A POSTERIORI ERROR ANALYSIS OF A FINITE ELEMENT METHOD FOR THE TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

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ABSTRACT. We consider the discontinuous Galerkin finite element method for the time-dependent Ginzburg-Landau model of superconductivity. The model consists of an initial-boundary value problem for a system of nonlinear parabolic partial differential equations. We prove an a posteriori error estimate in terms of computable residuals and stability factors related to a linearised dual problem. The a posteriori error estimate is suitable for the design of an adaptive finite element method. We also construct an interpolation operator, based on local averages, that preserves the boundary condition in a relevant function space.

1. Introduction

In 1908, H. Kammerlingh-Onnes discovered that certain metals lose there electrical resistance when cooled to temperatures just above the absolute zero. This was the discovery of superconductivity.

In 1933, W. Meissner and R. Ochsenfeld discovered that superconductors not only are perfect conductors, but also are perfect diamagnets. This was to become known as the Meissner effect.

An acceptable microscopic theory for low-temperature superconductivity came in 1957 with the BCS theory, named after its inventors J. Bardeen, L. Cooper and J. Schrieffer. This is a quantum-mechanical theory, based on the idea that the electrons form pairs, so called Cooper-pairs, that behave as bosons instead of fermions.

But even before this, several macroscopic models were proposed, most importantly, the homogeneous theory of F. London and H. London in 1935, and the model of V.L. Ginzburg and L.D. Landau in 1957.

The Ginzburg-Landau (GL) model is a phenomenological model; Ginzburg and Landau thought about the conducting electrons as a fluid that can be in two states, normal and superconducting. They used a general theory of second-order phase transitions introduced by Landau in 1937, to which they added certain effects to account for the magnetic field and its interaction with the electrons. Because of this, it was not clear if the GL model is a valid model for superconductors. But in 1959, L.P. Gor'kov showed that the Ginzburg-Landau model can be derived from the BCS theory by taking appropriate averages. Since then the GL model has been considered as a valid macroscopic model for low-temperature superconductors.

One of the early successes of the GL theory was the prediction of a new type of superconductor. In 1957, A.A. Abrikosov found that, if negative surface energy is associated with the phase transitions, then there will be relatively many phase

Date: May 22, 1998.

1991 *Mathematics Subject Classification.* 65M15, 65M60.

Key words and phrases. discontinuous Galerkin method, adaptive finite element method, a posteriori error estimate, duality, residual, Ginzburg-Landau equations, superconductivity.

transitions in the material; the normal and the superconducting state existing together in a so called mixed state. Ten years later this type of superconductor was discovered experimentally. They were named type II, in contrast to the first discovered type I superconductors, which have positive surface energy associated with the phase transitions, and hence relatively few phase transitions.

The superconductors found so far, had very limited usefulness; they all lose their superconducting properties above a temperature of around 2 K. But in 1987, high-temperature superconductors were discovered; certain metal oxide ceramics become superconducting at temperatures as high as 100 K (compare to 77 K, the temperature of liquid nitrogen). It was experimentally observed that they are extreme cases of type II superconductors.

Today, a variety of high-temperature superconductors have been discovered, that are, or may be, useful in industrial applications. There is no satisfactory theory to explain high-temperature superconductivity. It is not known if a mechanism similar to that in the low-temperature BCS theory is responsible for the superconductivity, or not. However, it is believed that the phenomenological GL model (or a variant of it) to some extent is a valid model even for high-temperature superconductors.

After this brief introduction to superconductors, we now turn to the formulation of the time-dependent GL model and the numerical approximation of it. For more reading about superconductors, we refer to [2], [8], [20] and [24].

The time-dependent GL model, obtained by taking averages in the BCS theory, consists of the following initial-boundary value problem for a system of nonlinear parabolic partial differential equations,

$$
\begin{aligned}
&\eta\psi_t + \Big(\frac{i}{\kappa}\nabla + \mathbf{A}\Big)^2\psi + i\eta\kappa\phi\psi + \big(|\psi|^2-1\big)\psi = 0, && \text{in } \Omega_T,\\
&\mathbf{A}_t + \nabla\times\nabla\times\mathbf{A} + \nabla\phi + \operatorname{Re}\Big\{\frac{i}{\kappa}\overline{\psi}\nabla\psi\Big\} + |\psi|^2\mathbf{A} = \nabla\times H, && \text{in } \Omega_T,\\
&\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\cdot\mathbf{n} = 0, \quad \nabla\times\mathbf{A} = H, && \text{on } \partial\Omega_T,\\
&\psi|_{t=0} = \psi_0, \quad \mathbf{A}|_{t=0} = \mathbf{A}_0, && \text{in } \Omega.
\end{aligned}
\tag{1}
$$

Here ψ is the complex-valued order parameter, $\mathbf{A}$ is the magnetic vector potential, ϕ is the electric potential, H is the applied external magnetic field, and ψ_0, $\mathbf{A}_0$ are the initial values. Further, κ is the Ginzburg-Landau parameter, η is another model parameter, and Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and outward unit normal $\mathbf{n}$. Also, $\nabla\times v = \big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\big)$ and $\nabla\times\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$ for $\mathbf{v} = (v_1, v_2)$. Finally, T is a positive time, and we set $J_T = (0,T)$, $\Omega_T = \Omega\times J_T$, $\partial\Omega_T = \partial\Omega\times J_T$.

The square of the absolute value of the order parameter, $|\psi|^2$, measures the density of Cooper pairs, it is close to 1 where the sample is in the superconducting state, and close to 0 where the sample is in the normal state. The vector potential $\mathbf{A}$ is related to the magnetic field h by $h = \nabla\times\mathbf{A}$, as usual. The Ginzburg-Landau parameter distinguishes between types of superconductor; we have $\kappa < \frac{1}{\sqrt{2}}$ for type I, and $\kappa > \frac{1}{\sqrt{2}}$ for type II.

We note that the system of equations (1) is not complete, since there is one more unknown than equations. One way to cure this is to add a gauge relation. The most frequently used gauge relations for the GL equations are: (i) the Coloumb gauge, where $\mathbf{A}$ is taken to be divergence free, (ii) the zero electric potential gauge, $\phi = 0$, and (iii) the Lorenz gauge, $\phi = -\nabla\cdot\mathbf{A}$; all three supplemented with the boundary condition $\mathbf{A}\cdot\mathbf{n} = 0$.

Several authors have studied numerical approximation of (1) with different gauge relations. The Lorentz gauge is used in [4], combined with the backward Euler method method in time and standard finite element methods in space, and in [3], combined with a mixed finite element method for the vector potential and standard finite element methods for the order parameter, together with a difference method in time. The zero electric potential gauge is used in [6], combined with the backward Euler method in time and standard finite element methods in space.

In these papers, various types of a priori error estimates are derived for the numerical methods under consideration.

From the point of view of numerical approximation, the Lorentz gauge seems to be the best choice, even though the zero electric potential gauge makes the equations look "easier". The Lorentz gauge makes the equation for the vector potential coercive in the function space under consideration, which seems to be important for the wellposedness of the problem. Moreover, the authors who use the zero electric potential gauge have to add an extra term multiplied by a small parameter in the equation for the vector potential in order to make the equation coercive, and then let this parameter tend to zero. The Coloumb gauge does not seem to be an attractive gauge, since one has to add one extra equation for ϕ, and since one also has to keep the vector potential divergence free, which is difficult in a time-dependent computation.

In this paper, we use the discontinuous Galerkin finite element method for the system (1) together with the Lorentz gauge.

The discontinuous Galerkin method, together with a posteriori error estimates, is well suited for the design of adaptive algorithms (see [9], [10], [11], [12], [13], [14], [18], [15], [23]). We believe that this is useful in numerical simulations of the GL equations, where the solutions exhibit many interesting phenomena. For example, type II superconductors, which are the ones of practical interest, have vortices in the solutions. The vortices undergo various processes such as pattern formation, creation and annihilation, pinning, and nucleation; this means that variable space and time meshes might be useful for efficient computations.

The outline of the paper is as follows:

In section 2, we discuss gauge equivalence for the GL equations, and give the weak form of the GL equations under the Lorentz gauge.

In section 3 we introduce time and space discretisations, and formulate the discontinuous Galerkin finite element method for the GL equations.

In section 4, we first give a characterisation of the finite element space that we use to approximate the vector potential, and then construct an interpolation operator, based on local averages, that preserves the boundary condition $\mathbf{A}\cdot\mathbf{n}=0$ on $\partial\Omega$. We also prove error estimates for the interpolation operator.

In section 5, we state and prove the main result in this paper: an a posteriori error estimate for the discontinuous Galerkin method introduced in section 3. The estimate is in terms of stability factors and computable residuals. The stability factors are related to the stability properties of a linearised dual problem.

We conclude this section by introducing some notation that will be useful in the sequel. We let $L^p(\Omega)$ denote the usual real Lebesgue spaces with norms

$$\|v\|_{L^p(\Omega)} = \begin{cases} \Big(\int_\Omega |v|^p \, dx \Big)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_\Omega |v|, & p = \infty, \end{cases}$$

and we let $W^{k,p}(\Omega)$ be the standard real Sobolev spaces with norms $\|\cdot\|_{W^{k,p}(\Omega)}$ and corresponding seminorms $|\cdot|_{W^{k,p}(\Omega)}$ defined by

$$\|v\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha|\le k} \|D^\alpha v\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}, \quad |v|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha| = k} \|D^\alpha v\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}},$$

for $1 \le p < \infty$ with the usual modifications in the case $p = \infty$. Sobolev spaces of fractional order k are defined by interpolation.

In the case $p = 2$, we set $H^k(\Omega) = W^{k,2}(\Omega)$, which is a Hilbert space with the inner product

$$(v, w)_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha v D^\alpha w \, dx.$$

We also introduce the corresponding complex-valued spaces, denoted $\mathcal{L}^p(\Omega)$, $\mathcal{W}^{k,p}(\Omega)$, $\mathcal{H}^k(\Omega)$ with obvious norms, and vector-valued spaces, denoted $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{k,p}(\Omega)$, $\mathbf{H}^k(\Omega)$, with norms $\|\mathbf{v}\|_{L^p(\Omega)} = \| \, |\mathbf{v}|_p \, \|_{L^p(\Omega)}$, and so forth, where $|\mathbf{v}|_p = (v_1^p + v_2^p)^{\frac{1}{p}}$ if $\mathbf{v} = (v_1, v_2)$. We also modify the inner products in the natural way, i.e., for $v, w \in \mathcal{H}^k(\Omega)$ and $\mathbf{v}, \mathbf{w} \in \mathbf{H}^k(\Omega)$,

$$(v, w)_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha v \overline{D^\alpha w} \, dx, \quad (\mathbf{v}, \mathbf{w})_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha \mathbf{v} \cdot D^\alpha \mathbf{w} \, dx.$$

For $L^2(\Omega)$, $\mathcal{L}^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ we drop the subscripts on both norms and inner products, and write $\| \cdot \| = \| \cdot \|_{L^2(\Omega)}$ and $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\Omega)}$, when there is no ambiguity.

We also need the space $\mathbf{H}^1_{\mathbf{n}}(\Omega)$ defined by

$$\mathbf{H}^1_{\mathbf{n}}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

From [17, Theorem I.3.9], we quote an important inequality, which is valid when Ω is a convex polygon,

$$|\mathbf{v}|^2_{H^1(\Omega)} \le \|\nabla \times \mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2, \quad \forall \mathbf{v} \in \mathbf{H}^1_{\mathbf{n}}(\Omega). \tag{2}$$

If Ω is a non-convex polygon, then (2) is not necessarily true.

When X is a Banach space with norm $\| \cdot \|_X$ and I is an interval, we define

$$L^p(I; X) = \{v(\cdot, t) : I \to X \text{ is measurable and } \|v\|_{L^p(I;X)} < \infty\},$$

where

$$\|v\|_{L^p(I;X)} = \begin{cases} \left(\int_I \|u(\cdot, t)\|_X^p \, dt \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_I \|u(\cdot, t)\|_X, & p = \infty, \end{cases}$$

and

$$\mathcal{C}(I; X) = \{v(\cdot, t) : I \to X \text{ is continuous and } \|v\|_{\mathcal{C}(I;X)} < \infty\},$$

where $\|v\|_{\mathcal{C}(I;X)} = \sup_{t \in I} \|v(\cdot, t)\|_X$. Also, we let X^* denote the dual space to X. We often use the abbreviation $\| \cdot \|_I = \| \cdot \|_{L^\infty(I; L^2(\Omega))}$.

We finally quote a trace theorem, that will be useful; see [19, Théo̊reme 1.3.1] or [22, Lemma III.1.2].

