Elliptic Uniformly Degenerate Operators

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Regularity and Fredholm properties are established for elliptic uniformly degenerate operators with variable indicial roots.

§1. Introduction

A uniformly degenerate operator on a differentiable manifold with boundary is a linear partial differential operator with smooth coefficients and the following type of degeneracy at the boundary. Near the boundary the operator is assumed to be of the form

$$\sum_{j+|\alpha|\leq m} a_{j,\alpha}(x,y)\,(x\partial_x)^j(x\partial_y)^\alpha$$

where $(x,y)=(x,y_1,\dots,y_n)$ are local coordinates with $x>0$ in the interior of the manifold and $x=0$ on its boundary. The coefficients $a_{j,\alpha}$ are assumed to be complex $N\times N$-matrix valued and smooth up to the boundary. The operator is said to be elliptic uniformly degenerate if it is elliptic in the usual sense in the interior of the manifold, and the matrix

$$\sum_{j+|\alpha|=m} a_{j,\alpha}(0,y)\,\xi^j\eta^\alpha$$

is invertible for all boundary points $(0,y)$ and all $(\xi,\eta)\neq 0$. The indicial roots at a boundary point $(0,y)$ are defined as the roots γ of the degree mN polynomial

$$\det\sum_{j=0}^{m} a_{j,0}(0,y)\,\gamma^j.$$

In §7 we verify that these definitions are independent of the choice of coordinates.

Examples of elliptic uniformly degenerate operators include the Laplacian (Example 8.1), the Hodge Laplacians (Example 8.2), and the Dirac operator on conformally compact asymptotically hyperbolic manifolds. Other examples include the linearizations of the Yamabe equation [ACF], the Einstein equations [GL], and the monopole equations ([R] and joint work with R. Mazzeo in progress) on such manifolds. These examples all have constant indicial roots. Schrödinger operators on asymptotically hyperbolic manifolds (Example 8.3 and joint work with R. Mazzeo in progress) provide examples with variable indicial roots. Other global analysis problems on such manifolds also lead to elliptic uniformly degenerate operators.

Partial differential operators with various types of boundary degeneracies have been studied by several people using extensive microlocal machinery, such as R.B. Melrose' calculus of conormal distributions on manifolds with corners [Me1], [Me2], [Me3], [Me4] and B.-W. Schulze's calculus of pseudo-differential operators with operator-valued symbols [Sch]. In this spirit, elliptic uniformly degenerate operators with constant indicial roots have been studied by R. Mazzeo [MaMe], [Ma1], [Ma2] using the calculus of conormal distributions on manifolds with corners. Various second order elliptic uniformly degenerate operators have been studied by other methods in [GL], [A], and [R]. The results in [BCH] apply to uniformly degenerate operators only in rather special cases.

In this paper we present a new approach, in the spirit of classical analysis, to elliptic uniformly degenerate operators. Using this approach we extend Mazzeo's regularity and Fredholm results to elliptic uniformly degenerate operators with variable indicial roots. Our main results are Theorems 6.2 and 6.4 (Sobolev estimates and regularity), Corollary 6.6 (conormality), Theorems 6.7 and 6.8 and Remark 6.9 (polyhomogeneity), Theorem 7.1 (semi-Fredholm properties) and Corollary 7.2 (Fredholm properties). The techniques in this paper can be applied to elliptic partial differential operators with other types of boundary degeneracies.

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§2. Weighted Sobolev spaces

In this section we review some properties of weighted Sobolev spaces on upper half space

$$\overline{\mathbb{R}}^{n+1}_+ = \{ \ (x,y) \ \big| \ x \geq 0 \text{ and } y \in \mathbb{R}^n \ \}.$$

The proofs of these are quite standard, and are left to the reader.

The space $x^\delta H^s(\mathbb{R}^n)$. For $\delta \in \mathbb{R}$ and s a nonnegative integer, we define $x^\delta H^s(\mathbb{R}^{n+1}_+)$ as the space of all $u \in H^s_{\text{loc}}(\mathbb{R}^{n+1}_+)$ such that $x^{-\delta}(x\partial_x)^j(x\partial_y)^\alpha \in L^2(\mathbb{R}^{n+1}_+, x^{-n-1}dx\,dy)$ for all j and α with $j + |\alpha| \leq s$. This is a Hilbert space with norm

$$\|u\|^2_{\delta,s} = \sum_{j+|\alpha|\leq s} \|x^{-\delta}(x\partial_x)^j(x\partial_y)^\alpha u\|^2_{L^2(\mathbb{R}^{n+1}_+, x^{-n-1}\,dx\,dy)}. \tag{2.1}$$

We define $x^\delta H^{-s}(\mathbb{R}^{n+1}_+)$ as the space of all $u \in H^{-s}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ such that there exist $v_{j,\alpha} \in L^2(\mathbb{R}^{n+1}_+, x^{-n-1}dx\,dy)$ with $u = x^\delta \sum_{j+|\alpha|\leq s}(x\partial_x)^j(x\partial_y)^\alpha v_{j,\alpha}$. This is a Hilbert space with norm

$$\|u\|^2_{\delta,-s} = \inf_{(v_{j,\alpha})} \sum_{j+|\alpha|\leq s} \|v_{j,\alpha}\|^2_{L^2(\mathbb{R}^{n+1}_+, x^{-n-1}\,dx\,dy)} \tag{2.2}$$

where we take infimum over all collections of functions $v_{j,\alpha}$ as above.

Let ρ be a smooth function $\mathbb{R}_+ \to [0,1]$ with $\operatorname{supp}\rho \subseteq [1/2, 4]$ and $\rho = 1$ on $[1,2]$. We use the notation $u_\mu(x,y) = u(2^{-\mu}x, 2^{-\mu}y)$. Let $\delta \in \mathbb{R}$ and $s \in \mathbb{Z}$. It follows from the above definition that $u \in x^\delta H^s(\mathbb{R}^{n+1}_+)$ if and only if $u \in H^s_{\text{loc}}(\mathbb{R}^{n+1}_+)$, $\rho u_\mu \in H^s(\mathbb{R}^{n+1})$ for all $\mu \in \mathbb{Z}$, and

$$\sum_{\mu=-\infty}^{\infty} 4^{\delta\mu} \int_{\mathbb{R}^{n+1}} (1 + \xi^2 + |\eta|^2)^s \, |\widehat{\rho u_\mu}(\xi,\eta)|^2 \, d\xi\, d\eta < \infty. \tag{2.3}$$

Here $(\rho u_\mu)(x,y) = \rho(x)u_\mu(x,y)$. We take this to be our definition of $x^\delta H^s(\mathbb{R}^{n+1}_+)$ for $\delta, s \in \mathbb{R}$.

We topologize $\overline{\mathbb{R}}^{n+1}_+$ with the relative topology as a subset of $\mathbb{R}^{n+1}$. If U is an open subset of $\overline{\mathbb{R}}^{n+1}_+$, then we define the Fréchet space $x^\delta H^s_{\text{loc}}(U)$ as the space of all $u \in H^s_{\text{loc}}(U)$ such that $\varphi u \in x^\delta H^s(\mathbb{R}^{n+1}_+)$ for all $\varphi \in C^\infty_{\text{comp}}(U)$. If K is a closed subset of $\overline{\mathbb{R}}^{n+1}_+$, then we define the Hilbert space $x^\delta H^s_0(K)$ as the space of all $u \in x^\delta H^s(\mathbb{R}^{n+1}_+)$ such that $\operatorname{supp} u \subseteq K$.

Lemma 2.1(a) *For $\delta, s \in \mathbb{R}$,*

$$(x^\delta H^s(\mathbb{R}^{n+1}_+))^* = x^{-\delta} H^{-s}(\mathbb{R}^{n+1}_+).$$

The pairing is given by the $L^2(\mathbb{R}^{n+1}_+, x^{-n-1} dx\, dy)$ inner product.

(b) *For $\delta, s_1, s_2 \in \mathbb{R}$ with $s_2 \geq s_1$,*

$$x^\delta H^{s_2}(\mathbb{R}^{n+1}_+) \subseteq x^\delta H^{s_1}(\mathbb{R}^{n+1}_+).$$

(c) *For $\delta_1, \delta_2, s_1, s_2 \in \mathbb{R}$ and $x_0 > 0$ with $\delta_2 \geq \delta_1$ and $s_2 \geq s_1$,*

$$x^{\delta_2} H^{s_2}_0([0, x_0] \times \mathbb{R}^n) \subseteq x^{\delta_1} H^{s_1}_0([0, x_0] \times \mathbb{R}^n).$$

(d) *For $\delta_1, \delta_2, s_1, s_2 \in \mathbb{R}$ with $\delta_2 > \delta_1$ and $s_2 > s_1$, and K a closed subset of $\overline{\mathbb{R}}^{n+1}_+$, the inclusion*

$$x^{\delta_2} H^{s_2}_0(K) \to x^{\delta_1} H^{s_1}_0(K)$$

is a compact operator.

(e) *If $u \in x^\delta H^{s-1}(\mathbb{R}^{n+1}_+)$ and the difference quotients*

$$\Delta_{h,k} u(x, u) = \big(u(x + hx, x + ky) - u(x, y)\big)/(h^2 + |k|^2)^{1/2}$$

are uniformly bounded in $x^\delta H^{s-1}(\mathbb{R}^{n+1}_+)$ as $(h, k) \to (0, 0) \in \mathbb{R}^{n+1}$, then $u \in x^\delta H^s(\mathbb{R}^{n+1}_+)$.

(f) *If U and V are open subsets of $\overline{\mathbb{R}}^{n+1}_+$ and $f : U \to V$ is a diffeomorphism, then pull-back gives a bounded linear map*

$$f^* : x^\delta H^s_{\text{loc}}(V) \to x^\delta H^s_{\text{loc}}(U).$$

The space $x^\delta H^{s,t}(\mathbb{R}^n)$. For $\delta \in \mathbb{R}$ and s, t nonnegative integers, we define $x^\delta H^{s,t}(\mathbb{R}^{n+1}_+)$ as the space of all $u \in H^{s+t}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ such that $x^{-\delta}(x\partial_x)^j (x\partial_y)^\alpha \partial_y^\beta u \in L^2(\mathbb{R}^{n+1}_+, x^{-n-1} dx\, dy)$ for all j, k, α, β with $j + |\alpha| + |\beta| \leq s$ and $|\beta| \leq t$. This is a Hilbert space with norm

$$\|u\|^2_{\delta,s,t} = \sum_{\substack{j+|\alpha|+|\beta| \leq s+t \\ |\beta| \leq t}} \|x^{-\delta}(x\partial_x)^j (x\partial_y)^\alpha \partial_y^\beta u\|^2_{L^2(\mathbb{R}^{n+1}_+, x^{-n-1} dx\, dy)}.$$

This definition can be extended to $\delta \in \mathbb{R}$ and $s, t \in \mathbb{Z}$ as in the case of $x^\delta H^s(\mathbb{R}^{n+1}_+)$. There are three cases to consider depending on the signs of s and t; the details are left to the reader.

For $\delta \in \mathbb{R}$ and $s, t \in \mathbb{Z}$, $u \in x^\delta H^{s,t}(\mathbb{R}^{n+1})$ if and only if $u \in H^{s+t}_{\text{loc}}(\mathbb{R}^{n+1}_+)$, and

$$\sum_{\mu=-\infty}^{\infty} 4^{\delta\mu} \int_{\mathbb{R}^{n+1}} (1+\xi^2+|\eta|^2)^s \, (1+\xi^2+(1+4^\mu)|\eta|^2)^t \, |\widehat{\rho u_\mu}(\xi,\eta)|^2 \, d\xi \, d\eta < \infty$$

We take this to be our definition of $x^\delta H^{s,t}(\mathbb{R}^{n+1}_+)$ for $\delta, s, t \in \mathbb{R}$.

If U is an open subset of $\overline{\mathbb{R}}^{n+1}_+$, then we define the Fréchet space $x^\delta H^{s,t}_{\text{loc}}(U)$ as the space of all $u \in H^{s+t}_{\text{loc}}(U)$ such that $\varphi u \in x^\delta H^{s,t}(\mathbb{R}^{n+1}_+)$ for all $\varphi \in C^\infty_{\text{comp}}(U)$. If K is a closed subset of $\overline{\mathbb{R}}^{n+1}_+$, then we define the Hilbert space $x^\delta H^{s,t}_0(K)$ as the space of all $u \in x^\delta H^{s,t}(\mathbb{R}^{n+1}_+)$ such that $\operatorname{supp} u \subseteq K$.

Lemma 2.2(a) *For $\delta, s, t \in \mathbb{R}$,*

$$(x^\delta H^{s,t}(\mathbb{R}^{n+1}_+))^* = x^{-\delta} H^{-s,-t}(\mathbb{R}^{n+1}_+).$$

The pairing is given by the $L^2(\mathbb{R}^{n+1}_+, x^{-n-1}dx\,dy)$ inner product.

(b) *For $\delta, s_1, s_2, t_1, t_2 \in \mathbb{R}$ with $s_2 + t_2 \ge s_1 + t_1$ and $t_2 \ge t_1$,*

$$x^\delta H^{s_2,t_2}(\mathbb{R}^{n+1}_+) \subseteq x^\delta H^{s_1,t_1}(\mathbb{R}^{n+1}_+).$$

(c) *For $\delta_1, \delta_2, s_1, s_2, t_1, t_2 \in \mathbb{R}$ and $x_0 > 0$ with $\delta_2 \ge \delta_1$, $\delta_2 + t_2 \ge \delta_1 + t_1$, and $s_2 + t_2 \ge s_1 + t_1$,*

$$x^{\delta_2} H^{s_2,t_2}_0([0,x_0] \times \mathbb{R}^n) \subseteq x^{\delta_1} H^{s_1,t_1}_0([0,x_0] \times \mathbb{R}^n).$$

(d) *For $\delta_1, \delta_2, s_1, s_2, t_1, t_2 \in \mathbb{R}$ and $x_0 > 0$ with $\delta_2 > \delta_1$, $\delta_2 + t_2 > \delta_1 + t_1$, and $s_2 + t_2 > s_1 + t_1$ and K a compact subset of $\overline{\mathbb{R}}^{n+1}_+$, the inclusion*

$$x^{\delta_2} H^{s_2,t_2}_0(K) \to x^{\delta_1} H^{s_1,t_1}_0(K)$$

is a compact operator.

(e) *If $u \in x^\delta H^{s+1,t-1}(\mathbb{R}^{n+1}_+)$ and the difference quotients $\big(u(x,y+h) - u(x,y)\big)/|h|$ are uniformly bounded in $x^\delta H^{s,t-1}(\mathbb{R}^{n+1}_+)$ as $h \to 0 \in \mathbb{R}^n$, then $u \in x^\delta H^{s,t}(\mathbb{R}^{n+1}_+)$.*

(f) *If U and V are open subsets of $\overline{\mathbb{R}}^{n+1}_+$ and $f : U \to V$ is a diffeomorphism, then pull-back gives a bounded linear map*

$$f^* : x^\delta H^{s,t}_{\text{loc}}(V) \to x^\delta H^{s,t}_{\text{loc}}(U).$$

For $s \in \mathbb{R}$ we define the Sobolev space $H^s(\mathbb{R})$ as usual. We define $H^s(\mathbb{R}_+)$ as the Hilbert space of all u such that $u \circ \exp \in H^s(\mathbb{R})$. If s is a nonnegative integer, then $u \in H^s(\mathbb{R}_+)$ if and only if $(r\,d/dr)^j u \in L^2(\mathbb{R}_+, r^{-1}dr)$ for $j \le s$.

