}{r}}\int_{\partial\Omega^-,|v|\geq r} F^j(y,v)s^+(x,v)\mid v\cdot n(y)\mid dydv \\
&\leq e^{\frac{2D}{r}}\int_{\partial\Omega^-,|v|\geq r} F^j(y,v)\mid v\cdot n(y)\mid dydv,
\end{aligned}
$$

which is bounded by (3.3) uniformly in j. The proof of (3.2) can be derived analogously. □

Lemma 3.2 *The sequence of solutions (F^j) to (2.14) is weakly compact in $L^1(k\Omega\times I\!R^3)$.*

<u>Proof of Lemma 3.2.</u> By Lemma 3.1, $(\int_{|v|\geq r-\frac{1}{j}} F^j dxdv)$ is uniformly bounded. Also uniformly in j

$$
\begin{aligned}
\int_{|v|\leq r-\frac{1}{j}} F^j dxdv \leq (\sup_{\partial\Omega}\int_{|v|\leq r-\frac{1}{j},v\cdot n(x)>0} M(x,v)(\frac{v}{\mid v\mid}\cdot n(x))dv)& \\
(D\int_{\partial\Omega}\sigma^j * \varphi_x^n(x)dx) \leq c&,
\end{aligned}
$$

and analogously for $\int_{|v|\leq r-\frac{1}{j}} \mid v\mid^2 F^j dxdv$. Let us prove that (F^j) is uniformly equiintegrable. It follows from the exponential form of (2.14) for F^j that

$$
F^j(x,v)\leq F^j(x+s^-(x,v)v,v)e^{\int_0^{s^-(x,v)}\nu_j(x+\tau v,v)d\tau}, \quad (x,v)\in\Omega\times I\!R^3.
$$

Hence

$$
\begin{aligned}
&\int_{\Omega\times\{|v|\geq\delta\}} F^j log F^j(x,v)dxdv \\
&\leq e^{\frac{D}{\delta}}(\frac{D}{\delta}\int_{\Omega\times\{|v|\geq\delta\}} F^j(x+s^-(x,v)v,v)dxdv \\
&+\int_{\Omega\times\{|v|\geq\delta\}} F^j log F^j(x+s^-(x,v)v,v)dxdv) \\
&= e^{\frac{D}{\delta}}(\frac{D}{\delta}\int_{\partial\Omega^-}\mid v\cdot n(y)\mid F^j(y,v)s^+(y,v)dydv
\end{aligned}
$$

$$+\int_{\partial\Omega^-} \mid v\cdot n(y) \mid F^j logF^j(y,v)s^+(y,v)dydv)$$
$$\leq e^{\frac{D}{\delta}}\frac{D}{\delta}\int_{\partial\Omega^-} \mid v\cdot n(y) \mid F^j(y,v)(\frac{D}{\delta}+logF^j(y,v))dydv. \qquad (3.5)$$

Here by (3.3) the first term to the right is uniformly in j bounded. As for the second term the following holds.
Multiplying (2.14) by $logF^j$ and integrating over $\Omega\times I\!R^3$ implies

$$\frac{1}{4\int_{\Omega\times I\!R^3}F^j(x,v)dxdv}\int_{\Omega^2\times I\!R^6} e(F^j,F^j)(x,y,v,v_*)dxdydvdv_*$$
$$\leq \int_{\partial\Omega\times I\!R^3} v\cdot n(x)F^j logF^j(x,v)dxdv + X^j,$$

where

$$e(F^j,F^j)(x,y,v,v_*) :=$$
$$\chi\chi^j(F^j(x,v)F^j(y,v_*)-F^j(x,v')F^j(y,v'_*))$$
$$log\frac{F^j(x,v)F^j(y,v_*)}{F^j(x,v')F^j(y,v'_*)},$$

and

$$X^j := -\frac{1}{\int F^j dydv_*}\int_{\Omega^2\times I\!R^6}\chi\chi^j\frac{\frac{F^j}{j}}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(y,v'_*)$$
$$(F^j(x,v')+F^j(y,v'_*)+\frac{F^j(x,v')}{j}F^j(y,v'_*))logF^j(x,v)dxdydvdv_*.$$

Then

$$X^j \leq \frac{1}{\int F^j dydv_*}\int_{\Omega^2\times I\!R^6, F^j(x,v)\leq 1}\chi\chi^j\frac{\frac{F^j}{j}}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(y,v'_*)$$
$$(F^j(x,v')+F^j(y,v'_*)+\frac{F^j(x,v')}{j}F^j(y,v'_*)) \mid log^-F^j(x,v) \mid dxdydvdv_*.$$

Now using Lemma 2.3,

$$F^j(x,v)\geq cM\sigma^j * \varphi_x^n(x-s^+(x,v)v)\geq c\inf_{x\in\partial\Omega}M(x,v)c_1.$$

This is bounded from below by a positive Maxwellian M_0. So uniformly in j, the denominator in X^j is bounded from below, and $log^-F^j(x,v)$ is bounded from below by $c_1 + c_2 \mid v \mid^2$. It follows that uniformly in j,

$$X^j \leq c \int F^j (1+ \mid v_* \mid^2) dy dv_* \leq c.$$

This in turn implies that the outflow of entropy through $\partial\Omega^-$ (together with the entropy dissipation) is uniformly bounded with respect to j, since the inflow is uniformly bounded with respect to j (for n fixed). We conclude that also the second term to the right in (3.5) is uniformly bounded with respect to j. Since F^j is constant along characteristics for $\mid v \mid \leq r - \frac{1}{j}$ and uniformly bounded on $\partial\Omega^+$ with respect to x, v, j (for n fixed), this ends the proof of compactness. □

Lemma 3.3 *The sequence* $(\frac{1}{\int F^j dy dv_*} \int_{k\Omega \times I\!R^3} \chi\chi^j \frac{F^j}{1+\frac{F^j}{j}}(x,v') \frac{F^j}{1+\frac{F^j}{j}}(y,v'_*) dy dv_*)$ *is weakly compact in* $L^1(k\Omega \times I\!R^3)$.

