ON THE NUMBER OF NEGATIVE EIGENVALUES FOR THE TWO-DIMENSIONAL MAGNETIC SCHRÖDINGER OPERATOR

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To Mikhail Shlëmovich Birman on the occasion of his 70-th birthday, as a sign of friendship and admiration

Introduction. Eigenvalue estimates for operators of mathematical physics play an important part in the study of asymptotic properties of the discrete spectrum, scattering theory, stability of matter etc. Among such results, estimates having the semiclassical order in the coupling constant are of special interest. The best known example here is the CLR-estimate for the Schrödinger operator $H(V) = -\Delta - V$ with the real potential $V \in L_{1,\mathrm{loc}}$:

$$N_0(H(V)) \le C(d) \int\limits_{\mathbb{R}^d} V_+(x)^{\frac{d}{2}} dx, \qquad d \ge 3. \tag{1}$$

Here and below we denote by $N_\beta(H)$ the number of eigenvalues of a selfadjoint operator H in the interval $(-\infty, \beta)$, setting this to be ∞ if there are points of the essential spectrum below β; V_+ stands for the positive part of V. Introducing the coupling constant, i.e. replacing the potential V by qV, we get on the right-hand side of (1) the factor $q^{\frac{d}{2}}$, and this is exactly the asymptotic order that the left-hand side has as $q \to \infty$. Besides, the asymptotic coefficient is just $c(d) \int V_+^{\frac{d}{2}} dx$. In other words, the quantity $N_0(H(qV))$ is estimated by its own asymptotics (within the value of the constant factor in the estimate).

Evidently, (1) implies the same inequality for $N_{-\gamma}(H(V))$ with any $\gamma > 0$.

The estimate (1) holds only for $d \ge 3$. In the two-dimensional case it fails, and one has to change the right-hand side, in order to save the semiclassical order $\frac{d}{2} = 1$. Moreover, the estimates for $N_{-\gamma}(H(V))$ with $\gamma = 0$ and $\gamma > 0$ look quite different.

The first author was partially supported by NFR grant MMA 11610-300.
The second author was partially supported by NFR grant RRA 11610-302.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let f be a function in $L_{r,\mathrm{loc}}(\mathbb{R}^2)$ for some $r > 1$. For an arbitrary $\mathbf{n} = \{n_1, n_2\} \in \mathbb{Z}^2$ denote $Q_{\mathbf{n}} = (n_1, n_1 + 1) \times (n_2, n_2 + 1)$ and put

$$S_r(f) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \left(\int_{Q_{\mathbf{n}}} |f|^r dx \right)^{\frac{1}{r}}.$$

Then for $\gamma > 0$

$$N_{-\gamma}(H(V)) \leq C(\gamma, r) S_r(V_+), \qquad d = 2. \tag{2}$$

This result, which also has correct order in the coupling constant, was obtained in [BBor]. Much later, this estimate was improved in [Sol], where the L_r-norms in (2) were replaced by the norm in the Orlicz space $L\log(1 + L)$.

In the same paper [Sol], estimates for $N_0(H(V))$ were obtained. The main difference with the case $\gamma > 0$ is that for $d = 2$ the contribution of the subspace of radially symmetric functions has to be considered separately. As a result, $N_0(H(V))$ is estimated by the sum of two independent terms of a different nature, and neither of them may be removed. In [BL] the same idea was used for the study of the asymptotic behavior of $N_0(H(qV))$ as $q \to \infty$.

Let us now pass to the Schrödinger operator with magnetic field

$$H_{\mathbf{a}}(V) = -\sum_{j=1}^{d} (\partial_{x_j} - i a_j)^2 - V,$$

where $a_j = a_j(x) \in L_{2,\mathrm{loc}}(\mathbb{R}^d)$ are components of the magnetic potential $\mathbf{a}$. For $H_{\mathbf{a}}(V)$ the inequality (1) (the *magnetic CLR-estimate*) also holds. A proof was outlined in [S1]. Another proof was given later in [MRoz]. Both approaches do not apply to $d = 2$. An attempt to handle the case $d = 2$ was made in [Ra2]. However, the estimate obtained there involves rather restrictive conditions on the magnetic potential $\mathbf{a}$ and the constant in the estimate depends on $\mathbf{a}$.

In [RozSol], by establishing an abstract version of Lieb's approach to the proof of (1), its analog was obtained for $N_0(\mathcal{A} - V)$, where $\mathcal{A}$ is a positive selfadjoint operator in a rather wide class. In particular, this gave one more proof of the magnetic CLR-estimate for $d \geq 3$. In this note we show that the approach of [RozSol] allows one to treat the two-dimensional case as well. More exactly, in Sec.1 – 4 we obtain an analog of the Birman – Borzov estimate (2), with a constant not depending on the magnetic potential. Basing on this, we justify in Sec.5 the Weyl type asymptotics for $N_{-\gamma}(H_{\mathbf{a}}(qV))$ as $q \to \infty$. Combining this result with a statement of a rather general nature, established in [B2], we also study the discrete spectrum in the gaps for perturbations of the operator $H_{\mathbf{a}}(V)$. Thus, we extend to the two-dimensional case the results of [Ra1] and [BRa]. This, in particular, removes excessive restrictions on the magnetic potential set in [Ra2].

In the final Sec.6 we consider the much more subtle question on estimates and asymptotics for $N_0(H_{\mathbf{a}}(V))$. First we obtain an estimate for $N_0(H_{\mathbf{a}}(V)+h(x)|x|^{-2})$ with a slowly varying positive function h, i.e. with a regularizing term. For constant function h this extends to our case a result of [BL] for the non-magnetic operator. After that, we prove an estimate for $N_0(H_{\mathbf{a}}(V))$, without the regularizing term. This time, the constant in the estimate may depend on the magnetic potential. Nevertheless, our result gives conditions ensuring finiteness of the negative spectrum and is sufficient to prove the large coupling constant asymptotics.