Lemma 2.3. *For any $\delta, s_1, t_1 \in \mathbb{R}$ and $x_0 > 0$ there exists $s_2, t_2 \in \mathbb{R}$ such that if* $\operatorname{supp} u \subseteq [0, x_0] \times \mathbb{R}^n$*, then*

$$u \in x^\delta H^{s_1,t_1}(\mathbb{R}^{n+1}_+) \quad \Rightarrow \quad u \in H^{t_2}(\mathbb{R}^n, x^{\delta+n/2} H^{s_2}(\mathbb{R}_+)).$$

Conversely, for any $\delta, s_2, t_2 \in \mathbb{R}$ and any $x_0 > 0$ there exists $s_1, t_1 \in \mathbb{R}$ such that if $\operatorname{supp} u \subseteq [0, x_0] \times \mathbb{R}^n$*, then*

$$u \in H^{t_2}(\mathbb{R}^n, x^{\delta+n/2} H^{s_2}(\mathbb{R}_+)) \quad \Rightarrow \quad u \in x^\delta H^{s_1,t_1}(\mathbb{R}^{n+1}_+).$$

The space $x^\delta \mathcal{A}(U)$. Let $\delta \in \mathbb{R}$ and let U be an open subset of $\overline{\mathbb{R}}^{n+1}_+$. Then we define the Fréchet space $x^\delta \mathcal{A}(U)$ as the space of all $u \in C^\infty(U \cap \mathbb{R}^{n+1}_+)$ such that $x^{-\delta}(x\partial_x)^j \partial_y^\alpha u \in L^2_{\mathrm{loc}}(U, x^{-n-1} dx\, dy)$ for all j and α. This space is known as the space of (weighted) conormal functions on U. We will need the following:

Lemma 2.4. *Let $\delta \in \mathbb{R}$, $x_0 \geq 0$ and Y be an open subset of $\mathbb{R}^n$. If $u \in x^\delta \mathcal{A}(\overline{\mathbb{R}}_+ \times Y)$ and* $\operatorname{supp} u \subseteq [0, x_0] \times Y$*, then $u \in C^\infty(Y, x^{\delta+n/2} H^\infty(\mathbb{R}_+))$. Conversely, $C^\infty(Y, x^{\delta+n/2} H^\infty(\mathbb{R}_+)) \subseteq x^\delta \mathcal{A}(\overline{\mathbb{R}}_+ \times Y)$.*

The space $r^\delta W^s(\mathbb{R}_+)$. For $\delta \in \mathbb{R}$ and s a nonnegative integer, we define $r^\delta W^s(\mathbb{R}_+)$ as the space of all $u \in H^s_{\mathrm{loc}}(\mathbb{R}_+)$ such that $r^{-\delta+j}(r\, d/dr)^k u \in L^2(\mathbb{R}_+, r^{-1} dr)$ for all nonnegative integers j and k with $j + k \leq s$. Then $r^\delta W^s(\mathbb{R}_+)$ is a Hilbert space with norm

$$\|u\|^2_{r^\delta W^s(\mathbb{R}_+)} = \sum_{j+k \leq s} \left\| r^{-\delta+j}(r\, d/dr)^k \right\|^2_{L^2(\mathbb{R}_+, r^{-1} dr)}. \tag{2.4}$$

We define $r^\delta W^{-s}(\mathbb{R}_+)$ as the space of all $u \in H^{-s}_{\mathrm{loc}}(\mathbb{R}_+)$ for which there exist $v_{j,k} \in L^2(\mathbb{R}_+, r^{-1} dr)$ such that $u = \sum_{j+k \leq s} r^{\delta+j}(r\, d/dr)^k v_{j,k}$. This is a Hilbert space with norm

$$\|u\|^2_{r^\delta W^{-s}(\mathbb{R}_+)} = \inf_{(v_{j,k})} \sum_{j+k \leq s} \|v_{j,k}\|^2_{L^2(\mathbb{R}_+, r^{-1} dr)} \tag{2.5}$$

where we take infimum over all collections of functions $v_{j,k}$ as above.

Let φ_0 be a smooth function $\mathbb{R}_+ \to [0,1]$ such that $\varphi_0 = 1$ on $(0,1]$ and $\varphi_0 = 0$ on $[2,\infty)$. Let $\varphi_\infty = 1 - \varphi_0$. If s is a non-negative integer, then

$$u \in r^\delta W^s(\mathbb{R}_+) \Leftrightarrow \begin{cases} r^{-\delta}(r\,d/dr)^k(\varphi_0 u) \in L^2(\mathbb{R}_+, r^{-1}dr) \\ r^{-\delta+s-k}(r\,d/dr)^k(\varphi_\infty u) \in L^2(\mathbb{R}_+, r^{-1}dr) \end{cases} \quad \text{for } k = 0,\ldots,s$$

$$\Leftrightarrow \begin{cases} r^{-\delta}(r\,d/dr)^k(\varphi_0 u) \in L^2(\mathbb{R}_+, r^{-1}dr) \\ r^{-\delta+s}(d/dr)^k(\varphi_\infty u) \in L^2(\mathbb{R}_+, r^{-1}dr) \end{cases} \quad \text{for } k = 0,\ldots,s$$

$$\Leftrightarrow \begin{cases} (r\,d/dr)^k(\varphi_0 u) \in r^\delta L^2(\mathbb{R}_+, r^{-1}dr) \\ (d/dr)^k(\varphi_\infty u) \in \langle r\rangle^{\delta-s+1/2} L^2(\mathbb{R}, dr) \end{cases} \quad \text{for } k = 0,\ldots,s$$

$$\Leftrightarrow \begin{cases} \varphi_0 u \in r^\delta H^s(\mathbb{R}_+) \\ \varphi_\infty u \in \langle r\rangle^{\delta-s+1/2} H^s(\mathbb{R}) \end{cases} \tag{2.6}$$

$$\Leftrightarrow \begin{cases} (\varphi_0 u)\circ \exp \in e^{\delta r} H^s(\mathbb{R}) \\ \varphi_\infty u \in \langle r\rangle^{\delta-s+1/2} H^s(\mathbb{R}). \end{cases} \tag{2.7}$$

Similarly, the characterizations (2.6) and (2.7) of $r^\delta W^s(\mathbb{R}_+)$ hold for negative integers s. We take (2.6), or equivalently (2.7), to be our definition of $r^\delta W^s(\mathbb{R}_+)$ for $\delta, s \in \mathbb{R}$.

The space $r^\delta W^s(\mathbb{R}_+)$ can also be characterized as follows. Let ρ be as before and let $u_\mu(r) = u(2^{-\mu} r)$. Then $u \in r^\delta W^s(\mathbb{R}_+)$ if and only if $u \in H^s_{\mathrm{loc}}(\mathbb{R}_+)$ and

$$\sum_{\mu=-\infty}^{\infty} 4^{\delta\mu} \int_{\mathbb{R}} (1 + 4^{-\mu} + \xi^2)^s |\widehat{\rho u_\mu}(\xi)|^2 d\xi < \infty. \tag{2.8}$$

Lemma 2.5(a) *For $\delta, s \in \mathbb{R}$,*

$$(r^\delta W^s(\mathbb{R}_+))^* = r^{-\delta} W^{-s}(\mathbb{R}_+).$$

The pairing is given by the $L^2(\mathbb{R}_+, r^{-1}dr)$ inner product.

(b) *For $\delta_1, \delta_2, s_1, s_2 \in \mathbb{R}$ with $\delta_2 \geq \delta_1$ and $s_2 - \delta_2 \geq s_1 - \delta_1$,*

$$r^{\delta_2} W^{s_2}(\mathbb{R}_+) \subseteq r^{\delta_1} W^{s_1}(\mathbb{R}_+).$$

(c) *For $\delta_1, \delta_2, s_1, s_2 \in \mathbb{R}$ with $\delta_2 > \delta_1$ and $s_2 - \delta_2 > s_1 - \delta_1$, the inclusion in* (b) *is a compact operator.*

§3. An integral transform

We define an integral transform Θ as follows: u is a function on $\mathbb{R}_+^{n+1}$, Θu is a function on $\mathbb{R}_+ \times \mathbb{R}_+ \times S^{n-1}$, and

$$(\Theta u)(r_1, r_2, \omega) = r_1^n \, r_2^{-n/2} \, \widetilde{u}(r_2/r_1, r_1\omega)$$

where

$$\widetilde{u}(x, \eta) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iy\cdot\eta} u(x, y) \, dy.$$

Straightforward calculations show that

$$\text{(3.1)} \qquad \begin{cases} \Theta\left(x\partial_x + \sum_j y_j \partial_{y_j}\right) u = -r_1 \partial_{r_1} \Theta u \\ \Theta \, \partial_{y_j} u = i r_1 \omega_j \, \Theta u \\ \Theta \, (x\partial_x - n/2) u = r_2 \partial_{r_2} \Theta u \\ \Theta \, x\partial_{y_j} u = i r_2 \omega_j \, \Theta u \end{cases}$$

and

$$\text{(3.2)} \qquad \Theta \, xu = r_1^{-1} r_2 \, \Theta u.$$

We also have

$$\Theta \overline{u}(r_1, r_2, \omega) = \overline{\Theta u}(r_1, r_2, -\omega)$$

and

$$\Theta v(r_1, r_2, \omega) = \Theta u(r_2, r_1, -\omega).$$

for $v(x, y) = x^{n/2} \, u(1/x, -y/x)$.

It follows from (3.1) that if

$$L_0 = \sum_{j+|\alpha| \le m} a_{j,\alpha} (x\partial_x)^j (x\partial_y)^\alpha$$

is a uniformly degenerate operator with constant coefficients $a_{j,\alpha}$, then

$$\text{(3.3)} \qquad \Theta L_0 u = B_\omega \Theta u,$$

where

$$B_\omega = \sum_{j+|\alpha| \le m} a_{j,\alpha} \left(r_2 \partial_{r_2} + n/2\right)^j \left(i r_2 \omega\right)^\alpha.$$

A straightforward calculation using Plancherel's theorem on $\mathbb{R}^n$ shows that

$$(3.4) \quad \Theta \,:\, L^2(\mathbb{R}^{n+1}_+, x^{-n-1}dx\,dy) \longrightarrow L^2(\mathbb{R}_+ \times \mathbb{R}_+ \times S^{n-1}, r_1^{-1} r_2^{-1} dr_1\,dr_2\,d\omega)$$

is a unitary operator. For $\delta, s \in \mathbb{R}$,

$$(3.5) \qquad \Theta \,:\, x^\delta H^s(\mathbb{R}^{n+1}_+) \longrightarrow r_1^{-\delta} L^2(\mathbb{R}_+) \otimes r_2^\delta W^s(\mathbb{R}_+) \otimes L^2(S^{n-1})$$

is an invertible bounded operator. Here and in the following $L^2(\mathbb{R}_+)$ stands for $L^2(\mathbb{R}_+, r^{-1}\,dr)$. For integral s, (3.5) follows from (2.1), (2.2), (2.4), (2.5), (3.1), (3.2), and (3.4). For real s, (3.5) follows either by interpolation or by working directly with the definition of Θ and (2.3) and (2.8).

Similarly, for $\delta, s \in \mathbb{R}$ and $t \geq 0$,

$$(3.6) \qquad \begin{aligned} \Theta \,:\, x^\delta H^{s,t}(\mathbb{R}^{n+1}_+) \longrightarrow\ & r_1^{-\delta} L^2(\mathbb{R}_+) \otimes r_2^\delta W^{s+t}(\mathbb{R}_+) \otimes L^2(S^{n-1}) \\ & \cap\ r_1^{-\delta} \langle r_1 \rangle^{-t} L^2(\mathbb{R}_+) \otimes r_2^\delta W^s(\mathbb{R}_+) \otimes L^2(S^{n-1}) \end{aligned}$$

and

$$(3.7) \qquad \begin{aligned} \Theta \,:\, x^\delta H^{s,-t}(\mathbb{R}^{n+1}_+) \longrightarrow\ & r_1^{-\delta} L^2(\mathbb{R}_+) \otimes r_2^\delta W^{s-t}(\mathbb{R}_+) \otimes L^2(S^{n-1}) \\ & +\ r_1^{-\delta} \langle r_1 \rangle^{t} L^2(\mathbb{R}_+) \otimes r_2^\delta W^s(\mathbb{R}_+) \otimes L^2(S^{n-1}) \end{aligned}$$

are invertible bounded operators.

Representation theoretical background. The integral transform Θ can be understood in terms of unitary representations of Lie groups. This will not be used in the following, and the reader may skip the rest of §3 without loss of continuity. However, the representation theory was the original motivation behind this paper and is sometimes useful as a guide when applying the techniques in this paper to other classes of elliptic operators with boundary degeneracies.

Upper half space $\mathbb{R}^{n+1}_+$ has a Lie group structure with group operation

$$(x, y)(x', y') = (xx', y + xy').$$

The identity element is $(1, 0)$ and

$$(x, y)^{-1} = (1/x, -y/x).$$

The vector fields $x\partial_x$ and $x\partial_{y_j}$ form a basis for the left-invariant vector fields on $\mathbb{R}^{n+1}_+$. Thus the constant coefficient uniformly degenerate operators are the left-invariant partial differential operators on the group $\mathbb{R}^{n+1}_+$.

The group $\mathbb{R}_+^{n+1}$ is a semidirect product of the multiplicative group $\mathbb{R}_+$ and the additive group $\mathbb{R}^n$. It then follows from Mackey's theorem, see [Si] §§6–7 or [F] Theorem 6.42, that the irreducible unitary representations of the group $\mathbb{R}_+^{n+1}$ fall into two families: one family of one-dimensional representations $\mathbb{R}_+^{n+1} \times \mathbb{C} \to \mathbb{C}$, $(x,y).z = x^{-i\xi} z$ parametrized by $\xi \in \mathbb{R}$ and one family of infinite-dimensional representations

$$\begin{aligned} &\mathbb{R}_+^{n+1} \times L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \\ &\big((x,y).f\big)(r) = e^{-iy\cdot r\omega} f(xr) \end{aligned} \tag{3.8}$$

parametrized by $\omega \in S^{n-1}$.

For any Lie group G, the two-sided regular representation of G is defined as the unitary representation

$$\begin{aligned} G \times G \times L^2(G, d\lambda) &\to L^2(G, d\lambda) \\ \big((g_1, g_2).u\big)(g) &= \Delta(g_2)^{-1/2} u(g_1^{-1} g g_2) \end{aligned}$$

of $G \times G$. Here $d\lambda$ and Δ denote the left-invariant Haar measure and the modular function on G. In particular, the two-sided regular representation of $\mathbb{R}_+^{n+1}$ is given by

$$\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1} \times L^2(\mathbb{R}_+^{n+1}, x^{-n-1} dx\, dy) \to L^2(\mathbb{R}_+^{n+1}, x^{-n-1} dx\, dy)$$

$$\begin{aligned} \big(((x_1,y_1),(x_2,y_2)).u\big)(x,y) &= x_2^{-n/2} u((x_1^{-1}, -x_1^{-1} y_1)(x,y)(x_2,y_2)) \\ &= x_2^{-n/2} u(x_1^{-1} x x_2, -x_1^{-1} y_1 + x_1^{-1} y + x_1^{-1} x y_2). \end{aligned}$$

A straightforward calculation shows that

$$\Theta\left((x_1,y_1),(x_2,y_2)).u\,(r_1,r_2,\omega) = e^{-iy_1\cdot r_1\omega + iy_2\cdot r_2\omega} \Theta u(x_1 r_1, x_2 r_2, \omega). \tag{3.9}$$

The identities (3.1) can be obtained by differentiating (3.9) with respect to x_1, y_1, x_2, and y_2 respectively at $(x_1,y_1) = (x_2,y_2) = (1,0)$. In other words, (3.1) is the Lie algebra version of the Lie group identity (3.9).