Proof of Lemma 3.3. The sequence $(\frac{1}{\int_{\Omega\times I\!R^3} F^j(y,v_*) dy dv_*})$ is uniformly in j bounded from above, since by the exponential form of F^j and Lemma 2.3,

$$F^j(x,v) \geq M(x - s^+(x,v)v, v)\sigma^j * \varphi^n_{x-s^+}(x - s^+(x,v)v) e^{-s^+(x,v)},$$
$$\geq cM(x - s^+(x,v)v, v), \quad x \in \Omega, \quad \mid v \mid \geq \lambda.$$

Hence, it is sufficient to prove the weak L^1-compactness of

$$(\int_{\Omega\times I\!R^3} \chi\chi^j F^j(x,v') F^j(y,v'_*) dy dv_*).$$

By (3.2) the total mass of F^j in $\mid v \mid \geq \lambda$ tends to zero uniformly in j, when $\lambda \to \infty$. So it remains to study the equiintegrability for domains of integration, where (x,v'), $(y,v'_*) \in \Omega \times \{v \in I\!R^3; \mid v \mid \leq \lambda\}$. But there the equiintegrability in (x,v) is an immediate consequence of Lemma 3.2. □

We are now in a position to remove the asymmetry between the gain and the loss term by taking the limit $j \to \infty$. Let us start from the weak formulation of (2.14), i.e. for any test function $\zeta \in C^1_c(\bar\Omega \times I\!R^3)$ vanishing on $\partial\Omega^-$,

$$\begin{aligned}
&\int_{\Omega\times I\!R^3} (v \cdot \nabla_x \zeta) F^j(x,v) dx dv \\
&+ \frac{1}{\int F^j dy dv_*} \int_{\Omega^2 \times I\!R^6} \chi\chi^j (\frac{F^j}{1+\frac{F^j}{j}}(x,v') \frac{F^j}{1+\frac{F^j}{j}}(y,v'_*) \\
&- F^j(x,v) F^j(y,v_*)) \zeta(x,v) dx dy dv dv_* \\
&= - \int_{\partial\Omega^+} v \cdot n(x) M(x,v) (\sigma^j * \varphi^n_x)(x) \zeta(x,v) dx dv. \qquad (3.6)
\end{aligned}$$

First (for subsequences),

$$\lim_{j\to+\infty}\int_{\Omega^2\times I\!R^6}\chi\chi^j\frac{F^j}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(y,v'_*)\zeta(x,v)dxdvdydv_* =$$
$$\lim_{j\to+\infty}\int_{\Omega^2\times I\!R^6}\chi\chi^j F^j(x,v')F^j(y,v'_*)\zeta(x,v)dxdvdydv_*,$$

by the weak L^1-compactness of (F^j). Then, by the change of variables $(v,v_*)\to(v',v'_*)$,

$$\int_{\Omega^2\times I\!R^6}\chi\chi^j F^j(x,v')F^j(y,v'_*)\zeta(x,v)dxdvdydv_* =$$
$$\int_{\Omega^2\times I\!R^6}\chi\chi^j F^j(x,v)F^j(y,v_*)\zeta(x,v'(x,y,v,v_*))dxdvdydv_*.$$

(F^j), as well as $(v\cdot\bigtriangledown_x F^j)$ are weakly compact in $L^1(\Omega\times I\!R^3)$ by Lemmas 3.2-3. Consequently, via averaging, $(\int_{I\!R^3}F^j(y,v_*)\zeta(x,v'(x,y,v,v_*))dv_*dy)$ is compact in $L^1(\Omega\times I\!R^3)$ and converges to $\int_{I\!R^3}F(y,v_*)\zeta(x,v'(x,y,v,v_*))dv_*dy$, where F is a weak L^1 limit of (F^j). Hence

$$\lim_{j\to+\infty}\int_{\Omega^2\times I\!R^6}\chi\chi^j F^j(x,v)F^j(y,v_*)\zeta(x,v'(x,y,v,v_*))dxdydvdv_* =$$
$$\int_{\Omega^2\times I\!R^6}\chi F(x,v)F(y,v_*)\zeta(x,v'(x,y,v,v_*))dxdydvdv_*.$$

Moreover, (γ^+F^j) converges up to a subsequence to (γ^+F), since (F^j) and $(v\cdot\bigtriangledown_x F^j)$ are weakly compact in $L^1(\Omega\times I\!R^3)$. Hence we can pass to the limit when $j\to+\infty$ in (3.6) and obtain

$$\begin{aligned}&\int_{\Omega\times I\!R^3}(v\cdot\bigtriangledown_x\zeta)F(x,v)dxdv\\&+\frac{1}{\int_{\Omega\times I\!R^3}F(y,v_*)dydv_*}\int_{\Omega^2\times I\!R^6}\chi(F(x,v')F(y,v'_*)\\&-F(x,v)F(y,v_*))\zeta(x,v)dxdydvdv_*\\&=-\int_{\partial\Omega^+}v\cdot n(x)M(x,v)(\rho*\varphi^n_x)(x)\zeta(x,v)dxdv,\end{aligned}$$

which means that $F:=F^{r,n}$ is a weak solution to the stationary *sm*-transformed Povzner problem

$$v\cdot\bigtriangledown_x F^{r,n} =$$

$$\frac{1}{\int F^{r,n} dy dv_*} \int \chi^r (F^{r,n}(x,v')F^{r,n}(y,v'_*) - F^{r,n}(x,v)F^{r,n}(y,v_*)) dy dv_*$$
$$(x,v) \in \Omega \times I\!R^3,$$
$$F^{r,n}(x,v) = M(x,v)(\rho^{r,n} * \varphi^n_x)(x), \quad (x,v) \in \partial\Omega^+. \tag{3.7}$$

And so the aim of this section has been achieved, to obtain a solution for an approximate equation with gain and loss terms of the same type, and with the truncation χ^r a characteristic function.

4 End of proof of Theorem 1.2. The main theorem.

We now have solutions $F^{r,n}$ corresponding to the remaining approximations of (1.3-4) involving χ^r and $\varphi^n_\cdot$. We will work with subsequences (r_p) monotonically decreasing to zero. The boundary convolution with $\varphi^n_\cdot$, and the small velocity truncation χ^r will no longer be used to control weak L^1 compactness. Now that the gain-loss symmetry is reintroduced, that control will be taken over by the entropy production term and by estimating integrals along characteristics. In particular the boundary convolution will be removed through Jensen type arguments. In Theorem 4.1 the entropy production term will be used to gain r, n-independent mass control. There we carry out the analysis for an arbitrary element of the sequence $(\varphi^n_\cdot)$ (and uniformly over n), but in Proposition 4.3 about weak L^1 compactness, subsequences $(F^{r_n,n})$ are being considered. Proposition 4.3 is used for obtaining a last limit in the approximation scheme, thereby completing the proof of Theorem 1.2. Finally in the main theorem of the paper, Theorem 4.8, the generalization to hard forces is carried out.