1. The main result. We start with necessary definitions. Let $\mathbf{a} = \{a_1, a_2\}$ be a magnetic vector potential on $\mathbb{R}^2$, whose components are real-valued functions $a_j = a_j(x_1, x_2) \in L_{2,\mathrm{loc}}(\mathbb{R}^2)$, $j = 1, 2$. The non-negative quadratic form

$$A_{\mathbf{a}}[u] = \int_{\mathbb{R}^2} \left(|\partial_{x_1} u - ia_1 u|^2 + |\partial_{x_2} u - ia_2 u|^2\right) dx$$

with the natural domain $\mathrm{Quad}(A_{\mathbf{a}})$ is closed in $L_2(\mathbb{R}^2)$ (see, e.g., [S2]). By definition, $\mathcal{A}_{\mathbf{a}}$ is the self-adjoint operator in $L_2(\mathbb{R}^2)$, associated with this quadratic form. Consider the operator $H_{\mathbf{a}}(V) = \mathcal{A}_{\mathbf{a}} - V$, where V is a scalar (electric) potential. Our main goal is to prove the following result.

Theorem 1. *Let $\mathbf{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^2)$, $V \in L_{1,\mathrm{loc}}(\mathbb{R}^2)$, and $S_r(V_+) < \infty$ for some $r > 1$. Then the operator $\mathcal{A}_{\mathbf{a}} - V$ is well defined as the form-sum, its negative spectrum is discrete, and the following inequality is satisfied:*

$$N_{-1}(\mathcal{A}_{\mathbf{a}} - V) \leq C_r S_r(V_+). \tag{3}$$

The constant C_r does not depend on $\mathbf{a}$ and on V.

Incidentally, we obtain one more statement, which is then used in the proof of Theorem 1.

Theorem 2. *Let $d \geq 3$, $0 < \theta \leq 1$ or $d = 2$, $0 < \theta < 1$. Let $\mathbf{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^d)$, $V \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$, and $V_+ \in L_{\frac{d}{2\theta}}(\mathbb{R}^d)$. Then*

$$N_0(\mathcal{A}_{\mathbf{a}}^{\theta} - V) \leq C(d, \theta) \int_{\mathbb{R}^d} V_+(x)^{\frac{d}{2\theta}} dx. \tag{4}$$

The constant $C(d, \theta)$ does not depend on $\mathbf{a}$ and on V.

In fact, this Theorem is an immediate consequence of general results of [RozSol]. Another application of it was recently given in [BPu].

Due to the variational principle, it suffices to prove all estimates for nonnegative V only. We will treat this case in what follows.

2. Technical preliminaries. Endow $\mathrm{Quad}(A_{\mathbf{a}})$ with the metric form

$$\|u\|_{\mathbf{a}}^2 = A_{\mathbf{a}}[u] + \int_{\mathbb{R}^2} |u|^2 dx = \| \,(\mathcal{A}_{\mathbf{a}} + I)^{\frac{1}{2}}\, u\|^2;$$

here and further $\|\cdot\|$ without subscript stands for the L_2-norm. Denote by $\mathcal{H}_{\mathbf{a}}$ the resulting Hilbert space. For $\mathbf{a} \in L_\infty$ the space $\mathcal{H}_{\mathbf{a}}$ coincides with the Sobolev space $H^1(\mathbb{R}^2)$, up to equivalence of the norms. Along with $\mathcal{H}_{\mathbf{a}}$, consider also the family of Hilbert spaces $\mathcal{H}_{\mathbf{a},\theta} = \mathcal{H}_{\mathbf{a},\theta}(\mathbb{R}^2) = \mathrm{Dom}\left((\mathcal{A}_{\mathbf{a}} + I)^{\frac{\theta}{2}}\right)$, $0 \le \theta \le 1$, with the norms $\|u\|_{\mathbf{a},\,\theta} = \| \,(\mathcal{A}_{\mathbf{a}} + I)^{\frac{\theta}{2}}\, u\|$. So, $\mathcal{H}_{\mathbf{a},1} = \mathcal{H}_{\mathbf{a}}$ and $\mathcal{H}_{\mathbf{a},0} = L_2(\mathbb{R}^2)$.

We also need a local counterpart of the spaces $\mathcal{H}_{\mathbf{a},\theta}$. Let Q be any unit square in $\mathbb{R}^2$, whose edges are parallel to the coordinate axes. Let $\mathbf{g}$ be a vector field with real components $g_{1,2} \in L_\infty(Q)$; here we do not consider arbitrary magnetic potentials from L_2. The nonnegative quadratic form

$$A_{\mathbf{g},Q}[u] = \int_Q \left(|\partial_{x_1} u - ig_1 u|^2 + |\partial_{x_2} u - ig_2 u|^2\right) dx$$

with the domain $H^1(Q)$ is closed in L_2. Let $\mathcal{A}_{\mathbf{g},Q}$ be the corresponding self-adjoint operator. For $\theta \in [0,1]$, introduce the spaces $\mathcal{H}_{\mathbf{g},\theta}(Q) = \mathrm{Dom}\left((\mathcal{A}_{\mathbf{g},Q} + I)^{\frac{\theta}{2}}\right)$, with the norm $\|u\|_{\mathbf{g},\theta,Q} = \| \,(\mathcal{A}_{\mathbf{g},Q} + I)^{\frac{\theta}{2}}\, u\|$. Due to the assumption $\mathbf{g} \in L_\infty(Q)$, the space $\mathcal{H}_{\mathbf{g},\theta}(Q)$ coincides with the Sobolev space $H^\theta(Q)$ up to equivalence of the norms.

Given a square Q, we call *"x_j-edges"* its sides, parallel to the coordinate axis x_j, $j = 1, 2$. Extend the field $\mathbf{g}$ from Q to the whole of $\mathbb{R}^2$, in the following way. Let Π be the rectangle, obtained by the reflection of Q in each of its x_1-edges. First we extend $\mathbf{g}$ to Π. Namely, we take the even extension of g_1 and the odd extension of g_2 through each of two x_1-edges. Then we extend the resulting vector field through both x_2-edges of Π. This time, we take the odd extension of g_1 and the even extension of g_2. The new field, say $\mathbf{g}'$, is defined on the square Q', concentric with Q and homothetic to it with the coefficient 3. We set $\mathbf{g}' = 0$ outside Q'.