Comparing (3.8) and (3.9) we see, at least intuitively, that the transform Θ decomposes the two-sided regular representation of $\mathbb{R}_+^{n+1}$ into irreducible representations of $\mathbb{R}_+^{n+1}$. This decomposition can be made precise using the notion of direct integrals of fields of representations; see [F] §7.4. If we denote the representation (3.8) of $\mathbb{R}_+^{n+1}$ by $\mathcal{H}_\omega$, then (3.9) says that Θ is a unitary isomorphism between the two-sided regular representation of $\mathbb{R}_+^{n+1}$ and the representation $\int_{S^{n-1}}^{\oplus} \mathcal{H}_\omega \otimes \overline{\mathcal{H}}_\omega \, d\omega$ of $\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}$.

It is instructive to compare with the commutative group $\mathbb{R}^n$. The irreducible unitary representations of $\mathbb{R}^n$ form a single family of one-dimensional representations

$$\begin{aligned} &\mathbb{R}^n \times \mathbb{C} \to \mathbb{C} \\ &x.z = e^{-ix\cdot\xi} z \end{aligned} \tag{3.10}$$

parametrized by $\xi \in \mathbb{R}^n$. The regular representation of $\mathbb{R}^n$ is the unitary representation

$$\mathbb{R}^n \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$
$$(x.u)(y) = u(y-x).$$

The Fourier transform $u \mapsto \hat{u}$ on $\mathbb{R}^n$ gives a unitary operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$\widehat{x.u}(\xi) = e^{-ix\cdot\xi}\hat{u}(\xi). \tag{3.11}$$

Comparing (3.10) and (3.11) we see that the Fourier transform decomposes the regular representation of $\mathbb{R}^n$ into irreducible representations of $\mathbb{R}^n$. In fact, if we denote the representation (3.10) of $\mathbb{R}^n$ by $\mathcal{K}_\xi$, then (3.11) says that the Fourier transform is a unitary isomorphism between the regular representation and the representation $\int_{\mathbb{R}^n}^{\oplus} \mathcal{K}_\xi \, d\xi$ of $\mathbb{R}^n$. Due to this analogy between Θ and the Fourier transform on $\mathbb{R}^n$, the transform Θ is considered the Fourier transform on the group $\mathbb{R}_+^{n+1}$. For an introduction to the Fourier transform on general Lie groups, see [F] §7.5.

§4. Elliptic Bessel operators

By (3.3), Θ transforms a constant coefficient uniformly degenerate operator

$$L_0 = \sum_{j+|\alpha|\le m} a_{j,\alpha}(x,y)(x\partial_x)^j(x\partial_y)^\alpha$$

to a family

$$B_\omega = \sum_{j+|\alpha|\le m} a_{j,\alpha}(r\,d/dr + n/2)^j(ir\omega)^\alpha$$

of ordinary differential operators parametrized by $\omega \in S^{n-1}$. These are of a type known as Bessel operators. In this section we study the properties of such operators in detail.

A Bessel operator of order m is defined as an ordinary differential operator

$$B = \sum_{j+k\le m} b_{j,k}(r\,d/dr)^j r^k$$

with $b_{j,k}$ constant complex $N\times N$-matrices. The operator acts on functions $\mathbb{R}_+ \to \mathbb{C}^N$. A Bessel operator B is said to be elliptic if the principal symbol

$$\sigma(\xi,\eta) = \sum_{j+k=m} b_{j,k}\,(i\xi)^j\,\eta^k$$

is invertible for all $(\xi,\eta) \neq 0 \in \mathbb{R}^2$. The indicial matrix of B is defined as

$$J(\gamma) = \sum_{j=0}^{m} b_{j,0}\,\gamma^j .$$

This is an $N\times N$ matrix with entries that are polynomials of degree m in γ. We define the indicial roots of B as the roots of the degree mN polynomial $\det J(\gamma)$.

Elliptic Bessel operators have been studied in [Ma1] and [Ma2] §5 using the calculus of totally characteristic partial differential operators [MeM], [Ma2] §4, [Me2], [Me4]. In this section we review and give elementary proofs of relevant parts of these results. We also derive an index formula for elliptic Bessel operators.

At $r=0$ the equation $Bu=0$ has a singularity of the first kind. It is easy to show that the solutions of $Bu=0$ have asymptotic series expansions in powers of r and $\log r$ as $r\to 0$; see [CL] Chapter 4. (At $r=\infty$ the equation $Bu=0$ has a singularity of the second kind. Thus, as was first shown by H. Poincaré [P], the solutions of $Bu=0$ also have asymptotic expansions as $r\to\infty$; see [CL] Chapter 5. Although these expansions will not be used in the following, they still add some insight.)

We first prove two preliminary lemmas.

Lemma 4.1. *Let $P = \sum_{j=0}^{m} p_j (d/dr)^j$ where p_j are complex $N\times N$-matrices and p_m is invertible. Let $\delta, s \in \mathbb{R}$. If the matrices $\sum_{j=0}^{m} p_j (i\xi)^j$ are invertible for all $\xi \in \mathbb{R}$, then the operator $P : \langle r\rangle^{\delta} H^{s+m}(\mathbb{R}) \to \langle r\rangle^{\delta} H^{s}(\mathbb{R})$ is invertible.*

Proof. The case $\delta=0$ follows at once using the Fourier transform. In the case $\delta\neq 0$, the operator is unitarily equivalent to the operator $\langle r\rangle^{-\delta} P \langle r\rangle^{\delta} : H^{s+m}(\mathbb{R}) \to H^{s}(\mathbb{R})$. Now $\langle r\rangle^{-\delta} P \langle r\rangle^{\delta} = P + R$ where R is a differential operator of order $m-1$ with coefficients that decay as $\langle r\rangle^{-1}$ as $r\to\pm\infty$. Here P is invertible. The operator R is bounded $H^{s+m}(\mathbb{R}) \to \langle r\rangle^{-1} H^{s+1}(\mathbb{R})$, and thus compact $H^{s+m}(\mathbb{R}) \to H^{s}(\mathbb{R})$. Hence $P+R$ is Fredholm. □

Lemma 4.2. *Let $Q = \sum_{j=0}^{m} q_j (r\,d/dr)^j$ where q_j are complex $N\times N$-matrices and q_m is invertible.*

Let $\delta, s \in \mathbb{R}$. If the polynomial $\det \sum_{j=0}^{m} q_j \gamma^j$ does not have any root γ with real part δ, then the operator $Q : r^{\delta} H^{s+m}(\mathbb{R}_+) \to r^{\delta} H^{s}(\mathbb{R}_+)$ is invertible.

Let $\delta_1, \delta_2, s \in \mathbb{R}$. If $\delta_1 < \delta_2$ and the polynomial $\det \sum_{j=0}^{m} q_j \gamma^j$ does not have any root γ with real part in $[\delta_1, \delta_2]$, then the operator

$$Q : r^{\delta_1} H^{s+m}(\mathbb{R}_+) + r^{\delta_2} H^{s+m}(\mathbb{R}_+) \to r^{\delta_1} H^{s}(\mathbb{R}_+) + r^{\delta_2} H^{s}(\mathbb{R}_+)$$

is invertible.

Proof. The shifted Mellin transform

$$M_\delta u(\xi) = (2\pi)^{-1/2} \int_0^\infty r^{-\delta-1-i\xi} u(r)\, dr$$

gives a unitary operator $r^\delta W^s(\mathbb{R}_+) \to \langle\xi\rangle^{-s} L^2(\mathbb{R})$. We have

$$(M_\delta Q u)(\xi) = \sum_{j=0}^{m} q_j (\delta + i\xi)^j \, (M_\delta u)(\xi).$$

The first part of the lemma follows. In the second part, surjectivity follows from the first part and injectivity follows by inspection. □

We now return to the Bessel operators.

Proposition 4.3. *Let B be an elliptic Bessel operator of order m. Let $\delta, s_1, s_2 \in \mathbb{R}$ with $s_1 \le s_2$. If $u \in r^\delta W^{s_1+m}(\mathbb{R}_+)$ and $Bu \in r^\delta W^{s_2}(\mathbb{R}_+)$, then $u \in r^\delta W^{s_2+m}(\mathbb{R}_+)$. There exists $c > 0$ such that*

$$\|u\|_{r^\delta W^{s_2+m}(\mathbb{R}_+)} \le c\big(\|Bu\|_{r^\delta W^{s_2}(\mathbb{R}_+)} + \|u\|_{r^\delta W^{s_1+m}(\mathbb{R}_+)}\big)$$

for all $u \in r^\delta W^{s_2+m}(\mathbb{R}_+)$.

Let Ω be a compact topological space. Let $\{B_\omega\}_{\omega\in\Omega}$ be a continuous family of elliptic Bessel operators. Then the above estimate holds uniformly for all B_ω.

Proof. We may assume that $s_2 \le s_1 + 1$; the general case follows from this case by iteration. It follows from (2.6) that $\varphi_0 u \in r^\delta H^{s_1+m}(\mathbb{R}_+)$ and $\varphi_0 Bu \in r^\delta H^{s_2}(\mathbb{R}_+)$. Hence

$$b_{m,0}(r\, d/dr)^m(\varphi_0 u) = \big(b_{m,0}(r\, d/dr)^m - B\big)(\varphi_0 u) + [B, \varphi_0]u + \varphi_0 Bu \in r^\delta H^{s_2}(\mathbb{R}_+).$$

As B is elliptic, the matrix $b_{m,0}$ is invertible. Hence $(r\, d/dr)^m(\varphi_0 u) \in r^\delta H^{s_2}(\mathbb{R}_+)$. It follows that

$$\varphi_0 u \in r^\delta H^{s_2+m}(\mathbb{R}_+). \tag{4.1}$$

It also follows from (2.6) that

$$\varphi_\infty u \in \langle r\rangle^{\delta-s_1-m+1/2} H^{s_1+m}(\mathbb{R}). \tag{4.2}$$

Let $B_\infty = \sum_{j=0}^m b_{j,m-j}(d/dr)^j$. Then $B - r^m B_\infty$ is a Bessel operator of order $m-1$. Hence $(B - r^m B_\infty)u \in r^\delta W^{s_1+1}(\mathbb{R}_+)$. Thus $r^m B_\infty u \in r^\delta W^{s_2}(\mathbb{R}_+)$, so $B_\infty u \in r^{\delta-m} W^{s_2}(\mathbb{R}_+)$. By (2.6), $\varphi_\infty B_\infty u \in \langle r\rangle^{\delta-s_2-m+1/2} H^{s_2}(\mathbb{R})$. Hence

$$B_\infty(\varphi_\infty u) = [B_\infty, \varphi_\infty]u + \varphi_\infty B_\infty u \in \langle r\rangle^{\delta-s_2-m+1/2} H^{s_2}(\mathbb{R}). \tag{4.3}$$

As B is elliptic, the matrix $\sum_{j=0}^m b_{j,m-j}(i\xi)^j$ is invertible for all $\xi \in \mathbb{R}$. It then follows from Lemma 4.1 that the operators

$$B_\infty : \langle r\rangle^{\delta-s_2-m+1/2} H^{s_2+m}(\mathbb{R}) \to \langle r\rangle^{\delta-s_2-m+1/2} H^{s_2}(\mathbb{R})$$

and

$$B_\infty : \langle r\rangle^{\delta-s_1-m+1/2} H^{s_1+m}(\mathbb{R}) \to \langle r\rangle^{\delta-s_1-m+1/2} H^{s_1}(\mathbb{R})$$

are invertible. It then follows from (4.2) and (4.3) that

$$\varphi_\infty u \in \langle r\rangle^{\delta-s_2-m+1/2} H^{s_2+m}(\mathbb{R}). \tag{4.4}$$

It follows from (2.6), (4.1), and (4.4) that $u \in r^\delta W^{s_2+m}(\mathbb{R}_+)$. By examining the above argument step by step, we get the desired estimate of u in $r^\delta W^{s_2+m}(\mathbb{R}_+)$ norm.

It is straightforward to reduce the case of a continuous family of Bessel operators parametrized by a compact space to the case of a single Bessel operator. ☐

Corollary 4.4. *Let B be an elliptic Bessel operator. Let $\delta, s \in \mathbb{R}$. If $u \in r^\delta W^s(\mathbb{R}_+)$ and $Bu = 0$, then, for any j and k, $d^j u/dr^j$ decays faster than r^{-k} as $r \to \infty$.*

Proof. It follows from Proposition 4.3 that $u \in r^\delta W^s(\mathbb{R}_+)$ for any s. ☐

Proposition 4.5. *Let B be an elliptic Bessel operator of order m. Let $\delta_1, \delta_2, s_1, s_2 \in \mathbb{R}$ with $\delta_1 < \delta_2$. Assume that B does not have any indicial roots with real parts in $[\delta_1, \delta_2]$. If $u \in r^{\delta_1} W^{s_1+m}(\mathbb{R}_+)$ and $Bu \in r^{\delta_2} W^{s_2}(\mathbb{R}_+)$, then $u \in r^{\delta_2} W^{s_2+m}(\mathbb{R}_+)$. There exists $c > 0$ such that*

$$\|u\|_{r^{\delta_2} W^{s_2+m}(\mathbb{R}_+)} \le c\big(\|Bu\|_{r^{\delta_2} W^{s_2}(\mathbb{R}_+)} + \|u\|_{r^{\delta_1} W^{s_1+m}(\mathbb{R}_+)}\big)$$

for all $u \in r^{\delta_2} W^{s_2+m}(\mathbb{R}_+)$.