Theorem 1.1. *Let V, H, V^* be three Hilbert spaces such that $V \subset H \subset V^*$, and $I = (t_1, t_2)$, $t_1 < t_2$. If $v \in L^2(I; V)$ and $v_t \in L^2(I; V^*)$, then $v \in \mathcal{C}(\overline{I}; H)$, after possibly being redefined on a set of measure zero.*

2. The Ginzburg-Landau equations

Following [5] (see also [7]), we first discuss gauge equivalence for the GL equations. We introduce $\hat{\mathbb{H}}(\Omega)$ defined by

$$\hat{\mathbb{H}}(\Omega) = \{(\mathbf{v}, w) \in \mathbf{H}^1_{\mathbf{n}}(\Omega) \times L^2(\Omega) : \nabla \cdot \mathbf{v} + w = 0\}.$$

We say that two triples $(\psi, \mathbf{A}, \phi)$ and $(\zeta, \mathbf{B}, \varphi)$ are gauge equivalent, if there exists $\chi \in L^2(J_T; H^2(\Omega))$, with $\chi_t \in L^2(J_T; L^2(\Omega))$ and $\chi|_{t=0} = 0$, such that $(\psi, \mathbf{A}, \phi) =$

$G_\chi(\zeta, \mathbf{B}, \varphi)$, where G_χ is the gauge χ-transformation from $L^2\big(J_T; \mathcal{H}^1(\Omega)\big) \times L^2\big(J_T; \mathbf{H}^1(\Omega)\big) \times L^2\big(J_T; L^2(\Omega)\big)$ into itself defined by

$$\psi = \zeta e^{i\kappa\chi}, \quad \mathbf{A} = \mathbf{B} + \nabla\chi, \quad \phi = \varphi - \frac{\partial \chi}{\partial t}.$$

We have the following lemmas (for the proofs, see [5]).

Lemma 2.1. *Any* $(\zeta, \mathbf{B}, \varphi) \in L^2\big(J_T; \mathcal{H}^1(\Omega)\big) \times L^2\big(J_T; \mathbf{H}^1(\Omega)\big) \times L^2\big(J_T; L^2(\Omega)\big)$ *is gauge equivalent to a unique element in* $L^2\big(J_T; \mathcal{H}^1(\Omega)\big) \times L^2\big(J_T; \hat{\mathbb{H}}(\Omega)\big)$.

Lemma 2.2. *Let* $\chi \in L^2\big(J_T; H^2(\Omega)\big)$ *with* $\chi_t \in L^2\big(J_T; L^2(\Omega)\big)$ *and* $\chi|_{t=0} = 0$. *If* $(\psi, \mathbf{A}, \phi)$ *is a solution to (1), then* $G_\chi(\psi, \mathbf{A}, \phi)$ *also is a solution to (1).*

The above lemmas justify that we look for solutions to (1) under the gauge defined in Lemma 2.1, which obviously is the Lorentz gauge. We thus use

$$\nabla \cdot \mathbf{A} + \phi = 0, \text{ in } \Omega_T, \quad \mathbf{A} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega_T,$$

which results in the following system of equations

$$\begin{aligned}
&\eta\psi_t + \Big(\frac{i}{\kappa}\nabla + \mathbf{A}\Big)^2\psi - i\eta\kappa(\nabla\cdot\mathbf{A})\psi + \big(|\psi|^2 - 1\big)\psi = 0, && \text{in } \Omega_T,\\
&\mathbf{A}_t + \nabla\times\nabla\times\mathbf{A} - \nabla(\nabla\cdot\mathbf{A}) + \operatorname{Re}\Big\{\frac{i}{\kappa}\overline{\psi}\nabla\psi\Big\} + |\psi|^2\mathbf{A} = \nabla\times H, && \text{in } \Omega_T,\\
&\frac{\partial\psi}{\partial\mathbf{n}} = 0, \quad \nabla\times\mathbf{A} = H, \quad \mathbf{A}\cdot\mathbf{n} = 0, && \text{on } \partial\Omega_T,\\
&\psi|_{t=0} = \psi_0, \quad \mathbf{A}|_{t=0} = \mathbf{A}_0, && \text{in } \Omega.
\end{aligned} \tag{3}$$

We now consider the weak formulation of (3). In order to do so, we introduce the spaces

$$\begin{aligned}
\mathcal{W}(J_T) &= \big\{v \in L^2\big(J_T; \mathcal{H}^1(\Omega)\big) : v_t \in L^2\big(J_T; \mathcal{H}^1(\Omega)^*\big)\big\},\\
\mathbf{W_n}(J_T) &= \big\{\mathbf{v} \in L^2\big(J_T; \mathbf{H}^1_\mathbf{n}(\Omega)\big) : \mathbf{v}_t \in L^2\big(J_T; \mathbf{H}^1_\mathbf{n}(\Omega)^*\big)\big\}.
\end{aligned}$$

The weak formulation of (3) is:

Find $\psi \in \mathcal{W}(J_T)$ and $\mathbf{A} \in \mathbf{W_n}(J_T)$ such that

$$\begin{aligned}
&\eta\int_{J_T}(\psi_t, v)\,dt + \int_{J_T}\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi, \frac{i}{\kappa}\nabla v + \mathbf{A}v\Big)\,dt\\
&\quad + \int_{J_T}\big(-i\eta\kappa(\nabla\cdot\mathbf{A})\psi + (|\psi|^2 - 1)\psi, v\big)\,dt = 0, \quad \forall v \in L^2\big(J_T; \mathcal{H}^1(\Omega)\big),
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\int_{J_T}(\mathbf{A}_t, \mathbf{v})\,dt + \int_{J_T}\{(\nabla\times\mathbf{A}, \nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{A}, \nabla\cdot\mathbf{v})\}\,dt\\
&+\int_{J_T}\Big(\operatorname{Re}\Big\{\frac{i}{\kappa}\overline{\psi}\nabla\psi\Big\} + |\psi|^2\mathbf{A}, \mathbf{v}\Big)\,dt = \int_{J_T}(H, \nabla\times\mathbf{v})\,dt, \ \forall\mathbf{v} \in L^2\big(J_T; \mathbf{H}^1_\mathbf{n}(\Omega)\big),
\end{aligned} \tag{5}$$

with initial values

$$\psi|_{\Omega\times\{t=0\}} = \psi_0, \quad \mathbf{A}|_{\Omega\times\{t=0\}} = \mathbf{A}_0. \tag{6}$$

It is obvious that (4) and (5) are well-defined, and so also (6), since we have that $\mathcal{H}^1(\Omega) \subset \mathcal{L}^2(\Omega) \subset \mathcal{H}^1(\Omega)^*$, so we see from Theorem 1.1, that if $v \in \mathcal{W}(J_T)$, then $v \in \mathcal{C}\big(\overline{J}_T; \mathcal{L}^2(\Omega)\big)$, and similarly, if $\mathbf{v} \in \mathbf{W_n}(J_T)$, then $\mathbf{v} \in \mathcal{C}\big(\overline{J}_T; \mathbf{L}^2(\Omega)\big)$.

Throughout this paper, we will assume that the problem (4)–(6) has unique solutions $\psi \in \mathcal{W}(J_T)$ and $\mathbf{A} \in \mathbf{W_n}(J_T)$.

3. The discontinuous Galerkin Method

In this section, we introduce space and time discretisations, and the discontinuous Galerkin method for the problem (4)–(6).

To discretise in space, we let $\mathcal{F}$ be a family of triangulations of Ω. With each triangulation $\mathcal{T} \in \mathcal{F}$, we associate a mesh function $h_{\mathcal{T}}$, which we define to be a piecewise constant function on Ω such that $h_{\mathcal{T}}|_K = h_K$ for $K \in \mathcal{T}$, where h_K is the diameter of K. We assume that the family of triangulations $\mathcal{F}$ is nondegenerate, i.e., there exists a constant γ_0 such that

$$\max_{K\in\mathcal{T}} \frac{h_K}{\rho_K} \le \gamma_0, \quad \forall \mathcal{T} \in \mathcal{F}, \tag{7}$$

where ρ_K denotes the diameter of the largest closed ball contained in the triangle $\overline{K}$. With each triangulation $\mathcal{T} \in \mathcal{F}$ we associate a finite element space $V_{\mathcal{T}}$ consisting of continuous piecewise linear functions,

$$V_{\mathcal{T}} = \big\{ v \in \mathcal{C}(\Omega) : v|_K \text{ is linear, } \forall K \in \mathcal{T} \big\}.$$

Further, we let $\mathcal{V}_{\mathcal{T}}$, $\mathbf{V}_{\mathcal{T}}$ be the corresponding complex-valued and vector-valued spaces, and $\mathbf{V}_{\mathcal{T},\mathbf{n}} = \{\mathbf{v} \in \mathbf{V}_{\mathcal{T}} : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$.

For the discretisation in time, we let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be a partition of $\mathbb{R}^+$ into time intervals $I_n = (t_{n-1}, t_n)$ of lengths $k_n = t_n - t_{n-1}$. For each such time interval, we take a triangulation $\mathcal{T}_n \in \mathcal{F}$ and corresponding finite element spaces $\mathcal{V}_{\mathcal{T}_n}$, $\mathbf{V}_{\mathcal{T}_n,\mathbf{n}}$ as above, with mesh functions $h_{\mathcal{T}_n}$, and for a fixed non-negative integer q, we define (the subscript $\mathcal{D}$ denotes discrete)

$$\begin{aligned} \mathcal{W}^n_{\mathcal{D}} &= \Big\{ v : v = \sum_{j=0}^{q} t^j \varphi_j,\ \varphi_j \in \mathcal{V}_{\mathcal{T}_n} \Big\}, \\ \mathbf{W}^n_{\mathcal{D},\mathbf{n}} &= \Big\{ \mathbf{v} : \mathbf{v} = \sum_{j=0}^{q} t^j \varphi_j,\ \varphi_j \in \mathbf{V}_{\mathcal{T}_n,\mathbf{n}} \Big\}. \end{aligned} \tag{8}$$

The discontinuous Galerkin finite element method for the Ginzburg-Landau equations now reads:

Find $\psi_{\mathcal{D}}$, $\mathbf{A}_{\mathcal{D}}$ with $\psi^-_{\mathcal{D},0} = \psi_0$, $\mathbf{A}^-_{\mathcal{D},0} = \mathbf{A}_0$, such that, for $n = 1, 2, \ldots$, we have $\psi_{\mathcal{D}}|_{\Omega\times I_n} \in \mathcal{W}^n_{\mathcal{D}}$, $\mathbf{A}_{\mathcal{D}}|_{\Omega\times I_n} \in \mathbf{W}^n_{\mathcal{D},\mathbf{n}}$, and

$$\begin{aligned} &\eta \int_{I_n} (\psi_{\mathcal{D},t}, v)\,dt + \eta([\psi_{\mathcal{D}}]_{n-1}, v^+_{n-1}) \\ &\quad + \int_{I_n} \Big(\frac{i}{\kappa}\nabla\psi_{\mathcal{D}} + \mathbf{A}_{\mathcal{D}}\psi_{\mathcal{D}}, \frac{i}{\kappa}\nabla v + \mathbf{A}_{\mathcal{D}} v \Big)\,dt \\ &\quad + \int_{I_n} \big(-i\eta\kappa(\nabla\cdot\mathbf{A}_{\mathcal{D}})\psi_{\mathcal{D}} + (|\psi_{\mathcal{D}}|^2 - 1)\psi_{\mathcal{D}}, v \big)\,dt = 0, \quad \forall v \in \mathcal{W}^n_{\mathcal{D}}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\int_{I_n} (\mathbf{A}_{\mathcal{D},t}, \mathbf{v})\,dt + ([\mathbf{A}_{\mathcal{D}}]_{n-1}, \mathbf{v}^+_{n-1}) \\ &\quad + \int_{I_n} \big\{ (\nabla\times\mathbf{A}_{\mathcal{D}}, \nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{A}_{\mathcal{D}}, \nabla\cdot\mathbf{v}) \big\}\,dt \\ &\quad + \int_{I_n} \Big(\mathrm{Re}\Big\{ \frac{i}{\kappa}\overline{\psi}_{\mathcal{D}}\nabla\psi_{\mathcal{D}} \Big\} + |\psi_{\mathcal{D}}|^2\mathbf{A}_{\mathcal{D}}, \mathbf{v} \Big)\,dt \\ &= \int_{I_n} (H, \nabla\times\mathbf{v})\,dt, \quad \forall \mathbf{v} \in \mathbf{W}^n_{\mathcal{D},\mathbf{n}}, \end{aligned} \tag{10}$$

where

$$[w]_n = w_n^+ - w_n^-, \quad w_n^{+(-)} = \lim_{s\to 0^{+(-)}} w(t_n+s).$$

We only analyse the case $q=0$, in which case $\psi_{\mathcal{D}}$ and $\mathbf{A}_{\mathcal{D}}$ are constant in time on each time interval I_n, and we set $\psi_{\mathcal{D},n} = \psi_{\mathcal{D}}|_{\Omega\times I_n}$ and $\mathbf{A}_{\mathcal{D},n} = \mathbf{A}_{\mathcal{D}}|_{\Omega\times I_n}$.