Consider also a natural extension procedure for functions $u \in H^1(Q)$. Given such a u, we first take its even extension to Π through x_1-edges of Q, and then the even extension of the resulting function to Q' through x_2-edges of Π. The new function, say v, belongs to $H^1(Q')$ and, by our construction,

$$\|v\|^2_{\mathcal{H}_{\mathbf{g}'}(Q')} = 9\|u\|^2_{\mathcal{H}_{\mathbf{g}}(Q)}. \tag{5}$$

Then, fix a cut-off function φ, which equals 1 on Q and 0 in a vicinity of $\partial Q'$. Extend the product φv outside Q' by zero and denote the resulting function by Γu. It is clear that the operator Γ acts from $H^1(Q)$ to $H^1(\mathbb{R}^2)$ and from $L_2(Q)$ to $L_2(\mathbb{R}^2)$. By interpolation, it also acts from $H^\theta(Q)$ to $H^\theta(\mathbb{R}^2)$ for any $0 < \theta < 1$. We are interested in the properties of Γ as an operator from $\mathcal{H}_{\mathbf{g},\theta}(Q)$ to $\mathcal{H}_{\mathbf{g}',\theta}(\mathbb{R}^2)$.

Lemma 3. *Let* $\mathbf{g} \in L_\infty(Q)$ *be a vector field with real components. Let the vector field* $\mathbf{g}'$ *on* $\mathbb{R}^2$ *and the extension operator* Γ *be as above. Then*

$$\|\Gamma u\|_{\mathcal{H}_{\mathbf{g}',\theta}(\mathbb{R}^2)} \le C(\Gamma)\|u\|_{\mathcal{H}_{\mathbf{g},\theta}(Q)}, \qquad u \in \mathcal{H}_{\mathbf{g}}(Q),\ 0 \le \theta \le 1,$$

where $C(\Gamma)$ *does not depend on* $\mathbf{g}$.

Proof. It is enough to handle $\theta = 1$ because the result is obvious for $\theta = 0$ and extends to other θ by interpolation. Denote $\max|\nabla\varphi| = m$. We have

$$\|\Gamma u\|^2_{\mathcal{H}_{\mathbf{g}'}(\mathbb{R}^2)} = \int_{Q'} \left(|\partial_{x_1}(\varphi v) - ig_1\varphi v|^2 + |\partial_{x_2}(\varphi v) - ig_2\varphi v|^2 + |\varphi v|^2\right) dx$$
$$\le 2\left(\int_{Q'} |\varphi|^2 \left(|\partial_{x_1} v - ig_1 v|^2 + |\partial_{x_2} v - ig_2 v|^2 + |v|^2\right) dx + \int_{Q'} |\nabla\varphi|^2|v|^2 dx\right)$$
$$\le 2\left(\|v\|^2_{\mathcal{H}_{\mathbf{g}'}(Q')} + m^2\|v\|^2_{L_2(Q')}\right),$$

and the result follows from (5). □

3. Semigroup considerations. Proof of Theorem 2. Consider the fractional powers $(-\Delta)^\theta$, $\theta > 0$, of the operator $-\Delta$ on $\mathbb{R}^d$. The semigroup $Q_\theta(t) = e^{-t(-\Delta)^\theta}$ can be represented as an integral operator with continuous kernel, say $Q_\theta(t; x, y)$. Its value on the diagonal does not depend on x:

$$Q_\theta(t; x, x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t|\xi|^{2\theta}} d\xi = c(d, \theta)t^{-d/2\theta}. \tag{6}$$

It is well known that the semigroup $Q_1(t)$ is *positivity preserving*, that is $Q_1(t)f \ge 0$ a.e. on $\mathbb{R}^d$ for any $t > 0$ and any nonnegative element $f \in L_2$. The same is true for the operators $Q_\theta(t)$ with any $0 < \theta < 1$, which follows from [BrKiRo], and can also be seen from the analytical expression for $Q_\theta(t; x, y)$.

Along with $Q_\theta(t)$, consider the family of semigroups $P_{\mathbf{a},\theta}(t) = e^{-t\mathcal{A}^\theta_{\mathbf{a}}}$, $\theta > 0$. For $0 < \theta \le 1$ the semigroup $P_{\mathbf{a},\theta}(t)$ is *dominated* by $Q_\theta(t)$; this means that

$$|P_{\mathbf{a},\theta}(t)f| \le Q_\theta(t)|f| \qquad \text{a.e. on } \mathbb{R}^d, \qquad \text{any } t > 0 \text{ and any } f \in L_2(\mathbb{R}^d).$$

For $\theta = 1$ this is a classical fact, established in [AHS]; see also [S1]. For $\theta < 1$, the domination property follows from [BrKiRo].

Our proof of both Theorems 1 and 2 will be based upon [RozSol, Theor.2.4]; see also Sec.2.3 there. Here we give the formulation for the particular case we need for our purposes in the present paper. We specially wish to stress here that in [RozSol] we do not need the continuity assumption, appearing in the formulation below.

