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Mathematics
Department of Mathematics
Chalmers University of Technology
and Göteborg University
S-412 96 Göteborg, Sweden

ABSTRACT

This report is a licentiate thesis in mathematics at Göteborg University. It consists of two parts; an introductory survey of the subject, and the main part being the paper '*L^p integrability of the second order derivatives of Green potentials in convex domains*'. The paper concerns the regularity properties, in a convex domain, of solutions to the Dirichlet boundary value problem with zero boundary data for an elliptic second order divergence form operator L with Lipschitz continuous coefficients. It generalises previous results for the Laplacian. Given that $Lu = f \in L^1$ it establishes a weak-type $(1, 1)$ estimate for the second order derivatives of a solution operator, using the Hörmander condition for singular integrals. Utilising this weak-type result and the classical result on boundedness in L^2, interpolation yields L^p-boundedness for the operator, where $1 < p \leq 2$.

Key words : elliptic, divergence form, Lipschitz coefficients, weak-type $(1, 1)$ estimate, Hörmander condition, convex domain

Introduction

This report is a licentiate thesis in mathematics at Göteborg University. It consists of two parts; an introductory survey of the subject, and the main part being the paper '*L^p integrability of the second order derivatives of Green potentials in convex domains*'. The paper concerns the regularity properties, in a convex domain Ω, of solutions to the Dirichlet boundary value problem with zero boundary data for an elliptic second order divergence form operator L with Lipschitz continuous coefficients. It generalises previous results for the Laplacian. It is well-known that when the Laplacian of a solution u is required to be in $L^2(\Omega)$, the solution to the Dirichlet boundary value problem with zero boundary data has additional regularity when the domain Ω is convex, i.e. $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega)$. Recently the corresponding result for $L^p(\Omega)$ has been established for $1 < p \leq 2$. We generalise these results for the Laplacian to any elliptic second order divergence form operator L with Lipschitz continuous coefficients by establishing a suitable weak-type $(1,1)$ estimate and using interpolation.

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WHAT THIS THESIS IS ABOUT

The aim of this brief historical note is to shortly describe the setting into which this thesis fits. For a fuller account of the general history see [1], [2] or [3]. Most of the later development pertaining to elliptic partial differential equations, (PDEs), in nonsmooth domains can be found in [4].

Mathematical modelling in science and engineering frequently takes the form of a PDE of some kind. It is often non-linear but if we are lucky it can be a linear type or by some approximation can be considered as a linear equation. Studies of the PDEs goes a long way back in the history of mathematics. Daniel Bernoulli (1700-1782) pioneered in PDE's, see [1]. Jean-Le-Rond D'Alembert published his Traite' de dynamique in 1743, based upon the great principle of kinetics that now bears his name. In 1744 he applied his principle in a treatise on the equilibrium and motion of fluids, and in 1746 in a treatise on the causes of winds. In each of these works, and also in one of 1747 devoted to vibrating strings, he was led to PDEs and he became a pioneer in the study of such equations. Joseph Louis Lagrange is another name associated to fundamental principles of physics formulated in the language of mathematics, viz. as PDE's, see [2]. His, Euler's and the Bernoulli's work in differential equations, based on Newton's infinitesimal calculus, implemented the Principle of Least Action in mathematical terms and laid the analytical foundation of the Calculus of Variations.

Later on Carl Friedrich Gauss (1777-1855), George Green, William Thomson, Dirichlet and others began to study the existence of solutions of Laplace's equation in a given domain with given values or given normal derivatives on the boundary. It gradually became clear that the type of initial or boundary values one could impose depended on the type of equation under investigation. This feeling was made precise by Du Bois-Reymond. In a book of 1864, he clarified Monge's theory of characteristics and this led him (in a posthumous paper of 1889) to classify second-order PDEs into hyperbolic equations (e.g. the wave equation), parabolic equations (e.g. the heat-conduction equation) and elliptic equations (e.g. Laplace equation). Hyperbolic equations have real and distinct characteristics, parabolic equations have real and coinciding characteristics and elliptic equations have imaginary characteristics. This classification allowed a generalisation to general classes of PDEs of the results obtained previously on the boundary value problems of the physically important PDEs. In 1890 Emile Picard began a general study of boundary value problems of elliptic PDEs, showing that many of the results concerning Laplace's equation could be carried over to the general setting. Subsequently the PDEs subject became strongly influenced by the introduction of functional analysis and topics in mathematics studying in a more abstract way the structures and general relations. For example, the Lax-Milgram Theorem in functional analysis is concerned with existence and uniqueness of the solution to an elliptic operator. We state (without any details) their well-known result.

Theorem 0.1. *Let V be a Hilbert space and $a(\cdot,\cdot)$ a continuous V-elliptic bilinear form. Then given $f \in V$, there exists a unique $u \in V$ such that*

$$a(u,v) = (f,v),$$

for every $v \in V$. If the form $a(\cdot,\cdot)$ is also symmetric then the functional $J : V \to R$ defined by

$$J(v) = \frac{1}{2}a(v,v) - (f,v)$$

attains its minimum at u. □

The creation of the theory of distributions by Laurent Schwartz and its development by Lars Hörmander made it possible to introduce the weak type solution, which opened a new door for development.

Studying boundary value problems it is natural to ask about the influence of the regularity of the boundary of the domain, on the solvability, uniqueness and regularity properties of the solution. To be concrete, consider the Classical Dirichlet Problem (CDP) for a domain Ω in $\mathbf{R}^n$. Given $f \in C^0(\Omega)$, is there a $u \in C^0(\overline{\Omega})$ such that

$$\left\{ \begin{array}{rcll} -\Delta u &=& 0 & \text{in } \Omega \\ u &=& f & \text{on } \partial\Omega? \end{array} \right. \tag{0.1}$$

In case Ω is a ball, this is answered by explicitly finding the Green's function for a ball and thereby the Poisson kernel, known more generally as a kernel function. The solution is then given by the Poisson integral of u on $\partial\Omega$. Further one can show that the CDP is solvable if all boundary points are regular, i.e. there is a barrier at the boundary point. Let $\xi \in \partial\Omega$. Then a function $w = w_\xi$ is called a barrier at ξ relative to Ω if

i) $w \in C^0(\bar{\Omega})$

ii) w is superharmonic in Ω.

iii) $w > 0$ in $\bar{\Omega} \setminus \{\xi\}$; $w(\xi) = 0$.

The following result exhibits barriers in case of a relatively simple domain.

Proposition 0.2. *Suppose $Q_0 \in \partial\Omega$ and that Ω satisfies an outer ball condition there, i.e., there exists a ball $B = B_R(y)$ such that $\bar{B} \cap \bar{\Omega} = Q$. Then $w(x) = R^{2-n} - |x-y|^{2-n}$, $n \geq 3$, is a barrier at Q_0.* □

A similar result applies to Lipschitz domains.

Theorem 0.3 (Zaremba-Lebesgue)**.** *If Ω verifies an exterior cone condition at Q_0, then Ω has a barrier at Q_0.* □

Having seen that the classical Dirichlet problem (CDP) is solvable in a Lipschitz domain the foremost question becomes as to the solvability of the same problem but with boundary data given in some space $L^p(\partial\Omega)$, $p > 1$. Thus, finding a solution includes finding a suitable sense in which the solution takes on its boundary values. Here the results going back to Lebesgue on non-tangential convergence are crucial.

Determining a right class of domains for this boundary value problem was solved by Dahlberg in [9] where it is shown that for a Lipschitz domain harmonic measure and surface measure on the boundary are mutually absolutly continuous. Studying the (global) regularity properties of the solutions requires a good control on the boundary behavior of solutions, (which in turn presupposes knowledge on the regularity properties of solutions in the interior. Essential results in this direction were achieved in the works of De Giorgi, [6], Nash, [7], and Moser, [8], who established the Hölder continuity of solutions). The results of Dahlberg concerned the Laplacian. Flattening the boundary, i.e. mapping the boundary of the domain locally onto the boundary of a half-space, shows that there is a relation between the coefficients of PDEs and the regularity of the domain. So it can be seen that studying the Laplacian in a Lipschitz domain can be achived as studying a variable coefficient operator in the unit ball. The boundary value problem theory for variable coefficient operators (in divergence form) in a Lipschitz domain can be studied as pertubations of the Laplacian, [10] and [11]. Moreover, it is possible to extend most of the theory for the class of Lipschitz domains to the non-tangentially accessible domains, (NTA domains), see [12].

Naturally, there is a correspondence between the homogeneous problem with inhomogeneous boundary data, i.e. the boundary value problem (0.1) considered above, and the inhomogeneous problem with homogeneous boundary data, i.e. the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{0.2}$$

The last situation is the situation dealt with in this thesis. The solution to this problem is given by the Green potential of f, $Gf(x) = \int_\Omega G(x,y)f(y)\,dy$, where $G(\cdot,\cdot)$ is the Green's function for the Laplacian in Ω. The fundamental regularity result on the Green potential in a Lipschitz domain is the following result concerning the first order derivatives, see [13].

Theorem 0.4. *Let $D \subset \mathbf{R}^n$ be a Lipschitz domain and set $p_2 = 4/3, p_n = 3n(n+3)^{-1}$ for $n \geq 3$. There is a number $\epsilon = \epsilon(D) > 0$ such that if $1 < p < p_n + \epsilon$ and q is given by $1/q = 1/p - 1/n$ then*

$$\Big(\int_D |\nabla G f|^q dx\Big)^{1/q} \leq C\Big(\int_D |f|^p dx\Big)^{1/p}$$

where C only depends on p and D. Also, there is a constant C only depending on D such that

$$|\{x \in D : \nabla G f > \lambda\}| \leq C\big(\lambda^{-1}\int_D |f| dx\big)^{\frac{n}{n-1}},$$

where $|E|$ denotes the Lebesgue measure of a set E, and Gf is the Green potential of f. □

Addressing the question of higher order regularity it is well-known that the presence of inwardly directed corners on the boundary, forces the second order derivatives of the solution to behave badly, i.e. they are not integrable in general. The situation is favourable for a domain with a boundary with only outwardly directed singularities on the boundary. A particular case of interest is the case of a convex domain which is crucial when working on numerically estimating the solution, i.e. using (convex) triangulations. Surprisingly enough, it was not until 1964 that Kadlec, [14], showed that for Ω smooth and convex one has for all $f \in L^2(\Omega)$ that the corresponding solution u with zero boundary data and fulfilling $\Delta u = f$, i.e. $u = Gf$, obeys the estimate

$$\|\nabla_2 u\|_2 \leq C\|f\|_2, \tag{0.3}$$

with a constant C not depending on the smoothness of the domain. Thus, it follows by approximating with smooth convex domains that the same result is valid for general (bounded) convex domains.

Having the result of Kadlec above the question of L^p-integrability, $p \neq 2$, of the second order derivatives arises. It is an unpublished result of B. Dahlberg, G. Verchota and T. Wolff, [15], that if $f \in L^2(\Omega) \cap L^1(\Omega)$ then $\nabla_2(Gf)$ satisfies a weak-type $(1,1)$ estimate. Consequently, interpolation of this estimate and the classical L^2-estimate (0.3) above, yields

$$\|\nabla_2 u\|_p \leq C\|f\|_p,$$

for $1 < p \leq 2$. The method of proof in the weak-type $(1,1)$ estimate relies on estimates of singular integrals using the Hörmander condition and is achieved through estimates of the (derivatives of the) Green's function, using the maximum principle. Hence the method of proof is not suitable for other types of boundary conditions, e.g. Neumann condition. A sought after method applicable not only to the Dirichlet problem (for the Laplacian) was found by V. Adolfsson [16] and V. Adolfsson and D. Jerison [17]. It relies on estimates for solutions with data in atomic H^1, replacing L^1. The main result in [16] for the Dirichlet problem of the Laplacian is,

Theorem 0.5. *Suppose D is a convex domain above a Lipschitz graph in $\mathbf{R}^n$, i. e. $D = \{x_n > \phi(x')\}$ where ϕ is a convex Lipschitz function with Lipschitz constant bounded by M and $x' \in \mathbf{R}^{n-1}$. Let G be the Green function for D and let the Green potential for $f \in L^p(D)$, $1 < p \leq 2$, be denoted by Gf. Then we have that*

$$\int_D |\nabla_2 Gf|^p dx \leq c \int_D |f|^p dx,$$

where ∇_2 denotes the second order derivatives and the constant c only depends on the Lipschitz constant M. □

Subsequently S. Fromm in [18] reproved the result of Dahlberg-Verchota-Wolff. It was also shown that the estimate is not valid for any $p > 2$ and that there are even higher order regularity in some cases, measured in terms of potential spaces. The result of Dahlberg-Verchota-Wolff proved in [18] is

Theorem 0.6. *Let $f \in L^1(\Omega) \cup L^2(\Omega)$, where $\Omega \subseteq \mathbf{R}^n$ is bounded and convex. Then for all $\lambda > 0$,*

$$|\{x \in \Omega : |\nabla^2(Gf)(x)| > \lambda\}| \leq \frac{c}{\lambda}\|f\|_{L^1(\Omega)}.$$

□

This thesis is about extending the results for the Laplacian cited above to more general second order elliptic divergence form opertors. For such operators with Lipschitz coefficients we have the analog of Kadlec's result for the Laplacian (0.3), i.e. the L^2-estimate in the case of the Dirichlet problem in a convex domain, see [19]. We will here prove that the weak-type $(1,1)$ estimate is also valid in this situation, consequently obtaining the esimate for the second order derivatives in the range $1 < p \leq 2$. Furthermore, we will prove this result not only by a straightforward Cacciopoli or energy estimate technique but also using techniques relying on representation formulas. This will be helpful when considering potential space estimates and is of independent interest in the case of higher regularity of the coefficients.

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L^p INTEGRABILITY OF THE SECOND ORDER DERIVATIVES OF GREEN POTENTIALS IN CONVEX DOMAINS

1. Introduction

The purpose of this paper is to investigate the regularity of the second order derivatives of the solution to the boundary value problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ \gamma(u) = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

with $f \in L^p(\Omega)$, for $1 < p \leq 2$. Here γ is the trace operator and the divergence form operator

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) \tag{1.2}$$

with $a_{ij} = a_{ji}$, satisfies a uniform ellipticity condition. It is to say that there is some $\lambda \geq 1$ such that for all $\xi \in \mathbf{R}^n$

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \qquad \text{for all } x \in \Omega. \tag{1.3}$$

We also assume that $a_{ij}(x) \in L^\infty(\Omega)$.

It is well-known that the second order derivatives are not generally in L^1 in case the boundary of Ω has low regularity and inward corners, e.g. [8]. We will here investigate the case of Ω convex. When $L = -\Delta$ and Ω convex, the L^p-estimates, $1 < p \leq 2$,

$$\|\nabla^2 u\|_p \leq C\|\Delta u\|_p \tag{1.4}$$

have been obtained in [1] and [4]. Those estimates are obtained from interpolation of the classical estimate for $p = 2$, due to Kadlec [9], and an atomic H^1 estimate in [1], and a weak type (1,1) estimate, (an unpublished result of B. Dahlberg, G. Verchota and T. Wolff), proved in [4]. The weak type (1,1) estimate gives a better control of the constant C in the estimate (1.4) and also gives an atomic H^1 estimate. However, the advantage of the atomic H^1 estimate is that it is also applicable to other boundary value problems, e.g. the Neuman problem $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0$, see [1] and [2]. The weak type estimate relies on estimates of the Green function using the maximum principle and is therefore not suitable for the Neuman problem. In the case of a general divergence form elliptic L with Lipschitz coefficients and a convex domain Ω the main estimate, $p = 2$, is still valid, [9], see e.g. [6], and the following theorem is well-known.