To write the equations (9)–(10) in a convenient way, we introduce

$$\begin{aligned} B_1 &: \tilde{\mathcal{W}}(J_N)\times\tilde{\mathbf{W}}_{\mathbf{n}}(J_N)\times\tilde{\mathcal{W}}(J_N)\to\mathbb{C},\\ B_2 &: \tilde{\mathcal{W}}(J_N)\times\tilde{\mathbf{W}}_{\mathbf{n}}(J_N)\times\tilde{\mathbf{W}}_{\mathbf{n}}(J_N)\to\mathbb{R}, \end{aligned}$$

defined by

$$\begin{aligned} B_1(w,\mathbf{w},v) = {} & \eta\sum_{n=1}^{N}\int_{I_n}(w_t,v)\,dt + \eta(w_0^+,v_0^+) + \eta\sum_{n=1}^{N-1}([w]_n,v_n^+)\\ & + \int_{J_N}\Big(\frac{i}{\kappa}\nabla w + \mathbf{w}w, \frac{i}{\kappa}\nabla v + \mathbf{w}v\Big)\,dt\\ & + \int_{J_N}\big(-i\eta\kappa(\nabla\cdot\mathbf{w})w + (|w|^2-1)w, v\big)\,dt, \end{aligned} \tag{11}$$

and

$$\begin{aligned} B_2(w,\mathbf{w},\mathbf{v}) = {} & \sum_{n=1}^{N}\int_{I_n}(\mathbf{w}_t,\mathbf{v})\,dt + (\mathbf{w}_0^+,\mathbf{v}_0^+) + \sum_{n=1}^{N-1}([\mathbf{w}]_n,\mathbf{v}_n^+)\\ & + \int_{J_N}\big\{(\nabla\times\mathbf{w},\nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{w},\nabla\cdot\mathbf{v})\big\}\,dt\\ & + \int_{J_N}\Big(\mathrm{Re}\Big\{\frac{i}{\kappa}\overline{w}\nabla w\Big\} + |w|^2\mathbf{w},\mathbf{v}\Big)\,dt, \end{aligned} \tag{12}$$

where $J_N \equiv J_{t_N} = (0,t_N)$. We also define spaces of piecewise (in time) differentiable functions,

$$\begin{aligned} \tilde{\mathcal{W}}(J_N) &= \big\{v\in L^2\big(J_N;\mathcal{H}^1(\Omega)\big) : v_t\in L^2\big(I_n;\mathcal{H}^1(\Omega)^*\big),\quad n=1,\dots,N\big\},\\ \tilde{\mathbf{W}}_{\mathbf{n}}(J_N) &= \big\{\mathbf{v}\in L^2\big(J_N;\mathbf{H}^1_{\mathbf{n}}(\Omega)\big) : \mathbf{v}_t\in L^2\big(I_n;\mathbf{H}^1_{\mathbf{n}}(\Omega)^*\big),\quad n=1,\dots,N\big\}, \end{aligned} \tag{13}$$

and the global finite element spaces

$$\begin{aligned} \mathcal{W}_{\mathcal{D}}(J_N) &= \big\{v : v|_{\Omega\times I_n}\in\mathcal{W}^n_{\mathcal{D}},\ n=1,\dots,N\big\},\\ \mathbf{W}_{\mathcal{D},\mathbf{n}}(J_N) &= \big\{\mathbf{v} : \mathbf{v}|_{\Omega\times I_n}\in\mathbf{W}^n_{\mathcal{D},\mathbf{n}},\ n=1,\dots,N\big\}. \end{aligned}$$

Observe that B_1 and B_2 are well-defined, since if $w,v\in\tilde{\mathcal{W}}(J_N)$, then Theorem 1.1 implies that $w,v\in\mathcal{C}\big(\overline{I}_n;\mathcal{L}^2(\Omega)\big)$, $n=1,\dots,N$. Thus, (w_0^+,v_0^+) and $([w]_n,v_n^+)$ in B_1 are well-defined, and similarly for B_2.

We can now write the equations (9)–(10), that determine the solutions $\psi_{\mathcal{D}}$ and $\mathbf{A}_{\mathcal{D}}$ up to time t_N, as follows:

Find $\psi_{\mathcal{D}}\in\mathcal{W}_{\mathcal{D}}(J_N)$ and $\mathbf{A}_{\mathcal{D}}\in\mathbf{W}_{\mathcal{D},\mathbf{n}}(J_N)$ such that

$$\begin{aligned} B_1(\psi_{\mathcal{D}},\mathbf{A}_{\mathcal{D}},v) &= \eta(\psi_0,v_0^+), && \forall v\in\mathcal{W}_{\mathcal{D}}(J_N),\\ B_2(\psi_{\mathcal{D}},\mathbf{A}_{\mathcal{D}},\mathbf{v}) &= (\mathbf{A}_0,\mathbf{v}_0^+) + \int_{J_N}(H,\nabla\times\mathbf{v})\,dt, && \forall \mathbf{v}\in\mathbf{W}_{\mathcal{D},\mathbf{n}}(J_N). \end{aligned} \tag{14}$$

Similarly, the exact solutions $\psi\in\mathcal{W}(J_N)$ and $\mathbf{A}\in\mathbf{W}_{\mathbf{n}}(J_N)$ of (4)–(6) satisfy

$$\begin{aligned} B_1(\psi,\mathbf{A},v) &= \eta(\psi_0,v_0^+), && \forall v\in\tilde{\mathcal{W}}(J_N),\\ B_2(\psi,\mathbf{A},\mathbf{v}) &= (\mathbf{A}_0,\mathbf{v}_0^+) + \int_{J_N}(H,\nabla\times\mathbf{v})\,dt, && \forall \mathbf{v}\in\tilde{\mathbf{W}}_{\mathbf{n}}(J_N), \end{aligned} \tag{15}$$

since $\psi\in\mathcal{C}\big(\overline{J}_N;\mathcal{L}^2(\Omega)\big)$, $\mathbf{A}\in\mathcal{C}\big(\overline{J}_N;\mathbf{L}^2(\Omega)\big)$, and so that $[\psi]_n=0$, $[\mathbf{A}]_n=0$.

4. The finite element spaces and interpolation

In this section, we prove a characterisation of the finite element space $\mathbf{V}_{\mathcal{T},\mathbf{n}}$, and construct an interpolation operator that preserves the boundary condition in $\mathbf{V}_{\mathcal{T},\mathbf{n}}$.

Take a triangulation $\mathcal{T} \in \mathcal{F}$, let $\{a_i\}_{i\in\mathcal{N}}$, where $\mathcal{N}$ is an index set, be the set of nodes of $\mathcal{T}$, and let $\{\phi_i\}_{i\in\mathcal{N}}$ be the corresponding nodal basis for $V_{\mathcal{T}}$. Recall that Ω is a polygonal domain in $\mathbb{R}^2$, so that every corner of Ω is a node a_i of the triangulation. Define the index sets $\mathcal{N}_b$ and $\mathcal{N}_c$ by

$$\begin{aligned}\mathcal{N}_b &= \{i \in \mathcal{N} : a_i \in \partial\Omega,\ a_i \text{ is not a corner of } \Omega\},\\ \mathcal{N}_c &= \{i \in \mathcal{N} : a_i \text{ is a corner of } \Omega\}.\end{aligned}$$

The space $\mathbf{V}_{\mathcal{T},\mathbf{n}}$ is characterised in the following lemma.

Lemma 4.1. *We have*

$$\mathbf{V}_{\mathcal{T},\mathbf{n}} = \Big\{\mathbf{v} = \sum_{i\in\mathcal{N}} \mathbf{b}_i\phi_i : \mathbf{b}_i \in \mathbb{R}^2;\ \mathbf{b}_i\cdot\mathbf{n}_i = 0,\ i \in \mathcal{N}_b;\ \mathbf{b}_i = 0,\ i\in\mathcal{N}_c\Big\}, \tag{16}$$

where $\mathbf{n}_i = \mathbf{n}(a_i)$ for $i \in \mathcal{N}_b$.

Proof. Let $\tilde{\mathbf{V}}$ be the right hand side of (16). We first show that $\mathbf{V}_{\mathcal{T},\mathbf{n}} \subset \tilde{\mathbf{V}}$. Take $\mathbf{v} = \sum_{j\in\mathcal{N}} \mathbf{b}_j\phi_j \in \mathbf{V}_{\mathcal{T},\mathbf{n}}$. If $i \in \mathcal{N}_b$, then $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$ implies that $\mathbf{b}_i\cdot\mathbf{n}_i = 0$. For $i \in \mathcal{N}_c$, let $\mathbf{n}_i^1$ and $\mathbf{n}_i^2$ be the outward unit normal vectors on each side of the corner. Continuity of $\mathbf{v}$ implies that $\mathbf{b}_i\cdot\mathbf{n}_i^1 = 0$ and $\mathbf{b}_i\cdot\mathbf{n}_i^2 = 0$. But $\mathbf{n}_i^1$ and $\mathbf{n}_i^2$ are linearly independent, so $\mathbf{b}_i = 0$. Thus, $\mathbf{V}_{\mathcal{T},\mathbf{n}} \subset \tilde{\mathbf{V}}$.

We finish the proof by showing that $\tilde{\mathbf{V}} \subset \mathbf{V}_{\mathcal{T},\mathbf{n}}$. Take $\mathbf{v} \in \tilde{\mathbf{V}}$. Let e be an edge of Ω, $\mathcal{N}(e) = \{i \in \mathcal{N} : a_i \in e\}$, and $\mathbf{n}_e$ the outward normal of the edge e. Then, for $x \in e$, $\mathbf{v}(x) = \sum_{i\in\mathcal{N}(e)\setminus\mathcal{N}_c} \mathbf{b}_i\phi_i(x)$, so $\mathbf{v}(x)\cdot\mathbf{n}(x) = \sum_{i\in\mathcal{N}(e)\setminus\mathcal{N}_c} \mathbf{b}_i\cdot\mathbf{n}_i\phi_i(x) = 0$, since $\mathbf{n}(x) = \mathbf{n}_e$ for $x \in e$ and $\mathbf{n}_e = \mathbf{n}_i$ for $i \in \mathcal{N}(e)\setminus\mathcal{N}_c$. But e is an arbitrary edge, so we have $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, and thus $\tilde{\mathbf{V}} \subset \mathbf{V}_{\mathcal{T},\mathbf{n}}$. $\square$

Following [21], we shall construct an interpolation operator on $\mathbf{W}^{l,p}(\Omega)$, based on local averages, that preserves the boundary condition $\mathbf{v}\cdot\mathbf{n} = 0$, where l and p satisfy

$$1 \le p < \infty, \quad \text{and} \quad l \ge 1, \text{ if } p = 1, \quad \text{and} \quad l > 1/p, \text{ if } p \ne 1. \tag{17}$$

We first summarize the main results of [21] in the following theorem, and also observe that this theorem can be directly extended to complex-valued functions.

Theorem 4.2. *Let l and p satisfy (17). Then there exists a projection operator $\Pi : W^{l,p}(\Omega) \to V_{\mathcal{T}}$ (or $\Pi : \mathcal{W}^{l,p}(\Omega) \to \mathcal{V}_{\mathcal{T}}$) that preserves the boundary condition $v|_{\partial\Omega} = 0$, and for $0 \le m \le l \le 2$,*

$$\Big(\sum_{K\in\mathcal{T}} \big(h_K^{-(l-m)}\|v - \Pi v\|_{W^{m,p}(K)}\big)^p\Big)^{\frac{1}{p}} \le c|v|_{W^{l,p}(\Omega)}, \quad \forall v \in W^{l,p}(\Omega)\ \big(or\ \mathcal{W}^{l,p}(\Omega)\big).$$

We now describe the construction of the interpolation operator for $\mathbf{V}_{\mathcal{T}}$. For each node $a_i \in \mathcal{T}$, depending on the type of node, we choose one triangle edge σ_i (or two triangle edges σ_i^1, σ_i^2) in the following way: If $i \notin \mathcal{N}_c$, then we let σ_i be any edge K' of any triangle K such that $a_i \in \overline{K'}$, with the restriction $K' \subset \partial\Omega$ if $a_i \in \partial\Omega$. If $i \in \mathcal{N}_c$, then we let σ_i^1, σ_i^2 be edges K_1', K_2' of a triangle (or two triangles), one on each side of the corner such that $K_1', K_2' \subset \partial\Omega$ and $a_i \in \overline{K_1'} \cap \overline{K_2'}$. Further, let $\mathbf{n}_i^k, \mathbf{t}_i^k$, $k = 1, 2$, be unit normal, respectively tangent, vectors, to σ_i^1 and σ_i^2.

To simplify the notation, we set $\sigma_i^1 = \sigma_i$ for $i \notin \mathcal{N}_c$. For each node a_i we have thus chosen edges σ_i^k, $k \in \mathcal{K}_i$, where $\mathcal{K}_i = \{1\}$ if $i \notin \mathcal{N}_c$, and $\mathcal{K}_i = \{1, 2\}$ if $i \in \mathcal{N}_c$.