Proposition 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonnegative self-adjoint operators in $L_2(\mathbb{R}^d)$. Suppose that the semigroup $e^{-t\mathcal{B}}$ is positivity preserving and can be represented as an integral operator with the continuous kernel $Q_{\mathcal{B}}(t;x,y)$, such that*

$$\sup_{x\in\mathbb{R}^d} Q_{\mathcal{B}}(t;x,x) \le Kt^{-\alpha}, \qquad \alpha > 1,\ K < \infty. \tag{7}$$

Suppose also that the semigroup $e^{-t\mathcal{A}}$ is dominated by $e^{-t\mathcal{B}}$. Then for any nonnegative $V \in L_\alpha(\mathbb{R}^d)$ the operator $\mathcal{A} - V$ is well defined as the form-sum, its negative spectrum is discrete, and

$$N_-(\mathcal{A} - V) \le C(\alpha)K \int_{\mathbb{R}^d} V^\alpha dx.$$

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. The equality (6) shows that for $\mathcal{B} = (-\Delta)^\theta$ the relation (7) is satisfied with $\alpha = \frac{d}{2\theta}$. For $d \ge 3$, the value $\theta = 1$ is admissible and we obtain from Proposition 4 the inequality (4) for any $\theta \in (0, 1]$. For $d = 2$, the value $\theta = 1$ is not allowed by (6), (7), and we get the same inequality only for $\theta < 1$:

$$N_0(\mathcal{A}_{\mathbf{a}}^\theta - V) \le C(\theta) \int_{\mathbb{R}^2} V^{\frac{1}{\theta}} dx, \qquad 0 < \theta < 1,\ d = 2. \tag{8}$$

□

Since $\mathcal{A}_{\mathbf{a}}^\theta < (\mathcal{A}_{\mathbf{a}} + I)^\theta$, inequality (8) implies

$$N_0((\mathcal{A}_{\mathbf{a}} + I)^\theta - V) \le C(\theta) \int_{\mathbb{R}^2} V^{\frac{1}{\theta}} dx, \qquad 0 < \theta < 1,\ d = 2. \tag{9}$$

Theorem 1 will be derived from (9), but this requires some more preparations.

4. Proof of Theorem 1. It is convenient to reduce the problem to singular numbers estimates for a certain compact operator. This reduction is based upon the Birman – Schwinger principle. Below we give one of its many equivalent formulations (for the simplest case), see [B1], or an exposition in [BSol].

Proposition 5. *Let $\mathcal{K}$ be a positive definite self-adjoint operator in a Hilbert space, and $\mathcal{C}$ be a nonnegative self-adjoint operator, form-bounded with respect to $\mathcal{K}$. Then the following two assertions are equivalent.*

1 . The operator $\mathcal{C}^{\frac{1}{2}}\mathcal{K}^{-\frac{1}{2}}$ is compact.

2 . For all $t > 0$ the operator $\mathcal{K} - t\mathcal{C}$ is well defined as the form-sum and its negative spectrum is finite.

Moreover, for all $t > 0$

$$N_0(\mathcal{K} - t\mathcal{C}) = n(t^{-1/2}; \mathcal{C}^{\frac{1}{2}}\mathcal{K}^{-\frac{1}{2}}). \qquad (10)$$

Here $n(s; \mathcal{T})$ stands for the singular numbers distrubution function of a compact operator $\mathcal{T}$; further, singular numbers are defined as square roots of the eigenvalues of the self-adjoint operator $\mathcal{T}^*\mathcal{T}$, see [GoKr].

Now, let the electric potential $V \geq 0$ be a function in $L_r(\mathbb{R}^2)$, $r > 1$. The estimate (9) applies with $\theta = r^{-1}$. Substituting $s^{-1}V$ for V, we obtain

$$N_0((\mathcal{A}_{\mathbf{a}} + I)^\theta - s^{-1}V) \leq C(\theta)s^{-r}\int_{\mathbb{R}^2} V^r dx, \qquad \theta = r^{-1}. \qquad (11)$$

Apply Proposition 5 to $\mathcal{K} = (\mathcal{A}_{\mathbf{a}} + I)^\theta$ and $\mathcal{C} = V$. Then $\mathcal{C}^{\frac{1}{2}}\mathcal{K}^{-\frac{1}{2}}$ turns into the operator

$$T_{\mathbf{a},\theta,V} := V^{\frac{1}{2}}(\mathcal{A}_{\mathbf{a}} + I)^{-\frac{\theta}{2}}.$$

Combining (10) and (11), we get $n(s; T_{\mathbf{a},\theta,V}) \leq C(\theta)s^{-2r}\int_{\mathbb{R}^2} V^r dx$ or, equivalently,

$$s_n(T_{\mathbf{a},\theta,V}) \leq C_1(r)n^{-\frac{\theta}{2}}\|V\|^{\frac{1}{2}}_{L_r(\mathbb{R}^2)}, \qquad \theta = r^{-1}.$$

It is important that $C_1(r)$ does not depend on the magnetic field $\mathbf{a}$.

In particular, the last estimate holds for V replaced by $V_Q := V\mathbf{1}_Q$, where $\mathbf{1}_Q$ is the characteristic function of Q:

$$s_n(T_{\mathbf{a},\theta,V_Q}) \leq C_1(r)n^{-\frac{\theta}{2}}\|V\|^{\frac{1}{2}}_{L_r(Q)}, \qquad \theta = r^{-1}. \qquad (12)$$

We need a similar inequality for the operator in $L_2(Q)$

$$T_{\mathbf{g},\theta,Q,V} = V_Q^{\frac{1}{2}}(\mathcal{A}_{\mathbf{g},Q} + I)^{-\frac{\theta}{2}},$$

with $\mathbf{g} = \mathbf{a} \restriction Q$. To establish it, consider the field $\mathbf{g}'$ and the extension operator Γ, constructed in Sec.2. Write down the Rayleigh quotients for the squares of singular numbers of the operators $T_{\mathbf{g}',\theta,V_Q}$ and $T_{\mathbf{g},\theta,Q,V}$. They are, respectively,

$$Z_{\mathbb{R}^2}(u) = \frac{\int_Q V|u|^2 dx}{\|u\|^2_{H_{\mathbf{g}',\theta}(\mathbb{R}^2)}}, \ u \in H^1(\mathbb{R}^2) \qquad \text{and} \qquad Z_Q(u) = \frac{\int_Q V|u|^2 dx}{\|u\|^2_{H_{\mathbf{g},\theta}(Q)}}, \ u \in H^1(Q).$$

By Lemma 3,

$$Z_Q(u) \leq C(\Gamma)^2 Z_{\mathbb{R}^2}(\Gamma u), \qquad u \in H^1(Q).$$

According to the variational principle, this implies

$$s_n(T_{\mathbf{g},\theta,Q,V}) \le C(\Gamma) s_n(T_{\mathbf{g}',\theta,V_Q}) \le C_2(r) n^{-\frac{\theta}{2}} \|V\|_{L_r(Q)}^{\frac{1}{2}}, \qquad \theta = r^{-1}. \tag{13}$$

The second inequality in (13) follows from (12); the constant $C_2(r) = C(\Gamma)C_1(r)$ is the same for all unit squares $Q \in \mathbb{R}^2$.

