ESTIMATES OF SOLUTIONS OF THE H^p AND $BMOA$ CORONA PROBLEM

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Abstract. We prove new sharper estimates of solutions to the H^p-corona problem in strictly pseudoconvex domains, in particular we show that the constant is independent of the number of generators. We also obtain sharper estimates for solutions to the $BMOA$ corona problem. The proofs also lead to new results about the Taylor spectrum of analytic Toeplitz operators on H^p and $BMOA$.

1. Introduction and background

In this paper we obtain new sharper estimates of solutions of the H^p-corona problem and the $BMOA$-corona problem in strictly pseudoconvex domains in $\mathbb{C}^n$, and furthermore we make an extension to each level of the Koszul complex of the generators. The new estimates consist in a sharper control of the dependence of δ, and independence of the number of generators. To explain the generalization we first discuss this type of division problems in the frame of spectral analysis.

Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary and let Λ^ℓ denote elements of degree ℓ of the exterior algebra of the basis $e_1, \dots, e_m$. Then a function (or a form) f in D with values in Λ^ℓ looks like

$$f = \sum_{|I|=\ell}{}' f_I e_I,$$

where $e_I = e_{I_1} \cap \ldots \cap e_{I_l}$ (we use $\cap$ instead of $\wedge$ in order not to confuse with the exterior multiplication in D) and the prime means that the summation is performed over increasing multiindices I of length ℓ. The pointwise norm of f is

$$|f|^2 = \sum_{|I|=\ell}{}' |f_I|^2. \tag{1.1}$$

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The Hardy space $H^p(\Lambda^\ell)$ consists of holomorphic Λ^ℓ-valued functions f such that

$$\|f\|_{H^p} = \limsup_{\epsilon\to 0} \int_{\partial D_\epsilon} |f|^p d\sigma < \infty ,$$

where $D_\epsilon = \{\rho < -\epsilon\}$ for some defining function ρ of D. If $g_1, \dots, g_m$ are bounded holomorphic functions, then, for each fixed $w \in \mathbb{C}^m$, the mapping

$$\delta_{w-g}\colon H^p(\Lambda^{\ell+1}) \to H^p(\Lambda^\ell) \tag{1.2}$$

is defined as interior multiplication with $\sum_1^m (w_j - g_j)e_j^*$, where e_j^* is the dual basis of e_j. Since $\delta_{w-g}\circ\delta_{w-g} = 0$ we have a complex

$$0 \leftarrow H^p(\Lambda^0) \leftarrow H^p(\Lambda^1) \leftarrow \dots \leftarrow H^p(\Lambda^m) \leftarrow 0. \tag{1.3}$$

The next to leftmost arrow is just the mapping $(H^p)^m \to H^p$ defined by

$$(u_1, \dots, u_m) \mapsto \sum (w_j - g_j)u_j. \tag{1.4}$$

The Taylor spectrum $\sigma(g, H^p)$ of the (commuting set of) multiplication operators g_j on H^p is defined as the set of $w \in \mathbb{C}^m$ such that (1.3) is not exact. The right spectrum $\sigma_r(g, H^p)$ is the set of w such that (1.3) is not exact at the next to leftmost point, i.e., such that (1.4) is not surjective. It is a consequence of the open mapping theorem that $\sigma_r(g, H^p)$ and $\sigma(g, H^p)$ are closed sets. Obviously (1.4) is not surjective if $w \in g(D)$, and therefore we have the inclusions

$$\overline{g(D)} \subset \sigma_r(g, H^p) \subset \sigma(g, H^p). \tag{1.5}$$

Suppose that $u = \sum u_j e_j \in H^\infty(\Lambda^1)$ satisfies $\delta_{w-g}u = 1$. Then the mapping $U\colon H^p(\Lambda^\ell) \to H^p(\Lambda^{\ell+1})$ defined by $f \mapsto u \cap f$ satisfies $\delta_{w-g}U + U\delta_{w-g} = 1$ and hence (1.3) is exact. In particular it follows that $\sigma(g, H^p)$ is contained in any Stein compact K that contains $g(D)$. In fact, if w is outside such a compact, then we can find $\phi_j \in \mathcal{O}(K)$ such that $\sum \phi(\xi)(\xi_j - w_j) \equiv 1$ and therefore $\delta_{w-g}u = 1$ if $u_j = \phi_j(g)$.

The corona problem is equivalent to the question whether the inclusions in (1.5) are equalities if $p = \infty$. This is true, for instance, in all finitely connected domains in the plane. In higher dimensions there are counterexamples in some smoothly bounded weakly pseudoconvex domains, see [22] and [13], but the question is still unsettled in strictly pseudoconvex domains, even in the ball.

In this language the H^2-corona problem, introduced in [3], amounts to decide whether $\overline{g(D)} = \sigma_r(g, H^2)$, and it was proved that this is in fact true when D is the ball in $\mathbb{C}^n$. The proof relies on the L^2-technique to handle division problems (or more precisely, to prove existence of holomorphic sections to certain holomorphic vector bundles) due to Skoda [24], but modified in order to obtain boundary estimates. The same arguments work for a wide class of weakly pseudoconvex domains,

[4], and by introducing certain weight factors we obtained the analogous result for all $p \leq 2$, [5], and also a good control of the dependence of the distance δ from w to $g(D)$. By an appropriate normalization we can always assume that $w = 0$ and that

$$0 < \delta \leq |g| \leq 1. \tag{1.6}$$

Theorem 1.1 ([3][4][5]). *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ that admits a C^2 plurisubharmonic defining function, and suppose that $g_1, \dots, g_m \in H^\infty$ satisfy* (1.6). *Then to each $\phi \in H^p$, $0 < p \leq 2$, there are $u_j \in H^p$ such that $\sum g_j u_j = \phi$ and $\|u\|_{H^p} \leq C_\delta \|\phi\|_{H^p}$, where $C_\delta \leq C \log(1/\delta)/\delta^{1+r}$, $r = \min(n, m-1)$, and C is independent of m and p.*

It is also proved in [5] that this result implies the corona theorem when $n = 1$, with the same dependence of δ. This estimate for the corona solution is the best known and due to Uchiyama, see [18] for a discussion.

The interpretation of this kind of division results as statements about the Taylor spectrum, was pointed out by Eschmeier and Putinar [12], and R. Wolff [30] proved the following result for $p = 2$.

