Some remarks on Lie bialgebra structures on simple complex Lie algebras

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Abstract

Let g be a complex simple Lie algebra with the root system Δ. We prove that any classical double $D(g)$ is graded by Δ. As a consequence of this fact we obtain that $D(g) \cong g \otimes A$, where A is a unital commuative associative algebra of dimension 2. Therefore we have two possibilities for A: nilpotent and semisimple. The first case leads to solutions of CYBE and the second case leads to solutions of mCYBE. We yield an explicit description of Lie bialgebra structures on g in both cases.

1 Introduction

Let g be a simple Lie algebra over $\mathbb{C}$ and let g^* be its dual vector space. Suppose g^* has also a fixed Lie algebra structure. Let $\varphi : g^* \otimes g^* \to g^*$ be defined by the rule: $\varphi(\ell_1 \otimes \ell_2) = [\ell_1, \ell_2]$. Then $\varphi^* : g \to g \otimes g$ satisfies the following conditions:

1) Im $\varphi^* \subset g \wedge g$

2) Per $(\varphi^* \otimes 1) \circ \varphi^*(a) = 0$ for any $a \in g$, where Per $(a \otimes b \otimes c) = a \otimes b \otimes c + c \otimes a \otimes b + b \otimes c \otimes a$ (co-Jacobi identity).

Let $\{e_i\}$ be a basis of g and $\{e^i\}$ be its dual basis of g^*. Let c_{ij}^k and f_k^{ij} be the corresponding structure constants. The follwoing result was proved in [D]:

Theorem 1.1. *The following conditions are equivalent:*

1) $c_{rs}^k f_k^{ij} = c_{\alpha r}^i f_s^{j\alpha} - c_{\alpha r}^j f_s^{i\alpha} - c_{\alpha s}^i f_r^{j\alpha} + c_{\alpha s}^j f_r^{i\alpha}$

2) *The map $\varphi^* : g \to g \otimes g$ is a 1-cocycle (g acts on $g \otimes g$ by means of the adjoint representation).*

3) *There is a Lie algebra structure on the vector space $g \oplus g^*$ inducing the given Lie algebra structures on g and g^*, which is such that the bilinear for Q on $g \oplus g^*$ defined by the formula*

$$Q(x_1 + \ell_1, x_2 + \ell_2) = \ell_1(x_2) + \ell_2(x_1)$$

is invariant with respect to the adjoint representation of $g \oplus g^*$. Moreover, such Lie algebra structure on $g \oplus g^*$ is unique if it exists. □

The Lie algebra structure on $g \oplus g^*$ satisfying the condition 3) of Theorem 1 is called the classical double of g and is denoted by $D(g)$. Here we note that $D(g)$ is not generally speaking unique for given g because there are many different Lie algebra structures on g^* and some of them satisfy the conditions of Theorem 1.

In the case of simple Lie algebras $\varphi^* : g \to g \otimes g$ is a co-boundary and hence is given by the formula $\varphi^*(a) = [r, a \otimes 1 + 1 \otimes a]$ for some $r \in g \otimes g$. It was proved in [D] that r can be chosen skew-symmetric and satisfying the following condition: $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$ is an invariant element of $g \otimes g \otimes g$. On the other hand, since g is simple, there is just one, up to a scalar factor, invariant tensor in $g \otimes g \otimes g$; namely, $a \cdot c_{ijk} I_i \otimes I_j \otimes I_k$, where $\{I_i\}$ is an orthonormal basis of g with respect to the Killing form, c_{ijk} are the structure constants in this basis and $a \in \mathbb{C}$.

In the case $a \neq 0$, all these tensors $r \in g \otimes g$ were found in [BD]. The aim of this paper is to refine the results of [BD] concerning the case $a = 0$ using a different method. We will not solve the equation

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = a \cdot c_{ij} I_i \otimes I_j \otimes I_k$$

as it was done in [BD].

Instead of this, we find $D(g)$, using the classification theory for Lie algebras graded by finite root systems ([BM, BZ]) and then we are looking for decompositions of $D(g)$ into the direct sum of g and g^*. It should be noted that the case $a = 0$ can be treated, using rational solutions of the classical YBE ([S1, S2, S3]). On the other hand, the author finds it worthy to describe a new method, which enables one to treat both cases simultaneously.

2 Preliminary results

Let Δ be a finite irreducible reduced root system and Γ be the integer lattice generated by Δ. The following definition was introduced in [BM].

Definition 2.1. A Lie algebra is graded by Δ if

(i) $L = \sum_{\gamma \in r} L_\gamma$, where $L_\gamma \neq (0)$ if and only if $\gamma \in \Delta \cup \{0\}$;

(ii) the split simple Lie algebra g whose root system is Δ, a subalgebra of L and relative to some split Cartan subalgebra $\mathcal{H}$ of g, we have $L_\gamma \supseteq g_\gamma$.

(iii) for all $h \in \mathcal{H}$ the operator adh acts diagonally on L_γ with eigenvalue $\gamma(h)$;

(iv) L is generated by its nonzero root spaces L_γ, where $\gamma \in \Delta$.

The following lemma was communicated to the author by E. Zelmanov:

Lemma 2.2. Let Δ be the root system of g. Then $D(g)$ is graded by Δ.

Proof. Since $g \subset D(g)$, $D(g) = \sum L_\gamma$, where $\gamma \in \mathcal{H}^*$ and $[h, \ell_j] = \gamma(h)\ell_\gamma$ for any $\ell_\gamma \in L_\gamma$. Assume that there exists $\gamma \notin \Delta \cup \{0\}$ such that $L_\gamma \neq 0$. Let $x_\gamma \in L_\gamma$. The invariancy of the bilinear form Q on $D(g)$ (see Theorem 1) implies that $Q(x_\gamma, g) = 0$. Then $x_\gamma \in g$ by the construction of Q. Hence $\gamma \in \Delta$.