Represent $T_{\mathbf{g},1,Q,V}$ as

$$T_{\mathbf{g},1,Q,V} = T_{\mathbf{g},\theta,Q,V}(\mathcal{A}_{\mathbf{g},Q} + I)^{-\frac{1-\theta}{2}}. \tag{14}$$

The second factor on the right-hand side can be treated as $T_{\mathbf{g},1-\theta,Q,\mathbf{1}_Q}$. Thus, the estimate (12) applies to it (with $1-\theta$ instead of θ), giving

$$s_n\left((\mathcal{A}_{\mathbf{g},Q} + I)^{-\frac{1-\theta}{2}}\right) \le C_2(r') n^{-\frac{1-\theta}{2}}, \qquad r' = r(r-1)^{-1}. \tag{15}$$

Now, apply to the product (14) Fan's inequality, see e.g. [GoKr, Cor.II.2.2]:

$$s_{k_1+k_2-1}(\mathfrak{T}_1\mathfrak{T}_2) \le s_{k_1}(\mathfrak{T}_1) s_{k_2}(\mathfrak{T}_2).$$

For n odd, setting here $k_1 = k_2 = \frac{n+1}{2}$, we derive from (13) and (15):

$$s_n(T_{\mathbf{g},1,Q,V}) \le C_2(r) C_2(r') n^{-\frac{1}{2}} \|V\|_{L_r(Q)}^{\frac{1}{2}}.$$

Evidently, this inequality extends to all $n \in \mathbb{N}$ (with a bigger constant factor). Returning to the distribution functions, we get

$$n(s; T_{\mathbf{g},1,Q,V}) \le C(r) s^{-2} \|V\|_{L_r(Q)}. \tag{16}$$

Now, for $\mathbf{a} \in L_\infty$ the estimate (3) easily follows from (16). Indeed, one has

$$Z_{\mathbb{R}^2}(u, \mathbf{a}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} Z_{Q_\mathbf{n}}(u, \mathbf{a} \restriction Q_\mathbf{n}), \qquad \mathbf{g_n} = \mathbf{a} \restriction Q_\mathbf{n}.$$

Therefore, by the Birman–Schwinger principle and "Neumann bracketing",

$$N_{-1}(\mathcal{A}_\mathbf{a} - V) = N_0(\mathcal{A}_\mathbf{a} + I - V) = n(1; T_{\mathbf{a},1,V}) \le \sum_{\mathbf{n}\in\mathbb{Z}^2} n(1; T_{\mathbf{g_n},1,Q_\mathbf{n},V}), \tag{17}$$

and we arrive at (3) applying (16) to each term of the last sum.

It remains to get rid of the assumption $\mathbf{a} \in L_\infty$. Given an arbitrary $\mathbf{a} \in L_{2,\mathrm{loc}}$, we can approximate it in $L_{2,\mathrm{loc}}$ by a sequence $\{\mathbf{a}_k\}$ of bounded magnetic potentials.

The corresponding magnetic Schrödinger operators converge in the strong resolvent sense; this result is basically due to Simon [S2], who proved it for positive potentials (in our notation, for $V \le 0$). In [MRoz, Prop.2.5] the result was extended to the general case. This convergence and independence of the coefficient C_r in (3) of $\mathbf{a}_k$ guarantee that the estimate extends to any $\mathbf{a} \in L_{2,\mathrm{loc}}$.

5. Eigenvalue asymptotics. For $d \ge 3$ the Weyl type asymptotic formula for $N_{-\gamma}(\mathcal{A}_{\mathbf{a}} + U - qV)$ as $q \to \infty$ was justified in [Ra1] and [BRa], assuming $V \in L_{\frac{d}{2}}$, $\mathbf{a} \in L_{d,\mathrm{loc}}$ and under certain conditions on the "background potential" U. Granted this, the abstract scheme developed in [B2] produces the asymptotic formula for eigenvalues of $\mathcal{A}_{\mathbf{a}} + U - qV$ in the gaps of the spectrum of $\mathcal{A}_{\mathbf{a}} + U$. Having the estimate (3) and following the reasoning in [BRa] (see also [BPu]), we can now obtain similar results for the two-dimensional case rather easily. We start with the asymptotics of the number of negative eigenvalues.

Denote by $L^{\mu}\log(1+L)_{\mathrm{loc}}$ the Orlicz class of functions f for which $|f|^{\mu}\log(1+|f|) \in L_{1,\mathrm{loc}}$.