Theorem 1.2. *Assume that $D \subset \mathbb{C}^n$ has a C^2 plurisubharmonic defining function. If $g_1, \dots, g_m \in H^\infty$, then $\sigma(g, H^p) = \overline{g(D)}$ for all $p \leq 2$.*

The proof uses the fact that the H^2-corona problem can be solved not only for m tuples g but for matrices g of constant rank, see [4]. This is also true for $p \leq 2$, following [5], and therefore Wolff's result can be generalized to any $p \leq 2$.

Probably it is possible to adapt the ideas of Sibony and Fornaess–Sibony, [22] and [13], to obtain counterexamples to Theorem 1.1 when $p > 2$. However, in the strictly pseudoconvex case the H^p-corona theorem is true for any $p < \infty$. This was proved by Amar, [1], (for two generators) in the ball, and in general strictly pseudoconvex domains in [8] and [9]. However, in these papers there is no (good) control of the dependence of δ. The proofs in [1] and [8] use the Koszul complex, whereas the proof in [9] uses explicit holomorphic division formulas. However, in all three papers some an analog of Wolff's approach to the corona problem is involved, and the starting point is the trivial pointwise minimal solution $\psi = \bar{g}/|g|^2$ to the division problem. In contrast, some years earlier Varopoulos [29] proved, in analogy to Carleson's original proof of the corona theorem, that one can make a delicate choice of a smooth solution ψ to $g\psi = 1$ such that $\bar{\partial}\psi$ is a Carleson measure. This was used to prove that one could find a holomorphic solution u in $H^\infty \cdot BMO$ to $gu = 1$ (at least for+ two generators). Using the same smooth solution ψ it is possible to prove the H^p-corona theorem for two generators just relying on results for $\bar{\partial}$ which follow from straightforward (i.e. non-Wolff) estimates of well-known integral formulas. However, as far as we know, this approach yields no information of the dependence of δ.

2. New results

The following theorem is one of our main results in this paper.

Theorem 2.1. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary, assume that $g_1, \dots, g_m \in H^\infty$ satisfy* (1.6) *and let $p < \infty$. For any $\phi \in H^p$ there are $u_j \in H^p$ such that $\sum g_j u_j = \phi$ and $\|u\|_{H^p} \leq C_\delta \|\phi\|_{H^p}$, where $C_\delta \leq C(\log(1/\delta))^{r/2}/\delta^{1+r}$ if $r = \min(n, m-1) \neq 1$, and $C_\delta \leq C \log(1/\delta)/\delta^2$ if $r = \min(n, m-1) = 1$, where C is independent of m (but unfortunately depends on p).*

If $p = \infty$ the proof yields a solution u in $H^\infty \cdot BMO$. In the proof of Theorem 2.1 we also start with the trivial pointwise minimal solution and use the Koszul complex. What is new here is the sharper dependence of δ and the independence of m. The latter is obtained by just being careful in the estimates and the organization of the Koszul complex, whereas for the estimate of the dependence of δ we use a recently found Carleson estimate for the complex Monge–Ampere operator, [7]. The crucial point is Proposition 5.2, which restricted to one variable is the observation made by Uchiyama (see [18]) that the Carleson norm of $(-\rho)|g'|^2/|g|^4$ in the disk is an absolute constant times $\log(1/\delta)/\delta^2$ (provided $\delta \leq |g| \leq 1$), whereas the standard, less subtle estimate gives the constant $1/\delta^4$. The proof of Theorem 2.1 gives the following more general result.

Theorem 2.2. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary, assume that $g_1, \dots, g_m \in H^\infty$ satisfy* (1.6) *and let $1 \leq p < \infty$. For any $\phi \in H^p(\Lambda^\ell)$ such that $\delta_g \phi = 0$ there is a $u \in H^p(\Lambda^{\ell+1})$ such that $\delta_g u = \phi$ and $\|u\|_{H^p} \leq C_\delta \|\phi\|_{H^p}$, where $C_\delta \leq C(\log(1/\delta))^{r/2}/\delta^{1+r}$ if $r = \min(n, m-1-\ell) \neq 1$, and $C_\delta \leq C \log(1/\delta)/\delta^2$ if $r = 1$, where C is independent of m.*

Corollary 2.3. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary and assume that $g_1, \dots, g_m \in H^\infty$. Then $\sigma(g, H^p) = \overline{g(D)}$ for all $p < \infty$.*

The case when $r = 0$, i.e. $\ell = m-1$, is trivial, since then the solution u is pointwise unique, and therefore the constant is just $1/\delta$.

Remark 1. From Theorems 2.1 and 2.2 one can obtain the corresponding results for the Bergman spaces $A^p(D) = L^p(D) \cap \mathcal{O}(D)$ instead of H^p. Given $D = \{\rho < 0\} \subset \mathbb{C}^n$, let $\widetilde{D} = \{(z, w) \in C^{n+1};\ \rho(z) + |w|^2 < 0\}$. Then A^p can be identified with the subspace of $H^p(\widetilde{D})$, consisting of functions that are independent of the variable w. Moreover, $\widetilde{D}$ is strictly pseudoconvex if D is. If $g_j \in H^\infty(D)$ satisfy (1.6), $\phi \in A^p(D, \Lambda^\ell)$ and $\delta_g \phi = 0$, then $\phi \in H^p(\widetilde{D}, \Lambda^\ell)$, and therefore it follows from Theorem 2.2 that there is a $\tilde{u}_j \in H^p(\widetilde{D}, \Lambda^{\ell+1})$ such that $\delta_g u = \phi$ in $\widetilde{D}$. If u is the mean values of all rotations in the w variable of $\tilde{u}$, then $u \in A^p(D, \Lambda^{\ell+1})$ and still $\delta_g u = \phi$. Therefore we also have that $\sigma(g, A^p) = \overline{g(D)}$ for $p < \infty$. □

In [20] Ortega and Fabrega prove the analogous statement of Theorem 2.1 for $BMOA$ instead of H^p in the ball, provided that the generators g_j are multiplicators on $BMOA$. In [19] they also consider some analogous results for Besov spaces. We give a proof here of the $BMOA$ result, including the Λ^ℓ case, and provide an estimate of the constant. Using the fact that the same solution operator K works in both Theorem 4.1 and Theorem 4.2 we avoid a certain difficulty from [20]. An essential part of their work consists in generalizing the characterization of the multiplicators on $BMOA$, due to Stegenga [25], to the multivariable case. In order to obtain the independence of m we need a vector-valued variant of this characterization that we include in Section 6.