Theorem 4.1 *The mass of $(F^{r,n})$ is bounded uniformly with respect to r, n.*

<u>Proof of Theorem 4.1.</u> Consider a sequence (r_p) monotonically decreasing to zero. If the mass of the corresponding sequence (F^p) is not uniformly bounded, then there is a subsequence (F^p) such that

$$\int_{\Omega \times I\!R^3} F^p(x,v) dx dv \geq 3c_a e^{2pD}, \tag{4.1}$$

with $c_a e^{2D} > 1$. Set $V_p = \{v \in I\!R^3;\ r_p \leq | v | \leq \frac{1}{p}\}$. Then using (3.1)

$$\int_{\Omega \times V_p} F^p(x,v) dx dv \geq 2c_a e^{2pD} \geq 2 \int_{\Omega \times \{|v| \geq \frac{1}{p}\}} F^p(x,v) dx dv,$$

so that

$$\int_{\Omega\times V_p} F^p(x,v)dxdv \geq \frac{2}{3}\int_{\Omega\times I\!R^3} F^p(x,v)dxdv.$$

By (3.1-2), ξF^p is integrable, for any component ξ of v. Green's formula can then be used for the equation (3.7). Let Π_{x_0} denote the plane through $x_0 \in \Omega$ and orthogonal to the ξ direction. By Green's formula for one of the subregions into which Ω is divided by Π_{x_0}, it holds that

$$\int_{\Pi_{x_0}\times I\!R^3} \xi^2 F^p(x,v)dxdv \leq c,$$

with c only depending on M of (1.4). Hence,

$$\int_{\Omega\times I\!R^3} \xi^2 F^p(x,v)dxdv \leq c.$$

Analogous estimates hold for the integrals of F^p time the square of any component of v. Hence

$$\int_{\Omega\times I\!R^3} | v |^2 F^p(x,v)dxdv \leq c_b. \tag{4.2}$$

Moreover,

$$\begin{aligned} F^p(y,v_*) &\geq F^p(y-s^+(y,v_*)v_*)e^{-s^+(y,v_*)} \\ = M(y-s^+(y,v_*)v_*,v_*)&(\rho^p*\varphi^p)(y-s^+(y,v_*)v_*)e^{-s^+(y,v_*)} \\ &\geq c, \quad 1 \leq| v_* |\leq \lambda, \end{aligned}$$

by Lemma 2.3. Also using the same type of estimate for $F^p(y,v'_*)$ but from above with respect to outgoing boundary, given x, for geometric reasons the following holds.For y in an x-dependent subset $\Omega_p \subset \Omega$ with $| \Omega_p |\geq \frac{|\Omega|}{2}$, there is $\bar\Omega_{py}$ with

$$| \bar\Omega_{py} |\geq \frac{4}{3}\pi(\lambda^3-1)-1,$$

and there is $\bar\Omega_{pyx} \subset \Omega_{py}$ with $| \bar\Omega_{pyx} |\gg 1$, such that,

$$\begin{aligned} v'_*(x,y,v,v_*) := v_* + (v-v_*)\cdot(\frac{x-y}{| x-y |})\frac{x-y}{| x-y |} \in \Omega_{py}, \\ v\in V_p, \quad v_* \in \bar\Omega_{pyx}. \end{aligned}$$

Here the following lemma is needed to complete the proof of Theorem 4.1.

Lemma 4.2 *Under the hypothesis (4.1) for the sequence* (F^p)*, there are* $S_p \subset \Omega$*, and for a.a.* $x \in S_p$*,* $y \in \Omega_p$*, there are* $S_{px} \subset V_p$ *and* $\Omega_{pyx} \subset \Omega_{py}$ *with* $\mid \Omega_{pyx} \mid > 1$*, such that*

$$\int_{S_p} \int_{S_{px}} F^p(x,v)dxdv \geq \frac{1}{6} \int_{\Omega \times I\!R^3} F^p(x,v)dxdv,$$

and

$$F^p(x,v) \geq p^3 F^p(x, v'(x,y,v,v_*)), \quad x \in S_p, \quad v \in S_{px}, \quad y \in \Omega_p, \quad v_* \in \Omega_{pyx},$$

where $v'(x,y,v,v_*) := v - (v - v_*) \cdot (\frac{x-y}{|x-y|}) \frac{x-y}{|x-y|}$.

Proof of Lemma 4.2. Using the previous discussion, it is enough to prove the existence of $S_p \subset \Omega$ and for a.a. $x \in S_p$ the existence of $S_{px} \subset V_p$ and $V'_{px} \subset \{v';\ 1 \leq \mid v' \mid \leq \lambda\}$ with $\mid V'^c_{px} \mid \leq 1$, such that

$$\int_{S_p} \int_{S_{px}} F^p(x,v)dxdv \geq \frac{1}{6} \int_{\Omega \times I\!R^3} F^p(x,v)dxdv,$$

and

$$F^p(x,v) \geq p^3 F^p(x,v'), \quad x \in S_p, \quad v \in S_{px}, \quad v' \in V'_{px}.$$

Let $U := \{v;\ 1 \leq \mid v \mid \leq \lambda\}$ and

$$S_p := \{x \in \Omega;\ \int_{V_p} F^p(x,v)dv \geq p \int_U F^p(x,v)dv\}.$$

Then

$$\begin{aligned}\int_{\Omega \setminus S_p} \int_{V_p} F^p(x,v)dxdv &\leq p \int_{\Omega \setminus S_p} \int_U F^p(x,v)dxdv \\ &\leq p c_e e^{2D} \leq p e^{2(1-p)D} \int_\Omega \int_{V_p} F^p(x,v)dxdv.\end{aligned}$$

Hence

$$\begin{aligned}\int_{S_p} \int_{V_p} F^p(x,v)dxdv &\geq (1 - p e^{2(1-p)D}) \int_\Omega \int_{V_p} F^p(x,v)dxdv \\ &\geq \frac{1}{3} \int_{\Omega \times I\!R^3} F^p(x,v)dxdv,\end{aligned}$$

for p large enough.