Theorem 6. *Let $d = 2$, $V \in L\log(1+L)_{\mathrm{loc}}$, and $S_r(V_+) < \infty$ for some $r > 1$. Suppose also that $U \ge 0$ and $|\mathbf{a}|^2 + U \in L\log(1+L)_{\mathrm{loc}}$. Then for any $\gamma > 0$*

$$\lim_{q\to+\infty} q^{-1} N_{-\gamma}(\mathcal{A}_{\mathbf{a}} + U - qV) = (4\pi)^{-1}\int_{\mathbb{R}^2} V_+ dx. \tag{18}$$

Proof. Our argument is rather standard, see e.g. [BBor] or [BRa]. For this reason, we only outline the main steps. Since $U \ge 0$, (3) implies $N_{-\gamma}(\mathcal{A}_{\mathbf{a}} + U - qV) \le CqS_r(V_+)$. In the usual way this estimate reduces the task of justifying the asymptotics to the case of bounded non-negative potential V, compactly supported in some square Q. Due to the variational principle (Dirichlet – Neumann bracketing), this problem, in its turn, reduces to the study of the spectrum of two operators, generated by the quadratic form $\int_Q V|u|^2dx$ in the spaces $H^1(Q)$ and $H^{1,0}(Q)$, equipped with the metric form

$$\begin{aligned} A_{\mathbf{a},Q}[u] + \int_Q U|u|^2dx + \gamma\int_Q |u|^2dx = \\ \int_Q (|\nabla u|^2 + \gamma|u|^2)dx + 2\sum_j \int_Q a_j \Im(\overline{u}\partial_j u)dx + \int_Q (|\mathbf{a}|^2 + U)|u|^2dx. \end{aligned} \tag{19}$$

The last term on the right-hand side of (19) is compact with respect to the first one, since $|\mathbf{a}|^2 + U \in L\log(1+L)(Q)$ (see, e.g., [Sol]). Relative compactness of the middle term follows from this via the inequality $2|ab| \le \varepsilon a^2 + \varepsilon^{-1}b^2$. Since relatively compact perturbations of the metric in the Hilbert space do not affect the leading term in asymptotics, one can leave on the right-hand side of (19) only the first term, and we are in the well-studied non-magnetic situation. $\square$

In order to include more general background potentials and consider eigenvalues in gaps, we need some more estimates.

By Σ_p we denote the class of compact operators $\mathfrak{T}$ satisfying the s-numbers estimate $|\mathfrak{T}|_p := \sup n^{\frac{1}{p}} s_n(\mathfrak{T}) < \infty$. The separable subclass $\Sigma_p^0 \subset \Sigma_p$ consists of operators with $s_n(\mathfrak{T}) = o(n^{-\frac{1}{p}})$.

Lemma 7. *Under the assumptions of Theorem 6, let $V \geq 0$. Then, for $\gamma > 0$,*

$$T_1 = V^{\frac{1}{2}}(\mathcal{A}_{\mathbf{a}} + U + \gamma)^{-\frac{1}{2}} \in \Sigma_2, \tag{20}$$

$$T_2 = V^{\frac{1}{2}}(\mathcal{A}_{\mathbf{a}} + U + \gamma)^{-1} \in \Sigma_2^0. \tag{21}$$

Proof. Taking into account (10), we can rewrite (3) as $|V^{\frac{1}{2}}(\mathcal{A}_{\mathbf{a}}+\gamma)^{-\frac{1}{2}}|_2 \leq CS_r(V)$. Since $U \geq 0$, this yields $|T_1|_2 \leq CS_r(V)$, thus establishing (20). Next, note that $S_r(V) < \infty$ implies $V \in L_1$. We will derive from this that the operator T_2 belongs to the Hilbert – Schmidt class $\mathfrak{S}_2$, which is a subset of Σ_2^0.

Indeed, it follows from the semigroup domination that $(\mathcal{A}_{\mathbf{a}} + U + \gamma)^{-1}$ is an integral operator with the kernel, dominated by that of $(-\Delta + \gamma)^{-1}$. The latter is $K(\gamma^{\frac{1}{2}}(x - y))$, where the Bessel – McDonald kernel $K(z)$ belongs to $L_2(\mathbb{R}^2)$. It follows that $V(x)^{\frac{1}{2}} K(\gamma^{\frac{1}{2}}(x - y))$ is square integrable over $\mathbb{R}^2 \times \mathbb{R}^2$, and the above domination implies that $T_2 \in \mathfrak{S}_2$. $\square$

Let F be one more term in the background potential and λ be a real regular point for the operator $\mathcal{B}_{\mathbf{a}} = \mathcal{A}_{\mathbf{a}} + U + F$, defined as the form-sum. For a non-negative potential V and $q > 0$ we denote by $\mathcal{N}_\lambda(q; \mathcal{B}, V)$ the number of eigenvalues of the operator $\mathcal{B}_{\mathbf{a}} - tV$ crossing the observation point λ as t grows from 0 to q.

Theorem 8. *Let the assumptions of Theorem 6 hold, $V \geq 0$ and multiplication by $|F|$ be form-bounded with respect to $\mathcal{A} + U$, with zero relative bound. Then*

$$\lim_{q \to +\infty} q\mathcal{N}_\lambda(q; \mathcal{B}_{\mathbf{a}}, V) = (4\pi)^{-1} \int_{\mathbb{R}^2} V dx. \tag{22}$$

Proof. Our argument repeats almost identically the reasoning in [B2] or [BPu]. By Theorem 1.2 in [B2], estimates (20), (21) imply that the leading term in the asymptotics of $\mathcal{N}_\lambda(q; \mathcal{B}_{\mathbf{a}}, V)$ does not change if we withdraw the term F in the background potential and replace the observation point λ by any real regular point μ of the operator $\mathcal{A}_{\mathbf{a}} + U$. If we take $\mu = -\gamma < 0$, the quantity $\mathcal{N}_\mu(q, \mathcal{A}_{\mathbf{a}} + U, V)$ becomes $N_{-\gamma}(\mathcal{A}_{\mathbf{a}} + U - qV)$, and it remains to refer to Theorem 6. $\square$

One can give sufficient analytical conditions on F guaranteeing the above form-boundedness. For example, it suffices to require that $\int_Q |F|^r dx \leq C$ for for some $r > 1$ and any unit square Q.

Remark. Following [Sa], one can relax the condition $V \geq 0$ in Theorem 8.

6. A generalisation: the case of $\gamma = 0$. In this final section we find conditions under which the results of the previous parts, especially the large coupling constant asymptotics from Theorem 6, can be extended to the case $\gamma = 0$. As in [BL], the conditions here are somewhat more restrictive than the ones for $\gamma > 0$.