Theorem 2.4. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary and let $g_1, \dots, g_m$ be multiplicators on $BMOA$ with multiplicator norm ≤ 1 and $|g| \geq \delta$. Then for any $\phi \in BMOA(\Lambda^\ell)$ with $\delta_g \phi = 0$ there is a $u \in BMOA(\Lambda^\ell)$ such that $\delta_g u = \phi$ and $\|u\|_{BMOA} \leq C_\delta \|\phi\|_{BMOA}$, where $C_\delta \leq C(\log(1/\delta))^{r/2}/\delta^{2+r}$, $r = \min(n, m - \ell - 1)$, and C is independent of m.*

Corollary 2.5. *If $g_1, \dots, g_m$ are multiplicators on $BMOA$, then $\sigma(g, BMOA) = \overline{g(D)}$.*

Remark 2. There are analogues of the above division theorems when g is a matrix instead of just one row, see [3] and [4] for the case $p = 2$. The same method works for $p \leq 2$. It is possible to obtain the matrix case in strictly pseudoconvex domain for all $p < \infty$ by the Koszul complex as in this paper but we omit that discussion here. □

3. Solution of division problems, the setup

If f and h are Λ^*-valued forms then

$$f \cap h = \sum_{I,K} f_I \wedge h_K e_I \cap e_K.$$

Let $\gamma = \sum_1^m \gamma_j e_j$ be a smooth section to Λ^1 such that $\delta_g \gamma \equiv 1$. Then $f \mapsto \gamma \cap f$ defines a linear mapping $C_q^\infty(\overline{D}, \Lambda^k) \to C_q^\infty(\overline{D}, \Lambda^{k+1})$ and since δ_g is an antiderivation it follows that $\delta_g(\gamma \cap f) = f$ if $\delta_g f = 0$. Let $K = K_q \colon C_{q+1}^\infty \to C_q^\infty$ be operators such that $\bar\partial K f = f$ if $\bar\partial f = 0$, $q > 0$, and extend them to Λ^ℓ-valued forms in the natural way. Then we have the following formula.

Theorem 3.1. *If ϕ is holomorphic with values in Λ^ℓ such that $\delta_g\phi = 0$ and*

$$(3.1)\quad u = \sum_{k=0}^{r}(-1)^k(\delta_g K)^k\gamma\cap(\bar\partial\gamma)^k\cap\phi$$
$$= \gamma\cap\phi - (\delta_g K)\gamma\cap\bar\partial\gamma\cap\phi + \cdots + (-1)^r(\delta_g K)^r\gamma\cap(\bar\partial\gamma)^r\cap\phi,$$

where $r = \min(n, m-\ell-1)$, then u is holomorphic and $\delta_g u = \phi$.

The idea to reduce division problems to systems of $\bar\partial$-equations by means of the Kozul complex is due to Hörmander [16]. In our formulation we use that our solution operator K is defined for any, not necessarily $\bar\partial$-closed, form. However, the solution u only depends on the action of K on $\bar\partial$-closed forms.

Sketch of proof. Since δ_g is an anti-derivation, $\delta_g\gamma = 1$ and $\delta\circ\delta = 0$, it is clear that $\delta_g u = \phi$. Since $I - \bar\partial K$ vanishes on the kernel of $\bar\partial$, there is an operator H such that $\bar\partial K + H\bar\partial = I$. Inductively one can verify that

$$\bar\partial(\delta K)^k = (-1)^{k+1}\left((\delta H)^{k-1}\delta - (\delta H)^k\bar\partial\right).$$

Therefore,

$$\bar\partial u = \sum -((\delta H)^{k-1}\delta - (\delta H)^k\bar\partial)\gamma\cap(\bar\partial\gamma)^k\cap\phi = 0.$$

□

The proofs of our theorems amount to obtain $L^p(\partial D)$-estimates and BMO-estimates of each term in the sum (3.1) when ϕ is in H^p and $BMOA$, respectively. To this end let $\gamma = \sum\overline{g_j}/|g|\, e_j$ and put $\omega_k = \gamma\cap(\bar\partial\gamma)^k$. We first (in Lemma 5.1) prove the pointwise estimate $|\omega_k\cap\phi| \le C|\omega_k|\,|\phi|$, where C is independent of m. From this it is easy to see that the term corresponding to $k = 0$ is in $L^p(\partial D)$. If $k \ge 2$ we prove in Proposition 5.2 that $(-\rho)^{\frac{k-2}{2}}|\omega_k|$ is a Carleson measure, with a sharp bound (in terms of δ) of its Carleson norm. Then by repeated use of Theorem 4.1, a non-Wolff estimate of the solution operator K, it follows that these terms are in $L^p(\partial D)$. It is harder to estimate the term corresponding to $k = 1$ and we need to prove that both $|\omega_1|^2$ and $|\sqrt{-\rho}\mathcal{L}\omega_1|$ are Carleson measures for a smooth $(1,0)$-vector field $\mathcal{L}$, see Theorem 5.3. Then by a Wolff-type estimate of K, Theorem 4.2, the term with $k = 1$ is in $L^p(\partial D)$ as well. In Section 6 we consider the modifications needed to prove the $BMOA$ result.

4. Estimates for the $\bar\partial$-equation

In the rest of this paper $D = \{\rho < 0\}$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ and ρ is a strictly plurisubharmonic defining function. For practical reasons we assume that ∂D and ρ are C^∞; with small modifications everything works equally well if ρ is just C^3.

It is well known that the $\bar\partial$-operator behaves differently in the complex tangential and normal directions near the boundary of a strictly pseudoconvex domain. This fact is reflected in the usual estimates of solutions to the $\bar\partial$-equation and therefore it is natural to measure forms with respect to a metric that takes this difference into account, such as

$$\Omega = (-\rho) i\partial\bar\partial \log(1/-\rho).$$

If f is a $(0,q)$-form we have that

$$|f|^2 = \frac{1}{B}(-\rho|f|^2_\beta + |\bar\partial\rho \wedge f|^2_\beta), \tag{4.1}$$

where $|\ \ |_\beta$ denotes the norm induced by the metric form $\beta = (i/2)\partial\bar\partial\rho$, which is equivalent to the Euclidean metric since ρ is strictly plurisubharmonic, and $B = -\rho + |\partial\rho|_\beta$ and hence ~ 1. Since $\Omega_n \sim \beta_n/(-\rho)$, we have

$$|f|^2\beta_n \sim c_q(-\rho) f \wedge \bar f \wedge \Omega_{n-q}\ , \tag{4.2}$$

for $(q,0)$-forms f.