Let $x \in S_p$ be fixed. By rearrangement, there are nonincreasing, non-negative and left-continuous functions $\tilde{F}^p_x$ and $\bar{F}^p_x$ respectively, defined on $[0, \mid V_p \mid]$ and $[0, \frac{4}{3}\pi(\lambda^3-1)]$ such that

$$\mid v \in V_p;\ \alpha - d\alpha \leq F^p(x,v) \leq \alpha + d\alpha \mid =$$
$$\mid \mu \in [0, \mid V_p \mid];\ \alpha - d\alpha \leq \tilde{F}^p_x(\mu) \leq \alpha + d\alpha \mid ,$$

and

$$\mid v \in U;\ \alpha - d\alpha \leq F^p(x,v) \leq \alpha + d\alpha \mid =$$
$$\mid \mu \in [0, \frac{4}{3}\pi(\lambda^3-1)];\ \alpha - d\alpha \leq \bar{F}^p_x(\mu) \leq \alpha + d\alpha \mid .$$

Let us divide the interval $0 \leq \mu \leq \mid V_p \mid$ into two intervals I_1, I_2 in increasing order, so that

$$\int_{I_\nu} \tilde{F}^p_x d\mu = \frac{1}{2}\int_{V_p} F^p(x,v)dv, \quad \nu \in \{1,2\}.$$

For any $\tilde{\mu}$ smaller than the right endpoint of I_1,

$$\begin{aligned}\frac{c}{2p^3}\tilde{F}^p_x(\tilde{\mu}) &\geq \tilde{F}^p_x(\tilde{\mu}) \mid I_2 \mid \geq \int_{I_2} \tilde{F}^p_x(\mu)d\mu \\ &= \frac{1}{2}\int_{V_p} F^p(x,v)dv \geq \frac{p}{2}\int_U F^p(x,v)dv \\ &= \frac{p}{2}\int_0^{\frac{4}{3}\pi(\lambda^3-1)} \bar{F}^p_x(\mu)d\mu.\end{aligned}$$

Hence, for any $\tilde{\mu}$ smaller than the right endpoint of I_1 and any $\bar{\mu} \in [1, \frac{4}{3}\pi(\lambda^3-1)]$,

$$\frac{c}{p^3}\tilde{F}^p_x(\tilde{\mu}) \geq p\int_0^1 \bar{F}^p_x(\mu)d\mu \geq p\bar{F}^p_x(\bar{\mu}),$$

which implies

$$\tilde{F}^p_x(\tilde{\mu}) \geq p^3\bar{F}^p_x(\bar{\mu}),$$

for p large enough. Let

$$S_{px} := \{v \in V_p;\ F^p(x,v) \geq \tilde{F}^p_x(\tilde{\mu}),\ for\ some\ \tilde{\mu}\ smaller\ than\ the\ right\ endpoint\ of\ I_1\}.$$

Then

$$\int_{S_p}\int_{S_{px}} F^p(x,v)dxdv = \int_{S_p}\int_{I_1} \tilde{F}^p_x(\mu)d\mu = \frac{1}{2}\int_{S_p}\int_{V_p} F^p(x,v)dxdv$$
$$\geq \frac{1}{6}\int_{\Omega\times I\!R^3} F^p(x,v)dxdv. \quad \Box$$

<u>End of proof of Theorem 4.1.</u>
Multiplying (3.7) with $logF^p$ and integrating over $\Omega\times I\!R^3$ implies

$$\frac{1}{4\int F^p dydv_*}\int_{\Omega^2\times I\!R^6} e(F^p,F^p)(x,y,v,v_*)dxdydvdv_* =$$
$$\int_{\partial\Omega\times I\!R^3} v\cdot n(x)F^p logF^p(x,v)dxdv =$$
$$\int_{\partial\Omega\times I\!R^3} v\cdot n(x)F^p log\frac{F^p}{M}(x,v)dxdv$$
$$+\int_{\partial\Omega\times I\!R^3} v\cdot n(x)F^p logM(x,v)dxdv.$$

Here

$$e(F,F) := \chi^r(F(x,v)F(y,v_*) - F(x,v')F(y,v'_*))log\frac{F(x,v)F(y,v_*)}{F(x,v')F(y,v'_*)}.$$

Also

$$\int_{\partial\Omega\times I\!R^3} v\cdot n(x)F^p logM(x,v)dxdv =$$
$$-\int_{\partial\Omega\times I\!R^3} v\cdot n(x)F^p(b(x)+c(x)\mid v\mid^2)dxdv \leq c\int_{\partial\Omega}\rho^p(x)dx = c,$$

and

$$\int_{\partial\Omega^+} v\cdot n(x)F^p log\frac{F^p}{M}(x,v)dxdv = \int_{\partial\Omega}\rho^p*\varphi^n_x(x)log(\rho^p*\varphi^n_x)(x)dx$$
$$\leq \int_{\partial\Omega}\varphi^n_x*(\rho^p log\rho^p)(x)dx = \int_{\partial\Omega}\rho^p log\rho^p(x)dx,$$

by Jensen's inequality. Again by Jensen's inequality (see [9]),

$$\int_{\partial\Omega}\rho^p(x)log\rho^p(x)dx$$
$$\leq \int_{\partial\Omega^-}\mid v\cdot n(x)\mid F^p log\frac{F^p}{M}(x,v)dxdv,$$

since

$$\rho(x) = \int_{v \cdot n(x) < 0} \mid v \cdot n(x) \mid M \frac{F^p}{M}(x,v) dv.$$

Hence

$$\frac{1}{\int F^p dy dv_*} \int_{\Omega^2 \times I\!R^6} e(F^p, F^p)(x, y, v, v_*) dx dy dv dv_* \leq c < \infty.$$

This together with Lemma 4.2 implies

$$\begin{aligned}
&\int_{S_p} \int_{S_{px}} F^p(x,v) dx dv \geq \frac{1}{6} \int_{\Omega \times I\!R^3} F^p(x,v) dx dv \\
&\geq c \int_{\Omega^2 \times I\!R^6} e(F^p, F^p)(x, y, v, v_*) dx dy dv dv_* \\
&\geq c \int_{x \in S_p, v \in S_{px}, y \in \Omega_p, v_* \in \Omega_{pyx}} e(F^p, F^p)(x, y, v, v_*) dx dy dv dv_*.
\end{aligned}$$

The inequalities

$$\begin{aligned}
&F^p(x,v) \geq p^3 F^p(x,v'), \quad c \leq F^p(y, v_*) \leq 1, \\
&c \leq F^p(y, v'_*) \leq 1, \quad x \in S_p, \quad v \in S_{px}, \quad y \in \Omega_p, \quad v_* \in \Omega_{pyx},
\end{aligned}$$

imply that

$$F^p(x,v) F^p(y, v_*) - F^p(x, v') F^p(y, v'_*) \geq c F^p(x,v),$$

and

$$\frac{F^p(x,v) F^p(y,v_*)}{F^p(x,v') F^p(y,v'_*)} \geq c p^3,$$

so that

$$e(F^p, F^p)(x, y, v, v_*) \geq c \chi^{r_p} F^p(x,v) logp.$$

Moreover, $\mid \Omega_p \mid$, and for a.a. $x \in S_p$, $y \in \Omega_p$, $\mid \Omega_{pyx} \mid$ are bounded from below, uniformly with respect to p and x, y. Hence,

$$\int_{S_p} \int_{S_{px}} \psi^{r_p} F^p(x,v) dx dv \geq c logp \int_{S_p} \int_{S_{px}} \psi^{r_p} F^p(x,v) dx dv,$$

which leads to a contradiction. This ends the proof of Theorem 4.1. □
Remark. In this step the condition of the diffuse reflection being Maxwellian was used to obtain

$$\int v \cdot n(x) F^p(x,v) log M(x,v) dxdv \leq c.$$

A number of generalizations are obviously also possible, including the one of replacing M by normalized functions ϕ with

$$\int v \cdot n(x) F^p(x,v) log\phi(x,v) dxdv \leq c.$$

The key Theorem 1.2 will easily follow from the previous results together with weak sequential compactness of $(F^{r,n})$.