We start with negative spectrum estimates for the operators

$$\mathcal{A}_{\mathbf{a}} + h(x)|x|^{-2} - V, \tag{23}$$

$$\mathcal{A}_{\mathbf{a}} + h(x)(1 + |x|^2)^{-1} - V \tag{24}$$

with a slowly varying bounded function $h(x) > 0$. In contrast to the original operator $\mathcal{A}_{\mathbf{a}} - V$, for these operators the quantity N_0 can be effectively estimated. For $h \equiv$ *const*, the estimate we give extends a result for the non-magnetic case obtained in [BL]. To describe the estimate, consider the family of annuli $\Omega_j = \{x \in \mathbb{R}^2 : e^{j-1} < |x| < e^j\}$, $j \in \mathbb{Z}$ and the disk $\Omega^\circ = \{x \in \mathbb{R}^2 : |x| < 1\}$. Given a number $r > 1$, we denote

$$\tilde{S}_r(f) = \sum_{j\in\mathbb{Z}} \left(\int_{\Omega_j} |x|^{2(r-1)} |f|^r dx \right)^{\frac{1}{r}};$$

$$\tilde{S}_r^\circ(f) = \left(\int_{\Omega^\circ} |f|^r dx \right)^{\frac{1}{r}} + \sum_{j>0} \left(\int_{\Omega_j} (1 + |x|^2)^{(r-1)} |f|^r dx \right)^{\frac{1}{r}}.$$

In what follows, M stands for sup h on $\mathbb{R}^2$.

Theorem 9. *(a) Let* $\mathbf{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^2)$, $V \in L_{1,\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$, *and* $\tilde{S}_r(h^{-1}V_+) < \infty$ *for some* $r > 1$. *Suppose that for some* $K > 0$ *the ratio* $h(x)/h(y)$ *is not greater than* K *when both* x *and* y *are in the same domain* Ω_j, $j \in \mathbb{Z}$. *Then the operator (23) is well defined as form-sum, its negative spectrum is finite, and*

$$N_0 \left(\mathcal{A}_{\mathbf{a}} + h(x)|x|^{-2} - V \right) \leq K\tilde{C}_1(r, M)\tilde{S}_r(h^{-1}V_+). \tag{25}$$

(b) Let $\mathbf{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^2)$, $V \in L_{1,\mathrm{loc}}(\mathbb{R}^2)$, *and* $\tilde{S}_r^\circ(h^{-1}V_+) < \infty$ *for some* $r > 1$. *Suppose that for some* K, *the ratio* $h(x)/h(y)$ *is not greater than* K *when both* x *and* y *are in the same domain* Ω_j, $j > 0$ *or in* Ω°. *Then the operator (24) is well defined as the form-sum, its negative spectrum is finite, and*

$$N_0 \left(\mathcal{A}_{\mathbf{a}} + \frac{h(x)}{1 + |x|^2} - V \right) \leq K\tilde{C}_2(r, M)\tilde{S}_r^\circ(h^{-1}V_+). \tag{26}$$

The constant factors $\tilde{C}_{1,2}(r, M)$ do not depend on $\mathbf{a}$ *or on* V.

For $\mathbf{a} = 0$ and $h \equiv const$, that is for the operator $-\Delta + c|x|^{-2} - V$, this is essentially the result of Lemma 3.3 in [BL].

We begin with a technical remark. The usage of the circular annuli Ω_j is not compulsory: one can start from an arbitrary bounded convex neighborhood X of the point 0 and any number $q > 1$. Such X and q being given, define the sets $X_j = q^j X \setminus q^{j-1} X$ and the norm

$$\widehat{S_r}(f) = \sum_{j \in \mathbb{Z}} \left(\int_{X_j} |x|^{2(r-1)} |f|^r dx \right)^{\frac{1}{r}}$$

(with $\tilde{S}_r^\circ$ modified in the same way). A simple geometric reasoning shows that all the norms of this type, corresponding to different choices of X and q, are mutually equivalent. For technical reasons, we make use of these norms, with X being the square $\{\max(|x_1|, |x_2|) < 1\}$ and with $q = 2$: this makes it possible to employ the estimate (16) directly.

Proof of Theorem 9. Both parts are proved in a similar way. Consider the part (a). For definiteness, we suppose $M \leq 1$. We follow the same line as in Theorem 1. It suffices to handle the case $V \geq 0$ and $\mathbf{a} \in L_\infty$. Consider the operator

$$\widehat{T}_{\mathbf{a},V} := V^{\frac{1}{2}} \left(\mathcal{A}_{\mathbf{a}} + h(x)|x|^{-2} \right)^{-\frac{1}{2}} .$$

We shall give an appropriate estimate of its singular numbers, then the desired result will follow by (10).

Along with $\widehat{T}_{\mathbf{a},V}$, consider a family $\{\widehat{T}_{j,\mathbf{a},V}\}$ of operators, acting in the spaces $L_2(X_j)$. To define them, consider positive definite quadratic forms

$$A_{j,\mathbf{a}}[u] = \int_{X_j} \left(|\partial_{x_1} u - i a_1 u|^2 + |\partial_{x_2} u - i a_2 u|^2 + h(x)|x|^{-2}|u|^2 \right) dx \qquad (27)$$

with the domains $H^1(X_j)$. Each $A_{j,\mathbf{a}}$ is closed in $L_2(X_j)$. Denote by $\mathcal{A}_{j,\mathbf{a}}$ the corresponding self-adjoint positive-definite operator in $L_2(X_j)$, and set

$$\widehat{T}_{j,\mathbf{a},V} = V^{\frac{1}{2}} \mathcal{A}_{j,\mathbf{a}}^{-\frac{1}{2}}.$$

As in (17), one has

$$N_0(\mathcal{A}_{\mathbf{a}} + h(x)|x|^{-2}|u|^2 - V) = n(1; \widehat{T}_{\mathbf{a},V}) \leq \sum_{j \in \mathbb{Z}} n(1; \widehat{T}_{j,\mathbf{a},V}). \qquad (28)$$

Denote by $\tilde{h}_j$ the minimal value of h in $\overline{X}_j$. Consider the term with $j = 1$ first. Replacing in (27) (for $j = 1$) the term $h(x)|x|^{-2}|u|^2$ by $|u|^2$, we obtain a new quadratic form, say $A'_{\mathbf{a}}$. Due to the assumption $M \le 1$, one has $A_{j,\mathbf{a}}[u] \ge \frac{\tilde{h}_1}{8} A'_{\mathbf{a}}[u]$.