It remains to show that $D(g)$ satisfies (iv). We have $D(g) = L_0 + \sum_{\gamma\in\Delta} L_\gamma$ and $\mathcal{H} \subset L_0$. Clearly, the restriction Q_1 of Q to L_0 is non-degenerate and $Q_1(\mathcal{H}, \mathcal{H}) = 0$. Let us show that $\mathcal{H} = \mathcal{H}^\perp$ with respect to Q_1. Indeed, if we have $x \in \mathcal{H}^\perp \subseteq L_0$, then $Q(x, L_\gamma) = 0$ and hence $Q(x, g_\gamma) = 0$ for $\gamma \neq 0$. Therefore $Q(x, g) = 0$ which implies that $x \in g$ and furtheremore, $x \in g \cap L_0 = \mathcal{H}$.

We see that $\mathcal{H}$ is a Lagrangian subspace of L_0 with respect to Q_1. Clearly, we can choose another Lagrangian subspace $V \subset L_0$ such that $V \oplus \mathcal{H} = L_0$.

It is sufficient to prove that V is contained in the vector space generated by nonzero root spaces L_γ since $\mathcal{H}$ clearly has this property. For any simple root $\alpha \in \Delta$ there exists $v_\alpha \in V$ such that $Q_1(v_\alpha, h) = \alpha(h)$. Since $\{v_\alpha\}$ is a basis in V, we have to prove that any v_α is contained in the space generated by nonzero L_γ.

To do this, we pick up any $e_\alpha \in g_\alpha \subset L_\alpha$ and $x \in L_{-\alpha}$ such that $Q(e_\alpha, c) = 1$. Clearly, $[e_\alpha, x] \in L_0$. Then $Q_1([e_\alpha, x], h) = Q([h, e_\alpha], x) = \alpha(h)$ and hence $Q_1([e_\alpha, x] - v_\alpha, h) = 0$ for any $h \in \mathcal{H}$. Since $\mathcal{H} = \mathcal{H}^\perp$ with respect to Q_1, we obtain that $[e_\alpha, x] - v_\alpha \in \mathcal{H}$.

This observation completes the proof. □

Now resutls from [BM, BZ] imply that there exists a commutative associative unital algebra A over $\mathbb{C}$ such that $D(g) = g \otimes A$. In our case $\dim_{\mathbb{C}} A = 2$ and we have up to an isomorphism just two possibilities:

$$1)\ A = \frac{\mathbb{C}[x]}{(x^2)}; \quad 2)\ A = \frac{\mathbb{C}[x]}{(\frac{1}{4} - x^2)}$$

Moreover, [Proposition 2.2 BZ] implies that $g \subset g + g^* = D(g) \cong g \otimes A$ is embedded as $g \otimes 1$.

Now let us recall that $g \otimes A$ has a non-degenerate symmetric invariant form Q. Then, it follows that $Q(g \otimes 1, g \otimes 1) \equiv 0$.

Proposition 2.3. 1) $Q(z \otimes \bar{x}, y \otimes \bar{x}) = 0$ for any $z, y \in g$. Here $\bar{x}$ is the image of x i A.

2) $Q(z \otimes 1, y \otimes \bar{x}) = K(z, y) \cdot \lambda$, where K is the Killing form on g, and $\lambda \in \mathbb{C}$ is a fixed number.

Proof. 1) Let us prove, for instance, that $Q(e_\alpha \otimes \bar{x}, y \otimes \bar{x}) = 0$ for $e_\alpha \in g_\alpha$. We have $[h \otimes 1, e_\alpha \otimes \bar{x}] = \alpha(h) e_\alpha \otimes \bar{x} = [h \otimes \bar{x}, e_\alpha \otimes 1]$.

Then

$$Q(e_\alpha \otimes \bar{x}, y \otimes \bar{x}) = \frac{1}{\alpha(h)} Q([h \otimes \bar{x}, e_\alpha \otimes 1], y \otimes \bar{x}) =$$
$$\frac{1}{\alpha(h)} Q([y \otimes \bar{x}, h \otimes \bar{x}], e_\alpha \otimes 1) = 0$$

since $\bar{x}^2 = 0$ or $0,25$. The case $Q(h_\alpha \otimes \bar{x}, y \otimes \bar{x})$ can be treated in the same way if we set $h_\alpha = [e_\alpha, e_{-\alpha}]$.

2) Put

$$t_\alpha(\bar{x}) = \frac{Q(e_\alpha \otimes \bar{x}, e_{-\alpha} \otimes 1)}{K(e_\alpha, e_{-\alpha})}$$

Here $e_\alpha \in g_\alpha$. Let $N_{\alpha\beta} \in \mathbb{C}$ satisfy $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$. If $e_{\alpha+\beta} \neq 0$, then $N_{\alpha\beta} \neq 0$ and

$$t_{\alpha+\beta}(\bar{x}) = \frac{Q([e_\alpha \otimes \bar{x}, e_\beta \otimes 1], e_{-\alpha-\beta} \otimes 1)}{K([e_\alpha, e_\beta], e_{-\alpha-\beta})} =$$

$$= \frac{Q(e_\alpha \otimes \bar{x}, e_{-\alpha} \otimes 1)}{K(e_\alpha, e_{-\alpha})} = t_\alpha(\bar{x})$$

since $N_{\beta,-\alpha-\beta} \neq 0$ if $N_{\alpha\beta} \neq 0$. Therefore $t_\alpha(\bar{x})$ does not depend on α. Let $t_\alpha(\bar{x}) = \lambda$. Similarly, one can show that

$$q_\alpha(\bar{x}) = \frac{Q(h_\alpha \otimes \bar{x}, h_\alpha \otimes 1)}{K(h_\alpha, h_\alpha)} = \lambda$$

The proof is complete.