Remark 3. In the ball Ω is just $-\rho(\zeta) = 1 - |\zeta|^2$ times the Bergman metric; the reason for the factor $-\rho$ is that otherwise we would have a power of ρ in (4.1) that depends on the degree of f, and this would make the formulation of the estimates below more involved. In particular, on the boundary, $|f|^2 \sim |\bar\partial\rho \wedge f|^2_\beta$, which is the natural norm for the complex tangential boundary values $f|_b$ of f. □

Let $d(p,q)$ be the Koranyi distance between points on ∂D and $B = B_r(p) = \{\zeta \in \partial D;\ d(\zeta,p) < r\}$ the corresponding ball. The tent $Q_r(p)$ over $B_r(p)$ is $Q_r(p) = \{z \in D;\ d(z',p) + d(\zeta) < r\}$ where $d(z) = d(z,\partial D)$ and z' is the projection of z on ∂D (defined in some appropriate way). A positive measure $d\mu$ on D is a Carleson measure if

$$\sup_B \frac{1}{|B|}\int_Q d\mu = C$$

is finite. The constant C is the Carleson norm of μ. It is well known, [17], that $d\mu$ is a Carleson measure with norm $\sim C$ if and only if

$$\int_D |\phi|^p d\mu \le C\|\phi\|^p_{H^p}, \quad \phi \in H^p. \tag{4.3}$$

We let W^α, $0 \le \alpha \le 1$, denote the interpolation spaces between the space of Carleson measures W^1 and the space of finite measures W^0 in D. If $0 < \alpha < 1$, then the measure f is in W^α if and only if $f = a d\tau$, where $d\tau \in W^1$, $a \in L^p(d\tau)$ and $1/p = 1 - \alpha$, see [2]. In view of (4.3), if $\phi \in H^p$ and $d\tau \in W^1$, then $\phi d\tau \in W^\alpha$ and

$$\|\phi d\tau\|_{W^\alpha} \lesssim \|\phi\|_{H^p}\|d\tau\|_{W^1} \text{ where } \alpha = 1 - 1/p\ . \tag{4.4}$$

If f is a section to Λ^ℓ, then $\|f\|_{W^1}$ means the Carleson norm of the function $|f|$.

A section f to Λ^ℓ in $L^2_{\rm loc}(\partial D)$ is in $BMO(\Lambda^\ell)$ if

$$\|f\|^2_{BMO}=\sup_B\frac{1}{|B|}\int_B|f-f_B|^2d\sigma+\int_{\partial D}|f|^2d\sigma$$

is finite, where the supremum is taken over all Koryanyi balls on ∂D, and f_B is the mean value of f over B. For holomorphic f, the square of this norm is equivalent to

$$\int_{\partial D}|f|^2d\sigma+\|\,|\partial f|^2\|_{W^1},$$

with constants that do not depend on m. This is well-known in the scalar valued case, see e.g. [8], and the proof extends verbatimly to the vector valued case. The space $\mathcal{H}^1(\Lambda^\ell)$ is defined by atoms a, where a is an atom if either $a\equiv 1$ or it is a section to Λ^ℓ that is supported in a Koranyi ball $B_r(p)$ with $\int a=0$ and

$$|B_r(p)|\int|a|^2\le 1.$$

The usual proof of the duality of $\mathcal{H}^1$ and BMO extends to this vector valued case and H^1 is naturally identified with the closed subspace of all CR-functions in $\mathcal{H}^1$.

Let $C^\infty_q(\overline{D},\Lambda^\ell)$ denote the space of Λ^ℓ-valued $(0,q)$ forms in D that are smooth up to the boundary.

Theorem 4.1. *Let $\tau>-1$ and $0\le\alpha\le 1$. There is an operator*

$$K\colon C^\infty_{0,q+1}(\overline{D},\Lambda^\ell)\to C_{0,q}(\overline{D},\Lambda^\ell),$$

such that $\bar\partial Kf=f$ if $\bar\partial f=0$, and which satisfies the estimates

$$\|(-\rho)^\tau Kf\|_{W^\alpha}\lesssim\|(-\rho)^{\tau+1/2}f\|_{W^\alpha}\tag{4.5}$$

for any (not necessarily $\bar\partial$ closed) f. Moreover, in the limit case $\tau=-1$ we have the following estimate, $1\le p\le *$, for the complex tangential boundary values of $K_\alpha f$,

$$\|Kf\|_{L^p(\partial D)}\lesssim\|(-\rho)^{-1/2}f\|_{W^\alpha},\tag{4.6}$$

if $1/p=1-1/\alpha$ for $\alpha<1$ and $\alpha=1$ corresponds to the BMO-norm. The constants in $\lesssim$ are independent of m.

Various cases of this theorem have been proved in [15], [23], [28] and [2]. The proof of all but the $*=BMO$-estimate only depends on the estimate

$$|Kf(z)|\lesssim\int_D\frac{(-\rho)^{r-1}|f|}{|v(\zeta,z)|^{r-1/2}}\left(\frac{|v|}{\sigma}\right)^{2n-1},$$

where r is some large number, $|v(\zeta,z)|\sim d(\zeta)+d(z)+d(\zeta',z')$ and $\sigma(\zeta,z)$ is like $d(\zeta',z')+|d(\zeta)-d(z)|+\sqrt{d(\zeta)+d(z)}c(\zeta',z)$, where $c(\zeta',z)$ is the distance between ζ' and z in the complex tangential directions. The BMO case also requires a certain smoothness property of the kernel, see e.g. (the proof of) Theorem 4 in [10]. We can for instance choose K as the operator K_α in [10], with α large enough, and extend it to $C^\infty_q(\overline{D},\Lambda^l)$ in the obvious way. These operators also satisfies the following Wolff type estimate of the boundary values of Kf. It can be formulated in terms of the W^α-norms, but for simplicity we state its dual formulation that fits our needs here. Let $\mathcal{L}$ denote a fixed smooth $(1,0)$-vector field on $\overline{D}$.

Theorem 4.2. *Let ω be a $Hom(\Lambda^\ell,\Lambda^{\ell+1})$-valued $(0,1)$-form such that the Carleson norm of $|\omega|^2$ ($|\omega|$ denotes the pointwise operator norm) is bounded by C^2 and the Carleson norm of $\sqrt{-\rho}|\mathcal{L}\omega|$ is bounded by C. Then there is an operator K such that $\bar\partial Kf=f$ if $\bar\partial f=0$, and for Λ^ℓ-valued holomorphic ϕ we have*

$$\|K\omega\phi\|_{L^p(\partial D)}\le C'C\|\phi\|_{H^p},\ 1\le p<\infty,$$

and

$$\|K\omega\phi\|_{BMO}\le C'C\|\phi\|_{H^\infty},$$

where C' just depends on the domain D (and $\mathcal{L}$).