Proof. We may assume that $\delta_2 \leq \delta_1 + 1$; the general case follows from this case by iteration. By Proposition 4.3, we may assume that $s_1 = s_2$. For simplicity we write s for s_1 and s_2. It follows from (2.6) that

$$\varphi_0 u \in r^{\delta_1} H^{s+m}(\mathbb{R}_+), \tag{4.5}$$

and

$$\varphi_0 B u \in r^{\delta_2} H^{s}(\mathbb{R}_+).$$

Now $B - J(r\,d/dr)$ is r times a Bessel operator of order $m-1$. Hence

$$(B - J(r\,d/dr))u \in r^{\delta_1+1} W^{s+1}(\mathbb{R}_+).$$

It then follows from (2.6) that $\varphi_0(B - J(r\,d/dr))u \in r^{\delta_1+1} H^{s+1}(\mathbb{R}_+)$. Hence

$$\begin{aligned} J(r\,d/dr)(\varphi_0 u) &= [J(r\,d/dr), \varphi_0]u \\ &\qquad + \varphi_0(J(r\,d/dr) - B)u + \varphi_0 B u \in r^{\delta_2} H^{s}(\mathbb{R}_+). \end{aligned} \tag{4.6}$$

By Lemma 4.2 and the assumptions on the indicial roots of B, the operators

$$J(r\,d/dr) : r^{\delta_2} H^{s+m}(\mathbb{R}_+) \to r^{\delta_2} H^{s}(\mathbb{R}_+)$$

and

$$J(r\,d/dr) : r^{\delta_1} H^{s+m}(\mathbb{R}_+) + r^{\delta_2} H^{s+m}(\mathbb{R}_+) \to r^{\delta_1} H^{s}(\mathbb{R}_+) + r^{\delta_2} H^{s}(\mathbb{R}_+)$$

are invertible. It then follows from (4.5) and (4.6) that

$$\varphi_0 u \in r^{\delta_2} H^{s+m}(\mathbb{R}_+).$$

By (2.6),

$$\varphi_\infty u \in \langle r\rangle^{\delta_1 - s - m + 1/2} H^{s+m}(\mathbb{R}) \subseteq \langle r\rangle^{\delta_2 - s - m + 1/2} H^{s+m}(\mathbb{R}).$$

It then follows from (2.6) that that $u \in r^{\delta_2} W^{s+m}(\mathbb{R}_+)$. By examining the above argument step by step, we get the desired estimate of u in $r^{\delta_2} W^{s+m}(\mathbb{R}_+)$ norm. $\square$

The adjoint $B^* : r^{-\delta} W^{-s}(\mathbb{R}_+) \to r^{-\delta} W^{-s-m}(\mathbb{R}_+)$ of B, see Lemma 2.5(a), is given by

$$B^* = \sum_{j+k\leq m} b^*_{j,k} r^k (-r\,d/dr)^j. \tag{4.7}$$

Corollary 4.6. *Let B be an elliptic Bessel operator of order m. Let $\delta, s \in \mathbb{R}$. If δ is not the real part of any indicial root of B, then the operator*

$$B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$$

is Fredholm.

Proof. Choose $\delta', s' \in \mathbb{R}$ such that $\delta' < \delta$, $s' - \delta' < s - \delta$, and B does not have any indicial roots with real parts in $[\delta', \delta]$. It then follows from Proposition 4.5 that

$$\|u\|_{r^\delta W^{s+m}(\mathbb{R}_+)} \le c\big(\|Bu\|_{r^\delta W^s(\mathbb{R}_+)} + \|u\|_{r^{\delta'} W^{s'+m}(\mathbb{R}_+)}\big).$$

By Lemma 2.5(c), the inclusion $r^\delta W^{s+m}(\mathbb{R}_+) \to r^{\delta'} W^{s'+m}(\mathbb{R}_+)$ is compact. It follows that B has finite-dimensional null space and closed range. Similarly, B^* has finite-dimensional null space and closed range. ☐

Corollary 4.7. *Let B be an elliptic Bessel operator of order m.*

The dimension of the null space of $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is independent of s and is a decreasing function of δ. Its discontinuities are located at real parts of the indicial roots of B. For sufficiently large δ the operator is injective.

The codimension of the closure of the range of $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is independent of s and is an increasing function of δ. Its discontinuities are located at real parts of the indicial roots of B. For sufficiently small δ the operator has dense range.

Proof. It follows from Proposition 4.3 the null space is independent of s. It follows from Lemma 2.5(b) that the space $\bigcap_{s\in\mathbb{R}} r^\delta W^s(\mathbb{R}_+)$ decreases with increasing δ. Hence the null space decreases with increasing δ. It follows from Proposition 4.5 that the dimension has discontinuities only at real parts of indicial roots. That the null space is trivial for δ large enough follows by examining the asymptotic expansions of the solutions of $Bu = 0$ as $r \to 0$; see [CL] Chapter 4. The second part of the corollary follows from the first part by duality. ☐

The following corollary also serves as the definition of $\underline{\delta}(B)$ and $\overline{\delta}(B)$.

Corollary 4.8. *Let B be an elliptic Bessel operator of order m.*

There exists $\underline{\delta}(B) \in \mathbb{R}$ such that the operator $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is injective for $\delta > \underline{\delta}(B)$ and not injective for $\delta < \underline{\delta}(B)$. The number $\underline{\delta}(B)$ is the real part of an indicial root of B.

There exists $\overline{\delta}(B) \in \mathbb{R}$ *such that the operator* $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ *has dense range for* $\delta < \overline{\delta}(B)$ *and does not have dense range for* $\delta > \overline{\delta}(B)$*. The number* $\overline{\delta}(B)$ *is the real part of an indicial root of* B*.*

In particular,

$$\begin{cases} \underline{\delta}(B^*) = -\overline{\delta}(B) \\ \overline{\delta}(B^*) = -\underline{\delta}(B). \end{cases} \tag{4.8}$$

Corollary 4.9. *Let* Ω *be a compact topological space. Let* $\{B_\omega\}_{\omega\in\Omega}$ *be a continuous family of elliptic Bessel operators. Let* $\delta, s \in \mathbb{R}$*. Assume that* $\delta > \underline{\delta}(B_\omega)$ *for all* $\omega \in \Omega$ *and that* δ *is not the real part of any indicial root of* B_ω *for any* $\omega \in \Omega$*. Then there exists* $c > 0$ *such that*

$$\|u\|_{r^\delta W^{s+m}(\mathbb{R}_+)} \leq c\|B_\omega u\|_{r^\delta W^s(\mathbb{R}_+)}$$

for all $\omega \in \Omega$ *and all* $u \in r^\delta W^{s+m}(\mathbb{R}_+)$*.*

Proof. In the case of a single Bessel operator B, it follows from Corollaries 4.6 and 4.8 that $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is injective with closed range. This proves the estimate for a single Bessel operator. It is straightforward to reduce the case of a continuous family of Bessel operators parametrized by a compact space to the case of a single Bessel operator. □

Proposition 4.10. *Let* B *be an elliptic Bessel operator of order* m*. Let* $\delta, s \in \mathbb{R}$*. Assume that* δ *is not the real part of any indicial root of* B*. Then the index of* $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ *is equal to the number of indicial roots of* B*, i.e. roots of the polynomial* $\det \sum_{j=0}^m b_{j,0}\,\gamma^j$*, with real part greater than* δ *minus the number of roots of the polynomial* $\det \sum_{j=0}^m b_{j,m-j}\,\gamma^j$ *with positive real part.*

The roots are counted with multiplicities. Note that if B is elliptic, then the polynomial $\det \sum_{j=0}^m b_{j,m-j}\,\gamma^j$ does not have any roots with vanishing real part. It follows from Proposition 4.10 that the index is a decreasing function of δ with discontinuities precisely at the real parts of the indicial roots.

Proof. First we consider operators $P = \sum_{j=0}^m p_j(r)(d/dr)^j$ on $\mathbb{R}$ where p_j are smooth complex $N\times N$-matrix valued functions on $\mathbb{R}$ and $\det p_m(r) \neq 0$ for all $r \in \mathbb{R}$. We assume that there exist complex $N\times N$-matrices $p_j^\pm$ such that

$$\begin{cases} p_j(r) = p_j^- + \mathrm{O}(|r|^{-1}) & \text{as } r \to -\infty \\ p_j(r) = p_j^+ + \mathrm{O}(|r|^{-1}) & \text{as } r \to +\infty \\ (d/dr)^k p_j(r) = \mathrm{O}(|r|^{-k-1}) & \text{as } r \to \pm\infty \qquad \text{for } k \geq 1 \end{cases}$$

and $\det p_m^{\pm} \neq 0$.

Let φ_- be a smooth function $\mathbb{R} \to [0,1]$ such that $\varphi_- = 1$ on $(-\infty,-1]$ and $\varphi_- = 0$ on $[1,\infty)$. Let $\varphi_+ = 1 - \varphi_-$. For $\delta_-, \delta_+, \lambda_-, \lambda_+ \in \mathbb{R}$ we define the Hilbert space $H^s_{\delta_-(\lambda_-),\delta_+(\lambda_+)}(\mathbb{R})$ as the space of all $u : \mathbb{R} \to \mathbb{C}^N$ such that $\varphi_- u \in e^{\delta_- r}\langle r\rangle^{\lambda_-} H^s(\mathbb{R})$ and $\varphi_+ u \in e^{\delta_+ r}\langle r\rangle^{\lambda_+} H^s(\mathbb{R})$. We then have the following:

If δ_- is not the real part of any root of the polynomial $\det \sum_{j=0}^m p_j^- \gamma^j$ and δ_+ is not the real part of any root of the polynomial $\det \sum_{j=0}^m p_j^+ \gamma^j$, then the operator

$$P : H^{s+m}_{\delta_-(\lambda_-),\delta_+(\lambda_+)}(\mathbb{R}) \to H^s_{\delta_-(\lambda_-),\delta_+(\lambda_+)}(\mathbb{R})$$

is Fredholm. Its index is equal to the number of roots of $\det \sum_{j=0}^m p_j^- \gamma^j$ with real part greater than δ_- minus the number of roots of $\det \sum_{j=0}^m p_j^+ \gamma^j$ with real part greater than δ_+.

That P is Fredholm is shown largely the same way as Corollary 4.6. The index is invariant under continuous deformations of $(P, \delta_-, \lambda_-, \delta_+, \lambda_+)$ as long as $\delta_\pm$ is never the real part of any root of the polynomial $\det \sum_{j=0}^m p_j^\pm \gamma^j$. Each $(P^0, \delta_-^0, \lambda_-^0, \delta_+^0, \lambda_+^0)$ can be deformed, subject to this constraint, to some $(P^1, \delta_-^1, 0, \delta_+^1, 0)$ with P^1 a constant coefficient operator. For such P^1 the index formula follows by inspection.

We now return to the proof of Proposition 4.10. Let ψ be a diffeomorphism $\mathbb{R} \to \mathbb{R}_+$ with $\psi(r) = e^r$ for $r \leq -1$ and $\psi(r) = r$ for $r \geq 1$. It then follows from (2.7) that

$$\psi^*(r^\delta W^{s+m}(\mathbb{R}_+)) = H^{s+m}_{\delta(0),0(\delta-s-m+1/2)}(\mathbb{R}).$$

Let $\tilde{r}_+$ be a smooth function $\mathbb{R} \to [1,\infty)$ such that $\tilde{r}_+(r) = 1$ for $r \leq -1$ and $\tilde{r}_+(r) = r$ for $r \geq 1$. By (2.7), $\psi^*(r^\delta W^s(\mathbb{R}_+)) = H^s_{\delta(0),0(\delta-s+1/2)}(\mathbb{R})$, so

$$\tilde{r}^{-m}\psi^*(r^\delta W^s(\mathbb{R}_+)) = H^s_{\delta(0),0(\delta-s-m+1/2)}(\mathbb{R}).$$

The index of $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is equal to the index of

$$\tilde{r}^{-m}\psi^* B : H^{s+m}_{\delta(0),0(\delta-s-m+1/2)}(\mathbb{R}) \to H^s_{\delta(0),0(\delta-s-m+1/2)}(\mathbb{R}).$$

The operator $\tilde{r}_+^{-m}\psi^* B$ is of the type considered above with $p_j^- = b_{j,0}$ and $p_j^+ = b_{j,m-j}$. The proposition follows. □

Corollary 4.11. *Let B be an elliptic Bessel operator. If $\underline{\delta}(B) < \overline{\delta}(B)$, then B does not have any indicial roots with real parts in $(\underline{\delta}(B), \overline{\delta}(B))$.*

Proof. If $\underline{\delta}(B) < \overline{\delta}(B)$, then the index of $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^{s+m}(\mathbb{R}_+)$ is zero for all $\delta \in (\underline{\delta}(B), \overline{\delta}(B))$ that are not real parts of indicial roots of B. Thus the index of B, as a function of δ, does not have any discontinuities in $(\underline{\delta}(B), \overline{\delta}(B))$. The corollary then follows from Proposition 4.10. □

Conjecture 4.12. *For a generic elliptic Bessel operator $\underline{\delta}(B) < \overline{\delta}(B)$.*

The motivation behind this is as follows: Let δ be a real number that is not the real part of any indicial root of B. By Corollary 4.6, the operator $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is Fredholm. Let φ_0 and φ_∞ be as in (2.6). Let $\mathcal{N}$ denote the space of all solutions of $Bu = 0$. Thus $\dim \mathcal{N} = mN$. Let $\mathcal{N}_\delta$ denote the space of solutions of $Bu = 0$ such that $\varphi_0 u \in r^\delta W^s(\mathbb{R}_+)$. Let $\mathcal{N}^0$ denote the space of solutions of $Bu = 0$ such that $\varphi_\infty u \in r^\delta W^s(\mathbb{R}_+)$. It follows from Proposition 4.3 that the spaces $\mathcal{N}_\delta$ and $\mathcal{N}^0$ do not depend on s. It follows from Corollary 4.4 that the space $\mathcal{N}^0$ does not depend on δ. Then

$$(4.9) \qquad \begin{cases} \dim \operatorname{Null} B = \dim(\mathcal{N}_\delta \cap \mathcal{N}^0) \\ \operatorname{codim} \operatorname{Range} B = \dim(\mathcal{N}_\delta \cap \mathcal{N}^0) + \dim \mathcal{N} - \dim \mathcal{N}_\delta - \dim \mathcal{N}^0. \end{cases}$$

The first part of (4.9) is obvious. The second part is seen as follows. By examining the asymptotic expansions of the solutions as $r \to 0$, as in [CL] Chapter 4, we see that *the number of indicial roots of P with real part greater than δ is* $\dim \mathcal{N}_\delta$. By examining the asymptotic expansions of the solutions as $r \to \infty$, as in [CL] Chapter 5, we see that the number roots of $\det \sum_{j=0}^m b_{j,m-j}\, \gamma^j$ with negative real part is $\dim \mathcal{N}^0$. It follows that *the number roots of* $\det \sum_{j=0}^m b_{j,m-j}\, \gamma^j$ *with positive real part is* $\dim \mathcal{N} - \dim \mathcal{N}^0$.

(The last statement can also be seen as follows, without using the asymptotic expansions as $r \to \infty$. Let $\delta' \in \mathbb{R}$ be smaller than the real parts of all the indicial roots. By Corollaries 4.6 and 4.7, the operator $B : r^{\delta'} W^{s+m}(\mathbb{R}_+) \to r^{\delta'} W^s(\mathbb{R}_+)$ is surjective. By examining the asymptotic expansions of the solutions as $r \to 0$, we see that its null space is $\mathcal{N}^0$. Hence its index is $\dim \mathcal{N}^0$. On the other hand, by Proposition 4.10 its index is $\dim \mathcal{N}$ minus the number of roots of $\det \sum_{j=0}^m b_{j,m-j}\, \gamma^j$ with positive real part.)

It then follows from Proposition 4.10 that the index of B is $\dim \mathcal{N}_\delta + \dim \mathcal{N}^0 - \dim \mathcal{N}$. The second part of (4.9) follows from the first part and this index formula.

Assume that $\overline{\delta}(B) < \underline{\delta}(B)$. Choose some $\delta \in (\overline{\delta}(B), \underline{\delta}(B))$ that is not the real part of any indicial root of B. Then $B : r^\delta W^{s+m}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is Fredholm and neither injective nor surjective. It then follows from (4.9) that $\mathcal{N}^0$ and $\mathcal{N}_\delta$ intersect nontransversely in $\mathcal{N}$. In order to establish Conjecture 4.12, one would have to show that such nontransverse intersections do not occur for generic B.