$\square$

3 Case $A = \frac{\mathbb{C}[x]}{(x^2)}$

In this case we see that $g \otimes A = g + g\bar{x} \quad (\bar{x}^2 = 0)$ and Q can be given by the following formula:

$$Q(a + b\bar{x}, c + d\bar{x}) = K(a, d) + K(b, c).$$

The problem is to define all subalgebras W satisfying the following conditions:

1) $g \oplus W = g \otimes A$

2) $W = W^{\perp}$ with respect to Q.

Example 3.1. $W = g\bar{x}$. In this case, $\varphi^* = 0$ (see Introduction).

Theorem 3.2. Let $W \subset g + g\bar{x}$ satisfy Conditions (1) and (2). Then $W = \{a + f(a)\bar{x} + a^{\perp}\bar{x} : a \in L, a^{\perp} \in L^{\perp}\}$. Here $L \subset g$ is a subalgebra, $L^{\perp}$ its orthogonal complement with respect to the Killing form, $f : L \to L^* = g/L^{\perp}$ is an isomorphism of vector spaces and a 1-cocycle of L with values in L^* (L^* is L-module with respect to the co-adjoint action).

Proof. Let $\pi : g + g\bar{x} \to g$ be the canonical projection. Then $L = \pi(w)$ is a subalgebra of g. Clearly, $W \subset L + g\bar{x}$. Then $W = W^{\perp} \supset (L + g\bar{x})^{\perp} = L^{\perp} \cdot \bar{x}$; moreover, we see that $\bar{W} = W \bmod (L^{\perp} \cdot \bar{x})$ becomes a Lagrangian subalgebra of $(L+g\bar{x})/(L^{\perp}\cdot\bar{x}) \cong L+L^*\cdot\bar{x}$ satisfying the following conditions:

$$\operatorname{Im} \bar{W} = L \text{ under the canonical projection } L + L^* \cdot \bar{x} \to L \qquad (*)$$
$$\bar{W} \oplus L = L + L^* \cdot \bar{x} \text{ (because } g \oplus W = g + g\bar{x}). \qquad (**)$$

The condition $(*)$ implies that $\bar{W} = \{\ell + f(\ell)\bar{x} : \ell \in L\}$ with a homomorphism $f : L \to L^*$ which should be a 1-cocycle.

The condition $(**)$ implies that $\operatorname{Ker} f = 0$. The theorem is proved. $\square$

Corollary 3.3. Any subalgebra $W \subset g + g\bar{x}$, satisfying the conditions (1), (2), is uniquely defined by the following data:

1) subalgebra $L \subset g$;

2) non-degenerae 2-cocycle B on g.

Proof. L was defined in theorem 3.2. Then $B(a,b)$ is defined by the following formula: $B(a,b) = f(a)(b) = K(f(a),b)$. One can easily check that B is a 2-cocycle since $f \in Z^1(L, L^*)$. $\square$

Remark 3.4. A Lie algebra L is called quasi-Frobenius if there exists a non-degenerate 2-cocycle on it. Such subalgebras were studied in [S2, E].

Remark 3.5. Let $Ad(g)$ the Lie group generated by $\{\exp(ad\, a) : a \in g\}$. Clearly, $Ad(g) \subset Ad(g + g\bar{x})$. By the way, $g + g\bar{x}$ has a simple matrix realization: let $g \subset s\ell(n)$, then any element $a+b\bar{x} \in g+g\bar{x}$ can be represented by the fololowing $2n \times 2n$ matrix $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

It is easily seen that if $W \subset g + g\bar{x}$ satisfies the conditions (1), (2), then $Ad(x)(W)$ satisfies these conditions too.

Definition. We say that two bialgebra structures are gauge equivalent if there exists $X \in Ad(g)$ such that the corresponding W_1 and W_2 are related by $W_1 = Ad(X)(W_2)$.

Our aim is to find conditions when (L, B) and (L_1, B_1) define equivalent bialgebra structures. Without loss of generality, we can assume that $Ad(X)(L_1) = L$. We define $N(L) = \{X \in Adg : Ad(X)(L) = L\}$ and $Ad(L) = \{X = \exp(ad\, a) : a \in L\} \subseteq N(L)$.

Lemma 3.6. Let (L, B) defines $W \subset g + g\bar{x}$ satisfying the conditions (1), (2). Let $T \in N(L)$. Then $Ad(T)(W)$ is defined by the pair $(L, Ad^*(T)(B))$, where $Ad^*(T)(B)(a,b) = B(Ad(T^{-1})(a), Ad(T^{-1})(b))$.

Proof. Straightforward computations. $\square$

Proposition 3.7. Let the pairs (L, B_1) and (L, B_2) define two bialgebra structures on g and $B_1 - B_2$ is a co-boundary. Then the corresponding structures are gauge equivalent.

Proof. Let $B_1(x,y) - B_2(x,y) = K(T, [x,y])$ and let $W_1, W_2 \subset g + g\bar{x}$ be the corresponding subalgebras. Here $T \in L^* = g/L^{\perp}$. One can check that $W_1 = \exp(ad(T\bar{x}))(W_2)$. Since $g \oplus W_1 = g \oplus W_2 = g + g\bar{x}$, [Lemma 4.2, S1] implies that there exists $X \in Ad\, g$ such that $W_1 = Ad(X)W_2$. Therefore, the corresponding bialgebra structures are gauge equivalent. $\square$

Taking into account that $Ad(X) \in Ad(g)$ constructed in the proof of Proposition 3.7 preserves L (and hence is an element of $N(L)$), we get the following

Corollary 3.8. A gauge equivalence class of bialgebra structures on g defines a quasi-Frobenius algebra $L \subset g$ (up to a conjugacy) and an orbit of $N(L)$ in $H^2(L, \mathbb{C})$. Any orbit of $N(L)$ in $H^2(L, \mathbb{C})$ corresponds to at most one gauge equivalence class of bialgebra structures. $\square$

Remark 3.9. It is well known that L acts trivially on $H^2(L, \mathbb{C})$ since for any 2-cocycle B and any $\ell \in L$ we have: $ad^*_\ell B(x, y) = B([\ell, x], y) + B(x, [\ell, y]) = B(\ell, [x, y])$ is a coboundary. Therefore $Ad(L)$ acts identically on $H^2(L, \mathbb{C})$.