This is the vector-valued version of the estimate for K discussed in §8 in [9], and it follows from the fact that

$$\Big|\int_{\partial D}\langle K\omega\phi,\psi\rangle\Big|\le CC'\|\psi\|_{\mathcal{H}^1}\|\phi\|_{H^\infty},$$

$$\Big|\int_{\partial D}\langle K\omega\phi,\psi\rangle\Big|\le CC'\|\psi\|_{L^q(\partial D)}\|\phi\|_{H^p}$$

and

$$\Big|\int_{\partial D}\langle K\omega\phi,\psi\rangle\Big|\le CC'\|\psi\|_{L^\infty}\|\phi\|_{H^1}.$$

These estimates, in turn are obtained as in [9] by means of the $T1$-theorem for Carleson measures of Christ and Journé, see [11].

5. Proof of Theorem 2.1

We first consider pointwise estimates of each form $\gamma\cap(\bar\partial\gamma)^k\cap\phi$ where $\gamma=\sum\overline{g_j}/|g|\,e_j$. The definition (1.1) extends to Λ^*-valued form as soon as we have chosen a metric for scalar valued forms. We then have

Lemma 5.1. *For Λ^*-valued forms we have*

$$|f\cap h|\le C|f||h|, \tag{5.1}$$

where C is independent of m (but depends on the degrees of f and h). If $g=\sum_1^m g_je_j^*$, where g_j are functions, then

$$|\delta_g h|\le|g||h|. \tag{5.2}$$

Proof. If f and h have degrees p and q, respectively, then

$$f \cap h = \sum_{|L|=p+q}{}' \left(\sum_{I \cup J = L}{}' \pm f_I h_J \right) e_L$$

and hence

$$|f \cap h|^2 = \sum_{|L|=p+q}{}' \left| \sum_{I \cup J = L}{}' \pm f_I h_J \right|^2 .$$

For each fixed L there is only a finite number of terms (in fact $(p+q)!/p!q!$) within the modulus signs, and hence we can estimate $|f \cap h|^2$ by a constant times

$$\sum_{|L|=p+q}{}' \sum_{I \cup J = L}{}' |f_I \cap h_J|^2 \le \sum_{I,J}{}' |f_I \cap h_J|^2 \le C|f|^2|h|^2.$$

For the second statement, we can make a change of basis e_j and normalize, and therefore assume that $g = e_1^*$ (at some fixed point in D). Now,

$$|\delta_g h|^2 = \left| \sum_{J \ni 1}{}' h_J e^{J \setminus 1} \right|^2 = \sum_{J \ni 1}{}' |h_J|^2 \le |h|^2.$$

□

Besides the $\bar\partial$-estimates, the key stone in the proof of Theorem 2.1 is the following result.

Proposition 5.2. *If $r \ge 2$ then*

$$\left\| (-\rho)^{\frac{r-2}{2}} \gamma \cap (\bar\partial\gamma)^r \right\|_{W^1} \le C \frac{1}{\delta^{r+1}} \left(\log \frac{1+\delta^2}{\delta^2} \right)^{r/2} . \tag{5.3}$$

Moreover,

$$\| \, |\partial g|^2/|g|^4 \|_{W^1} \le C \frac{1}{\delta^2} \log \frac{1+\delta^2}{\delta^2}. \tag{5.4}$$

It should be noted that it is the sharp estimate of the Carleson norm in (5.3) that is essential. Since $\| \, |\partial g|^2 \|_{W^1} \lesssim \|g\|^2_{H^\infty}$, and moreover $|\partial g| \le C/\sqrt{-\rho}$, it is immediate in view of (5.1) that (5.3) holds for some constant. Restricted to $n = 1$, (5.3) is exactly the key point in Uchiyama's sharp estimate for solutions to the corona theorem, cf. the remark preceding Theorem 2.2. The proof of Proposition 5.2 is based on the following result from [7], (with $\Omega = -\rho\omega$).

Theorem 5.3. *Let $u_1, \ldots, u_r$ be positive plurisubharmonic functions in D (and $r \ge 1$). Then*

$$d\tau = (-\rho)^r dd^c u_1 \wedge \ldots \wedge dd^c u_r \wedge \Omega_{n-r} \tag{5.5}$$

is a Carleson measure with norm $\le c_{r,n} \sup u_1 \ldots \sup u_r$, where $c_{n,r} = e(r-1)!n!/(n-r)!$.

Proof of Proposition 5.2. We first establish the inequality

$$\big\|(-\rho)^{r-1}|(\bar\partial\gamma)^r|^2\big\|_{W^1}\le C\frac{1}{\delta^{2r}}\left(\log\frac{1+\delta^2}{\delta^2}\right)^r \tag{5.6}$$

for $r\ge 1$. A direct computation gives that

$$\sum_j i\partial\bar\gamma_j\wedge\bar\partial\gamma_j=\frac{|g|^2 i\partial\bar\partial|g|^2-i\partial|g|^2\wedge\bar\partial|g|^2}{|g|^6}\le\frac{i\partial\bar\partial|g|^2}{|g|^4}.$$

Now,

$$\begin{aligned} i\partial\bar\partial\log\frac{|g|^2+\delta^2}{\delta^2} &= \frac{(|g|^2+\delta^2)i\partial\bar\partial|g|^2-i\partial|g|^2\wedge\bar\partial|g|^2}{(|g|^2+\delta^2)^2}\\ &\ge\frac{\delta^2}{(|g|^2+\delta^2)^2}i\partial\bar\partial|g|^2\ge\frac{\delta^2}{4}\frac{i\partial\bar\partial|g|^2}{|g|^4}.\end{aligned}\tag{5.7}$$

Here we have used that $i\partial|g|^2\wedge\bar\partial|g|^2=i\bar g\cdot\partial g\wedge g\cdot\overline{\partial g}\le|g|^2 i\partial\bar\partial|g|^2$, which is an inequality for positive forms; applying each side to $(v,\bar v)$ where v is any $(1,0)$ vector it follows by Schwarz' inequality. However, it also follows from the computation above since $\log(|g|^2+\delta^2)$ is plurisubharmonic and hence the second term in (5.7) has to be positive for any δ. Thus we have

$$\sum_j i\partial\bar\gamma_j\wedge\bar\partial\gamma_j\le\frac{4}{\delta^2}i\partial\bar\partial\log\frac{|g|^2+\delta^2}{\delta^2}. \tag{5.8}$$

Note that

$$|(\bar\partial\gamma)^r|^2=(r!)^2\sum_{|I|=r}{}'|\bar\partial\gamma_{i_1}\wedge\ldots\wedge\bar\partial\gamma_{i_r}|^2.$$