For $i \in \mathcal{N}$, let $\{a_{i,1}^k, a_{i,2}^k\}$ be the nodal points in σ_i^k, with $a_{i,1}^k = a_i$, $k \in \mathcal{K}_i$. Let $\{\phi_{i,1}^k, \phi_{i,2}^k\}$ be the corresponding nodal basis on σ_i^k. Then we have an $L^2(\sigma_i^k)$-dual basis $\{\psi_{i,1}^k, \psi_{i,2}^k\}$, defined by

$$\int_{\sigma_i^k} \phi_{i,m}^k(x)\psi_{i,l}^k(x)\,dx = \delta_{lm}, \quad l,m = 1,2,\ k \in \mathcal{K}_i.$$

Note that $\phi_i|_{\sigma_i^k} = \phi_{i,1}^k$, and set

$$\psi_i^k = \psi_{i,1}^k, \quad \forall i \in \mathcal{N},\ k \in \mathcal{K}_i.$$

Then, we have

$$\int_{\sigma_i^k} \phi_j(\xi)\psi_i^k(\xi)\,d\xi = \delta_{ij}, \quad i \in \mathcal{N},\ j \in \mathcal{N},\ k \in \mathcal{K}_i. \tag{18}$$

Now, we define the interpolation operator $\mathbf{\Pi} : \mathbf{W}^{l,p}(\Omega) \to \mathbf{V}_\mathcal{T}$ by

$$\mathbf{\Pi v} = \sum_{i \in \mathcal{N}} \mathbf{c}_i(\mathbf{v})\phi_i, \tag{19}$$

where

$$\mathbf{c}_i(\mathbf{v}) = \begin{cases} \displaystyle\int_{\sigma_i^1} \mathbf{v}(\xi)\psi_i^1(\xi)\,d\xi, & i \notin \mathcal{N}_c, \\ \displaystyle\sum_{k=1}^{2} \int_{\sigma_i^{k'}} \mathbf{v}(\xi) \cdot \frac{\mathbf{n}_i^{k'}}{\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k} \psi_i^{k'}(\xi)\,d\xi\,\mathbf{t}_i^k, & i \in \mathcal{N}_c, \end{cases} \tag{20}$$

and

$$k' = \begin{cases} 2, & k = 1, \\ 1, & k = 2. \end{cases}$$

Observe that $\{\frac{\mathbf{n}_i^{1'}}{\mathbf{n}_i^{1'}\cdot\mathbf{t}_i^1}, \frac{\mathbf{n}_i^{2'}}{\mathbf{n}_i^{2'}\cdot\mathbf{t}_i^2}\}$ is the dual basis to $\{\mathbf{t}_i^1, \mathbf{t}_i^2\}$, i.e.,

$$\frac{\mathbf{n}_i^{k'}}{\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k} \cdot \mathbf{t}_i^l = \delta_{kl}, \quad k,l = 1,2. \tag{21}$$

The operator $\mathbf{\Pi}$ has the following properties.

Lemma 4.3. *Let l and p satisfy (17). Then the operator $\mathbf{\Pi}$ defined by (19) is a projection from $\mathbf{W}^{l,p}(\Omega)$ onto $\mathbf{V}_\mathcal{T}$, that preserves the boundary condition $\mathbf{v}\cdot\mathbf{n} = 0$, i.e., if $\mathbf{v} \in \mathbf{W}^{l,p}(\Omega)$ satisfies $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, then $\mathbf{\Pi v} \in \mathbf{V}_{\mathcal{T},\mathbf{n}}$.*

Proof. $\mathbf{\Pi}$ is well-defined, since owing to the trace inequality and (17), we have for $i \in \mathcal{N}$,

$$|\mathbf{c}_i(\mathbf{v})|_1 \le c \max_{k \in \mathcal{K}_i} \|\psi_i^k\|_{L^\infty(\sigma_i^k)} \|\mathbf{v}\|_{W^{l,p}(\Omega)} < \infty, \quad \forall \mathbf{v} \in \mathbf{W}^{l,p}(\Omega).$$

To prove that $\mathbf{\Pi}$ is a projection, let $\mathbf{v} \in \mathbf{V}_\mathcal{T}$ and write $\mathbf{v} = \sum_{j\in\mathcal{N}} \mathbf{b}_j\phi_j$, where

$$\mathbf{b}_j = b_j^2\mathbf{t}_j^1 + b_j^2\mathbf{t}_j^2, \quad \text{if } j \in \mathcal{N}_c.$$

We write $\mathbf{c}_i(\mathbf{v})$ defined in (20) in the same way, i.e.,

$$\mathbf{c}_i(\mathbf{v}) = c_i^1(\mathbf{v})\mathbf{t}_i^1 + c_i^2(\mathbf{v})\mathbf{t}_i^2, \quad \text{if } i \in \mathcal{N}_c. \tag{22}$$

If $i \notin \mathcal{N}_c$, then (18) gives

$$\begin{aligned} c_i(\mathbf{v}) &= \int_{\sigma_i^1} \Big(\sum_{j \notin \mathcal{N}_c} \mathbf{b}_j \phi_j(\xi) + \sum_{j \in \mathcal{N}_c} (b_j^1 \mathbf{t}_j^1 + b_j^2 \mathbf{t}_j^2) \phi_j(\xi) \Big) \psi_i^1(\xi)\, d\xi \\ &= \sum_{j \notin \mathcal{N}_c} \mathbf{b}_j \int_{\sigma_i^1} \phi_j(\xi) \psi_i^1(\xi)\, d\xi + \sum_{j \in \mathcal{N}_c} (b_j^1 \mathbf{t}_j^1 + b_j^2 \mathbf{t}_j^2) \int_{\sigma_i^1} \phi_j(\xi) \psi_i^1(\xi)\, d\xi \\ &= \mathbf{b}_i. \end{aligned}$$

Further, if $i \in \mathcal{N}_c$, then (18) and (21) give

$$\begin{aligned} c_i^k(\mathbf{v}) &= \int_{\sigma_i^{k'}} \Big(\sum_{j \notin \mathcal{N}_c} \mathbf{b}_j \phi_j(\xi) + \sum_{j \in \mathcal{N}_c} (b_j^1 \mathbf{t}_j^1 + b_j^2 \mathbf{t}_j^2) \phi_j(\xi) \Big) \cdot \frac{\mathbf{n}_i^{k'}}{\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k} \psi_i^{k'}(\xi)\, d\xi \\ &= \sum_{j \notin \mathcal{N}_c} \mathbf{b}_j \cdot \frac{\mathbf{n}_i^{k'}}{\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k} \int_{\sigma_i^{k'}} \phi_j(\xi) \psi_i^{k'}(\xi)\, d\xi \\ &\quad + \sum_{j \in \mathcal{N}_c} (b_j^1 \mathbf{t}_j^1 + b_j^2 \mathbf{t}_j^2) \cdot \frac{\mathbf{n}_i^{k'}}{\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k} \int_{\sigma_i^{k'}} \phi_j(\xi) \psi_i^{k'}(\xi)\, d\xi \\ &= b_i^k, \quad k = 1, 2. \end{aligned}$$

Thus, $\mathbf{c}_i(\mathbf{v}) = \mathbf{b}_i$ for $i \in \mathcal{N}$, and we conclude that $\mathbf{\Pi v} = \mathbf{v}$ for $\forall \mathbf{v} \in \mathbf{V}_{\mathcal{T}}$.

We now prove that $\mathbf{\Pi}$ preserves the boundary condition. We will use the characterisation in Lemma 4.1. Let $\mathbf{v} \in \mathbf{W}^{l,p}(\Omega)$ be such that $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Clearly (20) implies that $\mathbf{c}_i(\mathbf{v}) = 0$, if $i \in \mathcal{N}_c$, and $\mathbf{c}_i(\mathbf{v}) \cdot \mathbf{n}_i = 0$, if $i \in \mathcal{N}_b$. Thus, $\mathbf{\Pi v} \in \mathbf{V}_{\mathcal{T},\mathbf{n}}$. ☐

The following stability result for $\mathbf{\Pi}$ is analogous to Theorem 3.1 in [21].

Lemma 4.4. *Let l and p satisfy (17), $1 \le q \le \infty$, and let m be any nonnegative integer. Then*

$$\|\mathbf{\Pi v}\|_{W^{m,q}(K)} \le c \sum_{k=0}^{l} h_K^{k-m+\frac{2}{q}-\frac{2}{p}} |\mathbf{v}|_{W^{k,p}(S_K)}, \quad \forall \mathbf{v} \in \mathbf{W}^{l,p}(\Omega), \tag{23}$$

for $K \in \mathcal{T}$, where $S_K = \operatorname{int}\Big(\bigcup \{ \overline{K_i} : \overline{K_i} \cap \overline{K} \neq \emptyset,\, K_i \in \mathcal{T} \} \Big)$.

Proof. Recall that $\|\mathbf{v}\|_{W^{m,q}(K)} = \big(\|v_1\|_{W^{m,q}(K)}^q + \|v_2\|_{W^{m,q}(K)}^q \big)^{\frac{1}{q}}$, if $\mathbf{v} = (v_1, v_2)$. Let $\mathcal{N}(K) = \{ i \in \mathcal{N} : a_i \in \overline{K} \}$. Then, we have

$$\begin{aligned} \|\mathbf{\Pi v}\|_{W^{m,q}(K)} &= \Big(\sum_{i \in \mathcal{N}(K)} |\mathbf{c}_i(\mathbf{v})|_q^q \|\phi_i\|_{W^{m,q}(K)}^q \Big)^{\frac{1}{q}} \\ &\le \max_{i \in \mathcal{N}(K)} \|\phi_i\|_{W_{m,q}(K)} \Big(\sum_{i \in \mathcal{N}(K)} |\mathbf{c}_i(\mathbf{v})|_q^q \Big)^{\frac{1}{q}} \\ &\le c h_K^{-m+\frac{2}{q}} \sum_{i \in \mathcal{N}(K)} |\mathbf{c}_i(\mathbf{v})|_q, \end{aligned} \tag{24}$$

where we have used the estimate $\|\phi_i\|_{W^{m,q}(K)} \le c h_K^{-m+\frac{2}{q}}$, for $i \in \mathcal{N}$, from [21]. We now estimate $|\mathbf{c}_i(\mathbf{v})|_q$. If $i \notin \mathcal{N}_c$,

$$\begin{aligned} |\mathbf{c}_i(\mathbf{v})|_q &= \Big| \int_{\sigma_i} \mathbf{v}(\xi) \psi_i^1(\xi)\, d\xi \Big|_q \le c \Big| \int_{\sigma_i} \mathbf{v}(\xi) \psi_i^1(\xi)\, d\xi \Big|_1 \\ &\le c \|\psi_i^1\|_{L^\infty(\sigma_i)} \|\mathbf{v}\|_{L^1(\sigma_i)}, \end{aligned}$$

and, if $i \in \mathcal{N}_c$,

$$\begin{aligned}
|\mathbf{c}_i(\mathbf{v})|_q &\le \sum_{k=1}^{2} \Big| \int_{\sigma_i^{k'}} \mathbf{v}(\xi) \cdot \frac{\mathbf{n}_i^{k'}}{\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k} \psi_i^{k'}(\xi)\, d\xi \Big| |\mathbf{t}_i^k|_q \\
&\le \sum_{k=1}^{2} \|\psi_i^{k'}\|_{L^\infty(\sigma_i^{k'})} \frac{|\mathbf{n}_i^{k'}|_\infty |\mathbf{t}_i^k|_q}{|\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k|} \int_{\sigma_i^{k'}} |\mathbf{v}|_1\, d\xi \\
&\le c \max_k \{ \|\psi_i^k\|_{L^\infty(\sigma_i^k)} \} \sum_{k=1}^{2} \|\mathbf{v}\|_{L^1(\sigma_i^k)},
\end{aligned}$$

where the constant c depends on Ω, since $|\mathbf{n}_i^{k'} \cdot \mathbf{t}_i^k|$ depends on angle of the corner. From [21], we have

$$\max_{i \in \mathcal{N}, k \in \mathcal{K}_i} \|\psi_i^k\|_{L^\infty(\sigma_i^k)} \le c h_K^{-1},$$

so together,

$$|\mathbf{c}_i(\mathbf{v})|_q \le c h_K^{-1} \sum_{k \in \mathcal{K}_i} \|\mathbf{v}\|_{L^1(\sigma_i^k)}, \quad i \in \mathcal{N}. \tag{25}$$

Finally, again from [21], we have

$$\|\mathbf{v}\|_{L^1(\tilde{K}')} \le c \sum_{k=0}^{l} h_K^{1+k-\frac{2}{p}} |\mathbf{v}|_{W^{k,p}(\tilde{K})}, \tag{26}$$

where $\tilde{K}'$ is any edge of an arbitrary triangle $\tilde{K} \subset S_K$. Putting together estimates (24), (25), and (26), we obtain (23). □

We are now in the position to prove the following approximation result for $\mathbf{\Pi}$, which is the main result in this section.