We remind that a quasi-Frobenius algebra is called Frobenius if the corresponding 2-cocyle B is a co-boundary.

Lemma 3.10. Let L be a Frobenius, Lie algebra. Then in any class of $H^2(L, \mathbb{C})$ there exists a non-degenerate representative.

Proof. If a bilinear form B is degenerate on L and B_1 is non-degenerate, then $B + \lambda B_1$ is non-degenerate for "big" λ. $\square$

Corollary 3.11. Let L be a Frobenius Lie subalgebra of g. Then the equivalence classes of bialgebra structures on g corresponding to L are in a one-to-oen correspondence with orbits of $N(L)$ in $H^2(L, \mathbb{C})$. In particular, if $N(L) = Ad(L)$, then such equivalence classes of bialgebra structures are in a one-to-one correspondence with $H^2(L, \mathbb{C})$. $\square$

Closing this section, we would like to discuss the problem how to reconstruct the corressponding solutions of the CYBE.

Let we have $W = \{\ell + f(\ell)\bar{x} + \ell^\perp \bar{x} : \ell \in L\}$ such that $g \oplus W = g + g\bar{x}$. If we choose any basis $\{e_i\}$ in g and its dual basis $\{f^j\}$ in W $(Q(e_i, f^j) = \delta_i^j)$, then $r = \sum e_i \otimes f^i \in D(g)^{\otimes 2}$ satisfies the CYBE and

$$r + r^{21} = t = \sum e_i \otimes e^i \bar{x} + \sum e^i \bar{x} \otimes e_i$$

where $\{e^i\}$ is the basis of g, dual to $\{e_i\}$ with respect to the Killing form. Let us consider $(\pi \otimes \pi)(r) \in g^{\otimes 2}$, where $\pi : g + g\bar{x} \to g$ is the canonical projection.

Theorem 3.12. $(\pi\otimes\pi)(r)$ is a skew-symmetric solution of the CYBE. If $\{a_i\}$ and $\{b^j\}$ are dual bases of L with respect to the 2-cocycle B, $(B(a_i, b^j) = \delta_i^j)$, then $(\pi \otimes \pi)(r) = \sum b^i \otimes a_i$.

Proof. The skew-symmetry of $(\pi \otimes \pi)(r)$ is clear since $(\pi \otimes \pi)(t) = 0$.

Let us choose any basis $\{a_i\}$ of L, and let us extend it in an arbitrary way to g. Let $\{c^i + f(c^i)\bar{x} + a_j^{\perp}\bar{x}\}$ be the dual basis of W with respect to Q. Here $a_j^{\perp}$ is a basis of $L^{\perp}$. So we have:

$$Q(a_i, c^i + f(c^j)\bar{x}) = K(a_i, f(c^j)) = \delta_i^j.$$

On the other hand, it follows from the construction of B (Corollary 3.3) that $K(a_i, f(c^j)) = B(c^j, a_i) = \delta_i^j$ and $c^j = -b^j$. Hence, $(\pi \otimes \pi)(r) = \sum a_i \otimes c^i = \sum b^i \otimes a_i$. The proof is complete. □

We note that this formula for solutions of the CYBE was obtained in [BFS].

Example 3.13. 1) Let $E_{ij} \in g\ell(n)$ be matirx units. Consider the subalgebra L of $s\ell(3)$ generated by E_{ij}, where $1 \leq i < j \leq 3$, and $h = E_{11} + E_{22} - 2E_{33}$. Then L is Frobenius with non-degenerate 2-coboundary $E_{13}^*([x, y])$. Here $\{E_{ij}^*\, 1 \leq i < j \leq 3,\, h^*\}$ is the dual basis of L^*.

It was shown in [S2] that $H^2(L) \cong \mathbb{C}$. The corresponding non-trivial 2-cocyle is $h^* \wedge E_{12}^*$. It can be checked that any 2-cocycle of the form $E_{13}^*([x, y]) + \lambda(h^* \wedge E_{12}^*)(x, y)$ is non-degenerate. However, any two such cocycles with non-zero λ_1, λ_2 lie in the same orbit of $N(L)$. Hence, we have two non-equivalent bialgebra structures on $s\ell(3)$ both corresponding to the L.

2) Let us consider a subalgebra M of $s\ell(4)$ generated by E_{ij}, $1 \leq i < j \leq 4$ and $h_1 = E_{11} - E_{44}$, $h_2 = E_{22} - E_{33}$.

It was shown in [SK, GG] that L is Frobenius with non-degenerate 2-cocycle $(E_{14}^* + E_{23}^*)([x, y]))$. Again, in this case there exists another 2-cocycle $h_1^* \wedge h_2^*$ and all the cocycles of the form $(E_{14}^* + E_{23}^*)([x, y]) + \lambda(h_1^* \wedge h_2^*)(x, y)$ are non-degenerate. We again have two non-equivalent bialgebra structures on $s\ell(4)$ corresponding to $\lambda = 0$ and $\lambda \neq 0$. So, we see that a Frobenius subalgebra does not define uniquely a bialgebra structure.