Proposition 4.3 *Any sequence* $(F^{r_n,n})$ *with* $\lim_{n\to\infty} r_n = 0$ *is weakly compact in* $L^1(k\Omega \times I\!R^3)$.

Proof of Proposition 4.3. The statement follows if the sequence $(F^{r_n,n})$ is uniformly equiintegrable. Given ϵ, there is K_ϵ such that

$$\int_\Omega \int_{|v|\geq K_\epsilon} F^{r_n,n}(x,v) dxdv < \epsilon.$$

So if the proposition does not hold, then there is $\epsilon > 0$, a subsequence (F^n) and a sequence of domains (A_n) with $(A_n) \subset \Omega \times \{r_n \leq \mid v \mid \leq K_\epsilon\}$, such that

$$\int_{A_n} F^n(x,v) dxdv \geq \epsilon, \quad \mid A_n \mid < n^{-3}.$$

Let us first discuss the case of

$$A_n \subset \Omega \times V_n = \Omega \times \{v \in I\!R^3;\ r_n \leq \mid v \mid \leq \frac{1}{n}\}.$$

By Theorem 4.1, for some $c_0 > 0$,

$$\int_{\Omega \times I\!R^3} F^{r,n}(x,v) dxdv \leq c_0 < \infty,$$

independently of r, n. And so

$$\int_{\Omega \times V_n} F^n(x,v) dxdv \geq \epsilon \geq \frac{\epsilon}{c_0} \int_{\Omega \times I\!R^3} F^n(x,v) dxdv.$$

We can then proceed as in the proof of Theorem 4.1 introducing Ω_n, Ω_{ny} and $\bar{\Omega}_{nyx}$. We will also need a variant of Lemma 4.2, namely

Lemma 4.4 *There are $S_n \subset \Omega$, and for a.a. $x \in S_n$, $y \in \Omega_n$, there are $S_{nx} \subset V_n$, $\Omega_{nyx} \subset \bar{\Omega}_{ny}$ with $| \Omega_{nyx} |> 1$, such that*

$$\int_{S_n}\int_{S_{nx}} F^n(x,v)dxdv \geq \frac{\epsilon}{4c_0}\int_{\Omega\times I\!\!R^3} F^n(x,v)dxdv,$$

and for $x \in S_n$, $v \in S_{nx}$, $y \in \Omega_n$, $v_* \in \Omega_{nyx}$,

$$F^n(x,v) \geq n^2 F^n(x, v'(x,y,v,v_*)).$$

<u>Proof of Lemma 4.4.</u> It is enough to prove the existence of $S_n \subset \Omega$ and for a.a. $x \in S_n$ the existence of $S_{nx} \subset V_n$ and $V'_{nx} \subset \{v' \in I\!\!R^3;\ 1 \leq | v' | \leq \lambda\}$ with $| V'^c_{nx} | \leq 1$ such that

$$\int_{S_n}\int_{S_{nx}} F^n(x,v)dxdv \geq \frac{\epsilon}{4c_o}\int_{\Omega\times I\!\!R^3} F^n(x,v)dxdv,$$

and

$$F^n(x,v) \geq n^2 F^n(x,v'), \quad x \in S_n, \quad v \in S_{nx}, \quad v' \in V'_{nx}.$$

Let $U := \{v \in I\!\!R^3;\ 1 \leq | v | \leq \lambda\}$ and

$$S_n := \{x \in \Omega;\ \int_{V_n} F^n(x,v)dv \geq \frac{\epsilon}{2c_b}\int_U F^n(x,v)dv\}.$$

Then by (4.2),

$$\int_{\Omega\setminus S_n}\int_{V_n} F^n(x,v)dxdv \leq \frac{\epsilon}{2c_b}\int_{\Omega\setminus S_n}\int_U F^n(x,v)dxdv \leq \frac{\epsilon}{2}$$
$$\leq \frac{1}{2}\int_\Omega\int_{V_n} F^n(x,v)dxdv.$$

Hence

$$\int_{S_n}\int_{V_n} F^n(x,v)dxdv \geq \frac{1}{2}\int_\Omega\int_{V_n} F^n(x,v)dxdv$$
$$\geq \frac{\epsilon}{2c_0}\int_{\Omega\times I\!\!R^3} F^n(x,v)dxdv.$$

Given $x \in S_n$, introduce the rearrangements $\tilde{F}^n_x$ and $\bar{F}^n_x$ as in the proof of Lemma 4.2 and conclude as there that for any $\tilde{\mu}$ smaller than the right endpoint of I_1 and any $\bar{\mu} \in [1, \frac{4}{3}\pi(\lambda^3 - 1)]$,

$$\frac{c}{n^3}\tilde{F}^n_x(\tilde{\mu}) \geq \frac{\epsilon}{4c_b}\bar{F}^n_x(\bar{\mu}).$$

Hence, for large n,

$$\tilde{F}_x^n(\tilde{\mu}) \geq n^2 \bar{F}_x^n(\bar{\mu}).$$

Let

$$S_{nx} := \{v \in V_n;\ F^n(x,v) \geq \tilde{F}_x^n(\tilde{\mu})\ \textit{for some } \tilde{\mu} \textit{ smaller than the right endpoint of } I_1\}.$$

Then

$$\int_{S_n}\int_{S_{nx}} F^n(x,v)dxdv = \int_{S_n}\int_{I_1} \tilde{F}_x^n(\mu)d\mu = \frac{1}{2}\int_{S_n}\int_{V_n} F^n(x,v)dxdv$$
$$\geq \frac{\epsilon}{4c_0}\int_{\Omega\times I\!R^3} F^n(x,v)dxdv. \quad \square$$