§5. Constant coefficient elliptic uniformly degenerate operators

A constant coefficient uniformly degenerate operator of order m on upper half space is defined as a partial differential operator

$$L_0 = \sum_{j+|\alpha|\le m} a_{j,\alpha}(x\partial_x)^j(x\partial_y)^\alpha$$

with $a_{j,\alpha}$ complex $N\times N$-matrices. The operator L_0 acts on functions $\mathbb{R}^{n+1}_+ \to \mathbb{C}^N$. We say that L_0 is a constant coefficient elliptic uniformly degenerate operator if the principal symbol

$$\sigma(\xi,\eta) = \sum_{j+|\alpha|=m} a_{j,\alpha}(i\xi)^j(i\eta)^\alpha$$

is invertible for all $(\xi,\eta)\neq 0 \in \mathbb{R}^{n+1}$. The indicial matrix of B is defined as

$$J(\gamma) = \sum_{j=0}^{m} a_{j,0}\,\gamma^j.$$

This is an $N\times N$ matrix with entries that are polynomials of degree m in γ. We define the indicial roots of L_0 as the roots of the degree mN polynomial $\det J(\gamma)$.

The model Bessel operators of L_0 are defined as the family

$$B_\omega = \sum_{j+|\alpha|\le m} a_{j,\alpha}(r\,d/dr + n/2)^j(ir\omega)^\alpha$$

of Bessel operators parametrized by $\omega \in S^{n-1}$. Then B_ω has principal symbol $\sigma(\xi,\eta\omega)$ and indicial matrix $J(\gamma+n/2)$. Thus L_0 is elliptic if and only if B_ω is elliptic for all ω and the indicial roots of B_ω are obtained by subtracting $n/2$ from the indicial roots of L_0. In particular, the indicial roots of B_ω are independent of ω.

Proposition 5.1. *Let L_0 be a constant coefficient elliptic uniformly degenerate operator of order m on $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta, s_1, s_2 \in \mathbb{R}$ with $s_1 \le s_2$. Then there exists $c > 0$ such that*

$$\|u\|_{\delta,s_2+m} \le c\big(\|L_0u\|_{\delta,s_2} + \|u\|_{\delta,s_1+m}\big)$$

for all $u \in x^\delta H^{s_2+m}(\mathbb{R}^{n+1}_+)$.

Proof. This follows from (3.3), (3.5), and Proposition 4.3. □

Let

$$\begin{cases} \underline{\delta}(L_0) = \sup_{\omega\in S^{n-1}} \underline{\delta}(B_\omega) \\ \overline{\delta}(L_0) = \inf_{\omega\in S^{n-1}} \overline{\delta}(B_\omega). \end{cases}$$

Proposition 5.2. *Let L_0 be a constant coefficient elliptic uniformly degenerate operator of order m on $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta, s \in \mathbb{R}$. Assume that $\delta > \underline{\delta}(L_0)$ and that $\delta + n/2$ is not the real part of any indicial root of L_0. Then there exists $c > 0$ such that*

$$\|u\|_{\delta,s+m} \leq c\|L_0 u\|_{\delta,s}$$

for all $u \in x^\delta H^{s+m}(\mathbb{R}^{n+1}_+)$.

Proof. This follows from (3.3), (3.5), and Corollary 4.9. □

Proposition 5.3. *Let L_0 be a constant coefficient elliptic uniformly degenerate operator of order m on $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta, s, t \in \mathbb{R}$. Assume that $\delta > \underline{\delta}(L_0)$ and that $\delta + n/2$ is not the real part of any indicial root of L_0. Then there exists $c > 0$ such that*

$$\|u\|_{\delta,s+m,t} \leq c\|L_0 u\|_{\delta,s,t}$$

for all $u \in x^\delta H^{s+m,t}(\mathbb{R}^{n+1}_+)$.

Proof. This follows from (3.3), (3.6), (3.7), and Corollary 4.9. □

§6. Variable coefficient elliptic uniformly degenerate operators

In the following U will be an open subset of $\overline{\mathbb{R}}^{n+1}_+$. By a slight abuse of notation, we will refer to the set $U \cap \partial\overline{\mathbb{R}}^{n+1}_+$ as the boundary ∂U of U. It is an open subset of $\mathbb{R}^n$. A uniformly degenerate operator on U of order m is defined as a partial differential operator

$$L = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(x,y)\,(x\partial_x)^j(x\partial_y)^\alpha$$

with coefficients $a_{j,\alpha}$ that are complex $N\times N$-matrix valued and smooth up to the boundary. The operator L acts on functions $U \to \mathbb{C}^N$. We say that L is an elliptic uniformly degenerate operator if the principal symbol

$$\sigma(x,y,\xi,\eta) = \sum_{j+|\alpha|=m} a_{j,\alpha}(x,y)\,\xi^j\eta^\alpha$$

is invertible for all $(x,y) \in U$ and all $(\xi,\eta) \neq 0 \in \mathbb{R}^{n+1}$. The indicial matrix of L at $(0,y) \in \partial U$ is defined as

$$J_y(\gamma) = \sum_{j=0}^{m} a_{j,0}(0,y)\,\gamma^j.$$

This is an $N\times N$-matrix with entries that are polynomials of degree m in γ. We define the indicial roots of L at $(0,y)\in\partial U$ as the roots γ of the degree mN polynomial $\det J_y(\gamma)$.

By freezing the coefficients of L at $(0,y_0)\in\partial U$, we get the constant coefficient uniformly degenerate operator

$$L_{y_0}=\sum_{j+|\alpha|\le m} a_{j,\alpha}(0,y_0)\,(x\partial_x)^j(x\partial_y)^\alpha$$

on $\overline{\mathbb{R}}^{n+1}_+$. The operator L_{y_0} has principal symbol $\sigma(0,y_0,\xi,\eta)$ and indicial matrix $J_{y_0}(\gamma)$.

We first prove two weighted Sobolev estimates for uniformly degenerate operators.

Proposition 6.1. *Let L be an elliptic uniformly degenerate operator of order m on an open subset U of $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta,s_1,s_2\in\mathbb{R}$ with $s_1<s_2$. For any $\varphi,\psi\in C^\infty_{\mathrm{comp}}(U)$ with $\psi=1$ on $\operatorname{supp}\varphi$ there exists $c>0$ such that*

$$\|\varphi u\|_{\delta,s_2+m}\le c\big(\|\psi Lu\|_{\delta,s_2}+\|\psi u\|_{\delta,s_1+m}\big)$$

for all $u\in H^{s_2+m}_{\mathrm{loc}}(U)$.

Proof. We may assume that $s_2\le s_1+1$; the general case follows from this case by iteration. It is enough to establish the estimate for φ and ψ supported in a small neighborhood of any given point (x_0,y_0) in U. For an interior point the estimate follows from standard elliptic theory. For a boundary point $(0,y_0)$, it follows from Proposition 5.1 that

$$\begin{aligned}\|\varphi u\|_{\delta,s_2+m}&\le c\big(\|L_{y_0}(\varphi u)\|_{\delta,s_2}+\|\varphi u\|_{\delta,s_1+m}\big)\\&\le c\big(\|(L_{y_0}-L)(\varphi u)\|_{\delta,s_2}+\|[L,\varphi](\psi u)\|_{\delta,s_2}+\|\varphi Lu\|_{\delta,s_2}+\|\varphi u\|_{\delta,s_1+m}\big).\end{aligned}$$

The first term is bounded by $\varepsilon\|\varphi u\|_{\delta,s_2+m}$, where we can make ε arbitrarily small by choosing the neighborhood small, and can thus be absorbed. Concerning the second term, note that $[L,\varphi]$ is a uniformly degenerate operator of order $m-1$ with coefficients that vanish on the boundary. Therefore

$$\|\varphi u\|_{\delta,s_2+m}\le c\big(\|\psi u\|_{\delta-1,s_2+m-1}+\|\varphi Lu\|_{\delta,s_2}+\|\varphi u\|_{\delta,s_1+m}\big).$$

The proposition now follows from Lemma 2.1(c). □

The model Bessel operators of L are defined as the family

$$B_{y_0,\omega} = \sum_{j+|\alpha|\le m} a_{j,\alpha}(0,y_0)\,\big(r\,d/dr + n/2\big)^j\,\big(ir\omega\big)^\alpha$$

of elliptic Bessel operators parametrized by $(0,y_0)\in\partial U$ and $\omega\in S^{n-1}$. Let

$$\begin{cases} \underline{\delta}(L) = \sup\limits_{(0,y_0)\in\partial U} \underline{\delta}(L_{y_0}) = \sup\limits_{(0,y_0)\in\partial U}\ \sup\limits_{\omega\in S^{n-1}} \underline{\delta}(B_{y_0,\omega}) \\ \overline{\delta}(L) = \inf\limits_{(0,y_0)\in\partial U} \overline{\delta}(L_{y_0}) = \inf\limits_{(0,y_0)\in\partial U}\ \inf\limits_{\omega\in S^{n-1}} \overline{\delta}(B_{y_0,\omega}). \end{cases}$$

Theorem 6.2. *Let L be an elliptic uniformly degenerate operator of order m on an open subset U of $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta_1,\delta_2,s_1,s_2\in\mathbb{R}$ with $\delta_1<\delta_2$. Assume that $\delta_2,\delta_1+1>\underline{\delta}(L)$. For any $\varphi,\psi\in C^\infty_{\mathrm{comp}}(U)$ with $\psi=1$ on $\operatorname{supp}\varphi$ there exists $c>0$ such that*

$$\|\varphi u\|_{\delta_2,s_2+m} \le c\big(\|\psi Lu\|_{\delta_2,s_2} + \|\psi u\|_{\delta_1,s_1+m}\big)$$

for all $u\in x^{\delta_2}H^{s_2+m}_{\mathrm{loc}}(U)$.

Proof. We may assume that $\delta_2\le\delta_1+1$ and $s_2\le s_1+1$; the general case follows from this case by iteration. Arguing as in the proof of Proposition 6.1 using Proposition 5.2 instead of Proposition 5.1 we get

$$\|\varphi u\|_{\delta_2,s_2+m} \le c\big(\|\psi u\|_{\delta_2-1,s_2+m-1} + \|\varphi Lu\|_{\delta_2,s_2}\big).$$

The theorem now follows from Lemma 2.1(c). □

In §7 we use Theorem 6.2 to establish Fredholm properties for uniformly degenerate operators on compact manifolds with boundary. We next establish the regularity results corresponding to the estimates Proposition 6.1 and Theorem 6.2.

Proposition 6.3. *Let L be an elliptic uniformly degenerate operator of order m on an open subset U of $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta,s_1,s_2\in\mathbb{R}$ with $s_1<s_2$. If $u\in x^\delta H^{s_1+m}_{\mathrm{loc}}(U)$ and $Lu\in x^\delta H^{s_2}_{\mathrm{loc}}(U)$, then $u\in x^\delta H^{s_2+m}_{\mathrm{loc}}(U)$.*

Proof. We may assume that $s_2\le s_1+1$. Let $\varphi\in C^\infty_{\mathrm{comp}}(U)$. The operators $\Delta_{h,k}$, as in Lemma 2.1(e), are uniformly bounded $x^\delta H^{s+1}(\mathbb{R}^{n+1}_+)\to x^\delta H^s(\mathbb{R}^{n+1}_+)$ as $(h,k)\to(0,0)$. The operators $[\Delta_{h,k},L]$ are uniformly bounded $x^\delta H^{s+m}(\mathbb{R}^{n+1}_+)\to x^\delta H^s(\mathbb{R}^{n+1}_+)$ as $(h,k)\to(0,0)$. It then follows from Proposition 6.1 that

$$\begin{aligned} \|\Delta_{h,k}(\varphi u)\|_{\delta,s_2+m-1} &\le c\big(\|L\Delta_{h,k}(\varphi u)\|_{\delta,s_2-1} + \|\Delta_{h,k}(\varphi u)\|_{\delta,s_1+m-1}\big) \\ &\le c\big(\|[L,\Delta_{h,k}](\varphi u)\|_{\delta,s_2-1} + \|\Delta_{h,k}L(\varphi u)\|_{\delta,s_2-1} + \|\varphi u\|_{\delta,s_1+m}\big) \\ &\le c\big(\|L(\varphi u)\|_{\delta,s_2} + \|\varphi u\|_{\delta,s_1+m}\big). \end{aligned}$$

The proposition now follows from Lemma 2.1(e). □

Theorem 6.4. *Let L be an elliptic uniformly degenerate operator of order m on an open subset U of $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta_1, \delta_2, s_1, s_2 \in \mathbb{R}$ with $\delta_1 < \delta_2$. Assume that $\delta_1 > \underline{\delta}(L)$ and that $[\delta_1 + n/2, \delta_2 + n/2]$ does not contain the real part of any indicial root of L at any point in ∂U. If $u \in x^{\delta_1} H^{s_1+m}_{\mathrm{loc}}(U)$ and $Lu \in x^{\delta_2} H^{s_2}_{\mathrm{loc}}(U)$, then $u \in x^{\delta_2} H^{s_2+m}_{\mathrm{loc}}(U)$.*

Lemma 6.5. *Let L be an elliptic uniformly degenerate operator of order m on an open subset U of $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta, s, t_1, t_2 \in \mathbb{R}$. Assume that $\delta > \underline{\delta}(L)$ and that $\delta + n/2$ is not the real part of any indicial root of L at any point in ∂U. If $u \in x^{\delta} H^{s+m,t_1}_{\mathrm{loc}}(U)$ and $Lu \in x^{\delta} H^{s,t_2}_{\mathrm{loc}}(U)$, then $u \in x^{\delta} H^{s+m,t_2}_{\mathrm{loc}}(U)$. For any $\varphi, \psi \in C^{\infty}_{\mathrm{comp}}(U)$ with $\psi = 1$ on* $\operatorname{supp}\varphi$ *there exists $c > 0$ such that*

$$\|\varphi u\|_{\delta,s+m,t_2} \leq c\big(\|\psi Lu\|_{\delta,s,t_2} + \|\psi u\|_{\delta,s+m,t_1}\big)$$

for all $u \in x^{\delta} H^{s+m,t_2}_{\mathrm{loc}}(U)$.

Proof of Lemma 6.5. We may assume that $t_2 \leq t_1 + 1$; the general case follows from this case by iteration. We prove the second part of the lemma first. Arguing as in the proof of Theorem 6.2 using Proposition 5.3 instead of Proposition 5.2 we get

$$\|\varphi u\|_{\delta,s+m,t_2} \leq c\big(\|\psi u\|_{\delta-1,s+m-1,t_2} + \|\varphi Lu\|_{\delta,s,t_2}\big).$$

The second part of the lemma now follows from Lemma 2.2(c). By Lemma 2.2(b), $Lu \in x^{\delta} H^{s+1,t_2-1}_{\mathrm{loc}}(U)$. By an argument similar to Proposition 6.3, $u \in x^{\delta} H^{s+m+1,t_2-1}_{\mathrm{loc}}(U)$. The first part of the lemma now follows as Proposition 6.3 using Lemma 2.2(e). ☐

Proof of Theorem 6.4. The two sets $U \cap (\overline{\mathbb{R}}_+ \times \partial U)$ and $U \cap \mathbb{R}^{n+1}_+$ form an open cover of U. By standard elliptic theory, the theorem holds with U replaced by $U \cap \mathbb{R}^{n+1}_+$. Thus we only have to prove it with U replaced by $U \cap (\overline{\mathbb{R}}_+ \times \partial U)$. In other words, we may assume that $U \subseteq \overline{\mathbb{R}}_+ \times \partial U$.