4 Case $A = \frac{\mathbb{C}[x]}{(1/4 - x^2)}$

In this case we have $A \cong \mathbb{C}e \oplus \mathbb{C}f$ and respectively $g \otimes A \cong ge \oplus gf$, where $e = \frac{1}{2} + x$ and $f = \frac{1}{2} - x$. It follows that in this case

$$Q(a_1 e + b_1 f, a_2 e + b_2 f) = K(a_1, a_2) - K(b_1, b_2),$$

where K is the Killing form on g. Our aim is to describe all Lagrangian subalgebras $W \subset g \otimes A \cong ge \oplus gf$ such that $W \oplus g(e+f) = g \otimes A$.

Lemma 4.1. Let $W \subset ge \oplus gf$ be a Lagrangian subspace with respect to Q. Then there exist subspaces $W_1 \subseteq g$ and $W_2 \subseteq g$ such that $W_i \supset W_i^{\perp_K}$, where $W_i^{\perp_K}$ is the orthogonal complement to W_i with respect to the Killing form and an isometry $F : W_1/W_1^{\perp} \to W_2/W_2^{\perp}$. Here the metric structure on $W_i/W_i^{\perp}$ is defined by K.

Proof. Let π_1, π_2 be the canonical projections $\pi_1 : ge \oplus gf \to ge$, $\pi_2 : ge \oplus gf \to gf$. Let $W_1 e = \pi_1(W)$ and $W_2 f = \pi_2(W)$. Then $W \subset W_1 e + gf$ and $W^{\perp_Q} = W \supset (W_1 e + gf)^{\perp_Q} = W_1^{\perp_K} e$. The signs $\perp_Q, \perp_K$ indicate the forms with respect to which we consider the orthogonal complements. Clearly

$$\dim g = \dim W = \dim W_i + \dim W_i^{\perp_K}. \tag{*}$$

On the other hand, Ker $\pi_2 = W \cap ge \supseteq W_1^{\perp_K} e$ and hence we see that dim $W \geq$ dim $W_2 +$ dim $W_1^{\perp_K}$. Similarly, dim $W \geq$ dim $W_1 +$ dim $W_2^{\perp_K}$.

Taking into account $(*)$, we conclude that dim $W =$ dim $W_1 +$ dim $W_2^{\perp_K}$ and therefore dim $W_1 =$ dim W_2, dim $(W_1/W_1^{\perp_K}) =$ dim $(W_2/W_2^{\perp_K})$. Moreover, we see that $W_i/W_i^{\perp_K} \cong W/(W_1^{\perp_K} e + W_2^{\perp_K} f)$ and W can be represented in the form

$$W = \{w_1^{\perp} e + w_1 e + F(w_1) f + w_2^{\perp} f\}.$$

Here $w_i^{\perp} \in W_i^{\perp_K}$, $w_1 \in W_1/W_1^{\perp_K}$ and $F : W_1/W_1^{\perp_K} \to W_2/W_2^{\perp_K}$ is an isomorphism. Obviously, the Killing form K on g generates a non-degenerate form on $W_i/W_i^{\perp_K}$, which we denote by K_i. The fact that W is Lagrangian with respect to Q implies that

$$K_1(W_1, W_2) = K_2(F(W_1), F(W_2)).$$

□

Lemma 4.2. Assume that $W \subset ge \oplus gf$ is a Lagrangian subspace and a subalgebra. Then the corresponding $W_i \subset g$ are subalgebras and $F : W_1/W_1^{\perp_K} \to W_2/W_2^{\perp_K}$ is an isomorphism of Lie algebras, preserving the forms on $W_i/W_i^{\perp_K}$ induced by the Killing form on g.

Proof. Clearly, π_i $(i = 1, 2)$ from Lemma 4.1 are Lie algebra homomorphisms. Hence W_i $(i = 1, 2)$ are subalgebras of g. Since $W_i^{\perp_K}$ are ideals of W_i, it is easily seen that $F : W_1/W_1^{\perp_K} \to W_2/W_2^{\perp_K}$ is a Lie algebra isomorphism. The lemma is proved. □

In the sequel, we will write $W = \{W_1, W_2, F\}$. The following question arises: How to describe all the subalgebras $L \subset g$ such that $L^{\perp_K} \subset L$? Let Γ be the set of all simple roots of g, Δ be the set of all roots g and Δ_+ be the set of all positive roots. Let $\Gamma_1 \subset \Gamma$ and $\Delta_1 = \{\gamma \in \Delta : \gamma = \sum_{\alpha\in\Gamma_1} \gamma_\alpha \cdot \alpha\}$. Set $L_{\Gamma_1} = \sum_{\alpha\in\Gamma_1} \mathbb{C}H_\alpha + \sum_{\gamma\in\Delta_1} g_\gamma$, where $H_\alpha = [E_\alpha, E_{-\alpha}]$.

The restriction of the Killing form K to $\mathcal{H}$ is non-degenerate and we denote this restriction by K_1. Then, clearly,

$$\mathcal{H} = (\sum_{\alpha\in\Gamma_1} \mathbb{C}H_\alpha) \oplus (\sum_{\alpha\in\Gamma_1} \mathbb{C}H_\alpha)^{\perp_{K_1}}$$

Morevoer, K_1 restricted to $(\sum_{\alpha\in\Gamma_1} \mathbb{C}H_\alpha)^{\perp_{K_1}}$ is non-degenerate and we denote this restriction by K_2. Let us choose any $\mathcal{L} \subset (\sum_{\alpha\in\Gamma_1} \mathbb{C}H_\alpha)^{\perp_{K_1}}$ such that $\mathcal{L}^{\perp_{K_2}} \subseteq \mathcal{L}$. The following result is known (see [BD]).