Therefore, in view of (4.2), we have

$$\begin{aligned} &(-\rho)^{r-1}|(\bar\partial\gamma)^r|^2\beta_n\\ \sim\ &(-\rho)^r\sum_{|I|=r}{}' i\partial\bar\gamma_{i_1}\wedge\bar\partial\gamma_{i_1}\wedge\ldots\wedge i\partial\bar\gamma_{i_r}\wedge\bar\partial\gamma_{i_r}\wedge\Omega_{n-r}\\ \le\ & r!(-\rho)^r\left(\sum_j i\partial\bar\gamma_j\wedge\bar\partial\gamma_j\right)^r\wedge\Omega_{n-r}\\ \le\ &\frac{4^r r!}{\delta^{2r}}(-\rho)^r\left(idd^c\log\frac{|g|^2+\delta^2}{\delta^2}\right)^r\wedge\Omega_{n-r},\end{aligned}$$

and the Carleson norm of the right hand side is bounded by

$$C\frac{4^r r!}{\delta^{2r}}\left(\log\left(\frac{1+\delta^2}{\delta^2}\right)\right)^r$$

according to Theorem 5.3. Thus we have proved (5.6).

It is now easy to deduce (5.3). By Lemma 5.1 we have that

$$\left|(-\rho)^{\frac{k-2}{2}}\gamma\cap(\bar\partial\gamma)^k\right| \lesssim (-\rho)^{\frac{k-2}{2}}|\gamma||\bar\partial\gamma||(\bar\partial\gamma)^{k-1}| \leq \frac{1}{\delta}\left((-\rho)^{k-2}|(\bar\partial\gamma)^{k-1}|^2\alpha+|\bar\partial\gamma|^2\frac{1}{\alpha}\right). \tag{5.9}$$

From (5.6) it follows that the Carleson norm of the right hand side of (5.9) is bounded by

$$\frac{1}{\delta}\left(\frac{\alpha}{\delta^{2k-1}}\left(\log\frac{1+\delta^2}{\delta^2}\right)^{k-1}+\frac{1}{\alpha\delta^2}\log\frac{1+\delta^2}{\delta^2}\right).$$

Choosing

$$\alpha=\delta^{k-2}\left(\log\frac{1+\delta^2}{\delta^2}\right)^{1-k/2}$$

we get the desired result. The inequality (5.4) is a consequence of the preceding arguments. □

Proof of Theorem 2.1. As usual, we may assume that the generators g as well as ϕ are holomorphic in a neighborhood of $\overline{D}$, and it is enough to obtain the à priori estimate

$$\|u\|_{L^p(\partial D)} \leq C_\delta\|\phi\|_{H^p}, \tag{5.10}$$

where u is given by (3.1). We estimate each term separately. The term corresponding to $k=0$ is trivial in view of (5.1). Next consider the terms $\omega_k\cap\phi$ corresponding to $k\geq 2$. Recall that $\omega_k=\gamma\cap(\bar\partial\gamma)^k$ and let

$$C_{\delta,k}=C\frac{1}{\delta^{k+1}}\left(\log\frac{1+\delta^2}{\delta^2}\right)^{k/2}. \tag{5.11}$$

In view of (5.3), (5.1) and (4.4), the W^α-norm of $(-\rho)^{k/2-1}\omega_k\cap\phi$ is bounded by $C_{\delta,k}$ times $\|\phi\|_{H^p}$. By $k-1$ applications of (4.5) and (5.2), we obtain $(-\rho)^{1/2}(\delta_g K)^{k-1}\omega_k\cap\phi\in W^\alpha$ and finally by (4.6), $(\delta_g K)^k\omega_k\cap\phi\in L^p(\partial D)$, with the norm bounded by $C_{\delta,k}$.

Thus it remains to estimate the term for $k=1$, which is the crucial one in the sense that one must use the Wolff type estimate in Theorem 4.2. A simple computation, holding in mind that $\sqrt{-\rho}|\mathcal{L}g|\lesssim|\partial g|$, reveals that

$$|\omega_1|^2\lesssim\frac{1}{\delta^2}\frac{|\partial g|^2}{|g|^4} \quad\text{and}\quad \sqrt{-\rho}|\mathcal{L}\omega_1|\lesssim\frac{|\partial g|^2}{|g|^4}. \tag{5.12}$$

In view of (5.4), therefore the Carleson norm of $|\omega_1|^2$ is $\lesssim C^2$ and the Carleson norm of $\sqrt{-\rho}|\mathcal{L}\omega_1|$ is $\lesssim C$, if $C=\delta^{-2}\log(1/\delta)$. From Theorem 4.2 we now get the estimate $\lesssim C\|\phi\|_{H^p}$ of the term corresponding to $k=1$. Thus Theorem 2.1 is proved. □

6. Multipliers on BMOA and proof of Theorem 2.4

As before D is a smoothly bounded strictly pseudoconvex domain. If V and W are vector spaces and g is a holomorphic $\mathrm{Hom}(V,W)$-valued function, then g is a multiplier for $BMOA$ if it maps $BMOA(V) \to BMOA(W)$ boundedly. The main result in this section is the characterization of such multipliers, see [25] and [20], adapted to our vector-valued setting. This result states that the multipliers on $BMOA$ are the bounded analytic functions such that

$$\sup_B \left(\log \frac{1}{|B|}\right)^2 \frac{1}{|B|} \int_B |g - g_B|^2$$

is finite. We also need an equivalent Carleson-type condition for the measure $|\partial g|$. This is contained in Theorem 6.2 below. Its statement and proof are similar to classical results for $BMOA$ that we first recall.

Theorem 6.1. *Let b be a vector valued holomorphic function. Then the following conditions are equivalent:*

$$\|b\|_{H^2} \leq C \text{ and } \sup_B \frac{1}{|B|} \int_B |b - b_B|^2 \leq C^2. \tag{6.1}$$

$$\|b\|_{H^2} \leq C \text{ and } \sup_{z \in D} d(z)^n \int_{\partial D} \frac{|b(\zeta) - b(z)|^2}{|v(z,\zeta)|^{2n}} \leq C^2. \tag{6.2}$$

$$\|b\|_{H^2} \leq C \text{ and } \sup_B \frac{1}{|B|} \int_Q |\partial b|^2 \leq C^2. \tag{6.3}$$

$$\|b\|_{H^2} \leq C \text{ and } \sup_{z \in D} d(z)^n \int_D \frac{|\partial b|^2}{|v(z,\zeta)|^{2n}} \leq C^2. \tag{6.4}$$

Moreover, the various constants C are equivalent up to constants that only depend on D.