We may assume that $\delta_2 \leq \delta_1 + 1$; the general case follows from this case by iteration. We have

$$L - J_y(x\partial_x) = xM + x\sum_{j=1}^{n} N_j \partial_{y_j} \tag{6.1}$$

for some uniformly degenerate operator M of order m and some uniformly degenerate operators N_j of order $m-1$. Hence

$$(L - J_y(x\partial_x))(\varphi u) \in x^{\delta_1+1} H^{s_1+1,-1}(\mathbb{R}^{n+1}_+).$$

The operator $[L,\varphi]$ is a uniformly degenerate operator of order $m-1$ with coefficients that vanish on the boundary. Hence

$$J_y(x\partial_x)(\varphi u) = (J_y(x\partial_x) - L)(\varphi u) + [L,\varphi]u + \varphi Lu \in x^{\delta_2} H^{\min\{s_1+1,s_2\},-1}(\mathbb{R}^{n+1}_+).$$

By assumption, $\varphi u \in x^{\delta_1} H^{s_1}(\mathbb{R}^{n+1}_+)$. It then follows from Lemma 2.3 that

$$\varphi u \in H^{t_1}_{\mathrm{comp}}\big(\partial U, x^{\delta_1+n/2} H^{s_3+m}(\mathbb{R}_+)\big) \tag{6.2}$$

and

$$J_y(x\partial_x)(\varphi u) \in H^{t_1}_{\mathrm{comp}}(\partial U, x^{\delta_2+n/2} H^{s_3}(\mathbb{R}_+)) \tag{6.3}$$

for some $s_3, t_1 \in \mathbb{R}$. It follows from Lemma 4.2 that the operators

$$J_y(x\partial_x) : H^{t_1}_{\mathrm{comp}}\big(\partial U, x^{\delta_2+n/2} H^{s_3+m}(\mathbb{R}_+)\big) \to H^{t_1}_{\mathrm{comp}}\big(\partial U, x^{\delta_2+n/2} H^{s_3}(\mathbb{R}_+)\big)$$

and

$$\begin{aligned} J_y(x\partial_x) : H^{t_1}_{\mathrm{comp}}\big(\partial U, x^{\delta_1+n/2} H^{s_3+m}(\mathbb{R}_+) + x^{\delta_2+n/2} H^{s_3+m}(\mathbb{R}_+)\big) \\ \to H^{t_1}_{\mathrm{comp}}\big(\partial U, x^{\delta_1+n/2} H^{s_3}(\mathbb{R}_+) + x^{\delta_2+n/2} H^{s_3}(\mathbb{R}_+)\big) \end{aligned}$$

are invertible. It then follows from (6.2) and (6.3) that

$$\varphi u \in H^{t_1}_{\mathrm{comp}}\big(\partial U, x^{\delta_2+n/2} H^{s_3+m}(\mathbb{R}_+)\big).$$

By Lemma 2.3, $\varphi u \in x^{\delta_2} H^{s_4+m,t_2}(\mathbb{R}^{n+1}_+)$ for some $s_4, t_2 \in \mathbb{R}$. We have shown that $u \in x^{\delta_2} H^{s_4+m,t_2}_{\mathrm{loc}}(U)$. By Lemma 6.5, $u \in x^{\delta_2} H^{s_4+m}_{\mathrm{loc}}(U)$. By Proposition 6.3, $u \in x^{\delta_2} H^{s_2+m}_{\mathrm{loc}}(U)$. □

In the rest of §6 we investigate the regularity of solutions of the homogeneous equation $Lu = 0$ in $x^{\delta} H^s_{\mathrm{loc}}(U)$ with $\delta > \underline{\delta}(L)$.

Corollary 6.6. *Let L be an elliptic uniformly degenerate operator on an open subset U of $\overline{\mathbb{R}}^{n+1}_+$. Let $\delta_1, \delta_2, s \in \mathbb{R}$ with $\delta_1 \le \delta_2$. Assume that $\delta_1 > \underline{\delta}(L)$ and that $[\delta_1 + n/2, \delta_2 + n/2]$ does not contain the real part of any indicial root of L at any point in ∂U. If $u \in x^{\delta_1} H^s_{\mathrm{loc}}(U)$ and $Lu = 0$, then $u \in x^{\delta_2} \mathcal{A}(U)$.*

The following is the most convenient way to state polyhomogeneity in the general case of variable indicial roots.

Theorem 6.7. *Let Y be an open subset of $\mathbb{R}^n$. Let L be a scalar ($N = 1$) elliptic uniformly degenerate operator on $[0,1) \times Y$. Let $\delta, s \in \mathbb{R}$. Assume that $\delta > \underline{\delta}(L)$ and that $\delta + n/2$ is not the real part of any indicial root of L at any point in $0 \times Y$. The polynomial $J_y(\gamma)$ can then be factored as $J_{y,-}(\gamma)J_{y,+}(\gamma)$ where the roots of $J_{y,-}(\gamma)$ have real parts $< \delta + n/2$ and the roots of $J_{y,+}(\gamma)$ have real parts $> \delta + n/2$ for any $y \in Y$. If $u \in x^\delta H^s_{\mathrm{loc}}([0,1) \times Y)$ and $Lu = 0$, then, for any nonnegative integer j,*

$$\left(\prod_{k=0}^{j-1} J_{y,+}(x\partial_x - k)^{k+1}\right) u \in x^{\delta+j}\mathcal{A}([0,1) \times Y). \tag{6.4}$$

If J_y is independent of y, then, for any nonnegative integer j,

$$\left(\prod_{k=0}^{j-1} J_{y,+}(x\partial_x - k)\right) u \in x^{\delta+j}\mathcal{A}([0,1) \times Y).$$

Proof. A totally characteristic partial differential operator on $[0,1) \times Y$ is defined as a partial differential operator of the form $\sum_{j,\alpha} a_{j,\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha$ with $a_{j,\alpha} \in C^\infty([0,1) \times Y)$. The algebra of totally characteristic operators on $[0,1) \times Y$ has a filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$$

where $P \in \mathcal{F}_j$ if and only if there exist uniformly degenerate operators L_α on $[0,1) \times Y$ such that $P = \sum_{|\alpha| \le j} L_\alpha \partial_y^\alpha$. Then

$$\begin{cases} \mathcal{F}_0 \text{ is the algebra of uniformly degenerate operators} \\ \mathcal{F}_j + \mathcal{F}_j \subseteq \mathcal{F}_j \\ \mathcal{F}_j \mathcal{F}_k \subseteq \mathcal{F}_{j+k} \\ \mathcal{F}_j x = x\mathcal{F}_j \\ [\mathcal{F}_0, \mathcal{F}_0] \subseteq x\mathcal{F}_1 \\ [\mathcal{F}_j, \mathcal{F}_k] \subseteq \mathcal{F}_{j+k-1} + x\mathcal{F}_{j+k+1} \qquad \text{for } (j,k) \neq (0,0). \end{cases} \tag{6.5}$$

It suffices to verify these relations for a filtered set of generators, such as $x\partial_x$, $x\partial_{y_j}$, and $C^\infty([0,1) \times Y)$ in $\mathcal{F}_0$ and ∂_{y_j} in $\mathcal{F}_1$.

We will now show that for any nonnegative integer j and any $Q \in \mathcal{F}_j$ there exists $R \in \mathcal{F}_{j+1}$ such that

$$J_y(x\partial_x)^{j+1} Q u = xRu. \tag{6.6}$$

This is shown by induction. Let $Q \in \mathcal{F}_0$. By (6.1),

$$J_y(x\partial_x)u = xS_1u \tag{6.7}$$

for some $S_1 \in \mathcal{F}_1$. By (6.5) and (6.7),

$$J_y(x\partial_x)Qu = [J_y(x\partial_x), Q]u + QJ_y(x\partial_x)u = [J_y(x\partial_x), Q]u + QxS_1u = xRu$$

for some $R \in \mathcal{F}_1$. The case $j = 0$ follows. Assume that (6.6) holds with j replaced by $j-1$. Let $Q \in \mathcal{F}_j$. By (6.5), (6.7), and the induction hypothesis,

$$\begin{aligned} J_y(x\partial_x)^{j+1}Qu &= J_y(x\partial_x)^j \left[J_y(x\partial_x), Q\right]u + J_y(x\partial_x)^j QJ_y(x\partial_x)u \\ &= J_y(x\partial_x)^j Q_1u + J_y(x\partial_x)^j xQ_2u + J_y(x\partial_x)^j QxS_1u \\ &= xRu \end{aligned}$$

for some $Q_1 \in \mathcal{F}_{j-1}$ and $Q_2, R \in \mathcal{F}_{j+1}$. We have established (6.6).

It follows from (6.6) that there exists a sequence $S_j \in \mathcal{F}_j$, $j = 1, 2, \ldots$, with S_1 as in (6.7) and $J_y(x\partial_x)^{j+1}S_ju = xS_{j+1}u$. Then

$$\left(\prod_{k=1}^{j} x^{-1}J_y(x\partial_x)^k\right)u = S_ju.$$

Now $J_y(x\partial_x - k) = x^kJ_y(x\partial_x)x^{-k}$. Hence

$$\left(\prod_{k=0}^{j-1} J_y(x\partial_x - k)^{k+1}\right)u = x^jS_ju \in x^{\delta+j}\mathcal{A}([0,1) \times Y). \tag{6.8}$$

We will now show (6.4) by induction. By assumption, (6.4) holds for $j = 0$. Assume that (6.4) holds with j replaced by $j-1$. Applying $J_{y,+}(x\partial_x - j + 1)^j$ to both sides we then get

$$\left(\prod_{k=0}^{j-1} J_{y,+}(x\partial_x - k)^{k+1}\right)u \in x^{\delta+j-1}\mathcal{A}([0,1) \times Y).$$

The operators $J_{\pm,y}(x\partial_x - k)$ commute, so it follows from (6.8) that

$$\left(\prod_{k=0}^{j-1} J_{y,-}(x\partial_x - k)^{k+1}\right)\left(\prod_{k=0}^{j-1} J_{y,+}(x\partial_x - k)^{k+1}\right)u \in x^{\delta+j}\mathcal{A}([0,1) \times Y).$$

Let φ be smooth function $\mathbb{R}_+ \to [0,1]$ such that $\varphi(x) = 1$ for $x \leq 1/3$ and $\varphi(x) = 0$ for $x \geq 2/3$. By Lemma 2.4,

$$\left(\prod_{k=0}^{j-1} J_{y,+}(x\partial_x - k)^{k+1}\right)(\varphi u) \in C^\infty(Y, x^{n/2+\delta+j-1}H^\infty(\mathbb{R}_+))$$

and

$$\left(\prod_{k=0}^{j-1} J_{y,-}(x\partial_x - k)^{k+1}\right)\left(\prod_{k=0}^{j-1} J_{y,+}(x\partial_x - k)^{k+1}\right)(\varphi u) \in C^\infty(Y, x^{n/2+\delta+j}H^\infty(\mathbb{R}_+)).$$

It follows from Lemma 4.2 that the operators

$$\prod_{k=0}^{j-1} J_{y,-}(x\partial_x - k)^{k+1} : C^\infty(Y, x^{n/2+\delta+j}H^\infty(\mathbb{R}_+)) \to C^\infty(Y, x^{n/2+\delta+j}H^\infty(\mathbb{R}_+))$$

and

$$\prod_{k=0}^{j-1} J_{y,-}(x\partial_x - k)^{k+1} : C^\infty\big(Y, x^{n/2+\delta+j-1}H^\infty(\mathbb{R}_+) + x^{n/2+\delta+j}H^\infty(\mathbb{R}_+)\big)$$
$$\to C^\infty\big(Y, x^{n/2+\delta+j-1}H^\infty(\mathbb{R}_+) + x^{n/2+\delta+j}H^\infty(\mathbb{R}_+)\big)$$

are invertible. We conclude that

$$\left(\prod_{k=0}^{j-1} J_{y,+}(x\partial_x - k)^{k+1}\right)(\varphi u) \in C^\infty(Y, x^{n/2+\delta+j}H^\infty(\mathbb{R}_+)).$$

(6.4) now follows from Lemma 2.4.

Finally, if $J_y(\gamma)$ is independent of y, then $[J_y(x\partial_x), \mathcal{F}_j] \subseteq x\mathcal{F}_{j+1}$. We can then omit the power $j+1$ in (6.6). $\square$

Theorem 6.8. *Let Y be an open subset of $\mathbb{R}^n$. Let L be an elliptic uniformly degenerate operator on $[0,1) \times Y$. Let $\delta, s \in \mathbb{R}$. Assume that $\delta > \underline{\delta}(L)$ and that $\delta + n/2$ is not the real part of any indicial root of L at any point in $0 \times Y$. Let $\widetilde{J}_y(\gamma)$ be a polynomial in γ with coefficients in $C^\infty(Y)$ such that $\widetilde{J}_y(\gamma)$ does not have any roots γ with real part $\delta + n/2$ for any $y \in Y$ and $\widetilde{J}_y I_N = J'_y J_y$ for some $N\times N$-matrix $J'_y(\gamma)$ with entries that are polynomials in γ with coefficients in $C^\infty(Y)$. Factor $\widetilde{J}_y$ as $\widetilde{J}_{y,-}\widetilde{J}_{y,+}$ as in Theorem 6.7. If $u \in x^\delta H^s_{\mathrm{loc}}([0,1) \times Y)$ and $Lu = 0$, then, for any nonnegative integer j,*

$$\left(\prod_{k=0}^{j-1} \widetilde{J}_{y,+}(x\partial_x - k)^{k+1}\right)u \in x^{\delta+j}\mathcal{A}([0,1) \times Y).$$

If $\widetilde{J}_y(\gamma)$ is independent of y, then, for any nonnegative integer j,

$$\left(\prod_{k=0}^{j-1} \widetilde{J}_{y,+}(x\partial_x - k)\right) u \in x^{\delta+j}\mathcal{A}([0,1) \times Y).$$

Such $\widetilde{J}_y$ always exist; one can take $\widetilde{J}_y = \det J_y$. For instance, for

$$J_y = \begin{pmatrix} (\gamma - a)^2 & 0 \\ 0 & (\gamma - b)^2 \end{pmatrix}$$

this gives $\widetilde{J}_y = (\gamma - a)^2(\gamma - b)^2$. That is not always the best choice however. For

$$J_y = \begin{pmatrix} (\gamma - a)(\gamma - b) & 0 \\ 0 & (\gamma - a)(\gamma - b) \end{pmatrix}$$

we can take $\widetilde{J}_y = (\gamma - a)(\gamma - b)$.