Theorem 4.5. *Let l and p satisfy (17), and $0 \le m \le l \le 2$. Then*

$$\Big(\sum_{K \in \mathcal{T}} \big(h_K^{-(l-m)} \|\mathbf{v} - \mathbf{\Pi}\mathbf{v}\|_{W^{m,p}(K)} \big)^p \Big)^{\frac{1}{p}} \le c |\mathbf{v}|_{W^{l,p}(\Omega)}, \quad \forall \mathbf{v} \in \mathbf{W}^{l,p}(\Omega).$$

Proof. Let $K \in \mathcal{T}$ and $\mathbf{v} \in \mathbf{W}^{l,p}(\Omega)$. For any polynomial $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2)$, $p_1, p_2 \in \mathcal{P}_1^2$ (linear functions of two variables), we have by Lemma 4.3 and Lemma 4.4,

$$\begin{aligned}
\|\mathbf{v} - \mathbf{\Pi}\mathbf{v}\|_{W^{m,p}(K)} &\le \|\mathbf{v} - \mathbf{p}\|_{W^{m,p}(K)} + \|\mathbf{\Pi}(\mathbf{p} - \mathbf{v})\|_{W^{m,p}(K)} \\
&\le \|\mathbf{v} - \mathbf{p}\|_{W^{m,p}(K)} + c \sum_{k=0}^{l} h_K^{k-m} |\mathbf{v} - \mathbf{p}|_{W^{k,p}(S_K)} \\
&\le c \sum_{k=0}^{l} h_K^{k-m} \|\mathbf{v} - \mathbf{p}\|_{W^{k,p}(S_K)}.
\end{aligned}$$

A routine application of the Bramble-Hilbert lemma, see [21], yields

$$\inf_{\mathbf{p}} \|\mathbf{v} - \mathbf{p}\|_{W^{k,p}(K)} \le c h_K^{l-k} |\mathbf{v}|_{W^{l,p}(S_k)}, \quad 0 \le k \le l \le 2,$$

so that

$$\|\mathbf{v} - \mathbf{\Pi}\mathbf{v}\|_{W^{m,p}(K)} \le c h_K^{l-m} |\mathbf{v}|_{W^{l,p}(S_K)}.$$

Summing over all triangles $K \in \mathcal{T}$, observing that

$$\sup_{K \in \mathcal{T}} \operatorname{card}\{\tilde{K} \in \mathcal{T} : \tilde{K} \cap S_K \ne \emptyset\},$$

is a constant depending only on γ_0 in (7), completes the proof. □

5. A posteriori error estimates

In this section, we state and prove an a posteriori error estimate for the discontinuous Galerkin method. We first state the theorem, and then prove it in the following subsections. More precisely, in subsection 5.1, we define the residuals of the computed solutions $\psi_{\mathcal{D}}$, $\mathbf{A}_{\mathcal{D}}$ as linear functionals acting on certain spaces of test functions, and derive an error representation in terms of the residuals and a linear dual problem. In subsection 5.2, we estimate the these abstractly defined residuals in terms of computable residuals and certain derivatives of the test functions, using the interpolation results from the previous section. Finally, in subsection 5.3, we discuss estimates of the solutions to the dual problem.

To state our a posteriori error estimate in a convenient form, we introduce a spatial mesh function h, a temporal mesh function k and a temporal jump function $[v]$ on $\Omega \times J_N$ defined by

$$h|_{\Omega\times I_n} = h_{\mathcal{T}_n}, \quad k|_{\Omega\times I_n} = k_n, \quad [v]|_{\Omega\times I_n} = [v]_{n-1}, \tag{27}$$

for $n = 1,\dots,N$. We also define the computable residuals R_1 and R_2 as functions on $\Omega \times J_N$ by

$$R_1 = |R_{1,i}| + R_{1,e}, \qquad R_2 = |\mathbf{R}_{2,i}| + R_{2,e}, \tag{28}$$

where (subscripts i and e denote "interior" respectively "edge")

$$\begin{aligned}
R_{1,i}|_{K\times I_n} &= \eta\frac{[\psi_{\mathcal{D}}]}{k} + \Big(\frac{i}{\kappa}\nabla + \mathbf{A}_{\mathcal{D}}\Big)^2\psi_{\mathcal{D}} - i\eta\kappa(\nabla\cdot\mathbf{A}_{\mathcal{D}})\psi_{\mathcal{D}} + \big(|\psi_{\mathcal{D}}|^2 - 1\big)\psi_{\mathcal{D}},\\
\mathbf{R}_{2,i}|_{K\times I_n} &= \frac{[\mathbf{A}_{\mathcal{D}}]}{k} - \Delta\mathbf{A}_{\mathcal{D}} + \mathrm{Re}\Big\{\frac{i}{\kappa}\overline{\psi}_{\mathcal{D}}\nabla\psi_{\mathcal{D}}\Big\} + |\psi_{\mathcal{D}}|^2\mathbf{A}_{\mathcal{D}} - \mathcal{J}(\nabla\times H),\\
R_{1,e}|_{K\times I_n} &= \big(h_K|K|\big)^{-\frac12}\frac{1}{\kappa^2}\Big(\Big\|\frac12\Big\lceil\frac{\partial\psi_{\mathcal{D}}}{\partial n}\Big\rceil\Big\|^2_{L^2(\partial K\setminus\partial\Omega)} + \Big\|\frac{\partial\psi_{\mathcal{D}}}{\partial n}\Big\|^2_{L^2(\partial K\cap\partial\Omega)}\Big)^{\frac12},\\
R_{2,e}|_{K\times I_n} &= \big(h_K|K|\big)^{-\frac12}\big(\|\tfrac12\lceil\nabla\cdot\mathbf{A}_{\mathcal{D}}\rceil\|^2_{L^2(\partial K\setminus\partial\Omega)} + \|\tfrac12\lceil\nabla\times\mathbf{A}_{\mathcal{D}}\rceil\|^2_{L^2(\partial K\setminus\partial\Omega)}\\
&\qquad + \|\nabla\times\mathbf{A}_{\mathcal{D}} - \mathcal{J}H\|^2_{L^2(\partial K\cap\partial\Omega)}\big)^{\frac12},
\end{aligned}$$

for $K \in \mathcal{T}_n$, $n = 1,\dots,N$, and $\lceil\cdot\rceil$ denote the jumps over the (interior) edges ∂K, $|K|$ the area of K, and $-\Delta\mathbf{v} = \nabla\times\nabla\times\mathbf{v} - \nabla(\nabla\cdot\mathbf{v})$. Also, $\mathcal{J}_n$ is the orthogonal projection of $L^2(I_n)$ onto P^q, where P^q is the space of polynomials of degree $\le q$ on I_n, and we set $\mathcal{J}|_{I_n} = \mathcal{J}_n$, $n = 1,\dots,N$.

Further, we introduce a temporal growth factor $G_{l,N}$ defined by

$$G_{l,N} = \begin{cases} \frac{t_N^{1-l/2}}{(2-l)^{1/2}}, & 1 \le l < 2,\\ \big(\log\frac{t_N}{k_N}\big)^{\frac12} + 1, & l = 2. \end{cases} \tag{29}$$

The main result in this paper is the following a posteriori error estimate. We use the abbreviation $\|\cdot\|_I = \|\cdot\|_{L^\infty(I;L^2(\Omega))}$.

Theorem 5.1. *Assume that $q = 0$, and for some l, l', $1 \le l, l' \le 2$ the stability assumptions (53) and (54) hold for the solutions to the dual problem (42)–(43), and that H is constant in time on $\partial\Omega$. Then, we have for the errors $e = \psi_{\mathcal{D}} - \psi$, $\mathbf{e} = \mathbf{A}_{\mathcal{D}} - \mathbf{A}$,*

$$\begin{aligned}
\big(\eta\|e_N^-\|^2 + \|\mathbf{e}_N^-\|^2\big)^{\frac12} &\le c_1 G_{l,N} S^{space}_{l,1,N}\|h^l R_1\|_{J_N} + c_2 G_{l',N} S^{space}_{l',2,N}\|h^{l'} R_2\|_{J_N}\\
&\quad + G_{2,N} S^{time}_{1,N}\|\eta[\psi_{\mathcal{D}}]\|_{J_N} + G_{2,N} S^{time}_{2,N}\big(\|[\mathbf{A}_{\mathcal{D}}]\|_{J_N} + \|k\nabla\times(H - \mathcal{J}H)\|_{J_N}\big).
\end{aligned}$$

Remark 5.1. We may write $\|h^l R_1\|_{J_N} = \max_{1\le n\le N}\|h^l_{\mathcal{T}_n} R_1\|_{I_n}$, and similarly for the other terms in the error estimate.

Remark 5.2. Observe that we only require that the triangulations are nondegenerate.

Remark 5.3. We use the interpolation operators introduced in the previous section in the proof of Theorem 5.1. However, one can use $L^2(\Omega)$-projections instead, as in [10], [12]. One may then replace $[\psi_{\mathcal{D}}]$ and $[\mathbf{A}_{\mathcal{D}}]$ in $R_{1,i}$ and $\mathbf{R}_{2,i}$ by $[\psi_{\mathcal{D}}]^\star$ and $[\mathbf{A}_{\mathcal{D}}]^\star$, where $[\cdot]^\star$ is a function on $\Omega \times J_N$ defined by $[\cdot]^\star|_{I_n} = [\cdot]$, if $\mathcal{T}_n \supset \mathcal{T}_{n-1}$, and $[\cdot]^\star = 0$ otherwise. But we then have to impose conditions on the variation of mesh size.

Remark 5.4. Theorem 5.1 also remains valid if we use two different triangulations $\mathcal{T}_{1,n}$ and $\mathcal{T}_{2,n}$, with corresponding finite element spaces $\mathcal{V}_{\mathcal{T}_{1,n}}$ and $\mathbf{V}_{\mathcal{T}_{2,n},\mathbf{n}}$, in the definition of $\mathcal{W}_{\mathcal{D}}^n$ and $\mathbf{W}_{\mathcal{D},\mathbf{n}}^n$ in (8). One then replace h^l with h_1^l and $h^{l'}$ with $h_2^{l'}$, where h_1 and h_2 are the corresponding spatial mesh functions.

5.1. **Error representation by duality.** We define the residuals as functionals $r_1 : \tilde{\mathcal{W}}(J_N) \to \mathbb{C}$ and $r_2 : \tilde{\mathbf{W}}_{\mathbf{n}}(J_N) \to \mathbb{R}$ by

$$\begin{aligned} r_1(v) &= B_1(\psi_{\mathcal{D}}, \mathbf{A}_{\mathcal{D}}, v) - \eta(\psi_0, v_0^+), \quad \forall v \in \tilde{\mathcal{W}}(J_N), \\ r_2(\mathbf{v}) &= B_2(\psi_{\mathcal{D}}, \mathbf{A}_{\mathcal{D}}, \mathbf{v}) - (\mathbf{A}_0, \mathbf{v}_0^+) - \int_{J_N} (H, \nabla \times \mathbf{v})\, dt, \ \forall \mathbf{v} \in \tilde{\mathbf{W}}_{\mathbf{n}}(J_N). \end{aligned} \tag{30}$$

Recall that the spaces of test functions, $\tilde{\mathcal{W}}(J_N)$, $\tilde{\mathbf{W}}_{\mathbf{n}}(J_N)$, were defined in (13). When restricted to the space $\mathcal{C}_0^\infty(\Omega \times J_N)$ of smooth test functions, r_1 is the distribution $\eta\psi_{\mathcal{D},t} + \big(\frac{i}{\kappa}\nabla + \mathbf{A}_{\mathcal{D}}\big)^2\psi_{\mathcal{D}} - i\eta\kappa(\nabla \cdot \mathbf{A}_{\mathcal{D}})\psi_{\mathcal{D}} + \big(|\psi_{\mathcal{D}}|^2 - 1\big)\psi_{\mathcal{D}}$, and similarly for r_2.

Observe that (14) means that

$$\begin{aligned} r_1(v) &= 0, \quad \forall v \in \mathcal{W}_{\mathcal{D}}(J_N), \\ r_2(\mathbf{v}) &= 0, \quad \forall \mathbf{v} \in \mathbf{W}_{\mathcal{D},\mathbf{n}}(J_N), \end{aligned} \tag{31}$$

that is, the residuals r_1 and r_2 are orthogonal to the corresponding approximation spaces $\mathcal{W}_{\mathcal{D}}(J_N)$ and $\mathbf{W}_{\mathcal{D},\mathbf{n}}(J_N)$.

We have the following error representation.

Proposition 5.2. *We have*

$$\eta\|e_N^-\|^2 + \|\mathbf{e}_N^-\|^2 = \mathrm{Re}\ r_1(z) + r_2(\mathbf{z}),$$

where z and $\mathbf{z}$ are the solutions to the dual problem (42)–(43).