Theorem 1.1. *Suppose that $\Omega \subset R^n$ is open, bounded and convex and assume that the elliptic divergence form operator L in (1.2) has coefficients $a_{ij} \in C^{0,1}(\Omega)$. Then there is a unique solution u in the Sobolev space $W^{2,2}(\Omega)$, to the boundary value problem (1.1) such that*

$$\|\nabla^2 u\|_2 \leq C\|Lu\|_2, \tag{1.5}$$

where the constant C depends only on the diameter of Ω and the Lipschitz constants of the coefficients a_{ij}.

Remark 1.2. We here use the standard notation $W^{k,p}(\Omega)$ to mean the space of functions having derivatives in $L^p(\Omega)$, of order $\ell \leq k$.

In this paper we show that the weak type (1,1) estimate is also valid in this situation. The solution to (1.1) is given as the Green potential of f,

$$Gf(x) = \int_\Omega G(x,y)f(y)dy,$$

where $G(\cdot,\cdot)$ is the Green function in Ω for the operator L. The existence of a (unique) continuous, symmetric Green function is well known, even for L^∞ coefficients; see [7]. We study the L^p-properties of the operator

$$f \longmapsto \nabla^2 Gf,$$

and show the following theorem, the main result of the paper.

Theorem 1.3. *Let $f \in L^1(\Omega) \cap L^2(\Omega)$ where $\Omega \subset R^n$ is open, bounded and convex and let G be the Green function for the elliptic divergence form operator L in (1.2). If the coefficients of L are Lipschitz continuous, then*

$$|\{x \in \Omega : |\nabla^2(Gf)(x)| > \lambda\}| \leq \frac{C}{\lambda} \parallel f \parallel_{L^1(\Omega)} \tag{1.6}$$

for all $\lambda > 0$.

Interpolation of the estimates obtained in the two theorems above, Theorem 1.1 and Theorem 1.3, gives in a standard fashion, see e.g. [1] or [4], the following result.

Theorem 1.4. *Suppose that $\Omega \subset R^n$ is open, bounded and convex. Let L be the elliptic divergence form operator in (1.2) and suppose L has coefficients $a_{ij} \in C^{0,1}(\Omega)$. Assume $1 < p \leq 2$. Then there is a unique solution to the boundary value problem (1.1), such that*

$$\|\nabla^2 u\|_p \leq C_p\|Lu\|_p. \tag{1.7}$$

The proof of Theorem 1.3 rests on estimating a singular integral using the Hörmander condition. We have that

$$\nabla^2(Gf)(x) = \int_\Omega \nabla_x^2 G(x,y) f(y)\, dy.$$

The Hörmander condition to verify, implying the weak $(1,1)$ property, is that

$$\int_{|x-y_0|\geq 5r} |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)|\, dx \leq C$$

for $|y-y_0| \leq r$ with C independent of $y_0 \in \mathbf{R}^n$ and $r > 0$. To verify the condition the domain of integration is split into two parts; the 'near' and the 'far away' parts, defining two integrals to be estimated. The proper estimation of the second integral follows from a use of the classical result Theorem 1.1, i.e. the case $p = 2$. To estimate the first integral we use properties of solutions to $Lu = 0$ where L is a second order divergence form operator as in (1.2). Specifically, $\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)$ can be viewed as $\nabla_x^2 \nabla_y G(x,\tilde{y})$. Noting that $\nabla_y G(\cdot,\tilde{y})$ is a solution of $Lu = 0$ in $\Omega \setminus \{\tilde{y}\}$ we need to estimate the second order derivatives of a solution, in the L^1-norm. Such a result can be proved rather painlessly using the observation that, in case the coefficients of the operator admits differentiation, i.e. are Lipschitz continuous, the derivatives of a solution will also fulfill an elliptic divergence form equation with a zeroth order term being of divergence form. Thus there should be a Cacciopoli type estimate; which is the content of the next result:

Lemma 1.5. *Suppose L in (1.2) is an elliptic divergence form operator with Lipschitz coefficients a_{ij}. If u is a solution of $Lu = 0$ in Ω, then for $B_{2d}(x_0) \subset\subset \Omega$, i.e. compactly contained, we have*

$$\int_{B_d(x_0)} |\nabla^2 u|^2 dx \leq C d^{-2} \int_{B_{2d}(x_0)} |\nabla u|^2 dx. \tag{1.8}$$

Remark 1.6. We take u a solution to $Lu = 0$, where L is given as in (1.2), to mean that u is a generalized or weak solution, i.e. $u \in W^{1,2}_{\text{loc}}(\Omega)$ and satisfies, using the customary summation convention,

$$\int_\Omega a_{i,j} D_i u(x) D_j \varphi(x)\, dx = 0,$$

for all $\varphi \in C_0^\infty(\Omega)$. It is a classical result that a solution is locally Hölder continuous in Ω even if the coefficients are just in $L^\infty(\Omega)$, see e.g. [10, Corollary 1.1.10]. If the domain is regular for L, there exists a unique (generalised) solution to the classical Dirichlet problem, i.e. a function $u \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ such that for given $f \in C(\partial\Omega)$

$$\begin{cases} Lu = 0 & \text{in} \quad \Omega \\ \gamma(u) = f & \text{on} \quad \partial\Omega. \end{cases}$$

Combining these facts above, we see that if $a_{ij} \in C^{0,1}$ so that

$$0 = D_i(a_{ij} D_j u) = a_{ij} D_i D_j u + D_i a_{ij} D_j u, \ \text{a.e.}$$

we have from [5, Cor. 9.18], that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $1 < p < \infty$. In particular $G(\cdot, y)$ for fixed $y \in \Omega$ is in $W^{2,p}_{\mathrm{loc}}(\Omega \setminus \{y\})$. Note also that for $v = D_k u$ we have that $D_i v \in L^p$ for any p large enough. Thus, $v \in C^{\lambda}(\Omega)$ for some $\lambda > 0$ by the Sobolev imbedding theorem. Furthermore, if the coefficients are even more regular, $a_{ij} \in C^{1,\alpha}(\Omega)$, then $u \in C^{2,\alpha}(\Omega)$, see [5, Thm. 9.19].

For many questions of regularity the technique of representation formulas are more informative. In fact the main part of the present paper is the presentation of such techniques. The aim is to facilitate an eventual study of these questions in a perspective of higher regularity and in this context the representation formulas are helpful. The use of representation formulas has a long history, being the original technique asserting existence and other properties of solutions. They have been used to obtain pointwise estimates of soluions and its derivatives. We will apply the technique in [7] to higher order derivatives which is possible if the coefficients are regular enough, i.e. not just that the coefficients are Lipschitz continuous so that the gradient of the coefficients are bounded, but that the gradient of the coefficients also fulfils an additional smoothness condition. For these types of results it is well-known that a Dini condition should be the right requirement. The operator L is said to satisfy a gradient Dini condition in Ω if the coefficients are in $C^1(\Omega)$ and has gradients that satisfy a Dini condition, i.e.

$$|\nabla a_{ij}(x) - \nabla a_{ij}(y)| \le \omega(|x-y|) \qquad \text{for any } x, y \in \Omega, \tag{1.9}$$

where $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ is non-decreasing and satisfies

$$\omega(2t) \le K_0\, \omega(t) \qquad \text{for some } K_0 > 0 \text{ and all } t > 0,$$

and

$$\int_0 \frac{\omega(t)}{t}\, dt < \infty.$$

Note that in this case ∇a_{ij} is uniformly continuous on Ω so it has a unique continuous extension to $\overline{\Omega}$ and consequently $\nabla a_{ij} \in L^\infty(\Omega)$. Since our main focus is on representation formulas in the less regular cases we will only consider the smooth cases for $C^{1,\alpha}$ coefficients. Note that the gradient of the coefficients satisfy a Dini condition in this case. We prove the following result through the use of representation formulas.

Theorem 1.7. *Suppose $u(x)$ is a solution of $Lu = 0$ in Ω, with coefficients in $C^{1,\alpha}(\Omega)$. Then there is a constant k such that for all $x \in \Omega$*

$$|\nabla^2 u(x)| \leq k d^{-1}(x) \sup_{\Omega} |\nabla u|, \tag{1.10}$$

where $k = k(\lambda, \sup \nabla a_{ij})$, and $d(\cdot)$ denotes the distance to the boundary $\partial\Omega$.

In particular we get estimates for the Green function of an elliptic divergence form operator with coefficients having Hölder continuous derivatives. Well-known estimates for Green functions state that if G is the Green function in Ω for a constant coefficient divergence form operator L as in (1.2) and if $\Omega \subseteq \mathbf{R}^n, n \geq 3$, is convex but not necessarily bounded, then for $x, y \in \Omega$

$$\begin{cases} (i) & G(x,y) \leq K|x-y|^{2-n} \\ (ii) & G(x,y) \leq Kd(x)|x-y|^{1-n} \\ (iii) & G(x,y) \leq Kd(x)d(y)|x-y|^{-n} \\ (iv) & |\nabla_x G(x,y)| \leq K\ |x-y|^{1-n} \\ (v) & |\nabla_x G(x,y)| \leq Kd(y)\ |x-y|^{-n} \\ (vi) & |\nabla_x \nabla_y G(x,y)| \leq K\ |x-y|^{-n} \\ (vii) & |\nabla_x^2 \nabla_y G(x,y)| \leq \frac{K\ |x-y|^{-n}}{min(|x-y|,d(x))}. \end{cases} \tag{1.11}$$

Here $d(\cdot)$ denotes the distance to the boundary $\partial\Omega$ and K is a constant that depends on the dimension n and the ellipticity constant λ, but not on Ω, see e.g. [4]. Furthermore, if L has coefficients a_{ij} fulfilling a Dini condition, (as for the gradient in (1.9)), and Ω is a bounded domain obeying a uniform external sphere condition, in particular if Ω is convex, then the estimates $(i) - (vi)$ of (1.11) are valid with a further dependence on the constant K, see [7]. The esimate of the following theorem, Theorem 1.8, complements the results in [7] and [4]. Specifically it shows that also the estimate (vii) in (1.11) is valid for domains with an outer uniform sphere condition and operators with coefficients whose gradient is in $C^{\alpha}(\Omega)$. Thus, we get immediately from Theorem 1.7 and the results in [7], the following result.

Theorem 1.8. *Suppose $G(x,y)$ is the Green function in Ω for an elliptic operator L as in (1.2), with coefficients a_{ij} in $C^{1,\alpha}(\Omega)$. Suppose*

$\Omega \subseteq \mathbf{R}^n, n \geq 3$ *is a bounded domain obeying a uniform external sphere condition. Then,*

$$|\nabla_x^2 \nabla_y G(x,y)| \leq C|x-y|^{-n}/\min(|x-y|, d(x)) \tag{1.12}$$

for all $x, y \in \Omega$.

In case Ω is convex the coefficients in the case of the last theorem are Lipschitz continuous and hence Theorem 1.3 applies. The theorem, Theorem 1.3, can also, by inspecting the proof, easily be derived directly from the estimate of Theorem 1.8, giving a better control of the constant.

As is well-known, it is not possible to obtain the estimate of Theorem 1.7 in case of the coefficients being merely Lipschitz continuous. As to the question of the applicability of the representation formula approach in this case, the coefficients being merely Lipschitz continuous, we obtain using that method the following result, Lemma 1.9 below. Note that Lemma 1.5 trivially implies Lemma 1.9; the point is that Lemma 1.9 is proved using the technique of representation formulas.

Lemma 1.9. *Suppose that* u *is a solution of* $Lu = 0$ *in* Ω*, where* L *is an elliptic divergence form operator with Lipschitz coefficients. Let* $B_d(x_0) \subset\subset \Omega$*. Then*

$$\int_{B_d(x_0)} |\nabla^2 u| dx \leq C d^{n-1} \sup_{B_{2d}(x_0)} |\nabla u|. \tag{1.13}$$

The plan of the paper is as follows. The proof of Theorem 1.3 is in Section 2. We will follow the proof given in [4] for the case $L = -\Delta$, and suitably modify it as to apply to more general divergence form operators. The proof relies on the L^2-result, Theorem 1.1, the Hörmander condition for singular integral operators and estimates for the third order mixed derivatives of the Green function. We will give an estimate of the second derivatives of the solution in terms of the first derivatives by a) deriving a Cacciopoli type estimate in Section 3 in case the coefficients are Lipschitz continuous, b) scouting out in Section 4, by standard representation formulas, a pointwise Schauder type estimate when the coefficients are in $C^{1,\alpha}(\Omega)$ and, c) in Section 5 generalising to the non-smooth case, i.e. Lipschitz coefficient case, the representation formula technique used in b). The representation formula technique employed in Section 4 uses in the weak formulation of a solution, a test function involving the Green function in a ball for an operator with constant coefficients, the coefficients being the coefficients of the divergence form operator L evaluated at the center of the ball. In case the coefficients of L are of higher regularity one obtains pointwise bounds for the second order derivatives of solutions, Theorem 1.7, proved in Section 4. The immediate consequence, of this result applied to the Green function, Theorem 1.8, is proved in the same section.

The techniques used for many of the estimates rely on a careful study of the Green function in a ball for a constant coefficient operator. This really amounts to a study of the Green function for the Laplacian in a ball. The explicit formulas for this function and its derivatives are completely elementary but often not that readily found in the litterature. For the convenience of the reader and our own reference we here state some of the formulas needed in the sequel.