Continuation of the proof of Proposition 4.3. From the proof of Theorem 4.1, we know that uniformly with respect to n,

$$\int_{\Omega^2\times I\!R^6} e(F^n,F^n)(x,y,v,v_*)dxdydvdv_* \leq c < \infty,$$

which this time using Lemma 4.4 leads to the same contradiction as in the proof of Theorem 4.1. We have thus proved that given $\epsilon > 0$, there is n_ϵ such that

$$\int_\Omega\int_{|v|\leq n_\epsilon^{-1}} F^n(x,v)dxdv < \epsilon.$$

It remains to prove the uniform ϵ-equiintegrability for (F^n) on $S_\epsilon = \Omega\times\{v \in I\!R^3;\ n_\epsilon^{-1} \leq\mid v\mid\leq K_\epsilon\}$. Assume that in the set S_ϵ there is a sequence (A_n) with $\mid A_n\mid\leq n^{-3}$ such that

$$\int_{A_n} F^n(x,v)dxdv \geq \epsilon.$$

Then $A_n = A_{n_1}\cup A_{n_2}$, with

$$A_{n_1} := \{(x,v)\in A_n; meas\{w \in I\!R^3; (x,w)\in A_n\} \leq \frac{1}{n}\},$$
$$A_{n_2} := A_n \setminus A_{n_1}.$$

Arguing similarly to the earlier case of $\mid v\mid\leq n_\epsilon^{-1}$, there is $n_0 \in I\!N$ such that

$$\int_{A_{n_1}} F^n(x,v)dxdv < \frac{\epsilon}{2}, \quad n > n_0.$$

It remains to exclude the possibility that $\int_{A_{n_2}} F^n(x,v)dxdv > \frac{\epsilon}{2}$ for an unbounded sequence of n. For this we first prove the weak $L^1(\partial\Omega)$ compactness of (ρ^n). That result is a direct consequence of the following two lemmas. Write $\rho^n = G^n + H^n$, where

$$G^n(x) := \int_{v\cdot n(x)<0} g_n(x,v)dv,$$
$$g_n(x,v) :=\mid v\cdot n(x)\mid M(x-s^+(x,v)v,v)\rho^n * \varphi_x^n(x-s^+(x,v)v)$$
$$e^{-\int_{-s^+(x,v)}^0 \nu_n(x+sv,v)ds},$$

$$H^n(x) := \frac{1}{\int F^n dydv_*}\int_{v\cdot n(x)<0}\mid v\cdot n(x)\mid \int_{-s^+(x,v)}^0 e^{-\int_s^0 \nu_n(x+\tau v,v)d\tau}$$
$$\int \chi^{r_n}(x+sv-y,v,v_*)F^n(x+sv,v')F^n(y,v'_*)dydv_*dvds.$$

Lemma 4.5 *(G^n) is strongly compact in $L^1(\partial\Omega)$.*

Lemma 4.6 *(H^n) is weakly compact in $L^1(\partial\Omega)$.*

Proof of Lemma 4.5. (G^n) is uniformly bounded in L^1 by $\int_{\partial\Omega}\rho^n(y)dy$. Next, (G^n) is an equicontinuous family, uniformly with respect to $x\in\partial\Omega$. Indeed, for a fixed $x\in\partial\Omega$, let $A_x := \{v\in I\!R^3; 0< -\frac{v}{|v|}\cdot n(x)\leq\delta\}$, $B_x := \{v\in I\!R^3; 0<-v\cdot n(x), \mid v\mid\leq\delta\}$, and
$C_x := \{v\in I\!R^3;\delta< -\frac{v}{|v|}\cdot n(x), \delta<\mid v\mid\leq K\}$. The M-factor makes the contribution from $\mid v\mid > K$ arbitrarily small for K large enough. Writing v in polar coordinates $v=\mid v\mid\omega$, $\omega\in S^2$, and bounding from above the exponential term by 1, implies that

$$\int_{A_x} g_n(x,v)dv \leq \int_{\omega\in S^2;0\leq-\omega\cdot n(x)\leq\delta}\mid\omega\cdot n(x)\mid(\rho^n*\varphi_x^n)(x-s^+(x,\omega)\omega)$$
$$\int_0^{+\infty}\mid v\mid^3 M(x-s^+(x,\omega)\omega,\mid v\mid\omega)d\mid v\mid d\omega$$
$$\leq c\delta\int_{\omega\in S^2;0\leq-\omega\cdot n(x)\leq\delta}(\rho^n*\varphi_x^n)(x-s^+(x,\omega)\omega)d\omega.$$

By the change of variables $\omega\to y = x-s^+(x,\omega)\omega\in\partial\Omega$,

$$\int_{A_x} g_n(x,v)dv \leq c\delta\int_{y\in\partial\Omega,0\leq-\frac{x-y}{|x-y|}\cdot n(x)\leq\delta}(\rho^n*\varphi_x^n)(y)\mid\frac{D\omega}{Dy}\mid dy$$
$$\leq c\delta\int_{\partial\Omega}\rho^n*\varphi_x^n = c\delta,$$

since by hypothesis $\mid \frac{D\omega}{Dy} \mid$ is uniformly bounded. The contribution of B_x to G_n is in the same way bounded by

$$\int_{\omega\in S^2;0\leq -\omega\cdot n(x)} \mid \omega\cdot n(x) \mid (\rho^n * \varphi_x^n)(x-s^+(x,\omega)\omega)$$
$$(\int_0^\delta \mid v \mid^3 M(x-s^+(x,\omega)\omega, \mid v \mid \omega) d \mid v \mid) d\omega$$
$$\leq c\delta \int_{S^2} \mid \omega\cdot n(x) \mid (\rho^n * \varphi_x^n)(x-s^+(x,\omega)\omega) d\omega$$
$$\leq c\delta.$$

There remains to study the contribution from C_x to G^n.

$$\int_{C_{x+h}} g_n(x+h,v)dv = \int_{-\omega\cdot n(x+h)>\delta} \mid \omega\cdot n(x+h) \mid$$
$$\rho^n * \varphi_{x+h}^n(x+h-s^+(x+h,\omega)\omega)(\int_\delta^{K_\epsilon} \mid v \mid^3 M(x+h-s^+(x+h,\omega)\omega, \mid v \mid \omega)$$
$$e^{-\frac{1}{\int F^n dy dv_*}\int_{-s^+(x+h,|v|\omega)}^0 \int \chi^{rn}(x+h+s|v|\omega-y,|v|\omega,v_*)F^n(y,v_*)dydv_*ds} d \mid v \mid) d\omega.$$