Proof. We start the proof by using (15) to rewrite (30) as

$$\begin{aligned} r_1(v) &= B_1(\psi_{\mathcal{D}}, \mathbf{A}_{\mathcal{D}}, v) - B_1(\psi, \mathbf{A}, v) \\ &= \eta \sum_{n=1}^{N} \int_{I_n} (\psi_{\mathcal{D},t} - \psi_t, v)\, dt + \eta(\psi_{\mathcal{D},0}^+ - \psi_0^+, v_0^+) + \eta \sum_{n=1}^{N-1} ([\psi_{\mathcal{D}} - \psi]_n, v_n^+) \\ &\quad + \int_{J_N} \Big\{ \frac{1}{\kappa^2}(\nabla\psi_{\mathcal{D}} - \nabla\psi, \nabla v) + \frac{i}{\kappa}(\mathbf{A}_{\mathcal{D}} \cdot \nabla\psi_{\mathcal{D}} - \mathbf{A} \cdot \nabla\psi, v) \Big\}\, dt \\ &\quad + \int_{J_N} \Big\{ -\frac{i}{\kappa}(\mathbf{A}_{\mathcal{D}}\psi_{\mathcal{D}} - \mathbf{A}\psi, \nabla v) + (\mathbf{A}_{\mathcal{D}}^2\psi_{\mathcal{D}} - \mathbf{A}^2\psi, v) \Big\}\, dt, \\ &\quad + \int_{J_N} \Big\{ -i\eta\kappa\big((\nabla \cdot \mathbf{A}_{\mathcal{D}})\psi_{\mathcal{D}} - (\nabla \cdot \mathbf{A})\psi, v\big) \Big\}\, dt \\ &\quad + \int_{J_N} \Big\{ \big((|\psi_{\mathcal{D}}|^2 - 1)\psi_{\mathcal{D}} - (|\psi|^2 - 1)\psi, v\big) \Big\}\, dt, \quad \forall v \in \tilde{\mathcal{W}}(J_N), \end{aligned} \tag{32}$$

and

$$
\begin{aligned}
r_2(\mathbf{v}) &= B_2(\psi_{\mathcal{D}}, \mathbf{A}_{\mathcal{D}}, \mathbf{v}) - B_2(\psi, \mathbf{A}, \mathbf{v}) \\
&= \sum_{n=1}^{N} \int_{I_n} (\mathbf{A}_{\mathcal{D},t} - \mathbf{A}_t, \mathbf{v})\, dt + (\mathbf{A}_{\mathcal{D},0}^{+} - \mathbf{A}_0^{+}, \mathbf{v}_0^{+}) + \sum_{n=1}^{N-1} ([\mathbf{A}_{\mathcal{D}} - \mathbf{A}]_n, \mathbf{v}_n^{+}) \\
&\quad + \int_{J_N} \big\{ (\nabla \times \mathbf{A}_{\mathcal{D}} - \nabla \times \mathbf{A}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{A}_{\mathcal{D}} - \nabla \cdot \mathbf{A}, \nabla \cdot \mathbf{v}) \big\}\, dt \\
&\quad + \int_{J_N} \Big(\mathrm{Re}\Big\{ \frac{i}{\kappa} \overline{\psi}_{\mathcal{D}} \nabla \psi_{\mathcal{D}} \Big\} - \mathrm{Re}\Big\{ \frac{i}{\kappa} \overline{\psi} \nabla \psi \Big\}, \mathbf{v} \Big)\, dt \\
&\quad + \int_{J_N} (|\psi_{\mathcal{D}}|^2 \mathbf{A}_{\mathcal{D}} - |\psi|^2 \mathbf{A}, \mathbf{v})\, dt, \quad \forall \mathbf{v} \in \tilde{\mathbf{W}}_{\mathbf{n}}(J_N).
\end{aligned} \tag{33}
$$

We linearise all the terms in (32) and (33). Let

$$
\begin{aligned}
\tilde{\psi} &= \frac{\psi + \psi_{\mathcal{D}}}{2}, & \tilde{\mathbf{A}} &= \frac{\mathbf{A} + \mathbf{A}_{\mathcal{D}}}{2}, \\
\tilde{\tilde{\psi}} &= \frac{|\psi|^2 + |\psi_{\mathcal{D}}|^2}{2}, & \hat{\mathbf{A}} &= \frac{\mathbf{A} - \mathbf{A}_{\mathcal{D}}}{2}.
\end{aligned}
$$

Easy calculations give

$$
\begin{aligned}
\mathbf{A}_{\mathcal{D}} \cdot \nabla \psi_{\mathcal{D}} - \mathbf{A} \cdot \nabla \psi &= \tilde{\mathbf{A}} \cdot \nabla e + \nabla \tilde{\psi} \cdot \mathbf{e}, \\
\mathbf{A}_{\mathcal{D}} \psi_{\mathcal{D}} - \mathbf{A} \psi &= \tilde{\mathbf{A}} e + \tilde{\psi} \mathbf{e}, \\
\mathbf{A}_{\mathcal{D}}^2 \psi_{\mathcal{D}} - \mathbf{A}^2 \psi &= (\tilde{\mathbf{A}}^2 + \hat{\mathbf{A}}^2) e + 2\tilde{\psi} \tilde{\mathbf{A}} \cdot \mathbf{e}, \\
(\nabla \cdot \mathbf{A}_{\mathcal{D}}) \psi_{\mathcal{D}} - (\nabla \cdot \mathbf{A}) \psi &= (\nabla \cdot \tilde{\mathbf{A}}) e + \tilde{\psi} (\nabla \cdot \mathbf{e}), \\
\big(|\psi_{\mathcal{D}}|^2 - 1\big) \psi_{\mathcal{D}} - \big(|\psi|^2 - 1\big) \psi &= (2\tilde{\tilde{\psi}} - 1) e + \psi \psi_{\mathcal{D}} \overline{e},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Re}\Big\{ \frac{i}{\kappa} \overline{\psi}_{\mathcal{D}} \nabla \psi_{\mathcal{D}} \Big\} - \mathrm{Re}\Big\{ \frac{i}{\kappa} \overline{\psi} \nabla \psi \Big\} &= \mathrm{Re}\Big\{ \frac{i}{\kappa} (\overline{\tilde{\psi}} \nabla e + \nabla \tilde{\psi} \overline{e}) \Big\}, \\
|\psi_{\mathcal{D}}|^2 \mathbf{A}_{\mathcal{D}} - |\psi|^2 \mathbf{A} &= \mathrm{Re}\Big\{ 2\overline{\tilde{\psi}} \tilde{\mathbf{A}} e \Big\} + \tilde{\tilde{\psi}} \mathbf{e}.
\end{aligned}
$$

We now insert these expressions into (32) and (33), to obtain

$$
\begin{aligned}
r_1(v) &= \eta \sum_{n=1}^{N} \int_{I_n} (e_t, v)\, dt + \eta (e_0^{+}, v_0^{+}) + \eta \sum_{n=1}^{N-1} ([e]_n, v_n^{+}) \\
&\quad + \int_{J_N} \Big\{ \frac{1}{\kappa^2} (\nabla e, \nabla v) + \frac{i}{\kappa} (\tilde{\mathbf{A}} \cdot \nabla e + \nabla \tilde{\psi} \cdot \mathbf{e}, v) \Big\}\, dt \\
&\quad + \int_{J_N} \Big\{ -\frac{i}{\kappa} (\tilde{\mathbf{A}} e + \tilde{\psi} \mathbf{e}, \nabla v) + \big((\tilde{\mathbf{A}}^2 + \hat{\mathbf{A}}^2) e + 2\tilde{\psi} \tilde{\mathbf{A}} \cdot \mathbf{e}, v \big) \Big\}\, dt \\
&\quad + \int_{J_N} \big\{ -i\eta\kappa \big((\nabla \cdot \tilde{\mathbf{A}}) e + \tilde{\psi} (\nabla \cdot \mathbf{e}), v \big) \big\}\, dt \\
&\quad + \int_{J_N} \big((2\tilde{\tilde{\psi}} - 1) e + \psi \psi_{\mathcal{D}} \overline{e}, v \big)\, dt, \quad \forall v \in \tilde{\mathcal{W}}(J_N),
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
r_2(\mathbf{v}) = & \sum_{n=1}^{N} \int_{I_n} (\mathbf{e}_t, \mathbf{v})\, dt + (\mathbf{e}_0^+, \mathbf{v}_0^+) + \sum_{n=1}^{N-1} ([\mathbf{e}]_n, \mathbf{v}_n^+) \\
& + \int_{J_N} \big\{ (\nabla \times \mathbf{e}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{v}) \big\}\, dt \\
& + \int_{J_N} \Big(\mathrm{Re} \Big\{ \frac{i}{\kappa} (\overline{\tilde{\psi}} \nabla e + \nabla \tilde{\psi} \overline{e}) \Big\}, \mathbf{v} \Big)\, dt \\
& + \int_{J_N} \Big(\mathrm{Re} \Big\{ 2\overline{\tilde{\psi}} \tilde{\mathbf{A}} e \Big\} + \tilde{\tilde{\psi}} \mathbf{e}, \mathbf{v} \Big)\, dt, \quad \forall \mathbf{v} \in \tilde{\mathbf{W}}_{\mathbf{n}}(J_N).
\end{aligned}
\tag{35}
$$

These equations are linearisations of (15). We now derive the dual problem corresponding to them. By Green's formula, integration by parts in time, and some rearrangements, we rewrite (34) and (35) as

$$
\begin{aligned}
r_1(v) = & \eta \sum_{n=1}^{N} \int_{I_n} (e, -v_t)\, dt + \eta \sum_{n=1}^{N-1} (e_n^-, -[v]_n) + \eta (e_N^-, v_N^-) \\
& + \int_{J_N} \Big\{ \frac{1}{\kappa^2} (\nabla e, \nabla v) + \frac{i}{\kappa} (\nabla e \cdot \tilde{\mathbf{A}}, v) - \frac{i}{\kappa} (\tilde{\mathbf{A}} e, \nabla v) + (\tilde{\mathbf{A}}^2 e, v) \Big\}\, dt \\
& + \int_{J_N} \Big\{ \big(e, \big(\hat{\mathbf{A}}^2 + i\eta\kappa(\nabla \cdot \tilde{\mathbf{A}}) + 2\tilde{\tilde{\psi}} - 1\big) v\big) + \big(\overline{e}, \overline{\psi \psi_{\mathcal{D}} v}\big) \Big\}\, dt \\
& + \int_{J_N} \Big(\mathbf{e}, -\frac{i}{\kappa} \nabla \overline{\tilde{\psi}} v + \frac{i}{\kappa} \overline{\tilde{\psi}} \nabla v + 2\overline{\tilde{\psi}} \tilde{\mathbf{A}} v - i\eta\kappa \nabla (\overline{\tilde{\psi}} v) \Big)\, dt \\
= & \eta \sum_{n=1}^{N} \int_{I_n} (e, -v_t)\, dt + \eta \sum_{n=1}^{N-1} (e_n^-, -[v]_n) + \eta (e_N^-, v_N^-) \\
& + \int_{J_N} \Big\{ \Big(\frac{i}{\kappa} \nabla e + \tilde{\mathbf{A}} e, \frac{i}{\kappa} \nabla v + \tilde{\mathbf{A}} v \Big) + (e, \gamma_{11} v) + (\overline{e}, \overline{\tilde{\gamma}}_{11} v) \Big\}\, dt \\
& + \int_{J_N} \Big(\mathbf{e}, -\frac{i}{\kappa} \nabla \overline{\tilde{\psi}} v + \frac{i}{\kappa} \overline{\tilde{\psi}} \nabla v + 2\overline{\tilde{\psi}} \tilde{\mathbf{A}} v - i\eta\kappa \nabla (\overline{\tilde{\psi}} v) \Big)\, dt,
\end{aligned}
\tag{36}
$$

where we write for short

$$
\begin{aligned}
\gamma_{11} &= \hat{\mathbf{A}}^2 + i\eta\kappa(\nabla \cdot \tilde{\mathbf{A}}) + 2\tilde{\tilde{\psi}} - 1, \\
\tilde{\gamma}_{11} &= \psi \psi_{\mathcal{D}},
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
r_2(\mathbf{v}) = & \sum_{n=1}^{N} \int_{I_n} (\mathbf{e}, -\mathbf{v}_t)\, dt + \sum_{n=1}^{N-1} (\mathbf{e}_n^-, -[\mathbf{v}]_n) + (\mathbf{e}_N^-, \mathbf{v}_N^-) \\
& + \int_{J_N} \big\{ (\nabla \times \mathbf{e}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{v}) + (\mathbf{e}, \tilde{\tilde{\psi}} \mathbf{v}) \big\}\, dt \\
& + \int_{J_N} \mathrm{Re} \Big\{ \Big(e, \frac{i}{\kappa} \nabla \cdot (\tilde{\psi} \mathbf{v}) + 2\tilde{\psi} \tilde{\mathbf{A}} \cdot \mathbf{v} \Big) + \Big(\overline{e}, -\frac{i}{\kappa} \nabla \overline{\tilde{\psi}} \cdot \mathbf{v} \Big) \Big\}\, dt \\
= & \sum_{n=1}^{N} \int_{I_n} (\mathbf{e}, -\mathbf{v}_t)\, dt + \sum_{n=1}^{N-1} (\mathbf{e}_n^-, -[\mathbf{v}]_n) + (\mathbf{e}_N^-, \mathbf{v}_N^-) \\
& + \int_{J_N} \big\{ (\nabla \times \mathbf{e}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{v}) + (\mathbf{e}, \tilde{\tilde{\psi}} \mathbf{v}) \big\}\, dt \\
& + \int_{J_N} \mathrm{Re} \Big(e, \frac{i}{\kappa} \nabla \cdot (\tilde{\psi} \mathbf{v}) + \frac{i}{\kappa} \nabla \tilde{\psi} \cdot \mathbf{v} + 2\tilde{\psi} \tilde{\mathbf{A}} \cdot \mathbf{v} \Big)\, dt,
\end{aligned}
\tag{38}
$$

since

$$\mathrm{Re}\,(\overline{e}, -\frac{i}{\kappa}\nabla\overline{\tilde{\psi}}\cdot\mathbf{v}) = \mathrm{Re}\,(e, \frac{i}{\kappa}\nabla\tilde{\psi}\cdot\mathbf{v}).$$