Let $F(x,y)$ be the Green function for the Laplacian in the ball $B = B_d(x_0)$. Then

$$F(x,y) = \frac{1}{|y-x|^{n-2}} - \Big(\frac{d}{|x-x_0|}\Big)^{n-2} \frac{1}{|y-x^*|^{n-2}},$$

where x^* is the inverse of x with respect to ∂B, i.e. $x^* = x_0 + \frac{d^2}{|x-x_0|^2}(x - x_0)$. Now

$$\frac{\partial}{\partial x_j}(x_i^*) = -2d^2 \frac{(x-x_0)_j (x-x_0)_i}{|x-x_0|^4} + \frac{d^2}{|x-x_0|^2}\delta_{ij},$$

so that

$$\begin{aligned} \frac{\partial}{\partial x_j}(|y-x^*|^{2-n}) &= \frac{(2-n)d^2}{|x-x_0|^2}|y-x^*|^{-n} \\ &\times \Big((\sum_{i=1}^{n} 2(y-x^*)_i (x-x_0)_i) \frac{(x-x_0)_j}{|x-x_0|^2} - (y-x^*)_j \Big). \end{aligned}$$

Since

$$\sum_{i=1}^{n} (y-x^*)_i (x-x_0)_i = \langle y-x_0, x-x_0 \rangle - d^2,$$

we get

$$\begin{aligned} \frac{\partial}{\partial x_j}(|y-x^*|^{2-n}) &= \frac{(2-n)d^2}{|x-x_0|^2}|y-x^*|^{-n} \\ &\times \Big(2(\langle y-x_0, x-x_0 \rangle - d^2) \frac{(x-x_0)_j}{|x-x_0|^2} - (y-x^*)_j \Big). \end{aligned}$$

Thus, letting D_j be differentiation with respect to the variable x we get

$$\begin{aligned}
D_jF(x,y) &= (n-2)|y-x|^{-n}(y-x)_j \\
&\quad + d^{n-2}(n-2)\frac{(x-x_0)_j}{|x-x_0|^n}\frac{1}{|y-x^*|^{n-2}} \\
&\quad + (n-2)\Big(\frac{d}{|x-x_0|}\Big)^n\frac{1}{|y-x^*|^n} \\
&\quad \times \Big(2(\langle y-x_0, x-x_0\rangle - d^2)\frac{(x-x_0)_j}{|x-x_0|^2} - (y-x^*)_j\Big).
\end{aligned}$$

In particular, for $\vec{e}\in S^{n-1}$ and $0<h$ we get

$$\begin{aligned}
&D_jF(x,x_0+h\vec{e}) - D_jF(x,x_0) = (n-2)\Big(\frac{(x_0+h\vec{e}-x)_j}{|x_0+h\vec{e}-x|^n} - \frac{(x_0-x)_j}{|x_0-x|^n}\Big) \\
&\quad + (n-2)d^{n-2}\frac{(x-x_0)_j}{|x-x_0|^n} \\
&\quad \times \Big(\frac{1}{|h\vec{e}-\frac{d^2}{|x-x_0|^2}(x-x_0)|^{n-2}} - \frac{1}{|\frac{d^2}{|x-x_0|^2}(x-x_0)|^{n-2}}\Big) \\
&\quad + (n-2)(\frac{d}{|x-x_0|})^n \\
&\quad \times \Big(\frac{1}{|h\vec{e}-\frac{d^2}{|x-x_0|^2}(x-x_0)|^n}(2\langle h\vec{e}, x-x_0\rangle\frac{(x-x_0)_j}{|x-x_0|^2} - h\vec{e}_j) \\
&\quad - (\frac{1}{|h\vec{e}-\frac{d^2}{|x-x_0|^2}(x-x_0)|^n} - \frac{1}{|\frac{d^2}{|x-x_0|^2}(x-x_0)|^n})\frac{d^2}{|x-x_0|^2}(x-x_0)_j\Big).
\end{aligned}$$

Put $G_\alpha(z) = \frac{1}{|z-\frac{d^2}{|x-x_0|^2}(x-x_0)|^{n-\alpha}}$. Then,

$$\frac{\partial G_\alpha}{\partial z_k}(z) = -(n-\alpha)\frac{(z-\frac{d^2}{|x-x_0|^2}(x-x_0))_k}{|z-\frac{d^2}{|x-x_0|^2}(x-x_0)|^{n-\alpha+2}}.$$

Consequently,

$$|G_\alpha(h\vec{e}) - G_\alpha(0)| = |\langle \nabla G_\alpha(\theta_\alpha h\vec{e}), h\vec{e}\rangle| \leq \frac{|n-\alpha|}{|\theta_\alpha h\vec{e} - \frac{d^2(x-x_0)}{|x-x_0|^2}|^{n-\alpha+1}}|h|,$$

where $\theta_\alpha = \theta_\alpha(h)$, $0 < \theta_\alpha < 1$, and

$$
\begin{aligned}
D_jF(x, x_0 + h\vec{e}) &- D_jF(x, x_0) \\
&= (n-2)\Big(\frac{(x_0 + h\vec{e} - x)_j}{|x_0 + h\vec{e} - x|^n} - \frac{(x_0 - x)_j}{|x_0 - x|^n}\Big) \\
&+ (n-2)d^{n-2}\frac{(x - x_0)_j}{|x - x_0|^n}\langle \nabla G_2(h\theta_2\vec{e}), h\vec{e}\rangle \\
&+ (n-2)\big(\frac{d}{|x - x_0|}\big)^n \\
&\times \Big(\frac{h}{|h\vec{e} - \frac{d^2}{|x-x_0|^2}(x - x_0)|^n}\big(2\langle \vec{e}, x - x_0\rangle \frac{(x - x_0)_j}{|x - x_0|^2} - \vec{e}_j\big) \\
&- \langle \nabla G_0(h\theta_0\vec{e}), h\vec{e}\rangle \frac{d^2}{|x - x_0|^2}(x - x_0)_j.\Big).
\end{aligned} \tag{1.14}
$$

Furthermore, for $0 \leq \theta \leq 1$

$$
\Big|\frac{d^2(x - x_0)}{|x - x_0|^2} - \theta h e\Big| \geq \frac{d^2}{|x - x_0|} - h,
$$

so that for $h \leq d/2$ and x such that $|x - x_0| \leq d$ we have

$$
\Big|\frac{d^2(x - x_0)}{|x - x_0|^2} - \theta h e\Big| \geq \frac{1}{2}\frac{d^2}{|x - x_0|}. \tag{1.15}
$$

Using the estimate (1.15) it is not hard to see from (1.14) that

$$
\begin{aligned}
\frac{1}{h}\Big(D_jF(x,x_0 + he) &- D_jF(x, x_0)\Big) \\
&= \frac{n-2}{h}\Big(\frac{(x_0 + he - x)_j}{|x_0 + he - x|^n} - \frac{(x_0 - x)_j}{|x_0 - x|^n}\Big) + C(d)\mathcal{O}(1),
\end{aligned} \tag{1.16}
$$

for $h \leq d/2$ and x such that $|x - x_0| \leq d$.

2. The main result

Proof of Theorem 1.3. Let

$$
f \longmapsto \nabla^2(Gf)(\cdot) = \int_\Omega \nabla_x^2 G(\cdot, y) f(y)\, dy \tag{2.1}
$$

While the Green function is a solution to an elliptic second order divergence form operator L as in (1.2) with Lipschitz continuous coefficients, it is in $W^{2,p}_{\mathrm{loc}}(\Omega \setminus \{y\})$ for any $1 < p < \infty$, see Remark 1.6. Thus, it is straightforward to see that the mapping (2.1) is well-defined. Moreover, from the estimates for the Green function, (1.11), and the discussion following there, we see that $\nabla_y G(\cdot, y)$ is also a solution to $Lu = 0$.

Hence also $\nabla_y G(\cdot, y)$ is in $W^{2,p}_{\mathrm{loc}}(\Omega \setminus \{y\})$ for any $1 < p < \infty$. We extend the operator to $\mathbf{R}^n$ by extending the kernel to be zero if x or $y \in \mathbf{R}^n \setminus \Omega$ and denote the extended operator by $\nabla^2 G$. The operator is bounded on $L^2(\mathbf{R}^n)$. To show the weak L^1 estimate we need only to show that the Hörmander condition is fulfilled, see e.g. [12]; it is to say that

$$\int_{|x-y_0|\geq 5r} |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)| dx \leq C$$

for $|y - y_0| \leq r$ with C independent of $y_0 \in \mathbf{R}^n$ and $r > 0$. To this end we split the integral into two parts $I = I_1 + I_2$ where

$$I_1 = \int_{\frac{1}{2}d(y_0)\geq|x-y_0|\geq 5r} |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)|\, dx,$$

and

$$I_2 = \int_{|x-y_0|\geq 5r,\ |x-y_0|>\frac{1}{2}d(y_0)} |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)|\, dx,$$

where $d(\cdot)$ denotes the distance to the boundary. If $y_0 \notin \Omega$ then $I_1 = 0$, since $\{x \in \Omega : |x - y_0| \leq \frac{1}{2}d(y_0)\} = \emptyset$. Let now $0 \leq \eta \leq 1$ and put $y' = \eta y + (1-\eta)y_0$. Then for x such that $5r \leq |x - y_0| \leq \frac{1}{2}d(y_0)$ we have that $|x - y'| + |y - y_0| \geq |x - y_0|$ and $|y - y_0| \leq r \leq \frac{1}{5}|x - y_0|$, which implies that

$$|x - y_0| \leq \frac{5}{4}|x - y'|.$$

Furthermore,

$$\begin{aligned} |x - y'| &\leq |x - y_0| + |y_0 - y'| \\ &\leq |x - y_0| + \eta|y_0 - y| \leq |x - y_0| + |y_0 - y|. \end{aligned}$$

Hence, $d(x) \geq \frac{1}{2}d(y_0)$ and $|x - y_0| \leq \frac{1}{2}d(y_0) \leq d(x)$ and also

$$|y_0 - y| \leq r \leq \frac{|x - y_0|}{5} \leq \frac{1}{5}d(x),$$

so that

$$|x - y'| \leq \frac{6}{5}d(x).$$

Writing a function as the integral of its derivative we obtain

$$\begin{aligned} &\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0) \\ &\qquad = \int_0^1 \langle \nabla_x^2 \nabla_y G(x, y_0 + t(y - y_0)), (y - y_0)\rangle\, dt. \end{aligned}$$

The line segment $\{y' = y_0 + t(y - y_0) :\ t \in [0,1]\}$ is contained in the closed ball $\overline{B_r(y_0)}$ since $|y - y_0| \leq r$. Let $D_j = B_{2^{j+3}r}(y_0) \setminus B_{2^{j+2}r}(y_0)$,

$j = 0, 1, 2, \ldots$. Then D_j can be covered by a finite number N of balls B_j^i, $i = 1, \ldots N$ of radii $2^j r$. Thus, using (1.8) and its direct consequence (1.13) together with the pointwise estimates for the Green function (1.11) and the discussion following there, we get for $y_0 \in \Omega$ and y such that $|y - y_0| \leq r$, that with $E = \{x : 5r \leq |x - y_0| < \frac{1}{2}d(y_0)\}$,

$$
\begin{aligned}
I_1 = \int_E & |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)|\, dx \\
& \leq \int_{\cup_{i,j} B_j^i} |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)|\, dx \\
& \leq \sum_{i,j} \int_{B_j^i} |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)|\, dx \\
& \leq \sum_{i,j} |y - y_0| \int_0^1 \int_{B_j^i} |\nabla_x^2 \nabla_y G(x, y_0 + t(y - y_0))|\, dx\, dt \\
& \leq C \sum_{i,j} |y - y_0| \int_0^1 \frac{1}{\operatorname{diam}(B_j^i)} (\operatorname{diam}(B_j^i))^n \qquad (2.2)\\
& \times \sup_{x \in 2B_j^i} |\nabla_x \nabla_y G(x, y_0 + t(y - y_0))|\, dt \\
& \leq C|y - y_0| \sum_j \frac{1}{2^{j+1} r} (2^{j+1} r)^n \frac{1}{(2^{j+1} r)^n} \\
& \leq C|y - y_0| \sum_j \frac{1}{2^{j+1} r} \leq C \sum_j \frac{1}{2^j} \leq C,
\end{aligned}
$$

where C is independent of r and y_0. Next we estimate I_2

$$
\begin{aligned}
I_2 & \leq \int_{|x-y_0| \geq 5r,\ |x-y_0| > \frac{1}{2}d(y_0)} |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)| dx \\
& \leq \int_{|x-y_0| \geq 5r,\ |x-y_0| > \frac{1}{2}d(y_0)} |\nabla_x^2 G(x,y)| dx \\
& + \int_{|x-y_0| > \frac{1}{2}d(y_0)} |\nabla_x^2 G(x,y_0)| dx.
\end{aligned}
$$

If $|x - y_0| \geq 5r$ and $|x - y_0| > \frac{1}{2}d(y_0)$, we have

- if $5r \leq \frac{1}{2}d(y_0)$, that

$$
\begin{aligned}
|x - y| \geq |x - y_0| - |y - y_0| & \geq \frac{1}{2}d(y_0) - r = \frac{1}{4}d(y_0) + \frac{1}{4}d(y_0) - r \\
& \geq \frac{1}{4}d(y_0) + \frac{5}{2}r - r,
\end{aligned}
$$

and since $d(y) \leq d(y_0) + r$ this leads to

$$|x-y| \geq \frac{1}{4}d(y_0) + \frac{5}{2}r - r \geq \frac{1}{4}d(y)$$

- if $5r > \frac{1}{2}d(y_0)$, that

$$\begin{aligned}|x-y| &\geq 4r = \frac{3}{2}r + \frac{5}{2}r \geq \frac{3}{2}r + \frac{1}{4}d(y_0)\\ &\geq \frac{3}{2}r + \frac{1}{4}d(y) - \frac{1}{4}r \geq \frac{1}{4}d(y).\end{aligned}$$

Thus,

$$\begin{aligned}I_2 &\leq \int_{|x-y|>\frac{1}{4}d(y)} |\nabla_x^2 G(x,y)|dx + \int_{|x-y_0|>\frac{1}{4}d(y_0)} |\nabla_x^2 G(x,y_0)|dx\\ &\leq 2\sup_{y\in\Omega} \int_{|x-y|>\frac{1}{4}d(y)} |\nabla_x^2 G(x,y)|dx,\end{aligned}$$

since if $y \notin \Omega$, then $\nabla_x^2 G(x,y) = 0$. We have to estimate

$$\int_{|x-y|>\frac{1}{4}d(y)} |\nabla_x^2 G(x,y)|dx.$$

Take $\phi \in C_0^\infty(\mathbf{R}^+)$ such that $0 \leq \phi \leq 1$, $\phi(t) = 0$ for $t \leq 1/2$ and $t \geq 4$, and such that $\phi(t) = 1$ for $1 \leq t \leq 2$. Put

$$\phi_j(x) = \phi(\frac{|x-y|}{2^j d(y)}).$$

Then,

$$\begin{aligned}\int_{|x-y|>\frac{1}{4}d(y)} |\nabla_x^2 G(x,y)|dx &\leq \sum_{j\geq -2} \int_\Omega |\nabla_x^2 \Big(\phi_j(x)G(x,y)\Big)|\, dx\\ &\leq C \sum_{j\geq -2} (2^j d(y))^{n/2} (\int_\Omega |\nabla_x^2 \Big(\phi_j(x)G(x,y)\Big)|^2)^{1/2}\, dx.\end{aligned} \tag{2.3}$$

From the classical L^2 result of Theorem 1.1 we have

$$\int_\Omega |\nabla_x^2 \Big(\phi_j(x)G(x,y)\Big)|^2\, dx \leq C \int_\Omega |L\Big(\phi_j(x)G(x,y)\Big)|^2\, dx, \tag{2.4}$$

while $\phi_j G(.,y) \in W^{2,2}(\Omega) \bigcap \overset{o}{W}{}^{2,1}(\Omega)$. Now,

$$\begin{aligned}L(\phi_j(x)G(x,y)) &= D_i(a_{ij}D_j(\phi_k G))\\ &= D_i((a_{ij}D_j G)\phi_k + ((a_{ij}D_j\phi_k)G)\\ &= a_{ij}D_j G D_i\phi_k + a_{ij}D_j\phi_k D_i G + D_i(a_{ij}D_j\phi_k)G\\ &= 2a_{ij}D_j G D_i\phi_k + D_i(a_{ij})D_j\phi_k G + a_{ij}D_i D_j\phi_k G.\end{aligned}$$

where we have used that $G(.,y)$ is a solution in $\Omega\setminus\{y\}$. Therefore,

(2.5)
$$\begin{aligned}\int_\Omega |L(\phi_j(x)G(.,y))|^2\,dx \le C(&\int_\Omega |\nabla\phi_j(x)|^2|\nabla_x G(x,y)|^2\,dx\\ &+\int_\Omega |\nabla^2\phi_j(x)|^2|G(x,y)|^2\,dx+\int_\Omega |\nabla\phi_j(x)|^2 G(x,y)^2\,dx\\ &\equiv C(I_a+I_b+I_c).\end{aligned}$$