Set

$$u_h = \frac{h+s^+(x+h,v)v}{\mid h+s^+(x+h,v)v \mid}.$$

Obviously $\frac{Du_h}{D\omega} = identity$ for $h=0$. By continuity, the convergence to this value when $h\to 0$ is uniform with respect to $x\in\partial\Omega$, $v\in C_x$. Also

$$\lim_{h\to 0} \omega\cdot n(x+h) - u_h\cdot n(x) = 0,$$

uniformly in $x\in\partial\Omega$,

$$\lim_{h\to 0}(e^{-\frac{1}{\int F^n dy dv_*}\int_{-s^+(x+h,|v|\omega)}^0 \int \chi^{rn}(x+h+s|v|\omega-y,|v|\omega,v_*)F^n(y,v_*)dydv_*ds}$$
$$-e^{-\frac{1}{\int F^n dy dv_*}\int_{-s^+(x,|v|\omega)}^0 \int \chi^{rn}(x+s|v|\omega-y,|v|\omega,v_*)F^n(y,v_*)dydv_*ds}) = 0,$$

uniformly with respect to $x\in\partial\Omega$, $v\in C_x$. Set

$$G_\delta^n(x) = \int_{C_x} g_n(x,v)dv,$$

and change variables from $d\omega$ to du_h in $G^n_\delta(x)$. It follows that $\lim_{h\to 0} G^n_\delta(x+h) - G^n_\delta(x) = O(\delta)$, uniformly with respect to $n \in I\!N$, $x \in \partial\Omega$, $0 \leq \rho \in L^1(\partial\Omega)$ with $\int_{\partial\Omega} \rho dx = 1$. $\square$

Proof of Lemma 4.6. It follows from the earlier discussion of 'small velocity mass for F^n', that in the definition of $H^n(x)$ it is enough to consider domains of integration with $\mid v \mid, \mid v_* \mid, \mid v' \mid, \mid v'_* \mid \geq \delta > 0$, and from the uniform bound for $\int_{|v|\geq 1} \mid v \mid^2 F^n(x,v)dxdv$ that it is enough to consider domains of integration with $\mid v \mid, \mid v_* \mid, \mid v' \mid, \mid v'_* \mid < \frac{1}{\delta}$. It also follows from the (A_{n_1}) part that it is enough to discuss as domains of integration, sets with $\mid \frac{v}{|v|} \cdot \frac{v'}{|v'|} \mid \geq \delta$. Let χ_δ be the characteristic function of the remaining v, v_*, v', v'_*. So with B_m a set in $\partial\Omega$ of measure $\leq m^{-2}$, instead of $I^m = \int_{B_m} H^n(x)dx$, it is enough to discuss

$$I^m_\delta = \frac{1}{\int F^n dydv_*} \int_{B_m \times I\!R^3} \mid v \cdot n(x) \mid \int_{-s^+}^0 \chi_\delta$$
$$F^n(x+sv, v')F^n(y, v'_*)e^{-\int_s^0 \nu_n d\tau} dsdydv_*dxdv,$$

since the contribution outside suppχ_δ tends to zero with δ uniformly in n. For s, v, v' fixed, $F^n(x+sv, v')$ can also be expressed in exponential form as an ingoing boundary value plus a gain term integral along a characteristic. The contribution from its boundary term to I^m_δ gives - similarly to the proof of Lemma 4.5 - a contribution to I^m_δ which tends to zero when $m \to \infty$, uniformly with respect to n. Another χ_δ truncation also removes an integral in the gain part, uniformly with respect to n small of order $o(1)$ in δ. Repeating once again the procedure of expressing the latest remaining gain term in exponential form, leads to a boundary contribution tending to zero when $m \to \infty$, uniformly in n, and an inner integral. Taking into account that all the occuring exponentials of collision frequencies are bounded by one, this inner integral is bounded by

$$K^m := \frac{1}{(\int F^n dydv_*)^3} \int_Z F^n(y_1, V'_{1*})F^n(y_2, V'_{2*})$$
$$F^n(y_3, V'_{3*})F^n(x + s_1 v + s_2 V'_1 + s_3 V'_2, V'_3)\chi^1_\delta \chi^2_\delta dxdvdy_{123}dv_{*123}ds_{123},$$

where

$$V'_1 := v'(x + s_1 v, y_1, v, v_*),$$

$$
\begin{aligned}
V_2' &:= v'(x + s_1 v + s_2 V_1', y_2, V_1', v_{*1}),\\
V_3' &:= v'(x + s_1 v + s_2 V_1' + s_3 V_2', y_3, V_2', v_{*2}),
\end{aligned}
$$

and analogous definitions for the $*$ variables, and

$$
\begin{aligned}
& Z := \{(x, v, y_i, v_{*i}, s_i);\ x \in B_m,\\
& v \in I\!R^3, y_i \in I\!R^3, v_{*i} \in I\!R^3, s_1 \in [-s^+(x, v), 0],\\
& s_2 \in [-s^+(x + s_1 v, V_1'), 0], s_3 \in [-s^+(x + s_1 v + s_2 V_1', V_2'), 0]\}.
\end{aligned}
$$

For (x, s_i, y_i) fixed, by the successive changes of variables $(v, v_*) \to (V_1', V_{1*}')$, $(V_1', v_{*1}) \to (V_2', V_{2*}')$, $(V_2', v_{*2}) \to (V_3', V_{3*}')$, $dvdv_{*123}$ is replaced by $JdV_3'dV_{1*}'dV_{2*}'dV_{3*}'$, where J is the product of the three successive Jacobians and is bounded from above due to the χ_δ truncations giving

$$
\mid \frac{v}{\mid v \mid} \cdot \frac{V_1'}{\mid V_1' \mid} \mid \geq \delta, \quad \mid \frac{v}{\mid v \mid} \cdot \frac{V_2'}{\mid V_2' \mid} \mid \geq \delta, \quad \mid \frac{V_1'}{\mid V_1' \mid} \cdot \frac{V_2'}{\mid V_2' \mid} \mid \geq \delta.
$$

Hence K^m is bounded by

$$
\begin{aligned}
& K^m \leq \frac{c}{(\int F^n dy dv_*)^3} \int F^n(y_1, V_{1*}') F^n(y_2, V_{2*}')\\
& F^n(y_3, V_{3*}') F^n(x + s_1 v + s_2 V_1' + s_3 V_2', V_3') dx dV_3' dV_{i*}' dy_i ds_i,
\end{aligned}
$$

where now v, V_1' and V_2' are functions of $(x, V_3', V_{i*}', y_i, s_i)$. For (x, V_3', V_{*i}') fixed, let us make the change of variables

$$
(s_1, s_2, s_3) \to z = x + s_1 v + s_2 V_1' + s_3 V_2'.
$$

The set $\{z = x + s_1 v + s_2 V_1' + s_3 V_2';\ (x, v, y_i, v_{*i}, s_i) \in Z\}$ is a volume in $I\!R^3$ and the Jacobian $\mid \frac{Ds}{Dz} \mid$ is bounded from above due to the χ_δ truncations, so that

$$
K^m \leq c \mid B_m \mid \int_{\Omega \times I\!R^3} F^n(z, V_3') dz dV_3' \leq c m^{-2}.
$$