Adding the real parts of (36) to (38), observing that

$$\mathrm{Re}\,(\overline{e}, \overline{\tilde{\gamma}}_{11}v) = \mathrm{Re}\,(e, \tilde{\gamma}_{11}\overline{v}),$$

and

$$\begin{aligned}\mathrm{Re}\,\Big(\mathbf{e}, -\frac{i}{\kappa}\nabla\overline{\tilde{\psi}}v + \frac{i}{\kappa}\overline{\tilde{\psi}}\nabla v + 2\overline{\tilde{\psi}}\tilde{\mathbf{A}}v - i\eta\kappa\nabla(\overline{\tilde{\psi}}v)\Big)\\ = \Big(\mathbf{e}, \mathrm{Re}\Big\{-\frac{i}{\kappa}\nabla\overline{\tilde{\psi}}v + \frac{i}{\kappa}\overline{\tilde{\psi}}\nabla v + 2\tilde{\psi}\tilde{\mathbf{A}}v - i\eta\kappa\nabla(\overline{\tilde{\psi}}v)\Big\}\Big),\end{aligned}$$

we obtain

$$\begin{aligned}&\mathrm{Re}\ r_1(v) + r_2(\mathbf{v})\\ &= \mathrm{Re}\Big[\eta\sum_{n=1}^{N}\int_{I_n}(e, -v_t)\,dt + \eta\sum_{n=1}^{N-1}(e_n^-, -[v]_n) + \eta(e_N^-, v_N^-)\\ &+ \int_{J_N}\Big(\frac{i}{\kappa}\nabla e + \tilde{\mathbf{A}}e, \frac{i}{\kappa}\nabla v + \tilde{\mathbf{A}}v\Big)\,dt\\ &+ \int_{J_N}\Big(e, \gamma_{11}v + \tilde{\gamma}_{11}\overline{v} + \frac{i}{\kappa}\nabla\cdot(\tilde{\psi}\mathbf{v}) + \frac{i}{\kappa}\nabla\tilde{\psi}\cdot\mathbf{v} + 2\tilde{\psi}\tilde{\mathbf{A}}\cdot\mathbf{v}\Big)\,dt\Big]\\ &+ \sum_{n=1}^{N}\int_{I_n}(\mathbf{e}, -\mathbf{v}_t)\,dt + \sum_{n=1}^{N-1}(\mathbf{e}_n^-, -[\mathbf{v}]_n) + (\mathbf{e}_N^-, \mathbf{v}_N^-)\\ &+ \int_{J_N}\big\{(\nabla\times\mathbf{e}, \nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{e}, \nabla\cdot\mathbf{v})\big\}\,dt\\ &+ \int_{J_N}\Big(\mathbf{e}, \tilde{\tilde{\psi}}\mathbf{v} + \mathrm{Re}\Big\{-\frac{i}{\kappa}\nabla\overline{\tilde{\psi}}v + \frac{i}{\kappa}\overline{\tilde{\psi}}\nabla v + 2\overline{\tilde{\psi}}\tilde{\mathbf{A}}v - i\eta\kappa\nabla(\overline{\tilde{\psi}}v)\Big\}\Big)\,dt,\end{aligned} \tag{39}$$

for $v \in \tilde{\mathcal{W}}(J_N)$ and $\mathbf{v} \in \tilde{\mathbf{W}}_{\mathbf{n}}(J_N)$. We define

$$\begin{aligned}\tilde{B}_1 &: \tilde{\mathcal{W}}(J_N)\times\tilde{\mathcal{W}}(J_N)\times\tilde{\mathbf{W}}_{\mathbf{n}}(J_N)\to\mathbb{C},\\ \tilde{B}_2 &: \tilde{\mathbf{W}}_{\mathbf{n}}(J_N)\times\tilde{\mathcal{W}}(J_N)\times\tilde{\mathbf{W}}_{\mathbf{n}}(J_N)\to\mathbb{R},\end{aligned}$$

by

$$\begin{aligned}\tilde{B}_1(w, v, \mathbf{v}) = \eta\sum_{n=1}^{N}\int_{I_n}(w, -v_t)\,dt + \eta\sum_{n=1}^{N-1}(w_n^-, -[v]_n) + \eta(w_N^-, v_N^-)\\ + \int_{J_N}\Big\{\Big(\frac{i}{\kappa}\nabla w + \tilde{\mathbf{A}}w, \frac{i}{\kappa}\nabla v + \tilde{\mathbf{A}}v\Big) + (w, \gamma_{11}v + \tilde{\gamma}_{11}\overline{v} + \gamma_{12}\mathbf{v})\Big\}\,dt,\end{aligned}$$

and

$$\begin{aligned}\tilde{B}_2(\mathbf{w}, v, \mathbf{v}) = \sum_{n=1}^{N}\int_{I_n}(\mathbf{w}, -\mathbf{v}_t)\,dt + \sum_{n=1}^{N-1}(\mathbf{w}_n^-, -[\mathbf{v}]_n) + (\mathbf{w}_N^-, \mathbf{v}_N^-)\\ + \int_{J_N}\big\{(\nabla\times\mathbf{w}, \nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{w}, \nabla\cdot\mathbf{v}) + (\mathbf{w}, \tilde{\tilde{\psi}}\mathbf{v} + \gamma_{21}v)\big\}\,dt,\end{aligned}$$

where, for short,

$$\begin{aligned}\gamma_{12}\mathbf{v} &= \frac{i}{\kappa}\nabla\cdot(\tilde{\psi}\mathbf{v}) + \frac{i}{\kappa}\nabla\tilde{\psi}\cdot\mathbf{v} + 2\tilde{\psi}\tilde{\mathbf{A}}\cdot\mathbf{v},\\ \gamma_{21}v &= \mathrm{Re}\Big\{-\frac{i}{\kappa}\nabla\overline{\tilde{\psi}}v + \frac{i}{\kappa}\overline{\tilde{\psi}}\nabla v + 2\overline{\tilde{\psi}}\tilde{\mathbf{A}}v - i\eta\kappa\nabla(\overline{\tilde{\psi}}v)\Big\},\end{aligned} \tag{40}$$

and observe that $\tilde{B}_1$ and $\tilde{B}_2$ are well-defined in view of Theorem 1.1. Then we can rewrite (39) as

$$\operatorname{Re} r_1(v) + r_2(\mathbf{v}) = \operatorname{Re} \tilde{B}_1(e, v, \mathbf{v}) + \tilde{B}_2(\mathbf{e}, v, \mathbf{v}), \tag{41}$$

for $v \in \tilde{\mathcal{W}}(J_N)$ and $\mathbf{v} \in \tilde{\mathbf{W}}_{\mathbf{n}}(J_N)$. We now introduce the following problem, suggested by the form of $\tilde{B}_1$ and $\tilde{B}_2$:

Find $z \in \mathcal{W}(J_N)$ and $\mathbf{z} \in \mathbf{W}_{\mathbf{n}}(J_N)$ such that

$$\begin{aligned} \eta \int_{J_N} (w, -z_t)\, dt + \int_{J_N} \Big(\frac{i}{\kappa}\nabla w + \tilde{\mathbf{A}} w, \frac{i}{\kappa}\nabla z + \tilde{\mathbf{A}} z\Big)\, dt \\ + \int_{J_N} (w, \gamma_{11} z + \tilde{\gamma}_{11}\overline{z} + \gamma_{12}\mathbf{z})\, dt = 0, \quad \forall w \in L^2\big(J_N; \mathcal{H}^1(\Omega)\big), \end{aligned} \tag{42}$$

and

$$\begin{aligned} \int_{J_N} (\mathbf{w}, -\mathbf{z}_t)\, dt + \int_{J_N} \big\{(\nabla\times\mathbf{w}, \nabla\times\mathbf{z}) + (\nabla\cdot\mathbf{w}, \nabla\cdot\mathbf{z})\big\}\, dt \\ + \int_{J_N} (\mathbf{w}, \tilde{\tilde{\psi}}\mathbf{z} + \gamma_{21} z)\, dt = 0, \quad \forall \mathbf{w} \in L^2\big(J_N; \mathbf{H}^1_{\mathbf{n}}(\Omega)\big), \end{aligned} \tag{43}$$

with initial conditions $z(\cdot, t_N) = e_N^-$ and $\mathbf{z}(\cdot, t_N) = \mathbf{e}_N^-$. This is the dual problem to (34)–(35). It is a linear problem with coefficients that depend on both ψ,$\mathbf{A}$ and $\psi_{\mathcal{D}}$,$\mathbf{A}_{\mathcal{D}}$. From Theorem 1.1 it follows that $z \in \mathcal{C}\big(\overline{J}_N; \mathcal{L}^2(\Omega)\big)$, $\mathbf{z} \in \mathcal{C}\big(\overline{J}_N; \mathbf{L}^2(\Omega)\big)$, and thus $\tilde{B}_1$ and $\tilde{B}_2$ are related to the dual problem through

$$\begin{aligned} \tilde{B}_1(w, z, \mathbf{z}) &= \eta(w_N^-, e_N^-), \quad \forall w \in \tilde{\mathcal{W}}(J_N), \\ \tilde{B}_2(\mathbf{w}, z, \mathbf{z}) &= (\mathbf{w}_N^-, \mathbf{e}_N^-), \quad \forall \mathbf{w} \in \tilde{\mathbf{W}}_{\mathbf{n}}(J_N), \end{aligned} \tag{44}$$

since the jump terms in $\tilde{B}_1$ and $\tilde{B}_2$ vanish. Also, observe that if in addition, z and $\mathbf{z}$ are smooth, then z and $\mathbf{z}$ are the (strong) solutions to

$$\begin{aligned} &-\eta z_t + \Big(\frac{i}{\kappa}\nabla + \tilde{\mathbf{A}}\Big)^2 z + \gamma_{11} z + \tilde{\gamma}_{11}\overline{z} + \gamma_{12}\mathbf{z} = 0, && \text{in } \Omega\times J_N, \\ &-\mathbf{z}_t + \nabla\times\nabla\times\mathbf{z} - \nabla(\nabla\cdot\mathbf{z}) + \tilde{\tilde{\psi}}\mathbf{z} + \gamma_{21} z = 0, && \text{in } \Omega\times J_N, \\ &\frac{\partial z}{\partial\mathbf{n}} = 0, \quad \nabla\times\mathbf{z} = 0, \quad \mathbf{z}\cdot\mathbf{n} = 0, && \text{on } \partial\Omega\times J_N, \\ &z|_{t_N} = e_N^-, \quad \mathbf{z}|_{t_N} = \mathbf{e}_N^-, && \text{in } \Omega. \end{aligned} \tag{45}$$

To obtain our representation for the L_2-errors at time level t_N, we take $v = z \in \mathcal{W}(J_N)$, $\mathbf{v} = \mathbf{z} \in \mathbf{W}_{\mathbf{n}}(J_N)$ in (41), $w = e \in \tilde{\mathcal{W}}(J_N)$, $\mathbf{w} = \mathbf{e} \in \tilde{\mathbf{W}}_{\mathbf{n}}(J_N)$ in (44), to get

$$\begin{aligned} \eta\|e_N^-\|^2 + \|\mathbf{e}_N^-\|^2 &= \operatorname{Re}\tilde{B}_1(e, z, \mathbf{z}) + \tilde{B}_2(\mathbf{e}, z, \mathbf{z}) \\ &= \operatorname{Re} r_1(z) + r_2(\mathbf{z}), \end{aligned}$$

which completes the proof of the proposition. □

5.2. **Estimates of the residuals.** We now estimate the abstract residuals $r_1(v)$ and $r_2(\mathbf{v})$ in terms of the computable residuals and certain derivatives of the test functions v and $\mathbf{v}$.