Furthermore, the estimates

$$|\nabla^2\phi_j(x)|\le C(2^j d(y))^{-2}$$
$$|\nabla\phi_j(x)|\le C(2^j d(y))^{-1},$$

are straightforward, and using (1.11) gives that

$$|G(x,y)|\le Cd(y)|x-y|^{1-n}\le Cd(y)(2^j d(y))^{1-n},$$

and

$$|\nabla_x G(x,y)|\le Cd(y)|x-y|^{-n}\le Cd(y)(2^j d(y))^{-n},$$

for $x\in\operatorname{supp}(\phi_j)$. Consequently,

$$I_a\le C(d(y)(2^j d(y))^{-n}(2^j d(y))^{-1})^2(2^j d(y))^{-n}\le C(2^j)^{-2}(2^j d(y))^{-n},$$

and

$$I_b\le C((2^j d(y))^{-2}d(y)(2^j d(y))^{1-n})^2(2^j d(y))^n\le C(2^j)^{-2}(2^j d(y))^{-n}.$$

For I_c we note that since $D_i\phi_j G(.,y)\in\overset{o}{W}{}^{2,1}(\Omega)$,

$$\begin{aligned}\int_\Omega (D_i\phi_j G(\cdot,y))^2\,dx &\le C\operatorname{diam}(\Omega)\int_\Omega(\nabla_x(D_i\phi_j G(\cdot,y)))^2\,dx\\ &\le C\operatorname{diam}(\Omega)\Big(\int_\Omega (D_kD_i\phi_j G(\cdot,y))^2\,dx+\int_\Omega (D_i\phi_j D_k G(\cdot,y))^2\,dx\Big)\\ &\le C\operatorname{diam}(\Omega)\Big(\int_\Omega (\nabla^2\phi_j G(\cdot,y))^2\,dx+\int_\Omega |\nabla\phi_j|^2|\nabla_x G(\cdot,y)|^2\,dx\Big)\\ &\le C(I_a+I_b)\le C(2^j)^{-2}(2^j d(y))^{-n}\end{aligned}$$

Thus, from (2.3), (2.4) and (2.5) together with the estimates for I_a, I_b and I_c we finally get,

$$\begin{aligned}\int_{|x-y|>\frac14 d(y)}|\nabla_x^2 G(x,y)|\,dx &\le C\sum_{j\ge -2}(2^j d(y))^{n/2}((2^j)^{-2}(2^j d(y))^{-n})^{1/2}\\ &\le C\sum_{j\ge-2}(2^{-1})^j\le C<\infty.\end{aligned}$$

This finishes the estimate of I_2, and the proof of the theorem. $\square$

Remark 2.1. In case the coefficients are smoother than Lipschitz, i.e. $C^{1,\alpha}$ coefficients, we use the estimate (1.10) instead of (1.13). Of course in this case the coefficients are still Lipschitz continuous so the theorem just proved still remains applicable. However, the estimate based on (1.10) instead gives a better behavior of the coefficient. To be specific

$$|\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)| \le |y-y_0| \sup_{0\le\eta\le 1} |\nabla_y \nabla_x^2 G(x,y')|,$$

and from Theorem 1.7, (or Theorem 1.8), we have

$$|\nabla_x^2 \nabla_y G(x,y)| \le C|x-y|^{-n}/\min(|x-y|, d(x)).$$

Thus as in (2.2) we obtain

$$\begin{aligned}\int_{\frac{1}{2}d(y_0)\ge|x-y_0|\ge 5r} & |\nabla_x^2 G(x,y) - \nabla_x^2 G(x,y_0)|dx \\ &\le \int_{|x-y_0|\ge 5r} C|y-y_0||x-y_0|^{-n-1}dx \\ &\le Cr\int_{5r}^{\infty} \frac{s^{n-1}}{s^{n+1}}ds = Cr(\frac{1}{5r}) \le C,\end{aligned}$$

where C is independent of r. This finishes the estimate of I_1 in the case of regular coefficients.

3. A Cacciopoli type estimate

Since the derivative of the solution to the boundary value problem (1.1) also fulfils an essentially elliptic equation there should be a Cacciopoli type estimate, energy estimate, for the gradient of a solution. In this section we derive that result.

Proof of Lemma 1.5. We have by the (weak) solution property that

$$\int_\Omega a_{ij} D_i u D_j \psi dx = 0 \qquad \forall \psi \in C_c^\infty(\Omega).$$

Put

$$\psi = -D_k^{-h}(\eta^2 D_k^h u),$$

where D_k^h denotes the difference quotient

$$D_k^h \omega(x) = \frac{\omega(x+he_k) - \omega(x)}{h}, \qquad h \in \mathbf{R},\ h \ne 0,$$

and η is a cut off function with support in $B_{3d/2}(x_0)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_d(x_0)$ and $|\nabla \eta| \le C d^{-1}$. Then,

$$0 = \int_\Omega a_{ij} D_i u D_j [D_k^{-h}(\eta^2 D_k^h u)]\, dx,$$

and for $0 < |h| < \frac{1}{2}\operatorname{dist}(B_{2d}(x_0), \Omega)$ we use

$$\int_\Omega v D_k^{-h} w dx = -\int_\Omega w D_k^h v dx$$

together with $D_k^h(vw) = v^h D_k^h w + w D_k^h v$, where $v^h(x) \equiv v(x + he_k)$, to derive,

$$\begin{aligned}
0 &= \int_\Omega D_k^h(a_{ij} D_i u) D_j(\eta^2 D_k^h u)\, dx \\
&= \int_\Omega a_{ij}^h (D_k^h D_i u) D_j(\eta^2 D_k^h u)\, dx + \int_\Omega (D_k^h a_{ij}) D_i u D_j(\eta^2 D_k^h u)\, dx \\
&= \int_\Omega a_{ij}^h (D_k^h D_i u)(D_k^h u) 2\eta D_j \eta\, dx + \int_\Omega a_{ij}^h (D_k^h D_i u)(D_j D_k^h u)\eta^2\, dx \\
&+ \int_\Omega (D_k^h a_{ij})(D_i u)(D_j D_k^h u)\eta^2\, dx + \int_\Omega (D_k^h a_{ij}) D_i u (D_k^h u) 2\eta D_j \eta\, dx.
\end{aligned}$$

From this follows that

$$\begin{aligned}
&\int_\Omega a_{ij}^h D_i(D_k^h u) D_j(D_k^h u)\eta^2\, dx \\
&\quad = -\int_\Omega a_{ij}^h (D_k^h D_i u)(D_k^h u) 2\eta D_j \eta\, dx - \int_\Omega (D_k^h a_{ij}) D_i u (D_j D_k^h u)\eta^2\, dx \\
&\quad - \int_\Omega (D_k^h a_{ij}) D_i u (D_k^h u) 2\eta D_j \eta\, dx.
\end{aligned}$$

As a consequence of the ellipticity condition for the coefficients a_{ij} we obtain,

$$\begin{aligned}
&\lambda \int_\Omega \eta^2 |\nabla D_k^h u|^2\, dx \\
&\qquad \le C\Big[\int_\Omega \eta |D_k^h D_i u||D_j \eta||D_k^h u|\, dx + \int_\Omega \eta |D_j D_k^h u| \eta |D_i u|\, dx \\
&\qquad + \int_\Omega \eta |D_i u||D_j \eta||D_k^h u|\, dx\,\Big],
\end{aligned}$$

where the constant C only depends on the L^∞-norm of the coefficients $a_{i,j}$, and their Lipschitz constants. Repeated applications of the estimate

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

for $\varepsilon, a, b > 0$, leads to

$$\lambda \int_\Omega \eta^2 |D_k^h \nabla u|^2 \, dx$$
$$\leq \varepsilon \int_\Omega \eta^2 |D_k^h \nabla u|^2 \, dx + \frac{C}{\varepsilon} \int_\Omega |\nabla \eta|^2 |D_k^h u|^2 \, dx + \frac{C}{\epsilon} \int_\Omega \eta^2 |\nabla u|^2 \, dx$$
$$+ \int_\Omega |Du| \eta| |D\eta| |D_k^h u| \, dx,$$

and we can choose $\varepsilon = \lambda/2$ to arrive at

$$\int_\Omega \eta^2 |D_k^h \nabla u|^2 \, dx$$
$$\leq C[\int_\Omega \eta^2 |\nabla u|^2 \, dx + \int_\Omega |\nabla \eta|^2 |D_k^h u|^2 \, dx + \int_\Omega \eta |\nabla u| |\nabla \eta| |D_k^h u| \, dx\,],$$

where the constant depends only on the ellipticity constant λ, the L^∞-norm of the coefficients $a_{i,j}$, and their Lipschitz constants. Using the properties of η we thus conclude that

$$\int_{B_d(x_0)} |D_k^h \nabla u|^2 \, dx$$
$$\leq C[\int_{B_{3d/2}(x_0)} |\nabla u|^2 \, dx + d^{-2} \int_{B_{3d/2}(x_0)} |D_k^h u|^2 \, dx$$
$$+ d^{-1} \int_{B_{3d/2}(x_0)} |\nabla u| |D_k^h u| \, dx\,].$$

Since u is a weak solution we have that $u \in W^{1,2}_{\text{loc}}(\Omega)$, and consequently for all small enough h,

$$\int_{B_{3d/2}(x_0)} |D_k^h u|^2 \, dx \leq \|\nabla u\|_{L^2(B_{2d}(x_0))}.$$

Choosing d small we can keep only the term with the lowest power of d and we get

$$\int_{B_d(x_0)} |D_k^h \nabla u|^2 \, dx \leq C d^{-2} \int_{B_{2d}(x_0)} |\nabla u|^2 \, dx.$$

Consequently, see e.g. [3, section 5.8.2],

$$\int_{B_d(x_0)} |\nabla^2 u|^2 \, dx \leq C d^{-2} \int_{B_{2d}(x_0)} |\nabla u|^2 \, dx,$$

and this finishes the proof of the lemma. □

4. $C^{1,\alpha}$ COEFFICIENTS

In this section we prove Theorems 1.7 and 1.8. We begin with the proof of Theorem 1.8 which easily follows by Theorem 1.7 just from the fact that $\nabla_y G(\cdot, y)$ is a solution in $\Omega \setminus \{y\}$.

Proof of Theorem 1.8. Let $x, y \in \Omega$ be fixed but arbitrary. Assume that $|x-y| < d(x)$. Then $B \equiv B_{\frac{1}{10}\ |x-y|}(x) \subset\subset \Omega \backslash \{y\}$. Since $\nabla_y G(\cdot, y)$ is a solution in B we get from Theorem 1.7 and the estimate (vi) of (1.11), that

$$\begin{aligned} |\nabla_x^2 \nabla_y G(x,y)| &\leq k(\frac{1}{10}\ |x-y|)^{-1} \sup_{z \in B} |\nabla_x \nabla_y G(z,y)| \\ &\leq C \frac{1}{|x-y|} \sup_{z \in B} |z-y|^{-n} \\ &\leq C \frac{1}{|x-y|} |x-y|^{-n} = C \frac{|x-y|^{-n}}{\min(|x-y|, d(x))}\ , \end{aligned}$$

since $|z-y| \geq \frac{9}{10}|x-y|$ for all $z \in B$. If on the other hand $|x-y| \geq d(x)$ then $B \equiv B_{\frac{1}{10}\ d(x)} \subset\subset \Omega \backslash \{y\}$ and estimates as before give

$$\begin{aligned} |\nabla_x^2 \nabla_y G(x,y)| &\leq k(\frac{1}{10} d(x))^{-1} \sup_{z \in B} |\nabla_x \nabla_y G(z,y)| \\ &\leq C \frac{1}{d(x)} \sup_{z \in B} |z-y|^{-n} \\ &\leq C \frac{1}{d(x)} |x-y|^{-n} = C \frac{|x-y|^{-n}}{\min(|x-y|, d(x))}, \end{aligned}$$

since also in this case $|z-y| \geq \frac{9}{10}|x-y|$. □

The result about the Green function just obtained followed easily from Theorem 1.7 which we now prove. The main idea of the proof is to realize that the gradient also fulfils an elliptic equation. This in turn makes it possible to obtain Schauder estimates since the coefficients are regular. Thus one obtains the counterpart of the Cacciopoli estimate having the second gradient estimated in terms of the gradient, but now in L^∞-norm. This is achieved by choosing a suitable test function involving the Green function for a constant coefficient operator. The Green functions of these operators have well established estimates. Using a Korn pertubation proceedure we will get a term

$$\int_{B_d(x_0)} a_0^{ij} D_i \psi D_j F(x,y) dx,$$

where $a_0^{ij} = a_{ij}(x_0)$, F the Green function of the constant coefficint operator with coefficients a_0^{ij}. Thus this term equals $\psi(y)$ and this test function will involve derivatives of u. Consequently we obtain a pointwise representation formula for the derivatives of u. Taking the derivative of this relation, which is possible because of the estimates of the Green function for a constant coefficient operator and its derivatives, gives a pointwise representation formula for the second derivatives of the solution u.

Proof of Theorem 1.7. We note that since the coefficients are regular a solution u to $Lu = 0$ is in $C^2_{loc}(\Omega)$, see Remark 1.6; in fact $u \in C^{2,\alpha}_{loc}(\Omega)$. By considering a slightly smaller domain we may assume

$$\sup_{x\in\Omega, 1\le i,j\le n} \delta(x)|D_iD_ju(x)| \equiv M_1 < \infty, \tag{4.1}$$

where $\delta(\cdot)$ denotes the distance to the boundary. Put

$$M_0 = \sup_{x\in\Omega, 1\le i\le n} |D_iu(x)|,$$

and choose $x_0 \in \Omega$ such that

$$\delta(x_0) \sup_{1\le i,j\le n} |D_iD_ju(x_0)| > \frac{M_1}{2}. \tag{4.2}$$

Let d be such that

$$d \le \frac{\delta(x_0)}{2}. \tag{4.3}$$

While u is a weak solution we have that

$$\int_\Omega a^{ij}D_iuD_j\phi = 0$$

for all test functions $\phi \in C_0^\infty(\Omega)$. Take now $\phi = D_k\psi$, where $\psi \in C_0^\infty(\Omega)$ and k with $1 \le k \le n$ is fixed. An integration by parts, using the summation convention, leads to

(4.4)

$$\begin{aligned} 0 &= \int_\Omega D_ka^{ij}(x)D_iu(x)D_j\psi(x)\,dx + \int_\Omega a^{ij}(x)D_kD_iu(x)D_j\psi(x)\,dx \\ &= \int_\Omega D_ka^{ij}(x)D_iu(x)D_j\psi(x)\,dx \\ &\quad + \int_\Omega (a^{ij}(x) - a_0^{ij})D_kD_iu(x)D_j\psi(x)\,dx \\ &\quad + \int_\Omega a_0^{ij}D_kD_iu(x)D_j\psi(x)\,dx. \end{aligned}$$

Pick $\eta \in C_0^\infty(B_d(x_0))$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{d/2}(x_0)$, $|\nabla\eta| \le kd^{-1}$ and $|\nabla^2\eta| \le kd^{-2}$. Let $F(x,y)$ be the Green function in the ball $B_{4d}(x_0)$ of the operator with constant coefficients $a_0^{ij} = a^{ij}(x_0)$.