Theorem 4.3. Let $L_\pm^{\Gamma_1} = L_{\Gamma_1} + \mathcal{N}_\pm + \mathcal{L}$, where $\mathcal{N}_\pm = \sum_{\gamma\in\Delta^\pm} g_\gamma$ (Δ^+ is the set of all positive roots of g and Δ^- is the set of negative roots). Then $L^{\perp_K} \subset L$ and any subalgebra of g, which contains its orthogonal complement with respect to the Killing form, can be transformed by a suitable inner automorphism of g to the form $L_+^{\Gamma_1}$ for some $\Gamma_1 \subset \Gamma$ and some $\mathcal{L}$. □

Recalling that $e + f = 1$, we see that a Lagrangian subalgebra $W \subset ge \oplus gf$, which generates a bialgebra structure on g, should satisfy the following condition $W \cap g(e + f) = \{0\}$. Clearly, the latter is equivalent to $W + g(e + f) = ge \oplus gf$. Let $X \in$ Aut g.

Then $X(e + f) \in$ Aut $(ge \oplus gf)$ and $X(W)$ is a Lagrangian subalgebra satisfying $X(W) \cap g(e+f) = \{0\}$. We will call W and $X(W)$ gauge equivalent.

Theorem 4.4. Let $W = \{W_1, W_2, F\}$ generates a bialgebra structure on g. Then W is gauge equivalent to $W_1 = \{L_+^{\Gamma_1}, L_-^{\Gamma_2}, F_1\}$ for some $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{H}$ and some subgraphs $\Gamma_1, \Gamma_2 \subset \Gamma$, which are isomorphic.

Proof. Theorem 4.3 implies that without loss of generality we can assume that $W_1 = L_+^{\Gamma_1}$. Let us consider the following decomposition of $ge \oplus gf$ into the following direct sum of Lagrangian subspaces: $ge \oplus gf = g(e+f) \oplus g(e-f)$. Let $\pi_- : ge + gf \to g(e-f)$ be the projection along $g(e+f)$. Then $\pi_-(W) = g(e-f)$ and this implies that $L_+^{\Gamma_1} + W_2 = g$.

Extending $L_+^{\Gamma_1}$ if it is necessary, we can assume that $L_+^{\Gamma_1} \supset \mathcal{H}$. Let $X \in Ad\, g$ be such that $(Ad\, X)(W_2) = L_-^{\Gamma_2}$ (it is possible by Theorem 4.3). Clearly $L_+^{\Gamma_1} + L_-^{\Gamma_2} = g$ and it follows from [Lemma 4.2, S1] that there exists $X_1 \in Ad\,(L_+^{\Gamma_1})$ such that $(Ad\, X_1)(W_2) = L_-^{\Gamma_2}$. Then $(Ad\, X_1)(e+f)$ is the required gauge equivalence. □

5 Examples of bialgebra structures

In this section we would like to describe the bialgebra structures found in [BD].

Let Γ be the set of simple roots of g. We say the following [BD] that a triple $(\Gamma_1, \Gamma_2, \tau)$ with $\Gamma_1, \Gamma_2 \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ is admissible if:

a) τ is a bijection,

b) $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$,

c) for any $\alpha \in \Gamma_1$ there exists k such that $\tau(\alpha), \tau^2(\alpha), \ldots, \tau^{k-1}(\alpha) \in \Gamma_1$ but $\tau^k(\alpha) \notin \Gamma_1$.

Here (α, β) is the standard scalar product on $\mathcal{H}^*$ induced by the Killing form. The condition c) means that there does not exist $\alpha \in \Gamma_1$ such that $\tau^k(\alpha) = \alpha$ for some k.

Obviously $\tau : \Gamma_1 \to \Gamma_2$ induces isomorphisms $\Delta_1 \to \Delta_2$ and $L_{\Gamma_1} \to L_{\Gamma_2}$, which we also denote by τ. Let $\{E_\alpha, H_\alpha,\ \alpha \in \Delta\}$ be a basis of g such that $K(E_\alpha, E_{-\alpha}) = 1$ and $[E_\alpha, E_{-\alpha}] = H_\alpha$. Then $\tau\,(E_\alpha) = E_{\tau(\alpha)}$ and $\tau\,(H_\alpha) = H_{\tau(\alpha)}$ for all $\alpha \in \Delta_1$.

Lemma 5.1. Let $(\Gamma_1, \Gamma_2, \tau)$ be an admissible triple. Let L_{Γ_i} be the semisimple subalgebra of g generated by Γ_i (see Section 4). Then $W_\tau = \{xe + \tau(x)f : x \in L_{\Gamma_1}\} \subset ge \oplus gf$ is an isotropic subalgebra of $ge \oplus gf$ such that $W_\tau \cap g(e+f) = \{0\}$.

Conversely, if τ does not satisfy condition c), then $W_\tau \cap g(e+f) \neq \{0\}$.

Proof. We prove, for instance, the second statement. Let τ do not satisfy c). Then there exists a cycle $\{\alpha, \tau(\alpha), \dots, \tau^k(\alpha) = \alpha\}$. Consider $P = E_\alpha + E_{\tau(\alpha)} \dots + E_{\tau^{k-1}(\alpha)} \in L_{\Gamma_1}$. Then $\tau(P) = P$ and $W_\tau \cap g(e+f)$ contains $P(e+f)$.