We conclude that (H^n) is weakly compact in $L^1(\partial\Omega)$.
<u>End of proof of Proposition 4.3.</u> It remains to study $\int_{A_{n_2}} F^n(x, v) dx dv$, where

$$
meas\{x \in \Omega;\ \exists v \in I\!R^3 s.t. (x, v) \in A_{n_2}\} \leq n^{-2}.
$$

Write F^n in exponential form as the sum of an ingoing boundary value term B^n and a characteristic gain term integral C^n. For the integral of B^n over

A_{n_2}, again given v, split the x'es of A_{n_2}, into those x_0 where $(x_0, v) \in A_{n_2}$ and the set of $x = x_0 + tv$ such that $(x, v) \in A_{n_2}$, has measure smaller than n^{-1} (small set along characteristic), and the rest which projected into a plane orthogonal to v have measure smaller than n^{-1} (small set of characteristics). The B^n integral for the first subset of A_{n_2} is bounded by cn^{-1}, with c independent of n. The B^n integral over the second subset of A_{n_2} tends to zero when $n \to \infty$ by the compactness of (ρ^n). So $\lim_{n\to\infty} \int_{A_{n_2}} B^n = 0$.
As for the (C^n) sequence, the same type of arguments that proved the compactness of (H^n) in Lemma 4.6, gives that $\lim_{n\to+\infty} \int_{A_{n_2}} C^n = 0$. This completes the proof of the proposition. □

Lemma 4.7 *The sequence* $(\frac{1}{\int F^n dydv_*} \int_{k\Omega\times I\!R^3} \chi^{r_n} F^n(x, v')F^n(y, v'_*)dydv_*)$ *is weakly compact in* $L^1(k\Omega \times I\!R^3)$.

Proof of Lemma 4.7. The proof of Lemma 4.7 is similar to the proof of Lemma 3.3. □

Proof of Theorem 1.2. Consider the weak L^1 limits of $(F^{r_n,n})$ and $\rho^{r_n,n} * \varphi^n_x := \rho^n * \varphi^n_x$. Using the weak L^1-compactness of Proposition 4.3 and Lemmas 4.5-7, we can pass to the limit in the weak formulation of (3.7), analogously to the end of Section 3. The weak limit of (ρ^n) is by the usual trace arguments equal to the outgoing flux of the limit F of (F^n). Hence F satisfies the sm-transformed problem (1.5-6), and so Theorem 1.2 follows by an application of Lemma 1.3. □

The discussion so far was restricted to the collision kernel in the Povzner collision operator being identically equal to 1. We shall now finally in the main theorem of the paper extend the results to more general collision kernels. Here test functions for the weak form of the Povzner equation are functions in $L^\infty(\Omega \times I\!R^3)$, continuously differentiable along characteristics and vanishing on $\partial\Omega^-$.

Theorem 4.8 *There exists a weak solution* f *in* $L^1(\Omega \times I\!R^3)$ *with given total mass* $k > 0$ *to the boundary value problem (1.1-2) when the collision kernel is a strictly positive function* $\tilde{B}(\frac{x-y}{|x-y|}, v - v_*)$, *bounded in* $L^\infty(S^2 \times I\!R^3)$ *(of pseudomaxwellian type), or is of hard force type*

$$B(\frac{x-y}{\mid x-y\mid}, v - v_*) = \tilde{B}(\frac{x-y}{\mid x-y\mid}, v - v_*) \mid v - v_* \mid^\beta, \ 0 \leq \beta < 2. \qquad (4.3)$$

Proof of Theorem 4.8. First, the existence of a weak L^1_+ solution to the Povzner equation in the case of a positive, bounded C^∞ collision kernel $B(\frac{x-y}{|x-y|}, v-v_*)$ can be proven in the same way as for Theorem 1.2 in the previous sections, where the collision kernel was identically equal to one. Let us next consider the case when the collision kernel B is bounded and measurable. Let (B_p) be a sequence of bounded C^∞ functions uniformly converging to B outside of sets of measure ϵ for all $\epsilon > 0$. The existence of a solution F^p to the Povzner equation with collision kernel B_p and total mass k^4 for the sm-transform is already clear. Then the arguments in the proof of Theorem 4.1 and Proposition 4.3 apply to (F^p), so that (F^p) is weakly compact in L^1. Using this weak compactness, as well as the uniform convergence of (B_p) outside of arbitrarily small sets, one can pass to the limit in the weak formulation satisfied by F^p.
Consider next the case of a collision kernel of the hard force type (4.3). Lemma 1.3 holds also in this case. Namely, solving the original Povzner problem (1.1) under (4.3), is equivalent to solving the following transformed problem

$$v \cdot \nabla_X F(X,v) = \frac{1}{\int (1+ \mid v_* \mid)^\beta F(Y,v_*) dY dv_*} \int B(\frac{X-Y}{\mid X-Y \mid}, v-v_*)$$
$$F(X,v')F(Y,v'_*) dY dv_* - F(X,v) \frac{\int BF(Y,v_*)dYdv_*}{\int (1+ \mid v_* \mid)^\beta F(Y,v_*) dY dv_*},$$

where

$$F(X,v) := f(\frac{X}{\int (1+ \mid v \mid)^\beta f(x,v) dx dv}, v), \quad \frac{X}{\int (1+ \mid v \mid)^\beta f(x,v) dx dv} \in \Omega, \quad v \in I\!R^3.$$

The proofs of the preceeding two steps imply the existence of solutions F^p to such transformed problem with collision kernels $B_p = min(B,p)$ and with $\int (1+ \mid v \mid^\beta) F(x,v) dX dv = k^4$. We can pass to the limit in the weak formulation satisfied by (F^p) when $p \to \infty$ by once more applying the proofs of Theorem 4.1 and Proposition 4.3. □
Remark. Other generalizations can also be treated by this approach, such as given indata boundary conditions, there also the soft force case with $-3 < \beta < 0$ by involving the renormalized solution concept, as well as more localized situations such as

$$B(x-y, v-v_*) := \tilde{B}(\frac{x-y}{\mid x-y \mid}, v-v_*) \mid v-v_* \mid^\beta \chi(x-y), \quad -3 < \beta < 2,$$

where χ is the characteristic function of a neighbourhood of zero. A paper in preparation by V.Panferov [29], studies the latter more physical situation.

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