Taking $\psi(x) = \eta(x)F(x,y)$ in (4.4) above and putting $v(x) = D_k u(x)$ we obtain that the third integral on the right hand side equals

$$\int_\Omega a_0^{ij} D_i v(x)(D_j\eta)(x)F(x,y)\,dx + \int_\Omega a_0^{ij} D_i v(x)\eta(x)(D_jF)(x,y)\,dx,$$

where for the last integral we find that it equals

$$\eta(y)v(y) - \int_{\mathbf{R}^n} a_0^{ij} v(x) D_i\eta(x) D_j F(x,y)\,dx.$$

Note that because of the regularity properties of u, i.e. $\nabla^2 u \in L^\infty_{\mathrm{loc}}$, the function $\psi(x) = \eta(x)F(x,y)$ is a suitable test function even in the case of F being the Green function for a general operator, see for example [7, Theorem 1.1]. Collecting the different expressions and inserting into (4.4) we obtain

$$\begin{aligned} \eta(y)v(y) = &\int_\Omega a_{ij}^0 v(x)(D_i\eta)(x)D_jF(x,y)\,dx \\ &- \int_\Omega a_{ij}^0 D_i v(x)(D_j\eta)(x)F(x,y)\,dx \\ &- \int_\Omega D_k a_{ij}(x) D_i u(x) D_j(\eta(x)F(x,y))\,dx \\ &- \int_\Omega (a_{ij}(x) - a_{ij}^0) D_i v(x) D_j(\eta(x)F(x,y))\,dx. \end{aligned} \tag{4.5}$$

We integrate by parts in the second term on the right hand side which then equals

$$\int_\Omega a_{ij}^0 v(x)(D_iD_j\eta)(x)F(x,y)\,dx + \int_\Omega a_{ij}^0 v(x)(D_j\eta)(x)D_iF(x,y)\,dx.$$

The third term on the right hand side of (4.5) we rewrite as

$$\begin{aligned} -\int_\Omega (D_k a_{ij} - D_k a_{ij}^0) D_i u(x) D_j(\eta(x)F(x,y))\,dx& \\ + \int_\Omega D_k a_{ij}^0 D_j D_i u(x)\eta(x)F(x,y)\,dx.& \end{aligned}$$

Inserting the expressions just obtained into (4.5) we get

$$
\begin{aligned}
\eta(y)v(y) = & \int_\Omega a^0_{ij} v(x)(D_i\eta)(x) D_j F(x,y)\,dx \\
& + \int_\Omega a^0_{ij} v(x)(D_iD_j\eta)(x) F(x,y)\,dx \\
& + \int_\Omega a^0_{ij} v(x)(D_j\eta)(x) D_i F(x,y)\,dx \\
& - \int_\Omega (D_k a_{ij} - D_k a^0_{ij}) D_i u(x) D_j(\eta(x)F(x,y))\,dx \\
& + \int_\Omega D_k a^0_{ij} D_j D_i u(x)\eta(x)F(x,y)\,dx \\
& - \int_\Omega (a_{ij}(x) - a^0_{ij}) D_i v(x) D_j(\eta(x)F(x,y))\,dx.
\end{aligned} \tag{4.6}
$$

We would next like to differentiate the last expression with respect to y. However, moving the differentiation inside the integral is not unproblematic. In the case of the first, third, fourth and sixth integral on the right hand side of (4.6) this will lead to potentially singular integrals since $\nabla\nabla_y F(x,y) \sim \frac{1}{|x-y|^n}$. In the first and third integrals this will not cause a problem for y close x_0 because of the factor $\nabla\eta$, remembering that $\eta = 1$ in $B_{d/2}(x_0)$. In the fourth and sixth integrals, computing the derivative in x of the product $\eta(x)F(x,y)$ we get terms that can be treated in the same way as the first and third integrals. The remaining terms are both of the form

$$
\int_{B_d(x_0)} (A(x) - A(x_0))B(x) D_j F(x,y)\,dx,
$$

where B is bounded with support in $B_d(x_0)$ and $A(x) - A(x_0) \sim |x - x_0|^\alpha$ for some α, $0 < \alpha \le 1$. Thus, the factor $A(x) - A(x_0)$ will make the integrals convergent for $y = x_0$. We briefly sketch a proof of these facts.

We have to show that

$$
\begin{aligned}
\lim_{h\searrow 0} \int_{B_d(x_0)} & (A(x) - A(x_0))B(x) \\
& \times \frac{1}{h}\Big(D_j F(x, x_0 + he) - D_j F(x, x_0)\Big)\,dx \\
& = \int_{B_d(x_0)} (A(x) - A(x_0))B(x)\langle D_j \nabla_y F(x,x_0), e\rangle\,dx,
\end{aligned} \tag{4.7}
$$

where F is the Green function in the ball $B_{4d}(x_0)$ for the constant coefficient divergence form operator with coefficients a^0_{ij}. This coefficient matrix defines a strictly positive symmetric matrix M and it follows that there exists a non-singular matrix T such that TMT is the identity matrix. Putting $v(x) = u(T^{-1}x)$ and $g(x) = f(T^{-1}x)$, the equation

$$\sum_{i,j=1}^{n} a_{ij}^{0} D_i D_j u = f,$$

is equivalent to

$$\Delta v = g.$$

As a consequence, in proving (4.7) it is enough to consider the case when F is the Green function for the Laplacian. Recalling (1.16), putting

$$E(x,y) = \frac{(y-x)_j}{|y-x|^n}$$

and

$$E_N(x,y) = \begin{cases} E(x,y), & x \in {}^cB_{1/N}(y) \\ N, & x \in B_{1/N}(y) \end{cases},$$

we see that it is sufficient to prove (4.7) with D_jF replaced with E and that

$$\begin{aligned} \int_{B_d(x_0)} & (A(x) - A(x_0))B(x) \\ & \times \frac{1}{h}\Big(E(x, x_0 + he) - E(x, x_0)\Big)\, dx \\ & = \int_{B_d(x_0)} (A(x) - A(x_0))B(x) \\ & \times \frac{1}{h}\Big(E(x, x_0 + he) - E_N(x, x_0 + he) \\ & - (E(x, x_0) - E_N(x, x_0))\Big)\, dx \\ & + \int_{B_d(x_0)} (A(x) - A(x_0))B(x) \\ & \times \frac{1}{h}\Big(E_N(x, x_0 + he) - E_N(x, x_0)\Big)\, dx. \end{aligned}$$

Choosing e.g. $6h = 1/N$ the first integral on the right hand side is easily seen to converge to zero as $h \searrow 0$. The second integral we split into four pieces integrating over $B_{1/N}(x_0 + he) \setminus B_{1/N}(x_0)$, $B_{1/N}(x_0) \setminus B_{1/N}(x_0{+}he)$, $B_{1/N}(x_0) \cap B_{1/N}(x_0{+}he)$ and ${}^c(B_{1/N}(x_0) \cup B_{1/N}(x_0{+}he))$, respectively. The first and second integrals are seen to converge to zero, the third integral is zero and for the fourth integral we note, again using the mean value theorem, that

$$
\begin{aligned}
I_4 = \int_{B_d(x_0)\cap^c(B_{1/N}(x_0)\cup B_{1/N}(x_0+he))} & (A(x)-A(x_0))B(x)\frac{1}{h}\sum_k \\
\times & \frac{\delta_{jk}|x_0+\theta he-x|^2 - n(x_0+\theta he-x)_k(x_0+\theta he-x)_j}{|x_0+\theta he-x|^{n+2}} \\
\times & \, he_k\,dx,
\end{aligned}
$$

for some θ, $0\le\theta\le 1$. Furthermore, for x in the domain of integration the simple estimate

$$
6h = 1/N \le |x_0-x| \le |x_0+\theta he-x| + h \le |x_0+\theta he-x| + \frac{1}{6}|x-x_0|
$$

shows that the integrand is uniformly bounded by $C/|x-x_0|^{n-\alpha}$ which is integrable. An application of Lebesgue's dominated convergence theorem shows that

$$
\lim_{h\searrow 0} I_4 = \int_\Omega (A(x)-A(x_0))B(x)\frac{\partial E}{\partial y}(x,x_0)\,dx,
$$

which finishes the proof of (4.7).

Recalling (4.6) we have seen that it is legitimate to differentiate that expression at $y=x_0$ which then, remembering that $\eta(x)=1$ for $x\in B_d(x_0)$, leads to

$$
\begin{aligned}
\nabla v(x_0) &= \int_\Omega a^0_{ij}v(x)(D_i\eta)(x)D_j\nabla_y F(x,x_0)\,dx \\
&+ \int_\Omega a^0_{ij}v(x)(D_iD_j\eta)(x)\nabla_y F(x,x_0)\,dx \\
&+ \int_\Omega a^0_{ij}v(x)(D_j\eta)(x)D_i\nabla_y F(x,x_0)\,dx \\
&- \int_\Omega (D_ka_{ij}-D_ka^0_{ij})D_iu(x)D_j\Big(\eta(x)\nabla_y F(x,x_0)\Big)\,dx \qquad (4.8)\\
&+ \int_\Omega D_ka^0_{ij}D_jD_iu(x)\eta(x)\nabla_y F(x,x_0)\,dx \\
&- \int_\Omega (a_{ij}(x)-a^0_{ij})D_iv(x)D_j\Big(\eta(x)\nabla_y F(x,x_0)\Big)\,dx \\
&\equiv I_1+\ldots+I_6.
\end{aligned}
$$

We will estimate the integrals $I_1, \dots I_6$. The integral I_1 is estimated remembering that $\nabla \eta = 0$ in $B_{d/2}(x_0)$ and consequently

$$
\begin{aligned}
|I_1| &\le \frac{C}{d} \sup_\Omega |v| \int_{B_d(x_0)\setminus B_{d/2}(x_0)} |\nabla_y D_j F(x, x_0)|\, dx \\
&\le \frac{CK}{d} \sup_\Omega |v| \int_{B_d(x_0)\setminus B_{d/2}(x_0)} \frac{1}{|x - x_0|^n}\, dx \\
&\le (\log 2) \frac{CK}{d} \sup_\Omega |v|.
\end{aligned}
$$

With similar considerations the integrals I_2 and I_3 are seen to obey the same estimate. For I_5 we get

$$
\begin{aligned}
|I_5| &\le CK\delta^{-1}(x_0) \int_{B_d(x_0)} \delta(x_0)|D_i D_j u(x)||\nabla_y F(x, x_0)|\, dx \\
&\le CK\delta^{-1}(x_0) M_1 \int_{B_d(x_0)} |\nabla_y F(x, x_0)|\, dx \\
&\le CK\delta^{-1}(x_0) M_1 d,
\end{aligned}
$$

using the definition of M_1, (4.1), the estimates for the Green function, (1.11), and the fact that for $x \in B_d(x_0)$ we have in view of (4.3) that $\delta(x_0) \le \delta(x) + d \le \delta(x) + \frac{1}{2}\delta(x_0)$. It remains to treat I_4 and I_6. Beginning with I_6 we see that

$$
\begin{aligned}
|I_6| &\le C\delta^{-1}(x_0) M_1 \int_\Omega |a_{ij}(x) - a_{ij}^0| \\
&\times |D_j\Big(\eta(x)\nabla_y F(x, x_0)\Big)|\, dx \\
&\le C\delta^{-1}(x_0) M_1 \int_{B_d(x_0)} |a_{ij}(x) - a_{ij}^0| \\
&\times \left(|D_j \nabla_y F(x, x_0)| + \frac{|\nabla_y F(x, x_0)|}{d}\right) dx \\
&\le C\delta^{-1}(x_0) M_1 d.
\end{aligned}
$$

Regarding I_4 we have

$$\begin{aligned}
|I_4| &\leq \int_{B_d(x_0)} (D_k a_{ij}(x) - D_k a^0_{ij}) D_i u(x) \\
&\times \Big(D_j \eta(x) \nabla_y F(x, x_0) + \eta(x) D_j \nabla_y F(x, x_0) \Big)\, dx \\
&\leq C \int_{B_d(x_0)} \omega(|x - x_0|) |D_i u(x)| \\
&\times |D_j \eta(x) \nabla_y F(x, x_0) + \eta(x) D_j \nabla_y F(x, x_0)|\, dx \\
&\leq C M_0 \int_0^d \omega(r) (\frac{K}{d} \frac{1}{r^{n-1}} + \frac{1}{r^n}) r^{n-1}\, dr \\
&\leq \frac{CKM_0}{d} + M_0 \int_0^d \frac{\omega(r)}{r}\, dr.
\end{aligned}$$

Summarising the estimates of the integrals $I_1, \ldots, I_6$ and using (4.8) we obtain

$$|D_i D_j u(x_0)| \leq \frac{C}{d} M_0 + \frac{Cd}{\delta(x_0)} M_1.$$

The choice of x_0, (4.2), implies that

$$\frac{M_1}{2} \leq \frac{C\delta(x_0)}{d} M_0 + CdM_1,$$

and choosing d in such a way that $Cd \leq 1/4$ we can hide on the left, the term on the right hand side involving M_1. In fact choosing

$$d_0 = \sup\{d : \ Cd \leq 1/4\},$$

we obtain

$$M_1 \leq \frac{4C\delta(x_0)}{d_0} M_0 \leq \frac{4C \operatorname{diam}(\Omega)}{d_0} M_0,$$

which finishes the proof of the theorem. □

5. Using representation formulas to estimate the second order derivatives of the solution when the coefficients are Lipschitz continuous

In this section we assume that the coefficients are Lipschitz continuous and we give an L^1 estimate of the second order derivatives of the solution. We will use a classical tecnique based on representation formulas using a Green function of a simpler operator.

Proof of Lemma 1.9. For $x_0 \in \Omega$ we put $a_0^{i,j} = a_0^{i,j}(x_0)$. Let $F(x, y)$ be the Green function of the operator $L_0 \equiv D_i(a_0^{i,j} D_j)$ in the set $B_{4d}(x_0)$.