Conversely, if $W_\tau \cap g(e+f) \ni P(e+f)$ for some $P \in L_{\Gamma_1}$, then $\tau(P) = P$. Writing $P = \sum_{\alpha\in\Gamma_1} c_\alpha H_\alpha + \sum_{\gamma\in\Delta_1} b_\gamma E_\gamma$, it is easily seen that for some root $\gamma \in \Delta_1$ we get a cycle: $\gamma, \tau(\gamma), \dots \tau^k(\gamma) = \gamma$. Since $\gamma = \sum_{\alpha\in\Gamma_1} c_\alpha \cdot \alpha$, we find that the triple $\{\Gamma_1, \Gamma_2, \tau\}$ is not admissible. $\square$

Lemma 5.2. Let $V = V_0 \otimes_{\mathbb{C}} (\mathbb{C}[x]/(1/4 - x^2)) = V_0 e + V_0 f$ be a vector space and let V_0 have a non-degenerate symmetric bilinear form B. Let $W_1, W_2 \subset V_0$ be subspaces such that B restricted to W_i is non-degenerate and let $\theta : V_1 \to V_2$ be an isometry, i.e., $B(\theta(a), \theta(b)) = B(a,b)$. Let also $X_0 \subset V$ be a Lagrangian subspace with respect to $\tilde{B}(ae+bf, ce+df) = B(a,c) - B(b,d)$. Assume $V_\theta \cap X_0 = \{0\}$, where $V_\theta = \{ae + \theta(a)f : a \in V_1\}$. If we choose any two vectors v_i satisfying $B(v_i, v_i) = 1$ and $B(V_i, v_i) = 0$, then we can extend θ to an isometry $\tilde{\theta} : V_1 \oplus \mathbb{C}v_1 \to V_2 \oplus \mathbb{C}v_2$ in such a way that $V_{\tilde{\theta}} = \{ae + \tilde{\theta}(a)f : a \in V_1 \oplus \mathbb{C}v_1\} \subset V$ does not intersect X_0.

Proof. Suppose $\{V_\theta \oplus \mathbb{C}\cdot(v_1 e + v_2 f)\} \cap X_0 = \{0\}$. Then we can extend θ setting $\tilde{\theta}(v_1) = v_2$. Otherwise, $\dim(\{V_\theta \oplus \mathbb{C}(v_1 e + v_2 f)\} \cap X_0) = 1$. Therefore, there exists $w \in V_\theta$ such that $w + v_1 e + v_2 f \in X_0$.

We claim that then $\{V_\theta \oplus \mathbb{C}\cdot(v_1 e - v_2 f)\} \cap X_0 = \{0\}$. Indeed,

$$\tilde{B}((w_1 + \lambda(v_1 e - v_2 f), w + v_1 e + v_2 f) = \lambda \tilde{B}(v_1 e, v_1 e) - \lambda \tilde{B}(v_2 f, v_2 f) = 2\lambda$$

since $\tilde{B}(w_1, w) = 0$ for all $w_1, w \in W_\theta$. Further, X_0 is Lagrangian with respect to $\tilde{B}$ and hence $w_1 + \lambda(v_1 e - v_2 f) \notin X_0$ for any $w_1 \in V_\theta$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

If $\lambda = 0$, then $w_1 \notin X_0$ since $V_\theta \cap X_0 = \{0\}$. Finally, we see that $\{V_\theta \oplus \mathbb{C}\cdot(v_1 e - v_2 f)\} \cap X_0 = \{0\}$. Then we can set $\tilde{\theta}(v_1) = -v_2$. $\square$

Now, let $\{\Gamma_1, \Gamma_2, \tau\}$ be an admissible triple. It follows from the previous two lemmas that $\tau : L_{\Gamma_1} \to L_{\Gamma_2}$ can be extended to an isometry

$$\bar{\tau} : L_{\Gamma_1} + \mathcal{H} \to L_{\Gamma_2} + \mathcal{H}$$

in such way that $V_{\bar{\tau}} \cap g(e+f) = \{0\}$. Observe that $\bar{\tau}(\mathcal{H}) = \mathcal{H}$ and if we denote this restriction by $\bar{\tau}_{\mathcal{H}}$ then $id - \bar{\tau}_{\mathcal{H}}$ is invertible. If we set $L_{\pm}^{\Gamma_i} = L_{\Gamma_i} + \mathcal{N}_{\pm} + \mathcal{H}$ then we have the following:

Corollary 5.3. Let $\{\Gamma_1, \Gamma_2, \tau\}$ be an admissible triple. Then the Lagrangian subalgebra $W_{\bar{\tau}} = \{L_+^{\Gamma_1},\, L_-^{\Gamma_2},\, \bar{\tau}\}$ yields a bialgebra structure on g. □

So, starting from an admissible triple we have constructed a bialgebra structure on g. Finally we would like to explain the construction of all other bialgebra structures. To do this we have to remind how to obtain the solution of the mCYBE, which corresponds to a given bialgebra structure.

Let W be a Lagrangian subalgebra satisfying the condition $W \oplus g(e+f) = ge \oplus gf$.

Let us choose dual bases $\{P_i\}$ for $g(e+f)$ and $\{S^i\}$ for W. Then $r = \sum_i P_i \otimes Q^i$ is a solution of mCYBE in $D(g)$ and $(\pi_1 \otimes \pi_1)(r) = r_W$ is a solution of mCYBE in g. Here $\pi_1 : D(g) \to ge$ is the canonical projection.

The next step is to find dual bases in case $W = W_{\bar{\tau}}$. For any $\gamma \in \Delta$ there exists $k(\gamma)$ such that $\tau^n(\gamma) \in \Delta_1$ for $n < k(\gamma)$ and $\tau^{k(\gamma)}(\gamma) \notin \Delta_1$. Note that $k(\gamma) = 0$ if $\gamma \notin \Delta_1$. Then we can choose the following basis in $W_{\bar{\tau}}$:
(i) $\{E_{\tau^k(\gamma)}\, e + E_{\tau^{k+1}(\gamma)}\, f,\ \gamma \in \Delta^+\}$, here we set $E_{\tau^{k+1}(\gamma)} = 0$ if $\tau^k(\gamma) \notin \Delta_1$.
(ii) $\{E_{\tau^{-k-1}(\gamma)}\, e + E_{\tau^{-k}(\gamma)}\, f,\ \gamma \in \Delta^-\}$, where $\tau^{-1} : \Delta_2 \to \Delta_1$ is inverse to τ. Here we set $E_{\tau^{-k-1}(\gamma)} = 0$ if $\tau^{-k}(\gamma) \notin \Delta_2$.
(iii) $\{H_i\, e + \bar{\tau}_{\mathcal{H}}(H_i)\, f\}$, where $\{H_i\}$ is a basis of $\mathcal{H}$ and $\bar{\tau}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ was constructed by means of Lemma 5.2.