Proof. By (6.7),

$$\widetilde{J}_y(x\partial_x)u = J_y'(x\partial_x)J_y(x\partial_x)u = J_y'(x\partial_x)xS_1u = xT_1u$$

for some $T_1 \in \mathcal{F}_1$. Note that $[\mathcal{F}_j, \mathcal{F}_k] \not\subseteq \mathcal{F}_{j+k-1} + x\mathcal{F}_{j+k+1}$ for $N \geq 2$. However, if either of the operators is a scalar operator, then the commutator is in $\mathcal{F}_{j+k-1} + x\mathcal{F}_{j+k+1}$. In particular, $[\widetilde{J}_y(x\partial_x), \mathcal{F}_j] \subseteq \mathcal{F}_{j-1} + x\mathcal{F}_{j+1}$. We can then argue as in the proof of Theorem 6.7. □

Remark 6.9. In the following we assume that u is as in Theorem 6.7 or Theorem 6.8. If $J_y(\gamma)$, as in Theorem 6.7, or $\widetilde{J}_y(\gamma)$, as in Theorem 6.8, is independent of y, then we can rearrange $\prod_{j=0}^{\infty} J_{y,+}(\gamma - k)$ or $\prod_{k=0}^{\infty} \widetilde{J}_{y,+}(\gamma - k)$ as

$$\prod_{j=0}^{\infty} (\gamma - \delta_j)^{\lambda_j}$$

with $\delta_j \in \mathbb{C}$ and $\delta_j \neq \delta_k$ for $j \neq k$. It then follows from Theorem 6.7 or Theorem 6.8 using the meromorphic Mellin transform that u has an asymptotic expansion

$$u(x,y) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\lambda_j - 1} a_{j,k}\, x^{\delta_j} (\log x)^k$$

with $a_{j,k} \in \mathbb{C}^N$.

If $J_y(\gamma)$, as in Theorem 6.7, or $\widetilde{J}_y(\gamma)$, as in Theorem 6.8, does depend on y, then the situation is more complicated. For any given $y_0 \in Y$, we can rearrange $\prod_{j=0}^{\infty} J_{y_0,+}(\gamma-k)^{j+1}$ or $\prod_{j=0}^{\infty} \widetilde{J}_{y_0,+}(\gamma-k)^{j+1}$ as above. It then follows from Theorem 6.7 or Theorem 6.8 using the meromorphic Mellin transform that $u(\cdot, y_0)$ has an asymptotic expansion

$$u(x,y_0) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\lambda_j-1} a_{j,k}\, x^{\delta_j} (\log x)^k$$

with $a_{j,k} \in \mathbb{C}^N$. However, the coefficients and exponents do not have to depend smoothly on y_0.

If $J_y(\gamma)$, as in Theorem 6.7, or $\widetilde{J}_y(\gamma)$, as in Theorem 6.7, does depend on y, then it may still happen that $\prod_{j=0}^{\infty} J_{y,+}(\gamma-j)^{j+1}$ or $\prod_{j=0}^{\infty} \widetilde{J}_{y,+}(\gamma-j)^{j+1}$ can be rearranged as

$$\prod_{j=0}^{\infty} \big(\gamma - \delta_j(y)\big)^{\lambda_j}$$

with $\delta_j \in C^\infty(Y)$ and $\delta_j(y) \neq \delta_k(y)$ for $j \neq k$ and all $y \in Y$. In that case it follows from Theorem 6.7 or Theorem 6.8 using the meromorphic Mellin transform that u has an asymptotic expansion

$$u(x,y) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\lambda_j-1} a_{j,k}(y)\, x^{\delta_j(y)} (\log x)^k$$

with $a_{j,k} \in C^\infty(Y, \mathbb{C}^N)$.

Adjoints. We finally discuss some properties of adjoints that will be needed in §7. Let $\mu \in C^\infty(U, \mathbb{R})$ with $\mu > 0$. Let L^* be the formal adjoint of L with respect to the volume element $x^{-n-1}\mu(x,y)\, dx\, dy$. Then

$$\begin{aligned} L^* &= \sum_{j+|\alpha|\le m} x^{n+1}\mu(x,y)^{-1}(-\partial_y x)^\alpha\,(-\partial_x x)^j a_{j,\alpha}^*(x,y)\, x^{-n-1}\mu(x,y) \\ &= \sum_{j+|\alpha|\le m} a_{j,\alpha}^*(x,y)\,(-x\partial_y)^\alpha\,(n - x\partial_x)^j + \cdots \end{aligned}$$

where the dots indicate terms with coefficients that vanish on the boundary. Hence $\gamma + n/2$ is an indicial root of L^* at $(0, y_0)$ if and only if $-\overline{\gamma} + n/2$ is an indicial root of L at $(0, y_0)$. We also see that L^* has model Bessel operators

$$\sum_{j+|\alpha|\le m} a_{j,\alpha}^*(0,y_0)\,\big(-ir\omega\big)^\alpha \big(n/2 - r\, d/dr\big)^j.$$

By (4.7), this simply is $B^*_{y_0,\omega}$. By (4.8), then

$$\text{(6.11)} \qquad \begin{cases} \underline{\delta}(L^*) = -\overline{\delta}(L) \\ \overline{\delta}(L^*) = -\underline{\delta}(L). \end{cases}$$

In particular, if L is selfadjoint, then the model Bessel operators $B_{y_0,\omega}$ are selfadjoint and

$$\underline{\delta}(L) = -\overline{\delta}(L).$$

This is the case in the examples discussed in §8.

§7. Elliptic uniformly degenerate operators on manifolds with boundary

In order to extend our results to compact manifolds with boundary, we have to check invariance under coordinate transformations. First we consider the case of a constant coefficient elliptic uniformly degenerate operator L_0 as in §5 and a linear coordinate transformation $(x'\, y') = (x\, y)\left(\begin{smallmatrix} A & C \\ 0 & D \end{smallmatrix}\right)$ on $\overline{\mathbb{R}}^{n+1}_+$. Here $A > 0$, $C \in \mathbb{R}^n$, and $D \in \mathrm{GL}(n, \mathbb{R})$. We have

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix}$$

and $x = A^{-1}x'$. Under the coordinate transformation, L_0 is therefore transformed to

$$\text{(7.1)} \qquad L_0' = \sum_{j+|\alpha|\le m} a_{j,\alpha}(x'\partial_{x'} + A^{-1}Cx'\partial_{y'})^j (A^{-1}Dx'\partial_{y'})^\alpha.$$

We see that L_0' is a constant coefficient elliptic uniformly degenerate operator with the same indicial matrices as L_0.

We have defined the model Bessel operators of L_0 as a family B_ω of elliptic Bessel operators parametrized by $\omega \in S^{n-1}$. At this point it is convenient to extend this to a family

$$\text{(7.2)} \qquad B_\eta = \sum_{j+|\alpha|\le m} a_{j,\alpha}(r\, d/dr + n/2)^j (ir\eta)^\alpha$$

of elliptic Bessel operators parametrized by $\eta \in \mathbb{R}^n \setminus 0$. If $\eta = s\omega$ with $s \in \mathbb{R}_+$ and $\omega \in S^{n-1}$, then B_η and B_ω are intertwined by a dilatation by a factor s. Hence they have the same mapping properties, so $\underline{\delta}(B_\eta) = \underline{\delta}(B_\omega)$ and $\overline{\delta}(B_\eta) = \overline{\delta}(B_\omega)$. Thus

$$\text{(7.3)} \qquad \begin{cases} \underline{\delta}(L_0) = \sup\limits_{\eta\in\mathbb{R}^n\setminus 0} \underline{\delta}(B_\eta) \\ \overline{\delta}(L_0) = \inf\limits_{\eta\in\mathbb{R}^n\setminus 0} \overline{\delta}(B_\eta). \end{cases}$$

It follows from (7.1) that, with the convention (7.2), the model Bessel operators $B'_{\eta'}$ of L'_0 are

$$(7.4)\qquad \begin{aligned} B'_{\eta'} &= \sum_{j+|\alpha|\le m} a_{j,\alpha}(r\,d/dr + iA^{-1}C\cdot r\eta')^j (iA^{-1}Dr\eta')^\alpha \\ &= e^{-ibr} B_\eta e^{ibr} \end{aligned}$$

where $b = A^{-1}C\cdot\eta'$ and $\eta = A^{-1}D\eta'$. Multiplication by e^{ibr} defines an automorphism of $r^\delta W^s(\mathbb{R}_+)$. Thus $\underline{\delta}(B'_{\eta'}) = \underline{\delta}(B_\eta)$ and $\overline{\delta}(B'_{\eta'}) = \overline{\delta}(B_\eta)$. It then follows from (7.3) that

$$(7.5)\qquad \begin{cases} \underline{\delta}(L'_0) = \underline{\delta}(L_0) \\ \overline{\delta}(L'_0) = \overline{\delta}(L_0). \end{cases}$$

Next we consider the case of a (variable coefficient) elliptic uniformly degenerate operator L on an open subset of $\overline{\mathbb{R}}^{n+1}_+$ as in §6. Under a coordinate transformation $(x', y') = (x'(x,y), y'(x,y))$ the operator L is transformed to

$$L' = \sum_{j+|\alpha|\le m} a_{j,\alpha}(x,y)\left(x\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + x\frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}\right)^j \left(x\frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + x\frac{\partial y'}{\partial y}\frac{\partial}{\partial y'}\right)^\alpha.$$

Now $x(\partial x'/\partial x) = x' + \mathrm{O}(x'^2)$ and $\partial x'/\partial y = \mathrm{O}(x')$. Hence

$$L' = \sum_{j+|\alpha|\le m} a_{j,\alpha}(x,y)\left(x'\frac{\partial}{\partial x'} + \left(\frac{\partial x'}{\partial x}\right)^{-1}\frac{\partial y'}{\partial x}\,x'\frac{\partial}{\partial y'}\right)^j \left(\left(\frac{\partial x'}{\partial x}\right)^{-1}\frac{\partial y'}{\partial y}\,x'\frac{\partial}{\partial y'}\right)^\alpha + \cdots,$$

where the dots indicate terms with coefficients that vanish on the boundary. We see that L' is an elliptic uniformly degenerate operator with the same indicial matrices as L. We also see that

$$\begin{aligned} L'_{y'_0} = \sum_{j+|\alpha|\le m} a_{j,\alpha}(0,y_0) &\left(x'\frac{\partial}{\partial x'} + \left(\frac{\partial x'}{\partial x}\right)^{-1}_{(0,y_0)}\left(\frac{\partial y'}{\partial x}\right)_{(0,y_0)} x'\frac{\partial}{\partial y'}\right)^j \\ &\times\left(\left(\frac{\partial x'}{\partial x}\right)^{-1}_{(0,y_0)}\left(\frac{\partial y'}{\partial y}\right)_{(0,y_0)} x'\frac{\partial}{\partial y'}\right)^\alpha. \end{aligned}$$

Comparing this with (7.1) we see that that $L'_{y'_0}$ is the operator obtained from L_{y_0} through the linear coordinate transformation

$$(x'\ y') = (x\ y)\begin{pmatrix} (\partial x'/\partial x)(0,y_0) & (\partial y'/\partial x)(0,y_0) \\ 0 & (\partial y'/\partial y)(0,y_0) \end{pmatrix}$$

on $\overline{\mathbb{R}}^{n+1}_+$. By (7.5), $\underline{\delta}(L'_{y'_0}) = \underline{\delta}(L_{y_0})$ and $\overline{\delta}(L'_{y'_0}) = \overline{\delta}(L_{y_0})$. Thus

$$\text{(7.6)} \qquad \begin{cases} \underline{\delta}(L') = \underline{\delta}(L) \\ \overline{\delta}(L') = \overline{\delta}(L). \end{cases}$$

By standard arguments, the notion of an elliptic uniformly degenerate operator L on a differentiable manifold M with boundary ∂M is now well-defined. The indicial matrix $J_p(\gamma)$ of L at point in $p \in \partial M$ is well-defined. It follows from (7.6) that $\underline{\delta}(L)$ and $\overline{\delta}(L)$ are well-defined. As usual, we can let L map sections of a smooth complex vector bundle E over M to sections of another smooth complex vector bundle F over M.

The model Bessel operators $B_{p,\eta}$ at a point in $p \in \partial M$ depend on a choice of coordinates in a neighborhood of p. However, it follows from the above discussion that if two coordinate systems give the same basis for T_pM, then they give the same Bessel operators $B_{p,\eta}$ for all $\eta \in \mathbb{R}^n \setminus 0$. Thus the model Bessel operators $B_{p,\eta}$ of L at p are well-defined once a basis for T_pM has been chosen. Under a change of basis for T_pM, given by a matrix $\left(\begin{smallmatrix} A & C \\ 0 & D \end{smallmatrix}\right)$, the operators $B_{p,\eta}$ are transformed as in (7.4).

By Lemma 2.1(f), the function spaces $x^\delta H^s_{\mathrm{loc}}(U)$ are invariant under coordinate transformations. Hence there are well-defined Hilbert spaces $x^\delta H^s(M)$. We then have the following:

Theorem 7.1. *Let L be an elliptic uniformly degenerate operator of order m on a compact differentiable manifold M with boundary. Let $\delta, s \in \mathbb{R}$. Assume that $\delta + n/2$ is not the real part of any indicial root of L at any point in ∂M. If $\delta > \underline{\delta}(L)$, then $L : x^\delta H^{s+m}(M) \to x^\delta H^s(M)$ has finite-dimensional null space and closed range. If $\delta < \overline{\delta}(L)$, then the range of $L : x^\delta H^{s+m}(M) \to x^\delta H^s(M)$ has finite codimension (and is hence closed).*

Proof. Assume that $\delta + n/2$ is not the real part of any indicial root of L at any point in ∂M and that $\delta > \underline{\delta}(L)$. Let $\delta' \in (\underline{\delta}(L) - 1, \delta)$ and $s' < s$. Cover M by coordinate charts. On the coordinate charts in the interior, we can use standard elliptic estimates. On the coordinate charts at the boundary, we can use Theorem 6.2. Using a partition of unity we then get

$$\|u\|_{\delta,s+m} \le c\big(\|Lu\|_{\delta,s} + \|u\|_{\delta',s'+m}\big)$$

for all $u \in x^\delta H^{s+m}(M)$. It follows from Lemma 2.1(d) that the inclusion

$$x^\delta H^{s+m}(M) \to x^{\delta'} H^{s'+m}(M)$$

is a compact operator. It follows that L has closed range and finite-dimensional null space.

Assume that $\delta + n/2$ is not the real part of any indicial root of L at any point in ∂M and that $\delta < \overline{\delta}(L)$. We choose a volume element on M that in local coordinates at the boundary is of the form $x^{-n-1}\mu(x,y)\,dx\,dy$ with μ smooth and positive up to the boundary. We let $L^* : x^{-\delta}H^{-s}(M) \to x^{-\delta}H^{-s-m}(M)$ be the adjoint of L with respect to this volume element. It then follows from (6.11) that $-\delta > \underline{\delta}(L^*)$. By the first part of the theorem, L^* has finite-dimensional null space and closed range. It follows that L has range of finite codimension. □

Corollary 7.2. *Let L be an elliptic uniformly degenerate operator of order m on a compact differentiable manifold M with boundary. If $\underline{\delta}(L) < \overline{\delta}(L)$, then the operator*

$$L : x^{\delta}H^{s+m}(M) \to x^{\delta}H^{s}(M)$$

is Fredholm for all $\delta \in (\underline{\delta}(L), \overline{\delta}(L))$ and all $s \in \mathbb{R}$.