Of course (6.1) is just a restatement of the fact that $b \in BMOA$ and $\|b\|_{BMO} \sim C$. Also recall that B denotes the Koranyi balls on ∂D and Q are the corresponding tent over B.

The important part of this theorem is the equivalence between (6.1) and (6.3), that was discussed in Section 4. The equivalence between (6.1) and (6.2) as well as that between (6.3) and (6.4) follow from simple size estimates of $|v|$; see [14], Theorem 1.2 and Lemma 3.3 in Chapter 6 for details in the classical case. Since we may choose $v(z,\zeta)$ to be holomorphic in ζ, that (6.3) implies (6.4) also follows immediately from (4.3) and the simple estimate

$$\int_{\partial D} \frac{d\sigma(\zeta)}{|v(z,\zeta)|^{n+\alpha}} \lesssim \frac{1}{d(z)^\alpha}, \quad \alpha > 0. \tag{6.5}$$

Theorem 6.2. *If g is a holomorphic function with values in $Hom(V,W)$, then the following conditions are equivalent:*

$$g \text{ is a multiplier on } BMOA \text{ with norm } C. \tag{6.6}$$

$$\|g\|_{H^\infty} \leq C \text{ and } \sup_B \left(\log\frac{1}{|B|}\right)^2 \frac{1}{|B|}\int_B |g-g_B|^2 \leq C^2. \tag{6.7}$$

(6.8)

$$\|g\|_{H^\infty} \leq C \text{ and } \sup_{z\in D} \left(\log\frac{1}{d(z)}\right)^2 d(z)^n \int_{\partial D} \frac{|g(\zeta)-g(z)|^2}{|v(z,\zeta)|^{2n}} \leq C^2.$$

$$\|g\|_{H^\infty} \leq C \text{ and } \sup_B \left(\log\frac{1}{|B|}\right)^2 \frac{1}{|B|}\int_Q |\partial g|^2 \leq C^2\ . \tag{6.9}$$

$$\|g\|_{H^\infty} \leq C \text{ and } \sup_{z\in D} \left(\log\frac{1}{d(z)}\right)^2 d(z)^n \int_D \frac{|\partial g|^2}{|v(z,\zeta)|^{2n}} \leq C^2. \tag{6.10}$$

Moreover, the constants C are equivalent up to constants that only depend on D.

The kernel $d^n/|v|^{2n}$ can be replaced with $d^\alpha/|v|^{n+\alpha}$ for any $\alpha > 0$. Our choice $\alpha = n$ is just because in the ball $-\rho(z)^n/|v|^{2n}$ is precisely the Poisson–Szegő kernel.

For the proof we recall that

$$\int_{\partial D} |\phi|^2 \sim \int_D |\partial\phi|^2 + |\phi(0)|^2 \tag{6.11}$$

if ϕ is holomorphic. We also need

Lemma 6.3. *If $b \in BMOA$, then $|b(z)| \lesssim \|b\|_{BMO} \log(1/d(z))$.*

Sketch of proof. Since b is holomorphic we have a representation

$$b(z) = c\int_{\partial D} \frac{b(\zeta)a(\zeta,z)d\sigma(\zeta)}{v(\zeta,z)^n},$$

where a is smooth. Since $v(\zeta,z)$ is approximately antiholomorphic in ζ it follows from the polarization of (6.11) that

$$|b(z)|^2 \lesssim \int_D \frac{|\partial b|^2}{|v|^n} \int_D \frac{|\bar\partial v|^2}{|v|^{n+2}} + |b(0)|^2.$$

By (6.3) and (4.3), the first factor is $\|b\|^2_{BMO}$ times $\int_{\partial D} |v|^{-n} d\sigma$ that by (6.5) is bounded by $\log(1/d(z))$. Since $|\partial\bar v|^2 \lesssim d(\zeta)$, by a variant of (6.5) the second factor is $\lesssim \log(1/d(z))$ as well. □

Sketch of proof of Theorem 6.2. As in Theorem 6.1 the equivalences between (6.7) and (6.8) and between (6.9) and (6.10) are easy.

To prove that (6.6) implies (6.8), let $f_z(\zeta) = \log(1/v(z,\zeta))$. Then f_z is holomorphic, and it is easy to check that $\| \, |\partial f_z|^2 \|_{W^1} \lesssim 1$ so that $\|f_z\|_{BMO} \lesssim 1$. Also

$$|g(\zeta)| \leq \sup_{|c|=1} \sup_{z\in D} \frac{|g(z) c f_\zeta(z)|}{|f_z(z)|} \,,$$

where c runs over vectors in V, since it is equality for $z = \zeta$. Now $f_z(z) \sim \log 1/d(z)$, and therefore Lemma 6.3 implies that

$$|g(\zeta)| \lesssim \sup_{|c|=1} \|g c f_\zeta\|_{BMO} \leq C \sup_{|c|=1} \|c f_\zeta\|_{BMO} \lesssim C.$$

To complete the proof of (6.8) note that

$$\begin{aligned} |(g-g(z))f(z)|^2 &\leq |gf - g(z)f(z)|^2 + |g(f-f(z))|^2 \\ &\leq |gf - g(z)f(z)|^2 + C^2|f-f(z)|^2 \,. \end{aligned}$$

Since gf as well as f are in $BMOA$, we get, in view of (6.2), that

$$\sup_z d(z)^n \int_{\partial D} \frac{|f(z)(g-g(z))|^2}{|v|^{2n}} \lesssim C^2 \|f\|^2_{BMO}.$$

Taking $f = cf_z$ for various $|c| = 1$ one gets the estimate (6.8).

To see that (6.8) implies (6.10), write

$$\frac{1}{v^n}\partial g = \partial\left(\frac{1}{v^n}(g-g(z))\right) - (g-g(z))\partial\frac{1}{v^n}.$$

Then,

$$\int_D \frac{|\partial g|^2}{|v|^{2n}} \lesssim \int_D \left|\partial\left(\frac{1}{v^n}(g-g(z))\right)\right|^2 + \int_D \left|\frac{g-g(z)}{v^n}\right|^2 |\partial \log(1/v)|^2.$$

The first integral is estimated by (6.11) and since $\| \, |\partial \log(1/v)|^2 \|_{W^1}$ is finite, (4.3) implies that the second integral is bounded by

$$\int_{\partial D} \left|\frac{g-g(z)}{v^n}\right|^2$$

as desired.