Further, X_1 splits into the union of 12 unit squares, say $Q_1, \dots, Q_{12}$. By the variational principle all this, together with the inequality (16) and an analog of (17), leads to the estimate

$$n(s; \widehat{T}_{1,\mathbf{a},V}) \le \sum_{k=1}^{12} n\left(s\left(\frac{\tilde{h}_1}{8}\right)^{\frac{1}{2}}; T_{\mathbf{a},1,Q_k,V} \right) \le C'(r) s^{-2} \tilde{h}_1^{-1} \sum_{k=1}^{12} \|V\|_{L_r(Q_k)}$$
$$\le C'(r) K s^{-2} \sum_{k=1}^{12} \|h^{-1}V\|_{L_r(Q_k)}$$
$$\le K C''(r) s^{-2} \left(\int_{X_1} |x|^{2(r-1)} (h^{-1}V)^r dx \right)^{\frac{1}{r}}. \quad (29)$$

Finally, taking into account that the constant $C''(r)$ is independent of the magnetic field , we derive from (29) a similar estimate for $n(s; \widehat{T}_{j,\mathbf{a},V})$ with any $j \in \mathbb{Z}$ using scaling. Now, the theorem follows from (28). The proof of part (b) goes exactly in the same way, we only have to consider the square X separately. $\square$

Consider now the operator $\mathcal{A} - V$ without the additional terms present in (23), (24). We suppose the magnetic potential to be *nontrivial*: the equation $\nabla\varphi = i\mathbf{a}\varphi$ has no nontrivial solutions in $H^1_{\text{loc}}(\mathbb{R}^2)$. In other words, this means that the magnetic potential can not be gauged away.

Proposition 10. *Let the magnetic potential* $\mathbf{a} \in L^2 \log(1 + L)_{\text{loc}}$ *be nontrivial. Then*

$$N_0(\mathcal{A}_{\mathbf{a}} - V) \le C(r, \mathbf{a}) \tilde{S}^\circ_r (h_0^{-1} V_+), \qquad h_0(x) = (\log(e + |x|))^{-2}. \quad (30)$$

The constant factor in (30) may depend on $\mathbf{a}$ *but does not depend on* V.

Proof. We make use of the following Hardy-type inequality:

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{|x|<1} |u(x)|^2 dx \ge c \int_{\mathbb{R}^2} \frac{h_0(x)}{1 + |x|^2} |u(x)|^2 dx, \quad (31)$$

which should be considered as essentially known. It can be easily derived from the inequality (2.19) of [B1], for example.

It follows from domination (see, e.g., [S3]) that for any $u \in \text{Dom}(\mathcal{A}_{\mathbf{a}})$ one has

$$(\mathcal{A}_{\mathbf{a}} u, u)_{L_2} \ge \int_{\mathbb{R}^2} |\nabla |u||^2 dx. \quad (32)$$

This yields

$$A_{\mathbf{a}}[u] + \int\limits_{|x|<1} |u(x)|^2 dx \geq c \int\limits_{\mathbb{R}^2} \frac{h_0(x)}{1+|x|^2} |u(x)|^2 dx, \qquad u \in \mathrm{Quad}(A_{\mathbf{a}}). \tag{33}$$

Indeed, for $u \in \mathrm{Dom}(\mathcal{A}_{\mathbf{a}})$ (33) follows from (32) and the inequality (31), applied to the function $|u|$ instead of u. Then it extends to the whole of $\mathrm{Quad}(A_{\mathbf{a}})$ by continuity.

We apply now Corollary 9.1 in [W]. According to this result, for nontrivial $\mathbf{a}$ the forms $A_{\mathbf{a}}[u]$ and $A_{\mathbf{a}}[u]+\int_{|x|<1}|u|^2dx$ are equivalent on H^1_{loc} (however the coefficients in the inequalities in this equivalence may depend on the magnetic field). Thus (33) implies

$$A_{\mathbf{a}}[u] \geq C\left(A_{\mathbf{a}}[u] + \int_{\mathbb{R}^d} \frac{h_0(x)}{1+|x|^2}|u(x)|^2 dx\right).$$

Therefore,

$$N_0(\mathcal{A}_{\mathbf{a}} - V) \leq N_0\left(\mathcal{A}_{\mathbf{a}} + \frac{h_0(x)}{1+|x|^2} - C^{-1}V\right)$$

and it remains to apply (26). □

Remark. The result in [W] was established only for $\mathbf{a} \in L_{2+\epsilon,\mathrm{loc}}$. However, what actually is used in the proof is the fact that the form $A_{\mathbf{a}}[u] + \|u\|^2$ is locally equivalent to the norm in H^1, and this, as one can see from (19), takes place for $\mathbf{a} \in L^2\log(1+L)_{\mathrm{loc}}$.

Now the asymptotics in the large coupling constant arises automatically. One only has to repeat word by word the reasoning in Theorem 6.

Theorem 11. *Under conditions of Theorem 6, let the magnetic potential be nontrivial and* $\tilde{S}^{\circ}_r(h_0^{-1}V_+) < \infty$. *Then*

$$\lim_{q\to\infty} q^{-1} N_0\left(\mathcal{A}_{\mathbf{a}} + U - qV\right) = (4\pi)^{-1}\int_{\mathbb{R}^2} V_+ dx.$$

Note that for trivial magnetic potentials the problem reduces by gauging to the non-magnetic case, studied in [BL].

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