Since u is a solution to $Lu = 0$ in Ω we have, using summation convention, that

$$\int_\Omega a^{i,j}(x) D_i u(x) D_j \phi(x)\, dx = 0, \qquad \forall \phi \in C_0^\infty(\Omega).$$

Let $\psi_\varepsilon(x) = \varphi_\varepsilon(\eta(\cdot)F(\cdot, y))(x)$, where φ_ε is an approximation of the identity and $\eta \geq 0$ is a cut-off function such that $\operatorname{supp} \eta \subset B_d(x_0) \subset\subset \Omega$, $\eta \equiv 1$ on $B_{d/2}(x_0)$, and

$$|\nabla^i \eta| \leq K d^{-i}, \qquad i = 1, 2.$$

Then $\operatorname{supp} \psi_\varepsilon \subset B_{d+\varepsilon}(x_0)$ so that for small $\varepsilon > 0$ we can take $\phi_\varepsilon = D_k \psi_\varepsilon$ so that

$$\begin{aligned} 0 = &-\int_\Omega a^{i,j} D_i u D_j D_k \psi_\varepsilon \, dx \\ &= \int_\Omega [D_k a^{ij}] D_i u D_j \psi_\varepsilon \, dx + \int_\Omega a^{ij} D_k D_i u D_j \psi_\varepsilon \, dx. \end{aligned}$$

We put for convenience $v = D_k u$ and get

$$\begin{aligned} 0 = &\int_\Omega (D_k a_{ij}) D_i u D_j \psi_\varepsilon \, dx \\ &+ \int_\Omega (a_{ij} - a_{ij}^0) D_i v D_j \psi_\varepsilon \, dx + \int_\Omega a_{ij}^0 D_i v D_j \psi_\varepsilon \, dx. \end{aligned} \tag{5.1}$$

Computing the derivative in the third integral, I, we get

$$\begin{aligned} I - &\int_\Omega a_{ij}^0 D_i v(x) (\varphi_\varepsilon * (D_j \eta(\cdot) F(\cdot, y)))(x)\, dx \\ &= \int_\Omega a_{ij}^0 D_i v(x) (\varphi_\varepsilon * (\eta(\cdot) D_j F(\cdot, y)))(x)\, dx \\ &= \int_\Omega a_{ij}^0 D_i v(x) \int \varphi_\varepsilon(z) \eta(x - z) D_j F(x - z, y)\, dz\, dx \\ &= \int \varphi_\varepsilon(z) \int_\Omega a_{ij}^0 D_i v(x) \eta(x - z) D_j F(x - z, y)\, dx\, dz, \end{aligned}$$

since $\operatorname{supp} \eta \subset B_d(x_0) \subset B_{d+\varepsilon}(x_0) \subset \Omega$, and for the same reason the integration over the domain Ω can be replaced by integration over all space $\mathbf{R}^n$. Thus,

$$\begin{aligned}
\int_\Omega & a_{ij}^0 D_i v(x)\eta(x-z)D_jF(x-z,y)\,dx \\
&= \int_{\mathbf{R}^n} a_{ij}^0 D_i v(w+z)\eta(w)D_jF(w,y)\,dw \\
&= \int_{\mathbf{R}^n} a_{ij}^0 D_i (v(w+z)\eta(w))D_jF(w,y)\,dw \\
&- \int_{\mathbf{R}^n} a_{ij}^0 v(w+z)D_i\eta(w)D_jF(w,y)\,dw \\
&= v(y+z)\eta(y) - \int_{\mathbf{R}^n} a_{ij}^0 v(w+z)D_i\eta(w)D_jF(w,y)\,dw.
\end{aligned}$$

Consequently

$$\begin{aligned}
I - & \int_\Omega a_{ij}^0 D_i v(x)(\varphi_\varepsilon * (D_j\eta(\cdot)F(\cdot,y)))(x)\,dx \\
&= \eta(y)((\tilde{\varphi})_\varepsilon * v)(y) - \int_{\mathbf{R}^n} a_0^{i,j}((\tilde{\varphi})_\varepsilon * v)(w)D_i\eta(w)D_jF(w,y)\,dw,
\end{aligned}$$

where $\tilde{\varphi}(x) = \varphi(-x)$. Using (5.1) we arrive at

$$\begin{aligned}
\eta(y)((\tilde{\varphi})_\varepsilon * v)(y) &= \int_{\mathbf{R}^n} a_0^{i,j}((\tilde{\varphi})_\varepsilon * v)(w)D_i\eta(w)D_jF(w,y)\,dw \\
&- \int_{\mathbf{R}^n} a_{ij}^0 D_i v(x)(\varphi_\varepsilon * (D_j\eta(\cdot)F(\cdot,y)))(x)\,dx \\
&- \int_{\mathbf{R}^n} (D_k a_{ij})D_i u D_j\psi_\varepsilon\,dx \\
&- \int_{\mathbf{R}^n} (a_{ij}-a_{ij}^0)D_i v D_j\psi_\varepsilon\,dx.
\end{aligned} \tag{5.2}$$

Now, u is a solution so $u \in W^{1,2}_{\text{loc}}(\Omega)$. Since $a_{ij} \in C^{0,1}$ it follows, see Remark 1.6, that $u \in W^{1,p}_{\text{loc}}(\Omega)$ for any $p > 1$. In particular, recalling that $v = D_k u$, $D_i v \in L^p$ for arbitrarily large p. Thus, the last term of the right hand side of (5.2) fulfils for some constant C

$$\int_{\mathbf{R}^n} |(a_{ij}-a_{ij}^0)D_i v D_j\psi_\varepsilon|\,dx \le C(\int_{B_{d+\varepsilon}(x_0)} |D_i v|^p)^{1/p}(\int_{\mathbf{R}^n} |D_j\psi_\varepsilon|^q)^{1/q},$$

where p and q are the dual exponents. For $y \in B_{2d}(x_0)$ we have

$$\|\eta D_j F\|_{L^q} \le C(\int_{B_d(x_0)} \frac{1}{|x-y|^{(n-1)q}}\,dx)^{1/q}.$$

The integral can be estimated by integration over the ball $B_{4d}(y)$. Then, using polar coordinates gives that

$$\|\eta D_j F\|_{L^q} \le C_{n,q}(4d)^{((n-1)(1-q)+1)/q}$$

which is bounded for d bounded if $q < 1 + \frac{1}{n-1}$. This together with similar but simpler estimates for the other term in $D_j(\eta F)$, show that $D_j(\eta F) \in L^q$. Thus $D_j \psi_\varepsilon \to D_j(\eta F)$ in L^q. Consequently,

$$\int_{\mathbf{R}^n} (a_{ij} - a_{ij}^0) D_i v D_j \psi_\varepsilon \, dx \to \int_{\mathbf{R}^n} (a_{ij} - a_{ij}^0) D_i v D_j(\eta F) \, dx,$$

as $\varepsilon \to 0$. Furthermore, $v \in C^\lambda(\Omega)$ by the Sobolev imbedding theorem and thus v is uniformly continuous on every compact subset of Ω. Therefore, utilising estimates for the remaining terms in (5.2) similar to the one just used it follows, letting $\varepsilon \to 0$, that

$$\begin{aligned}
\eta(y)v(y) = &\int_{\mathbf{R}^n} a_0^{i,j} v(x) D_i \eta(x) D_j F(x,y) \, dx \\
&- \int_{\mathbf{R}^n} a_{ij}^0 D_i v(x) D_j \eta(x) F(x,y) \, dx \\
&- \int_{\mathbf{R}^n} (D_k a_{ij}) D_i u D_j(\eta(x) F(x,y)) \, dx \\
&- \int_{\mathbf{R}^n} (a_{ij} - a_{ij}^0) D_i v D_j(\eta(x) F(x,y)) \, dx.
\end{aligned}$$

For the second integral in the right hand side we get by integration by parts that

$$\begin{aligned}
&- \int_{\mathbf{R}^n} a_{ij}^0 D_i v(x) D_j \eta(x) F(x,y) \, dx \\
&\qquad = \int_{\mathbf{R}^n} a_{ij}^0 v D_i D_j \eta F(x,y) \, dx + \int_{\mathbf{R}^n} a_{ij}^0 v D_j \eta D_i F(x,y) \, dx.
\end{aligned}$$

Thus, for $y \in B_{4d}(x_0)$ we have

$$\begin{aligned}
\eta(y)v(y) = &\int_{\mathbf{R}^n} a_0^{i,j} v(x) D_i \eta(x) D_j F(x,y) \, dx \\
&+ \int_{\mathbf{R}^n} a_{ij}^0 v D_i D_j \eta F(x,y) \, dx \\
&+ \int_{\mathbf{R}^n} a_{ij}^0 v D_j \eta D_i F(x,y) \, dx \\
&- \int_{\mathbf{R}^n} (D_k a_{ij}) D_i u D_j(\eta(x) F(x,y)) \, dx \\
&- \int_{\mathbf{R}^n} (a_{ij} - a_{ij}^0) D_i v D_j(\eta(x) F(x,y)) \, dx \\
&\equiv I_1 + \ldots + I_5,
\end{aligned}$$

i.e.

$$\eta(y)v(y) = \sum_{i=1}^{5} I_i(y) \qquad \text{for } y \in B_{4d}(x_0).$$

Thus we get,

$$\eta(y)\nabla v(y) = -\nabla\eta(y)v(y) + \sum_{i=1}^{5} \nabla I_i(y), \qquad \text{for } y \in B_{4d}(x_0). \tag{5.3}$$

It will be necessary for technical reasons to become clear, to multiply (5.3) by the cut-off function η so that we get

$$\eta^2(y)\nabla v(y) = -\eta(y)\nabla\eta(y)v(y) + \sum_{i=1}^{5} \eta(y)\nabla I_i(y),$$

for $y \in B_{4d}(x_0)$. Consequently,

$$\begin{aligned}
&(\int_{B_d(x_0)} |\eta^2(x)\nabla v(x)|^2\, dx)^{1/2} \\
&\qquad \le (\int_{B_d(x_0)} |\eta(x)\nabla\eta(x)v(x)|^2\, dx)^{1/2} \\
&\qquad + (\int_{B_d(x_0)} |\eta(x)\nabla(I_1 + I_2)(x)|^2\, dx)^{1/2} \\
&\qquad + \sum_{i=3}^{5} (\int_{B_d(x_0)} |\eta(x)\nabla I_i(x)|^2\, dx)^{1/2}.
\end{aligned} \tag{5.4}$$

Let $M(x) = (M^{i,j}(x))$ be a $n \times n$-matrix valued function and B_i, ψ_i functions defined on Ω. Put $B = (B_i)$, $\psi = (\psi_i)$ with $\operatorname{supp}\psi$ or $\operatorname{supp} B \subset B_d(x_0)$. Let

$$I \equiv I(M, B, \psi, y) \equiv \int_{\mathbf{R}^n} M^{i,j}(x)B_i(x)D_j(\psi_i(x)F(x,y))\, dx. \tag{5.5}$$

Then, letting $\mathcal{I} = \mathrm{id} = (1, \dots, 1)$ we obtain

(5.6)

$$I_1(y) + I_2(y) = \int_{\mathbf{R}^n} a_{ij}^0 v D_j(D_i\eta F(x,y))\, dx = I(A_0, v\cdot\mathcal{I}, \nabla\eta, y),$$

and since A is symmetric,

$$I_3(y) = \int_{\mathbf{R}^n} a_{ij}^0 v D_i\eta D_j F(x,y))\, dx = I(A_0, v\nabla\eta, \mathcal{I}, y), \tag{5.7}$$

(5.8)

$$I_4(y) = -\int_{\mathbf{R}^n} (D_k a_{ij}) D_i u D_j(\eta(x)F(x,y))\,dx = I(-D_k A, \nabla u, \eta\mathcal{I}, y),$$

$$\begin{aligned} I_5(y) &= -\int_{\mathbf{R}^n} (a_{ij} - a^0_{ij}) D_i v D_j(\eta(x)F(x,y))\,dx \\ &= I(-(A - A_0), \nabla v, \eta\mathcal{I}, y). \end{aligned} \tag{5.9}$$

To study $I(M, B, \psi, y)$ we introduce for small $\varepsilon > 0$, (i.e. $\varepsilon < 3d$),

$$I^\varepsilon(M, B, \psi, y) \equiv \int_{\mathbf{R}^n} M^{i,j}(x) B_i(x) D_j(\psi_i(x)(\varphi_\varepsilon(\cdot) * F(\cdot, y))(x)))\,dx,$$

where φ_ε is an approximative identity.

Proposition 5.1. *For each fixed $\varepsilon > 0$,*

$$I^\varepsilon(M, B, \psi, \cdot) \in \mathrm{Lip}(B_{2d}(x_0)),$$

if $M \in L^\infty(B_d(x_0))$, $B \in L^p(B_d(x_0))$ and $\psi \in C^1(B_d(x_0))$. Thus, $I^\varepsilon(M, B, \psi, \cdot) \in L^\infty(B_{2d}(x_0))$.

Proof. Put for notational convenience $I^\varepsilon(y) = I^\varepsilon(M, B, \psi, y)$. We have

$$\begin{aligned} &|I^\varepsilon(y_2) - I^\varepsilon(y_1)| \\ &\quad = |\int_{rn} M^{i,j}(x) B_i(x) D_j(\psi_i(x)(\varphi_\varepsilon * (F(\cdot, y_2) - F(\cdot, y_1))(x)))\,dx| \\ &\quad \le \sup_{B_d(x_0)} |M^{i,j}| \Big(\sup_{B_d(x_0)} |D_j \psi_i| + \sup_{B_d(x_0)} |\psi_i| \Big) \\ &\quad \times \int_{B_d(x_0)} |B_i(x)| \Big(|\varphi_\varepsilon * (F(\cdot, y_2) \\ &\quad - F(\cdot, y_1))(x)| + |D_j \varphi_\varepsilon * (F(\cdot, y_2) - F(\cdot, y_1))(x)| \Big)\,dx \\ &\quad \le C \|B_i\|_{L^p(B_d(x_0))} \Big(\|\varphi_\varepsilon\|_{L^q(\mathbf{R}^n)} \\ &\quad + \|D_j \varphi_\varepsilon\|_{L^q(\mathbf{R}^n)} \Big) \|F(\cdot, y_2) - F(\cdot, y_1)\|_{L^1(B_{d+\varepsilon}(x_0))}, \end{aligned}$$

where p, q are dual exponents. Furthermore,

$$\begin{aligned} &\|F(\cdot, y_2) - F(\cdot, y_1)\|_{L^1(B_{d+\varepsilon}(x_0))} \\ &\quad \le C \int_0^1 |y_2 - y_1| \int_{B_{d+\varepsilon}(x_0)} \frac{dx}{|x - (y_1 + t(y_2 - y_1))|^{n-1}}\,dt \\ &\quad \le C |y_2 - y_1| \int_0^1 \int_{B_{3d+\varepsilon}(y_1 + t(y_2 - y_1))} \frac{dx}{|x - (y_1 + t(y_2 - y_1))|^{n-1}}\,dt \\ &\quad \le C |y_2 - y_1| (3d + \varepsilon). \end{aligned}$$

Hence,

$$
\begin{aligned}
&|I^{\varepsilon}(y_2) - I^{\varepsilon}(y_1)| \\
&\qquad \le C\Big(\varepsilon, d, n, \|M\|_{L^{\infty}(B_d(x_0))}, \|B\|_{L^{p}(B_d(x_0))}, \|\psi\|_{C^1(B_d(x_0))}\Big) \\
&\qquad \times |y_2 - y_1|,
\end{aligned}
$$

for any $y_2, y_1 \in B_{2d}(x_0)$. $\square$

Estimates similar to the ones just employed in the proof of the last proposition show that,

Proposition 5.2. *Under the conditions of Proposition 5.1 we have that*

$$
\begin{aligned}
&\nabla_y I^{\varepsilon}(M,B,\psi,y) \\
&\qquad = \int_{\mathbf{R}^n} M^{i,j}(x) B_i(x) D_j(\psi_i(x)(\varphi_{\varepsilon}(\cdot) * \nabla_y F(\cdot,y))(x)))\, dx.
\end{aligned}
$$

Notice that the requirements of the last two propositions are fulfiled, with p sufficiently large, in the case of the integrals $I_1, \dots, I_5$, since $u \in W^{2,p}_{\mathrm{loc}}$ for $p > 1$, see Remark 1.6, page 10.