Finally we prove that (6.10) implies (6.6). Let $b \in BMOA(V)$ with $\|b\|_{BMO} \leq 1$. We will confirm that gb is in $BMOA(W)$ by verifying (6.4). Now

$$d(z)^n \int_D \frac{|\partial(gb)|^2}{|v|^{2n}} \lesssim \tag{6.12}$$

$$d(z)^n \int_D \frac{|\partial g \, (b-b(z))|^2}{|v|^{2n}} + d(z)^n \int_D \frac{|\partial g \, b(z)|^2}{|v|^{2n}} + d(z)^n \int_D \frac{|g \partial b|^2}{|v|^{2n}}.$$

In the first integral we use that $\| \, |\partial g|^2 \|_{W^1} \lesssim \|g\|^2_{H^\infty} \lesssim C^2$ and (4.3), to obtain the bound

$$C^2 d(z)^n \int_{\partial D} \frac{|b-b(z)|^2}{|v|^{2n}} \lesssim C^2.$$

By Lemma 6.3, the second integral in (6.12) is bounded by

$$\left(\log \frac{1}{d(z)}\right)^2 d(z)^n \int_D \frac{|\partial g|^2}{|v|^{2n}} \lesssim C^2.$$

The estimate of the third integral in (6.12) is immediate. Thus gb is in $BMOA(W)$. □

Remark 4. To prove Theorem 2.4, we also need that (6.7) implies (6.6) also for not necessarily holomorphic functions, i.e. if $\|g\|_{L^\infty} \leq C$ and

$$\sup_B \left(\log \frac{1}{|B|}\right)^2 \frac{1}{|B|} \int_Q |g - g_B|^2 \leq C^2,$$

then g is a multiplier on BMO.

To prove this, note that $gf - g_B f_B = g(f - f_B) + f_B(g - g_B)$. Hence

$$\frac{1}{|B|} \int |gf - g_B f_B|^2 \lesssim \frac{1}{|B|} \int |g(f - f_B)|^2 + \frac{1}{|B|} \int |f_B(g - g_B)|^2.$$

The first term is clearly bounded by $\|g\|^2_{L^\infty} \|f\|^2_{BMO}$. To estimate the second term we use the well-known estimate $|f_B| \lesssim \log 1/|B| \, \|f\|_{BMO}$, to obtain

$$\frac{1}{|B|} \int |f_B(g - g_B)|^2 \lesssim \|f\|^2_{BMO} \frac{\log^2 \frac{1}{|B|}}{|B|} \int |g - g_B|^2 \lesssim C^2 \|f\|^2_{BMO}$$

as desired.

□

Proof of Theorem 2.4. Again we have to estimate each term in the sum (3.1), assuming that $\phi \in BMOA(\Lambda^\ell)$. In what follows we assume that $\|\phi\|_{BMO} \leq 1$.

To estimate the term corresponding to $k = 0$, we claim that

$$\left(\log \frac{1}{|B|}\right)^2 \frac{1}{|B|} \int_B |\gamma - \gamma_B|^2 \lesssim \frac{1}{\delta^4}. \tag{6.13}$$

To prove this we may replace γ_B with any constants c_B. If we choose $c_B = \bar{g}_B / |g_B|^2$, we have

$$|\gamma - c_B| \leq \left| \frac{\bar{g}}{g\bar{g}} - \frac{\bar{g}_B}{g\bar{g}} \right| + \left| \frac{\bar{g}_B}{g\bar{g}} - \frac{\bar{g}_B}{g_B\bar{g}} \right| + \left| \frac{\bar{g}_B}{g_B\bar{g}} - \frac{\bar{g}_B}{g_B\bar{g}_B} \right| \lesssim \frac{1}{\delta^2} |g - g_B| \, ,$$

and (6.13) follows. By Remark 4 this implies that $\|\gamma \cap \phi\| \lesssim \delta^{-2}$.

To estimate the term with $k = 1$, note that

$$|\partial g|^2 |\phi|^2 = \sum_I{}' \sum_j |\partial g_j|^2 |\phi_I|^2 \lesssim \sum_I{}' \sum_j \left(|\partial(g_j \phi_I)|^2 + |g_j|^2 |\partial \phi_I|^2 \right),$$

which we for short can write as $|\partial g||\phi| \lesssim |g||\partial\phi| + |\partial(g\phi)|$. If $\tau = |\partial\phi| + |\partial(g\phi)|$, then by our assumptions the Carleson norm of $\tau^2 \lesssim 1$. If $\omega = \gamma \cap \bar{\partial}\gamma \cap \phi$, then

$$|\omega| \lesssim |g|^{-2}|\partial\phi| + |g|^{-3}|\partial(g\phi)| \leq |\delta|^{-3}\tau \ ,$$

and hence $\| \, |\omega|^2\|_{W^1} \lesssim \delta^{-6}$. Furthermore, using that $\sqrt{-\rho}|\mathcal{L}g| \lesssim |\partial g|$ and the corresponding inequality for ϕ, we have

$$\sqrt{-\rho}|\mathcal{L}\omega| \lesssim |g|^{-4}|\partial g||\partial(g\phi)| + |g|^{-3}|\partial g||\partial\phi| \leq \frac{1}{\delta^2}\frac{|\partial g|}{|g|^2}\tau \ ,$$

Using $|ab| \leq \alpha|a|^2 + |b|^2/\alpha$ with $\alpha = \delta/\sqrt{\log(1/\delta)}$, we obtain by (5.4) that

$$\|\sqrt{-\rho}|\mathcal{L}\omega|\|_{W^1} \lesssim \delta^{-3}\sqrt{\log(1/\delta)}.$$

In view of Theorem 4.2 we find that the term for $k = 1$ has BMO norm $\lesssim \delta^{-3}\sqrt{\log(1/\delta)}$.

Finally we consider a term where $k \geq 2$. We then have to estimate the Carleson norm of $I = (-\rho)^{\frac{k-2}{2}}|\gamma \cap (\bar{\partial}\gamma)^k \cap \phi|$. We have

$$I \lesssim (-\rho)^{\frac{k-2}{2}}|\bar{\partial}\gamma|^{k-1}|\omega| \leq \alpha(-\rho)^{k-2}|(\bar{\partial}\gamma)^{k-1}|^2 + \frac{1}{\alpha}|\omega|^2 \ .$$

Taking $\alpha = \delta^{k-4}\left(\log(1/\delta)\right)^{-\frac{k-1}{2}}$ we obtain, using (5.6) and $\| \, |\omega|^2\|_{W^1} \lesssim \delta^{-6}$, that $\|I\|_{W^1} \lesssim C_{\delta,k}$, with $C_{\delta,k}$ from (5.11). By repeated use of Theorem 4.1 we get the BMO-estimate $C_{\delta,k}$ for the term k. This proves Theorem 2.4. □

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