Let us recall that $id - \bar{\tau}_{\mathcal{H}}$ is invertible.

Lemma 5.4. The dual basis of $g(e+f)$ consists of the following elements:
(i) $\{\Sigma_{s=0}^{k} E_{-\tau^s(\gamma)}\,(e+f)\}$.
(ii) $\{-\Sigma_{s=0}^{k} E_{-\tau^{-s}(\gamma)}\,(e+f)\}$.
(iii) Further, if $\{H_i\}$ is an orthonormal basis of $\mathcal{H}$, then $\{(id - \bar{\tau}_{\mathcal{H}})^{-1}\,(H_i)\, e + \bar{\tau}_{\mathcal{H}}(id - \bar{\tau}_{\mathcal{H}})^{-1}\,(H_i)\, f\}$ are elements of the basis of $W_{\bar{\tau}}$ dual to $\{H_i\,(e+f)\}$.

Proof. Just the last statement requires a proof. Let $T_i\, e + \bar{\tau}_{\mathcal{H}}\,(T_i)\, f$ be the dual elements to $H_i\,(e+f)$. It follows immediately that $K\,(H_i,\, T_j - \bar{\tau}_{\mathcal{H}}\,(T_j)) = \delta_i^j$. Therefore, $T_j - \bar{\tau}_{\mathcal{H}}\,(T_j) = H_j$ and the proof is complete. □

Corollary 5.5. The tensor

$$r_\tau = \Sigma_{\alpha\in\Delta^+} E_\alpha \otimes E_{-\alpha} + \Sigma_{\gamma\in\Delta^+, p>0} E_{\tau^p(\gamma)} \wedge E_{-\gamma} + \Sigma_i\,(id - \bar{\tau}_{\mathcal{H}})^{-1}(H_i) \otimes H_i$$

satisfies mCYBE. Here $\bar{\tau}_{\mathcal{H}}(H_\alpha) = \tau(H_\alpha) = H_{\tau(\alpha)}$ for $\alpha \in \Gamma_1$. □

Let W_0 be another bialgebra structure corresponding to the same admissible triple. Analogously we construct the corresponding solution of mCYBE, namely r_0.

One can easily prove that $r_\tau - r_0$ is a skew-symmetric tensor from $\mathcal{H}_1 \otimes \mathcal{H}_1$, where the subspace $\mathcal{H}_1 \subset \mathcal{H}$ is defined by the following equation:
$(\alpha - \tau(\alpha))(h) = 0$ for any $\alpha \in \Gamma_1$.

Conversely, $r_\tau + S$ is a solution of mCYBE for any skew-symmetric tensor $S \in \mathcal{H}_1 \otimes \mathcal{H}_1$ and hence it defines a bialgebra structure on g.

Remark 5.1. We have constructed a family of Lie bialgebra structures on g using a very simple isomorphism $\tau : L_{\Gamma_1} \to L_{\Gamma_2}$. One might think that using an inner automorphism $Ad(F)$ of L_{Γ_2} we can obtain a new bialgebra structure. However it is not true. Let $W = \{L_+^{\Gamma_1}, L_-^{\Gamma_2}, Ad(F) \circ \bar{\tau}\}$ satisfy the condition $W \oplus g(e+f) = ge \oplus gf$. Then $Ad\,(Id \cdot e + F \cdot f)\,(W_{\bar{\tau}}) = W$. Further $W_{\bar{\tau}} \oplus g(e+f) = ge \oplus gf$ and it follows from [Lemma 4.2, S1] that there exists an inner automorphism $Ad(X) \in Ad(ge \oplus gf)$ such that $Ad(X)(W_{\bar{\tau}}) = W$ and $Ad(X)$ preserves $g(e+f)$. Therefore, the structers determined by W and $W_{\bar{\tau}}$ are gauge equivalent.

It follows from results of [BD] that any bialgebra structure on g is equivalent to one of those described above.

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References

[BD] Belavin, A., Drinfeld, V. On classical Yang-Baxter equation for simple Lie algebras, Funct. Anal. Appl. 16 (1982).

[BFS] Beidar, K., Fong, Y., Stolin, A. On Frobenius algebras and the Quantum Yang-Baxter equation, Trans. Amer. Math. Soc. 349 (1997).

[BZ] Benkart, G., Zelmanov, E. Lie algebras graded by finite root systems and intersection matrix algebras, Invent. Math. 126 (1996).

[BM] Berman, S., Moody, R. V. Lie algebras graded by finite root systems and the intersection matirx algebras of Slodowy, Invent. Math. 108 (1992).

[D] Drinfeld, V. Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation, Sov. Math. Dokl., 27 (1983).

[GG] Gerstenhaber, M., Giaquinto, A. Boundary solutions of the classical Yang-Baxter equation, Lett. Math. Phys. 40 (1997).

[S1] Stolin, A. On rational solutions of Yang-Baxter equation for $s\ell(n)$, Math. Scan. 69 (1991).

[S2] Stolin, A. Constant solutions of Yang-Baxter equation for $s\ell(2), s\ell(3)$, Math. Scan. 69 (1991).

[S3] Stolin, A. On rational solutions of Yang-Baxter equations. Maximal orders in loop algebra, Comm. Math. Phys. 141 (1991).

[SK] Stolin, A., Kulish, P. Yang-Baxter equation and deformed Yangians, Czechoslovak J. Phys. 47 (1997).