We next observe that $I^{\varepsilon}(y)$ is integration against a kernel being the Green function, so 'I^{ε} is a solution'. Thus we should be able to derive a Caccioppoli-like estimate for I^{ε}, i.e. an estimate of the gradient in L^2 in terms of the L^2-norm of the function. To this end we take a test function $\phi \in C_0^{0,1}(B_{2d}(x_0)))$. Then

$$
\begin{aligned}
&\int_{B_{2d}(x_0)} a^0_{k\ell} D^y_k I^{\varepsilon}(y) D^y_{\ell}\phi(y)\, dy \\
&\qquad = \int_{B_{2d}(x_0)} a^0_{k\ell}\Big(\int_{B_d(x_0)} M^{i,j}(x) B_i(x) \\
&\qquad \times D_j(\psi_i(x)(\varphi_{\varepsilon}(\cdot) * D^y_k F(\cdot,y))(x))\, dx\Big) D^y_{\ell}\phi(y)\, dy \\
&\qquad = \int_{B_d(x_0)} M^{i,j}(x) B_i(x)\Big(\int_{B_{2d}(x_0)} a^0_{k\ell} \\
&\qquad \times D_j(\psi_i(x)(\varphi_{\varepsilon}(\cdot) * D^y_k F(\cdot,y))(x)) D^y_{\ell}\phi(y)\, dy\Big)\, dx.
\end{aligned} \tag{5.10}
$$

Naming the inner integral J, we find

$$
\begin{aligned}
J &= D_j\psi_i(x)\int_{B_\varepsilon(0)}\varphi_\varepsilon(z)\int_{B_{2d}(x_0)} a^0_{k\ell}D^y_kF(x-z,y)D^y_\ell\phi(y)\,dy\,dz\\
&+\psi_i(x)\int_{B_\varepsilon(0)}(D_j\varphi_\varepsilon)(z)\int_{B_{2d}(x_0)} a^0_{k\ell}D^y_kF(x-z,y)D^y_\ell\phi(y)\,dy\,dz\\
&= D_j\psi_i(x)\int_{B_\varepsilon(0)}\varphi_\varepsilon(z)\phi(x-z)\,dz\\
&+\psi_i(x)\int_{B_\varepsilon(0)}(D_j\varphi_\varepsilon)(z)\phi(x-z)\,dz,
\end{aligned}
$$

where we have choosen $\varepsilon < d$ so that $x-z\in B_{2d}(x_0)$ for all $x\in B_d(x_0)$ and $z\in B_\varepsilon(0)$. Inserting J into (5.10) gives

$$
\begin{aligned}
&\int_{B_{2d}(x_0)} a^0_{k\ell}D^y_kI^\varepsilon(y)D^y_\ell\phi(y)\,dy\\
&\qquad=\int_{B_d(x_0)} M^{i,j}(x)B_i(x)\Big(D_j\psi_i(x)(\varphi_\varepsilon*\phi)(x)\\
&\qquad+\psi_i(x)(\varphi_\varepsilon*D_j\phi)(x)\Big)\,dx.
\end{aligned}
$$

Choosing a particular test function, $\phi(y)=\eta^2(y)I^\varepsilon(y)$, we obtain,

$$
\begin{aligned}
&\int_{B_{2d}(x_0)} a^0_{k\ell}\eta^2D_kI^\varepsilon(y)D_\ell I^\varepsilon(y)\,dy\\
&\qquad=-\int_{B_{2d}(x_0)} a^0_{k\ell}D_kI^\varepsilon(y)2\eta(y)D_\ell\eta(y)I^\varepsilon(y)\,dy\\
&\qquad+\int_{B_d(x_0)} M^{i,j}(x)B_i(x)D_j\psi_i(x)(\varphi_\varepsilon*\eta^2I^\varepsilon)(x)\,dx\\
&\qquad+\int_{B_d(x_0)} M^{i,j}(x)B_i(x)\psi_i(x)(\varphi_\varepsilon*(D_j\eta^2)I^\varepsilon)(x)\,dx\\
&\qquad+\int_{B_d(x_0)} M^{i,j}(x)B_i(x)\psi_i(x)(\varphi_\varepsilon*\eta^2D_jI^\varepsilon)(x)\,dx.
\end{aligned}
\tag{5.11}
$$

Using ellipticity and remembering the summation convention we get

$$
\begin{aligned}
\lambda \int_{B_{2d}(x_0)} & \eta^2(x)|\nabla I^\varepsilon(x)|^2\,dx \\
& \leq 2|a^0_{k\ell}|(\int_{B_{2d}(x_0)} \eta^2(x)|\nabla I^\varepsilon(x)|^2\,dx)^{1/2} \\
& \times (\int_{B_{2d}(x_0)} |\nabla\eta(x)|^2|I^\varepsilon(x)|^2\,dx)^{1/2} \\
& + \sup_{B_d(x_0)}|M^{i,j}| \sup_{B_d(x_0)}|D_j\psi_i|(\int_{B_d(x_0)} |B_i(x)|^2\,dx)^{1/2} \\
& \times (\int_{B_d(x_0)} |\eta^2(x)I^\varepsilon(x)|^2\,dx)^{1/2} \\
& + \sup_{B_d(x_0)}|M^{i,j}| \sup_{B_d(x_0)}|\psi_i|(\int_{B_d(x_0)} |B_i(x)|^2\,dx)^{1/2} \\
& \times (\int_{B_d(x_0)} |(D_j\eta^2(x))I^\varepsilon(x)|^2\,dx)^{1/2} \\
& + \sup_{B_d(x_0)}|M^{i,j}| \sup_{B_d(x_0)}|\psi_i|(\int_{B_d(x_0)} |B_i(x)|^2\,dx)^{1/2} \\
& \times (\int_{B_d(x_0)} |\eta^2(x)\nabla I^\varepsilon(x)|^2\,dx)^{1/2}.
\end{aligned}
$$

We estimate the terms containing ∇I^ε on the right hand side. Thus,

$$
\begin{aligned}
(\int_{B_d(x_0)} & |B_i(x)|^2\,dx)^{1/2}(\int_{B_d(x_0)} |\eta^2(x)\nabla I^\varepsilon(x)|^2\,dx)^{1/2} \\
& \leq a\int_{B_d(x_0)} |B_i(x)|^2\,dx + \frac{1}{a}\int_{B_d(x_0)} |\eta^2(x)\nabla I^\varepsilon(x)|^2\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
(\int_{B_{2d}(x_0)} & \eta^2(x)|\nabla I^\varepsilon(x)|^2\,dx)^{1/2} \\
& \times (\int_{B_{2d}(x_0)} |\nabla\eta(x)|^2|I^\varepsilon(x)|^2\,dx)^{1/2} \\
& \leq b\int_{B_d(x_0)} |\nabla\eta(x)|^2|I^\varepsilon(x)|^2\,dx + \frac{1}{b}\int_{B_d(x_0)} \eta^2(x)|\nabla I^\varepsilon(x)|^2\,dx,
\end{aligned}
$$

and we choose a and b so that

$$
2|a^0_{k\ell}|\frac{1}{b} + \sup_{B_d(x_0)}|M^{i,j}| \sup_{B_d(x_0)}|\psi_i|\frac{1}{a} = \frac{\lambda}{2}.
$$

Hiding the terms containing ∇I^ε leads to

$$\begin{aligned}
\frac{\lambda}{2}\int_{B_d(x_0)} &\eta^2(x)|\nabla I^\varepsilon(x)|^2\,dx \\
&\leq \frac{4}{\lambda}(2|a^0_{k\ell}|)^2\int_{B_d(x_0)}|\nabla\eta(x)|^2 I^\varepsilon(x)^2\,dx \\
&+ \sup_{B_d(x_0)}|M^{i,j}|\sup_{B_d(x_0)}|D_j\psi_i|(\int_{B_d(x_0)}|B_i(x)|^2\,dx)^{1/2} \\
&\times(\int_{B_d(x_0)}|\eta^2(x)I^\varepsilon(x)|^2\,dx)^{1/2} \\
&+ \sup_{B_d(x_0)}|M^{i,j}|\sup_{B_d(x_0)}|\psi_i|(\int_{B_d(x_0)}|B_i(x)|^2\,dx)^{1/2} \\
&\times(\int_{B_d(x_0)}|(D_j\eta^2(x))I^\varepsilon(x)|^2\,dx)^{1/2} \\
&+ \frac{4}{\lambda}\Big(\sum_{i,j}\sup_{B_d(x_0)}|M^{i,j}|\sup_{B_d(x_0)}|\psi_i|\Big) \\
&\times \sup_{B_d(x_0)}|M^{i,j}|\sup_{B_d(x_0)}|\psi_i|\int_{B_d(x_0)}|B_i(x)|^2\,dx.
\end{aligned}$$

Next we note that I^ε are uniformly bounded in $L^\infty(B_{4d}(x_0))$. Specifically,

$$\begin{aligned}
|I^\varepsilon(y)| \leq \sup_{B_d(x_0)} &\|B_i\|_{L^p(B_d(x_0))} \\
&\times\Big(\sup_{B_d(x_0)}|D_j\psi_i|\|F(\cdot,y)\|_{L^q(B_{2d}(x_0))} \\
&+\sup_{B_d(x_0)}|\psi_i|\|D_jF(\cdot,y)\|_{L^q(B_{2d}(x_0))}\Big),
\end{aligned}$$

which is uniformly bounded if p and q are dual exponents and q is sufficiently close to 1. We have thus proved

Lemma 5.3. *For*

$$I^\varepsilon(y)\equiv I^\varepsilon(M,B,\psi,y),$$

if $M\in L^\infty(B_d(x_0))$, $B\in L^p(B_d(x_0))$, where p is sufficiently large, and $\psi\in C^1(B_d(x_0))$, we have that $\{I^\varepsilon(y)\}$ and $\{\nabla I^\varepsilon(y)\}$ are uniformly, in $\varepsilon>0$, bounded in $L^\infty(B_{4d}(x_0))$ and $L^2(B_{d/2}(x_0))$, respectively.

Estimates similar to the one above showing the boundedness of I^ε, shows that

$$I^\varepsilon(y)\to I(M,B,\psi,y).$$

From the last lemma we immediately conclude that there is a subsequence of $\{\nabla I^\varepsilon(y)\}$, again denoted by $\{\nabla I^\varepsilon(y)\}$, which converges weakly in $L^2(B_{d/2}(x_0))$. Thus,

$$\nabla I(M, B, \psi, \cdot) \in L^2(B_d(x_0)),$$

where we have assumed without loss of generality, that the estimates are valid for radii larger than $d/2$. Moreover,

$$\int_{B_d(x_0)} |\nabla_y I(M, B, \psi, \cdot)|^2\, dy = \lim \int_{B_d(x_0)} \nabla_y I(M, B, \psi, \cdot) \nabla I^\varepsilon(y)\, dy,$$

so that a use of the Cauchy-Schwartz inequality implies together with a reasoning along the same lines as above, that

$$\int_{B_d(x_0)} |\eta(y) \nabla_y I(M, B, \psi, \cdot)|^2\, dy \le \lim \int_{B_d(x_0)} |\eta(y) \nabla I^\varepsilon(y)|^2\, dy. \tag{5.12}$$

We next study the limit as $\varepsilon \searrow 0$ in (5.12). This amounts, via ellipticity as on page 39, to studying the limit of the right hand side in (5.11) on page 38. We rewrite the first integral of the right hand side of (5.11). Accordingly,

$$\begin{aligned}
&-\int_{B_d(x_0)} a^0_{k\ell} D_k I^\varepsilon(y) 2\eta(y) D_\ell \eta(y) I^\varepsilon(y)\, dy \\
&\quad = -\int_{B_d(x_0)} a^0_{k\ell} D_k I^\varepsilon(y) 2\eta(y) D_\ell \eta(y) (I^\varepsilon(y) - I(M, B, \psi, y))\, dy \\
&\quad - \int_{B_d(x_0)} a^0_{k\ell} D_k I^\varepsilon(y) 2\eta(y) D_\ell \eta(y) I(M, B, \psi, y)\, dy.
\end{aligned}$$

Clearly, the last integral has a limit

$$-\int_{B_d(x_0)} a^0_{k\ell} D_k I(M, B, \psi, y) 2\eta(y) D_\ell \eta(y) I(M, B, \psi, y)\, dy,$$

and for the first we have that its absolute value is bounded independently of $\varepsilon > 0$, by

$$\begin{aligned}
|a^0_{k\ell}| \sup_{B_d(x_0)} |2\eta D_\ell \eta| &\Big(\int_{B_d(x_0)} |D_k I^\varepsilon(y)|^2\, dy\Big)^{1/2} \\
&\times \Big(\int_{B_d(x_0)} |I^\varepsilon(y) - I(M, B, \psi, y)|^2\, dy\Big)^{1/2} \\
&\le C\Big(\int_{B_d(x_0)} |I^\varepsilon(y) - I(M, B, \psi, y)|^2\, dy\Big)^{1/2} \to 0.
\end{aligned}$$

Thus, the first integral of the right hand side of (5.11) has a limit

$$-\int_{B_d(x_0)} a^0_{k\ell} (D_k I(M, B, \psi, y)) 2\eta(y) D_\ell \eta(y) I(M, B, \psi, y)\, dy.$$

Furthermore, the three remaining integrals of the right hand side of (5.11) are all of the form

$$J_\varepsilon \equiv \int_{B_d(x_0)} h(x)(\varphi_\varepsilon * f_\varepsilon)(x)\, dx,$$

where $h \in L^2(B_d(x_0))$ and f_ε are uniformly bounded in $L^2(B_d(x_0))$, have support in $B_d(x_0)$ and $f_\varepsilon \rightharpoonup f$ weakly in $L^2(B_d(x_0))$. Now

$$J_\varepsilon = \int_{B_\varepsilon(0)} \varphi_\varepsilon(z) g_\varepsilon(z)\, dz + \int_{B_d(x_0)} h(x)(\varphi_\varepsilon * f)(x)\, dx,$$

where

$$g_\varepsilon(z) = \int_{B_d(x_0)} h(x)\Big(f_\varepsilon(x-z) - f(x-z)\Big)\, dx \to 0.$$

Also,

$$|g_\varepsilon(z)| \le \|h\|_{L^2(B_d(x_0))}\Big(\|f_\varepsilon\|_{L^2(B_d(x_0))} + \|f\|_{L^2(B_d(x_0))}\Big).$$

Consequently, $\varphi_\varepsilon(z)g_\varepsilon(z) \to 0$ a.e. and $|\varphi_\varepsilon(z)g_\varepsilon(z)| \le c\varphi_\varepsilon(z) \in L^1$, so dominated convergence gives that

$$\lim_{\varepsilon \searrow 0} J_\varepsilon = \int_{B_d(x_0)} h(x)f(x)\, dx.$$

Combining the estimate in (5.12) with the ellipticity condition we get with $I(y) = I(M, B, \psi, y)$, that

$$\begin{aligned}
\int_{B_d(x_0)} |\eta \nabla I(y)|^2\, dy &\le \frac{1}{\lambda} \lim_{\searrow 0} \int_{B_d(x_0)} a^0_{k\ell}\eta^2(y) D_k I^\varepsilon(y) D_\ell I^\varepsilon(y)\, dy \\
&\le \frac{1}{\lambda}\Big(- \int_{B_{2d}(x_0)} a^0_{k\ell} D_k I^\varepsilon(y) 2\eta(y) D_\ell \eta(y) I^\varepsilon(y)\, dy \\
&+ \int_{B_d(x_0)} M^{i,j}(x) B_i(x) D_j \psi_i(x)(\varphi_\varepsilon * \eta^2 I^\varepsilon)(x)\, dx \\
&+ \int_{B_d(x_0)} M^{i,j}(x) B_i(x) \psi_i(x)(\varphi_\varepsilon * (D_j \eta^2) I^\varepsilon)(x)\, dx \\
&+ \int_{B_d(x_0)} M^{i,j}(x) B_i(x) \psi_i(x)(\varphi_\varepsilon * \eta^2 D_j I^\varepsilon)(x)\, dx\Big).
\end{aligned}$$