Proof. By Theorem 7.1, we only have to verify that L does not have any indicial root at any point in ∂M with real part in $(\underline{\delta}(L) + n/2, \overline{\delta}(L) + n/2)$. This follows from Corollary 4.11. □

§8. Examples

In the following examples M is an $(n+1)$-dimensional conformally compact asymptotically hyperbolic manifold. This means that M has a compactification $\overline{M}$ that is a smooth manifold with boundary, and that the metric on M appears, in local coordinates as in §7, as

$$x^{-2}\Big(g_{00}(x,y)\,dx^2 + \sum_{j=1}^{n} g_{0j}(x,y)\,(dx \otimes dy_j + dy_j \otimes dx) + \sum_{j,k=1}^{n} g_{jk}(x,y)\,dy_j \otimes dy_k\Big),$$

where the functions g_{00}, g_{0j}, and g_{jk} are smooth up to the boundary, $g_{jk} = g_{kj}$, and

$$g_{00}(x,y)\,\xi^2 + 2\sum_{j=1}^{n} g_{0j}(x,y)\,\xi\eta_j + \sum_{j,k=1}^{n} g_{jk}(x,y)\,\eta_j\eta_k > 0$$

for all (x,y) in the coordinate chart and all $(\xi,\eta) \neq 0 \in \mathbb{R}^{n+1}$. For such a metric the sectional curvatures converge to a common limit $-K(p) < 0$ at any boundary point p. Let

$$\begin{pmatrix} g^{00}(x,y) & g^{0j}(x,y) \\ g^{k0}(x,y) & g^{kj}(x,y) \end{pmatrix} = \begin{pmatrix} g_{00}(x,y) & g_{0k}(x,y) \\ g_{j0}(x,y) & g_{jk}(x,y) \end{pmatrix}^{-1}.$$

If p has local coordinates $(0,y)$, then $g^{00}(0,y) = K(p)^2$. For proofs of these statements, see [Ma1].

Example 8.1. Let Δ be the (scalar) Laplacian on an $(n+1)$-dimensional conformally compact asymptotically hyperbolic manifold M. Then Δ is a selfadjoint second order scalar elliptic uniformly degenerate operator. The equation $\Delta u = 0$ is the variational equation for the Lagrangian $\mathfrak{L}(u) = \frac{1}{2}\int_M |du|_g^2\, dV_g$. In local coordinates

$$\mathfrak{L}(u) = \frac{1}{2}\int\Big(g^{00}(x\partial_x u)^2 + 2\sum_{j=1}^{n} g^{0j}\,(x\partial_x u)\,(x\partial_{y_j}u) + \sum_{j,k=1}^{n} g^{jk}\,(x\partial_{y_j}u)\,(x\partial_{y_k}u)\Big)x^{-n-1}\sqrt{\det g}\,dx\,dy.$$

It follows that

$$\Delta = g^{00}\big((x\partial_x)^2 - n\,x\partial_x\big) + \sum_{j=1}^{n} g^{0j}\big(2(x\partial_x)(x\partial_{y_j}) - (n+1)x\partial_{y_j}\big) + \sum_{j,k=1}^{n} g^{jk}(x\partial_{y_j})(x\partial_{y_k}) + \cdots$$

where the dots indicate lower order terms. The coefficients of these lower order terms all contain $x\partial_x$ and $x\partial_y$ applied to the components of the metric tensor. Hence these coefficients vanish on the boundary. Recall that $g^{00}(0,y_0) = K(p)^2$. It follows that the indicial "matrix" of Δ at any $p \in \partial\overline{M}$ is

$$J_p(\gamma) = K(p)^2(\gamma^2 - n\gamma)$$

and the model Bessel operators of Δ at any $p \in \partial\overline{M}$ are

$$g^{00}\big((r\,d/dr)^2 - n^2/4\big) + i\sum_{j=1}^{n} g^{0j}\omega_j\big(2(r\,d/dr)r - r\big) - \sum_{j,k=1}^{n} g^{jk}\omega_j\omega_k r^2.$$

Near any $p \in \partial\overline{M}$ we can choose local coordinates (x,y) with $p = (0,y_0)$ such that $g^{0j}(0,y_0) = 0$ and $g^{jk}(0,y_0) = K(p)^2\delta^{jk}$. In such coordinates the model Bessel operators at p are

$$B_{p,\omega} = K(p)^2\big((r\,d/dr)^2 - r^2 - n^2/4\big).$$

In particular, $B_{p,\omega}$ has indicial roots $\pm n/2$ for all p and ω.

Let $\delta > -n/2$. Let u be in the null space of $B_{p,\omega} : r^\delta W^{s+2}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$. By Corollary 4.4, u and its derivatives decay rapidly as $r \to \infty$. By [CL] Chapter 4, u has an asymptotic expansion as $r \to 0$ with leading term $r^{n/2}$. Hence u and $r\,du/dr$ decay as $r^{n/2}$ as $r \to 0$. We can then multiply the equation $B_{p,\omega}u = 0$ by u and integrate by parts with respect to $r^{-1}dr$ to get

$$\int_0^\infty \big((r\,du/dr)^2 + r^2u + (n^2/4)u\big)r^{-1}dr = 0.$$

Hence $u = 0$. It follows that $B_{p,\omega} : r^\delta W^{s+2}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is injective for $\delta > -n/2$.

The Bessel operator $B_{p,\omega}$ is selfadjoint. It follows that $B_{p,\omega}$ has dense range for $\delta < n/2$. By Corollary 4.6, $B_{p,\omega}$ is then invertible for $-n/2 < \delta < n/2$. By Corollaries 4.8 and 4.11, $\underline{\delta}(B_{p,\omega}) = -n/2$ and $\overline{\delta}(B_{p,\omega}) = n/2$. Thus

$$\begin{cases} \underline{\delta}(\Delta) = -n/2 \\ \overline{\delta}(\Delta) = n/2. \end{cases}$$

The indicial roots of L at any boundary point are 0 and n. It then follows from Corollary 7.2 that

$$\Delta : x^\delta H^{s+2}(M) \to x^\delta H^s(M)$$

is Fredholm for $\delta \in (-n/2, n/2)$. Using the maximum principle it is not hard to show that, if M does not have any compact connected components, Δ is invertible for $\delta \in (-n/2, n/2)$.

Example 8.2. ([Ma1], [A]) Let Δ_k be the Hodge Laplacian for k-forms on an $(n+1)$-dimensional conformally compact asymptotically hyperbolic manifold M. Then Δ_k is a selfadjoint second order elliptic uniformly degenerate operator with $\binom{n+1}{k}$ components. A tedious but straightforward calculation, see [Ma1], shows that the indicial matrix of Δ_k at any $p \in \partial\overline{M}$ is

$$J_p(\gamma) = K(p)^2 \big(\underbrace{J^1 \oplus \cdots \oplus J^1}_{\binom{n}{k}} \oplus \underbrace{J^2 \oplus \cdots \oplus J^2}_{\binom{n}{k-1}} \big),$$

where

$$J^1(\gamma) = \gamma^2 - n\gamma + k(n-k)$$

and

$$J^2(\gamma) = \gamma^2 - n\gamma + (k-1)(n-k+1),$$

and that the model Bessel operators of Δ_k at any $p \in \partial\overline{M}$, expressed in suitable bases for $T_p\overline{M}$ and $\Lambda^k T_p^*\overline{M}$, are

$$B_{p,\omega} = K(p)^2 \big(\underbrace{B^1 \oplus \cdots \oplus B^1}_{\binom{n-1}{k}} \oplus \underbrace{B^2 \oplus \cdots \oplus B^2}_{\binom{n-1}{k-2}} \oplus \underbrace{B^3 \oplus \cdots \oplus B^3}_{\binom{n-1}{k-1}} \big),$$

where

$$\begin{aligned} B^1 &= (r\,d/dr)^2 - r^2 - (n/2-k)^2, \\ B^2 &= (r\,d/dr)^2 - r^2 - (n/2+1-k)^2, \end{aligned}$$

and

$$B^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (r\,d/dr)^2 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} r^2 + \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} r - \begin{pmatrix} (n/2-k)^2 & 0 \\ 0 & (n/2+1-k)^2 \end{pmatrix}.$$

In particular, B^1 has indicial roots $\pm(n/2-k)$, B^2 has indicial roots $\pm(n/2+1-k)$, and B^3 has indicial roots $\pm(n/2-k)$ and $\pm(n/2+1-k)$. We have

$$\begin{cases} \underline{\delta}(B^1) = -\left|n/2-k\right| \\ \overline{\delta}(B^1) = \left|n/2-k\right|, \end{cases}$$

$$\begin{cases} \underline{\delta}(B^2) = -\left|n/2+1-k\right| \\ \overline{\delta}(B^2) = \left|n/2+1-k\right|, \end{cases}$$

and

$$\begin{cases} \underline{\delta}(B^3) = 1/2 - \left|(n+1)/2-k\right| \\ \overline{\delta}(B^3) = -1/2 + \left|(n+1)/2-k\right|. \end{cases}$$

This follows by integration by parts largely as in Example 8.1, except for B^3 in the case $2k = n+1$. In that case the null space of B^3 is spanned by

$$\begin{pmatrix} \sqrt{r}\,e^{-r} \\ \sqrt{r}\,e^{-r} \end{pmatrix}, \quad \begin{pmatrix} \sqrt{r}\,e^{r} \\ -\sqrt{r}\,e^{r} \end{pmatrix}, \quad \begin{pmatrix} \sqrt{r}\,e^{r} \int r^{-2}e^{-2r}dr \\ -\sqrt{r}\,e^{r} \int r^{-2}e^{-2r}dr \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \sqrt{r}\,e^{-r} \int r^{-2}e^{2r}dr \\ \sqrt{r}\,e^{-r} \int r^{-2}e^{2r}dr \end{pmatrix};$$

and it follows by inspection that $\underline{\delta}(B^3) = 1/2$ and $\overline{\delta}(B^3) = -1/2$.

It then follows that

$$\begin{cases} \underline{\delta}(\Delta_k) = 1/2 - \left|(n+1)/2-k\right| \\ \overline{\delta}(\Delta_k) = -1/2 + \left|(n+1)/2-k\right|. \end{cases}$$

In particular,

$$\begin{cases} \underline{\delta}(\Delta_k) < 0 < \overline{\delta}(\Delta_k) & \text{for } 2k \neq n, n+1, n+2 \\ \underline{\delta}(\Delta_k) = 0 = \overline{\delta}(\Delta_k) & \text{for } 2k = n, n+2 \\ \underline{\delta}(\Delta_k) > 0 > \overline{\delta}(\Delta_k) & \text{for } 2k = n+1. \end{cases}$$

The indicial roots of Δ_k at any boundary point are $k-1$, k, $n-k$, and $n+1-k$. It then follows from Corollary 7.2 that

$$\Delta_k : H^{s+2}(M, \Lambda^k T^*M) \to H^s(M, \Lambda^k T^*M)$$

is Fredholm for $2k \neq n, n+1, n+2$.

It follows from Theorem 6.8 and Remark 6.9 that if u is in the L^2 null space of Δ_k with $2k < n$, then u has an asymptotic expansion

$$u(x,y) \sim u_0(y)\, x^{n-k} + \sum_{j=1}^{\infty} \big(u_j(y) + v_j(y) \log x\big) x^{n-k+j}.$$

at the boundary. Similarly, if u is in the L^2 null space of Δ_k with $2k > n+2$, then u has an asymptotic expansion

$$u(x,y) \sim u_0(y)\, x^{k-1} + \sum_{j=1}^{\infty} \big(u_j(y) + v_j(y) \log x\big) x^{k-1+j}$$

at the boundary. In both cases u_j and v_j are smooth.

In [Ma1] it is shown that the L^2 null space of Δ_k gives the k^{th} singular cohomology group of M for $2k < n$ and the k^{th} singular cohomology group of $\overline{M}$ relative to $\partial\overline{M}$ for $2k > n+2$.

Example 8.3. ([MaMe], [A].) Next we consider the stationary Schrödinger operators $-\Delta + V - \lambda$ on an $(n+1)$-dimensional conformally compact asymptotically hyperbolic manifold M. Here Δ is the scalar Laplacian as in Example 8.1, $V \in C^\infty(\overline{M}, \mathbb{R})$, and $\lambda \in \mathbb{R}$. Thus $-\Delta + V - \lambda$ is a selfadjoint second order scalar elliptic uniformly degenerate operator. As in Example 8.1, $-\Delta + V - \lambda$ has indicial "matrices"

$$J_p(\gamma) = K(p)^2\big(-\gamma^2 + n\gamma\big) + V(p) - \lambda$$

and model Bessel operators

$$\begin{aligned} B_{p,\omega} &= K(p)^2\big(-(r\,d/dr)^2 + r^2 + n^2/4\big) + V(p) - \lambda \\ &= K(p)^2\big(-(r\,d/dr)^2 + r^2 + \gamma(p,\lambda)^2\big) \end{aligned}$$

where

$$\gamma(p,\lambda) = \sqrt{\frac{n^2}{4} + \frac{V(p) - \lambda}{K(p)^2}}.$$

In particular, $B_{p,\omega}$ has indicial roots $\pm\gamma(p,\lambda)$.

Let

$$\begin{cases} \lambda_{\mathrm{I}} = \min\limits_{p \in \partial\overline{M}} \Big(\dfrac{n^2}{4} K(p)^2 + V(p)\Big) \\[2ex] \lambda_{\mathrm{II}} = \max\limits_{p \in \partial\overline{M}} \Big(\dfrac{n^2}{4} K(p)^2 + V(p)\Big). \end{cases}$$

In the following we consider the case $\lambda < \lambda_{\rm I}$. Then $\gamma(p,\lambda)$ is real for all $p \in \partial\overline{M}$. The operator $B_{p,\omega} : r^\delta W^{s+2}(\mathbb{R}_+) \to r^\delta W^s(\mathbb{R}_+)$ is injective for $\delta > -\gamma(p,\lambda)$. In fact, if u lies in the null space, then integration by parts as in Example 8.1 gives

$$\int_0^\infty \big((r\,du/dr)^2 + r^2 u + \gamma(p,\lambda)^2 u^2\big) r^{-1} dr = 0,$$

so $u = 0$. It then follows as in Example 8.1 that $\underline{\delta}(B_{p,\omega}) = -\gamma(p,\lambda)$ and $\overline{\delta}(B_{p,\omega}) = \gamma(p,\lambda)$. Thus

$$\begin{cases} \underline{\delta}(-\Delta + V - \lambda) = -\min\limits_{p\in\partial\overline{M}} \gamma(p,\lambda) \\ \overline{\delta}(-\Delta + V - \lambda) = \min\limits_{p\in\partial\overline{M}} \gamma(p,\lambda). \end{cases}$$

In particular,

$$\underline{\delta}(-\Delta + V - \lambda) < 0 < \overline{\delta}(-\Delta + V - \lambda).$$

The indicial roots of $-\Delta + V - \lambda$ at $p \in \partial\overline{M}$ are $n/2 \pm \gamma(p,\lambda)$. It then follows from Corollary 7.2 that

$$-\Delta + V - \lambda : H^2(M) \to L^2(M)$$

is Fredholm. By standard arguments, the part of the L^2-spectrum of $-\Delta + V$ that lies below $\lambda_{\rm I}$ is then discrete and consists of eigenvalues of finite multiplicities. The corresponding eigenfunctions describe bound states of energy λ for the quantum Hamiltonian $p^2/2 + V$. It follows from Theorem 6.7 and Remark 6.9 that any such eigenfunction u has an asymptotic expansion

$$u(x,y) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{j} u_{j,k}(y)\, x^{\gamma(y,\lambda)+n/2+j} (\log x)^k$$

with $u_{j,k}$ smooth.

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