Using the Cauchy-Schwartz inequality we obtain,

$$\begin{aligned}
&\int_{B_d(x_0)} |\eta(y)\nabla I(y)|^2\,dy \\
&\quad\le \frac{2a^0_{k\ell}}{\lambda}\Big(\int_{B_d(x_0)\setminus B_{d/2}(x_0)} \eta^2(y)|\nabla I(y)|^2\,dy)^{1/2} \\
&\quad\times (\int_{B_d(x_0)\setminus B_{d/2}(x_0)} |\nabla\eta(y)|^2|I(y)|^2\,dy)^{1/2} \\
&\quad+ \frac{\sup_{B_d(x_0)}|M^{i,j}|}{\lambda}\Big((\int_{B_d(x_0)} \eta^4(y)|B_i(y)|^2\,dy)^{1/2} \\
&\quad\times (\int_{B_d(x_0)} |\nabla\psi_i(y)|^2|I(y)|^2\,dy)^{1/2} \\
&\quad+ (\int_{B_d(x_0)} \eta^2(y)|\psi_i(y)|^2|B_i(y)|^2\,dy)^{1/2} \\
&\quad\times (\int_{B_d(x_0)} |\nabla\eta(y)|^2|I(y)|^2\,dy)^{1/2} \\
&\quad+ (\int_{B_d(x_0)} |\psi_i(y)|^2\eta^2(y)|B_i(y)|\,dy)^{1/2} \\
&\quad\times (\int_{B_d(x_0)} \eta^2(y)|\nabla I(y)|^2\,dy)^{1/2}\Big).
\end{aligned}$$

Hiding the terms of the right hand side containing ∇I we get for $I(y) = I(M, B, \psi, y)$ as defined in (5.5) on page 35, that

(5.13)

$$\begin{aligned}
&\int_{B_d(x_0)} |\eta(y)\nabla I(y)|^2\,dy \\
&\le C(\frac{\sum_{k,\ell} a^0_{k\ell}}{\lambda})^2\int_{B_d(x_0)} |\nabla\eta(y)|^2|I(y)|^2\,dy + \frac{\sup_{B_d(x_0)}|M^{i,j}|}{\lambda} \\
&\times \Big((\int_{B_d(x_0)} \eta^4(y)|B_i(y)|^2\,dy)^{1/2}(\int_{B_d(x_0)} |\nabla\psi_i(y)|^2|I(y)|^2\,dy)^{1/2} \\
&+ (\int_{B_d(x_0)} \eta^2(y)|\psi_i(y)|^2|B_i(y)|^2\,dy)^{1/2}(\int_{B_d(x_0)} |\nabla\eta(y)|^2|I(y)|^2\,dy)^{1/2}\Big) \\
&+ C\frac{\sum_{i,j}\sup_{B_d(x_0)}|M^{i,j}|}{\lambda}\sup_{B_d(x_0)}|M^{i,j}|\int_{B_d(x_0)} |\psi_i(y)|^2\eta^2(y)|B_i(y)|\,dy.
\end{aligned}$$

where C is a numerical constant. We next estimate, recalling the definition (5.5),

$$I(y) = I(M, B, \psi, y) = \int_{B_d(x_0)} M^{i,j}(x) B_i(x) D_j(\psi_i(x) F(x, y))\, dx,$$

assuming that $B \in L^\infty$.

$$\begin{aligned}
|I(y)| &\le C \sup_{B_d(x_0)} |M^{i,j}| \int_{B_d(x_0)} |B_i(x)| \Big(|D_j \psi_i(x)| \frac{1}{|x-y|^{n-2}} \\
&\quad + |\psi_i(x)| \frac{1}{|x-y|^{n-1}} \Big)\, dx \\
&\le C \sup_{B_d(x_0)} |M^{i,j}| \sup_{B_d(x_0)} |B_i| \sup_{B_d(x_0)} (d|D_j \psi_i| + |\psi_i|) \\
&\quad \times \int_{B_d(x_0)} \frac{1}{|x-y|^{n-1}}\, dx \\
&\le C \sup_{B_d(x_0)} |M^{i,j}| \sup_{B_d(x_0)} |B_i| \sup_{B_d(x_0)} (d|D_j \psi_i| + |\psi_i|) d \\
&\equiv Q(M, B, \psi)
\end{aligned} \tag{5.14}$$

Note that the estimate just given applies to the case of the integrals $I_1 + I_2$, I_3 and I_4, since in those cases we have $B_i \in L^\infty$. For I_5 we note that

$$I_5(y) = \eta(y) v(y) - \sum_{i=1}^{4} I_i(y),$$

so

$$\sup_{B_d(x_0)} |I_5(y)| \le \sup_{B_d(x_0)} (|v| + \sum_{i=1}^{4} |I_i(y)|). \tag{5.15}$$

From (5.4) and (5.6) – (5.9) we get that

$$
\begin{aligned}
(\int_{B_d(x_0)} |\eta^2(x)\nabla v(x)|^2\,dx)^{1/2} \le \frac{C}{d} \sup_{B_d(x_0)} |v| d^{n/2} \\
&+ (\int_{B_d(x_0)} |\eta^2(x)\nabla_y I(A_0, v(y)\cdot\mathcal{I}, \nabla\eta(y), y)|^2\,dy)^{1/2} \\
&+ (\int_{B_d(x_0)} |\eta^2(x)\nabla_y I(A_0, v(y)\nabla\eta(y), \mathcal{I}, y)|^2\,dy)^{1/2} \\
&+ (\int_{B_d(x_0)} |\eta^2(x) \\
&\times \nabla_y I(-D_k A(y), \nabla u(y), \eta(y)\cdot\mathcal{I}, y)|^2\,dy)^{1/2} \\
&+ (\int_{B_d(x_0)} |\eta^2(x) \\
&\times \nabla_y I(-(A(y)-A_0), \nabla v(y), \eta(y)\cdot\mathcal{I}, y)|^2\,dy)^{1/2}.
\end{aligned}
\tag{5.16}
$$

Using (5.13), (5.14) and the notation $Q(M,B,\psi)$ introduced in (5.14), we see that the second term on the right hand side of (5.16) can be estimated by

$$
\begin{aligned}
C(n,\|A_0\|,\lambda)d^{n/2-1}\Big(&Q(A_0, v\cdot\mathcal{I}, \nabla\eta) \\
&+ (\sup_{B_d(x_0)} |v|)^{1/2}(Q(A_0, v\cdot\mathcal{I}, \nabla\eta))^{1/2} + \sup_{B_d(x_0)} |v|\Big),
\end{aligned}
$$

which in turn can be estimated by

$$
C(n,\|A_0\|,\lambda)d^{n/2-1} \sup_{B_d(x_0)} |v| \tag{5.17}
$$

since

$$
Q(A_0, v\cdot\mathcal{I}, \nabla\eta) = \|A_0\| \sup_{B_d(x_0)} |v|.
$$

In the same manner the third term in (5.16) can be estimated by

$$
\begin{aligned}
C(n,\|A_0\|,\lambda)d^{n/2-1}\Big(&Q(A_0, v\nabla\eta, \mathcal{I}) \\
&+ (\sup_{B_d(x_0)} |v|)^{1/2}(Q(A_0, v\nabla\eta, \mathcal{I}))^{1/2} + \sup_{B_d(x_0)} |v|\Big),
\end{aligned}
$$

which too can be estimated by (5.17) since also

$$
Q(A_0, v\nabla\eta, \mathcal{I}) = \|A_0\| \sup_{B_d(x_0)} |v|.
$$

For the fourth term of the right hand side of (5.16), similarly it can be estimated by

$$C(n,\|A_0\|, \sup_{B_d(x_0)} \|\nabla A(x)\|, \lambda) d^{n/2} \Big(\frac{1}{d} Q(-D_k A, \nabla u, \eta \cdot \mathcal{I}) + \frac{1}{d^{1/2}} (\sup_{B_d(x_0)} |\nabla u|)^{1/2} (Q(-D_k A, \nabla u, \eta \cdot \mathcal{I}))^{1/2} + \sup_{B_d(x_0)} |\nabla u| \Big),$$

and using that

$$Q(-D_k A, \nabla u, \eta \cdot \mathcal{I}) \leq d \sup_{B_d(x_0)} \|\nabla A(x)\| \sup_{B_d(x_0)} |\nabla u|,$$

we see that the fourth term can be estimated by

$$C(n, \|A_0\|, \sup_{B_d(x_0)} \|\nabla A(x)\|, \lambda) d^{n/2} \sup_{B_d(x_0)} |\nabla u|. \tag{5.18}$$

Finally, for the fifth and last term on the right hand side of (5.16) we note, using (5.13), that it is bounded by

(5.19)

$$\begin{aligned} & C(n,\|A_0\|, \lambda) \frac{d^{n/2}}{d} \sup_{B_d(x_0)} |I(-(A(y) - A_0), \nabla v(y), \eta(y) \cdot \mathcal{I}, y)| \\ & + (\sup_{B_d(x_0)} \|A(y) - A_0\|)^{1/2} \Big((\int_{B_d(x_0)} \eta^4(y) |\nabla v(y)|^2 \, dy)^{1/2} \\ & \times (\int_{B_d(x_0)} |\nabla \eta(y)^2| \\ & \times |I(-(A(y) - A_0), \nabla v(y), \eta(y) \cdot \mathcal{I}, y)|^2 \, dy)^{1/2} \Big)^{1/2} \\ & + C(n) \frac{1}{\lambda^{1/2}} \sup_{B_d(x_0)} \|A(y) - A_0\| (\int_{B_d(x_0)} \eta^4(y) |\nabla v(y)|^2 \, dy)^{1/2}. \end{aligned}$$

Thus, summarising the estimates in (5.16), (5.17), (5.18) and (5.19), we obtain

$$
\begin{aligned}
&(\int_{B_d(x_0)} |\eta^2(x)\nabla v(x)|^2\,dx)^{1/2}\\
&\qquad \le C(n, \|A_0\|, \sup_{B_d(x_0)} \|\nabla A(x)\|, \lambda)\\
&\qquad \times \Big(\frac{d^{n/2}}{d} \sup_{B_d(x_0)} |v| + d^{n/2} \sup_{B_d(x_0)} |\nabla u|\Big)\\
&\qquad + C(n, \|A_0\|, \lambda)\frac{d^{n/2}}{d}\\
&\qquad \times \sup_{B_d(x_0)} |I(-(A(y)-A_0), \nabla v(y), \eta(y)\cdot \mathcal{I}, y)|\\
&\qquad + (\sup_{B_d(x_0)} \|A(y)-A_0\|)^{1/2}\Big((\int_{B_d(x_0)} \eta^4(y)|\nabla v(y)|^2\,dy)^{1/2}\\
&\qquad \times (\int_{B_d(x_0)} |\nabla \eta(y)^2|\\
&\qquad \times |I(-(A(y)-A_0), \nabla v(y), \eta(y)\cdot \mathcal{I}, y)|^2\,dy)^{1/2}\Big)^{1/2}\\
&\qquad + C(n)\frac{1}{\lambda^{1/2}} \sup_{B_d(x_0)} \|A(y)-A_0\|\\
&\qquad \times (\int_{B_d(x_0)} \eta^4(y)|\nabla v(y)|^2\,dy)^{1/2}.
\end{aligned} \tag{5.20}
$$

Note now, using (5.14) and (5.15), that

$$
\begin{aligned}
&|I(-(A(y)-A_0), \nabla v(y), \eta(y)\cdot \mathcal{I}, y)|\\
&\quad \le \sup_{B_d(x_0)} |v(y)| + Q(A_0, v(y)\cdot \mathcal{I}, \nabla \eta(y))\\
&\quad + Q(A_0, v(y)\nabla\eta(y), \mathcal{I}) + Q(-D_k A(y), \nabla u(y), \eta(y)\cdot \mathcal{I})\\
&\quad \le C(n, \|A_0\|) \sup_{B_d(x_0)} |v(y)| + d \sup_{B_d(x_0)} \|\nabla A(x)\| \sup_{B_d(x_0)} |\nabla u(y)|.
\end{aligned} \tag{5.21}
$$

Thus, ∇v appears only in two integrals on the right hand side of (5.20). We will next be quite explicit about the hiding of these terms on the left hand side. In a standard fashion we have for $\alpha > 0$

(5.22)
$$
\begin{aligned}
&\Big(\Big(\int_{B_d(x_0)} \eta^4(y) |\nabla v(y)|^2 \, dy \Big)^{1/2} \\
&\quad \times \Big(\int_{B_d(x_0)} |\nabla \eta(y)^2 | I(-(A(y) - A_0), \nabla v(y), \eta(y) \cdot \mathcal{I}, y)|^2 \, dy \Big)^{1/2} \Big)^{1/2} \\
&\quad \leq \Big(\alpha \int_{B_d(x_0)} \eta^4(y) |\nabla v(y)|^2 \, dy \Big)^{1/2} \\
&\quad + \Big(\frac{1}{\alpha} \int_{B_d(x_0)} |\nabla \eta(y)^2 | I(-(A(y) - A_0), \nabla v(y), \eta(y) \cdot \mathcal{I}, y)|^2 \, dy \Big)^{1/2}.
\end{aligned}
$$

Using (5.22) in (5.20), collecting the terms containing ∇v and choosing $\alpha = 1/\lambda$ gets us in front of

$$
\Big(\int_{B_d(x_0)} \eta^4(x) |\nabla v(x)|^2 \, dx \Big)^{1/2}
$$

a coefficient being

$$
\begin{aligned}
&C(n) \Big(\frac{1}{\lambda^{1/2}} \sup_{B_d(x_0)} \|A(x) - A_0\| + \alpha^{1/2} \big(\sup_{B_d(x_0)} \|A(x) - A_0\| \big)^{1/2} \Big) \\
&\quad \leq C(n) \frac{1}{\lambda^{1/2}} \big(\sup_{B_d(x_0)} \|A(x) - A_0\| \big)^{1/2} \leq 1/2,
\end{aligned}
\tag{5.23}
$$

if $d < d_0$ for some $0 < d_0 = d_0(n, \lambda, \sup_\Omega \|\nabla A(x)\|)$. Consequently, from (5.20) through (5.23) we get that

$$
\begin{aligned}
&\Big(\int_{B_d(x_0)} \eta^4(x) |\nabla v(x)|^2 \, dx \Big)^{1/2} \\
&\qquad \leq C(n, \sup_{B_d(x_0)} \|\nabla A(x)\|, \lambda,) \\
&\qquad \times \Big(\frac{1}{d} \sup_{B_d(x_0)} |v| + \sup_{B_d(x_0)} |\nabla u| \Big) d^{n/2}.
\end{aligned}
$$

Recalling that $v = D_k u$ and summing over k finally gives

$$
\begin{aligned}
&\int_{B_{d/2}(x_0)} |D_k D_\ell u(x)| \, dx \\
&\qquad \leq C d^{n/2} \Big(\int_{B_{d/2}(x_0)} |D_k D_\ell u(x)|^2 \, dx \Big)^{1/2} \leq C d^{n-1} \sup_{B_d(x_0)} |\nabla u|.
\end{aligned}
$$

This finishes the proof of Lemma 1.9. $\square$

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