Proposition 5.3. *Let $q = 0$, $1 \le l, l' \le 2$, and H be constant in time on $\partial\Omega$. Then*

$$\begin{aligned} |r_1(v)| \le c_1\|h^l R_1\|_{J_N} \sum_{n=1}^{N} \Big|\int_{I_n} v\, ds\Big|_{H^l(\Omega)} \\ + \|\eta[\psi_{\mathcal{D}}]\|_{J_N} \sum_{n=1}^{N} \min\Big\{\int_{I_n} \|v_t\|\, ds, 2\|v\|_{I_n}\Big\}, \end{aligned}$$

and

$$|r_2(\mathbf{v})| \le c_2 \|h^{l'} R_2\|_{J_N} \sum_{n=1}^{N} \Big| \int_{I_n} \mathbf{v}\, ds \Big|_{H^{l'}(\Omega)}$$
$$+ \big(\|[\mathbf{A}_{\mathcal{D}}]\|_{J_N} + \|k\nabla \times (H - \mathcal{J}H)\|_{J_N} \big) \sum_{n=1}^{N} \min \Big\{ \int_{I_n} \|\mathbf{v}_t\|\, ds, 2\|\mathbf{v}\|_{I_n} \Big\},$$

for v and $\mathbf{v}$ sufficiently smooth.

Remark 5.5. The constants c_1, c_2 depend only on the constant in the trace inequality (48), and the constants in Theorem 4.2 and Theorem 4.5.

To prepare for the proof, we define $\Pi_n : \mathcal{H}^1(\Omega) \to \mathcal{V}_{\mathcal{T}_n}$ to be the interpolant in Theorem 4.2, and we define $\mathbf{\Pi}_n : \mathbf{H}^1_{\mathbf{n}}(\Omega) \to \mathbf{V}_{\mathcal{T}_n,\mathbf{n}}$ to be the interpolant defined by (19).

We also split up the residuals on the time intervals I_n, more precisely, by using the definition of the residuals (30), and also (11)–(12), we write,

$$r_1(v) = \sum_{n=1}^{N} r_{1,n}(v), \qquad r_2(\mathbf{v}) = \sum_{n=1}^{N} r_{2,n}(\mathbf{v}),$$

where

$$\begin{aligned} r_{1,n}(v) = {} & \eta \int_{I_n} \big(\psi_{\mathcal{D},t}, v\big)\, dt + \eta\big([\psi_{\mathcal{D}}]_{n-1}, v^+_{n-1}\big) \\ & + \int_{I_n} \Big(\frac{i}{\kappa}\nabla\psi_{\mathcal{D}} + \mathbf{A}_{\mathcal{D}}\psi_n, \frac{i}{\kappa}\nabla v + \mathbf{A}_{\mathcal{D}} v\Big)\, dt \\ & + \int_{I_n} \big(- i\eta\kappa(\nabla\cdot\mathbf{A}_{\mathcal{D}})\psi_{\mathcal{D}} + (|\psi_{\mathcal{D}}|^2 - 1)\psi_{\mathcal{D}}, v\big)\, dt, \end{aligned} \tag{46}$$

and

$$\begin{aligned} r_{2,n}(\mathbf{v}) = {} & \int_{I_n} \big(\mathbf{A}_{\mathcal{D},t}, \mathbf{v}\big)\, dt + \big([\mathbf{A}_{\mathcal{D}}]_{n-1}, \mathbf{v}^+_{n-1}\big) \\ & + \int_{I_n} \big\{ \big(\nabla\times\mathbf{A}_{\mathcal{D}}, \nabla\times\mathbf{v}\big) + \big(\nabla\cdot\mathbf{A}_{\mathcal{D}}, \nabla\cdot\mathbf{v}\big)\big\}\, dt \\ & + \int_{I_n} \Big(\mathrm{Re}\Big\{\frac{i}{\kappa}\overline{\psi}_{\mathcal{D}}\nabla\psi_{\mathcal{D}}\Big\} + |\psi_{\mathcal{D}}|^2\mathbf{A}_{\mathcal{D}}, \mathbf{v}\Big)\, dt - \int_{I_n} \big(H, \nabla\times\mathbf{v}\big)\, dt. \end{aligned} \tag{47}$$

We also set $R_{1,n} = R_1|_{\Omega\times I_n}$, $n = 1, \dots, N$, and similarly for $R_{1,e}$, $R_{1,i}$, R_2, etc.

Our proof of Proposition 5.3 is based on the following two lemmas; the first one concerning the error in the time discretisation, and the second one the error in the spatial discretisation.

Lemma 5.4. *If $q = 0$, and H is constant in time on $\partial\Omega$, then*

$$\begin{aligned} |r_{1,n}(v - \mathcal{J}_n v)| &\le \|\eta[\psi_{\mathcal{D}}]_{n-1}\| \min\Big\{ \int_{I_n} \|v_t\|\, ds, 2\|v\|_{I_n}\Big\}, \\ |r_{2,n}(\mathbf{v} - \mathcal{J}_n\mathbf{v})| &\le \big(\|[\mathbf{A}_{\mathcal{D}}]_{n-1}\| + k_n\|\nabla\times(H - \mathcal{J}_n H)\|_{I_n}\big) \\ &\qquad \times \min\Big\{ \int_{I_n} \|\mathbf{v}_t\|\, ds, 2\|\mathbf{v}\|_{I_n}\Big\}, \end{aligned}$$

for v and $\mathbf{v}$ sufficiently smooth.

Lemma 5.5. *If $q = 0$ and $1 \le l, l' \le 2$, then*

$$|r_{1,n}(\mathcal{J}_n v - \mathcal{J}_n \Pi_n v)| \le c_1 \|h^l_{\mathcal{T}_n} R_{1,n}\| \Big| \int_{I_n} v\, ds \Big|_{H^l(\Omega)},$$

$$|r_{2,n}(\mathcal{J}_n \mathbf{v} - \mathcal{J}_n \mathbf{\Pi}_n \mathbf{v})| \le c_2 \|h^{l'}_{\mathcal{T}_n} R_{2,n}\| \Big| \int_{I_n} \mathbf{v}\, ds \Big|_{H^{l'}(\Omega)},$$

for v and $\mathbf{v}$ sufficiently smooth.

Proof of Proposition 5.3. Let $V|_{\Omega\times I_n} = J_n \Pi_n v$, $\mathbf{V}|_{\Omega\times I_n} = J_n \mathbf{\Pi}_n \mathbf{v}$. Then $V \in \mathcal{W}_{\mathcal{D}}(J_N)$, $\mathbf{V} \in \mathbf{W}_{\mathcal{D},\mathbf{n}}(J_N)$, so by using the orthogonality (31), we can decompose the residuals as follows

$$r_1(v) = r_1(v - V) = \sum_{n=1}^N \big\{ r_{1,n}(v - \mathcal{J}_n v) + r_{1,n}(\mathcal{J}_n v - \mathcal{J}_n \Pi_n v) \big\},$$

$$r_2(\mathbf{v}) = r_2(\mathbf{v} - \mathbf{V}) = \sum_{n=1}^N \big\{ r_{2,n}(\mathbf{v} - \mathcal{J}_n \mathbf{v}) + r_{2,n}(\mathcal{J}_n \mathbf{v} - \mathcal{J}_n \mathbf{\Pi}_n \mathbf{v}) \big\}.$$

The proof now follows from Lemma 5.5 and Lemma 5.4. □

Proof of Lemma 5.4. By using (46)–(46), and the definition of $\mathcal{J}_n$, observing that $\psi_{\mathcal{D}}$ and $\mathbf{A}_{\mathcal{D}}$ are constant in time on each interval I_n, we find

$$\begin{aligned} r_{1,n}(v - \mathcal{J}_n v) &= \eta \int_{I_n} (\psi_{\mathcal{D},t}, v - \mathcal{J}_n v)\, dt + \eta\big([\psi_{\mathcal{D}}]_{n-1}, (v - \mathcal{J}_n v)^+_{n-1}\big) \\ &\quad + \int_{I_n} \Big(\frac{i}{\kappa}\nabla\psi_{\mathcal{D}} + \mathbf{A}_{\mathcal{D}}\psi_{\mathcal{D}}, \frac{i}{\kappa}\nabla(v - \mathcal{J}_n v) + \mathbf{A}_{\mathcal{D}}(v - \mathcal{J}_n v) \Big)\, dt \\ &\quad + \int_{I_n} \big(-i\eta\kappa(\nabla\cdot\mathbf{A}_{\mathcal{D}})\psi_{\mathcal{D}} + (|\psi_{\mathcal{D}}|^2 - 1)\psi_{\mathcal{D}}, v - \mathcal{J}_n v \big)\, dt \\ &= \big(\eta[\psi_{\mathcal{D}}]_{n-1}, (v - \mathcal{J}_n v)^+_{n-1}\big), \end{aligned}$$

and

$$\begin{aligned} &r_{2,n}(\mathbf{v} - \mathcal{J}_n \mathbf{v}) \\ &\quad = \int_{I_n} (\mathbf{A}_{\mathcal{D},t}, \mathbf{v} - \mathcal{J}_n \mathbf{v})\, dt + \big([\mathbf{A}_{\mathcal{D}}]_{n-1}, (\mathbf{v} - \mathcal{J}_n \mathbf{v})^+_{n-1}\big) \\ &\qquad + \int_{I_n} \big\{ \big(\nabla\times\mathbf{A}_{\mathcal{D}}, \nabla\times(\mathbf{v} - \mathcal{J}_n \mathbf{v})\big) + \big(\nabla\cdot\mathbf{A}_{\mathcal{D}}, \nabla\cdot(\mathbf{v} - \mathcal{J}_n \mathbf{v})\big) \big\}\, dt \\ &\qquad + \int_{I_n} \Big(\mathrm{Re}\Big\{ \frac{i}{\kappa}\overline{\psi}_{\mathcal{D}}\nabla\psi_{\mathcal{D}} \Big\} + |\psi_{\mathcal{D}}|^2\mathbf{A}_{\mathcal{D}}, \mathbf{v} - \mathcal{J}_n \mathbf{v} \Big) - \big(H, \nabla\times(\mathbf{v} - \mathcal{J}_n \mathbf{v})\big)\, dt \\ &\quad = \big([\mathbf{A}_{\mathcal{D}}]_{n-1}, (\mathbf{v} - \mathcal{J}_n \mathbf{v})^+_{n-1}\big) - \int_{I_n} \big(H - \mathcal{J}_n H, \nabla\times(\mathbf{v} - \mathcal{J}_n \mathbf{v})\big)\, dt. \end{aligned}$$

By Green's formula,

$$\begin{aligned} &\big(H - \mathcal{J}_n H, \nabla\times(\mathbf{v} - \mathcal{J}_n \mathbf{v})\big) \\ &\quad = \int_{\partial\Omega} (H - \mathcal{J}_n H)(\mathbf{v} - \mathcal{J}_n \mathbf{v})\cdot\mathbf{t}\, ds + \big(\nabla\times(H - \mathcal{J}_n H), \mathbf{v} - \mathcal{J}_n \mathbf{v}\big). \end{aligned}$$

Since H is constant in time on $\partial\Omega$, the boundary integral vanishes, and we have

$$\begin{aligned} \Big| \int_{I_n} \big(H - \mathcal{J}_n H, \nabla\times(\mathbf{v} - \mathcal{J}_n \mathbf{v})\big)\, ds \Big| &\le \|\nabla\times(H - \mathcal{J}_n H)\|_{I_n} \int_{I_n} \|\mathbf{v} - \mathcal{J}_n \mathbf{v}\|\, ds \\ &\le k_n \|\nabla\times(H - \mathcal{J}_n H)\|_{I_n} \|\mathbf{v} - \mathcal{J}_n \mathbf{v}\|_{I_n}, \end{aligned}$$

and also

$$\big|\big(\eta[\psi_{\mathcal{D}}]_{n-1}, (v - \mathcal{J}_n v)^+_{n-1}\big)\big| \le \|\eta[\psi_{\mathcal{D}}]_{n-1}\| \|v - \mathcal{J}_n v\|_{I_n},$$

and

$$\big|\big([\mathbf{A}_{\mathcal{D}}]_{n-1}, (\mathbf{v}-\mathcal{J}_n\mathbf{v})_{n-1}^{+}\big)\big| \le \|[\mathbf{A}_{\mathcal{D}}]_{n-1}\|\|\mathbf{v}-\mathcal{J}_n\mathbf{v}\|_{I_n}.$$

We complete the proof of the lemma by estimating $\|v-\mathcal{J}_n v\|_{I_n}$ and $\|\mathbf{v}-\mathcal{J}_n\mathbf{v}\|_{I_n}$. For $s\in I_n$, we have by Minkowski's inequality for integrals (see, for example, [16, Theorem (6.19)]),

$$\begin{aligned}\|v(s)-\mathcal{J}_n v\| &= \frac{1}{k_n}\Big\|\int_{I_n}\int_{s'}^{s} v_t(s'')\,ds''\,ds'\Big\| \\ &\le \frac{1}{k_n}\int_{I_n}\int_{I_n}\|v_t(s'')\|\,ds''\,ds' = \int_{I_n}\|v_t(s'')\|\,ds'',\end{aligned}$$

and

$$\|v(s)-\mathcal{J}_n v\| \le \|v(s)\| + \frac{1}{k_n}\int_{I_n}\|v(s')\|\,ds' \le 2\|v\|_{I_n}.$$

Thus,

$$\|v-\mathcal{J}_n v\|_{I_n} \le \min\Big\{\int_{I_n}\|v_t\|\,ds, 2\|v\|_{I_n}\Big\},$$

and in the same way,

$$\|\mathbf{v}-\mathcal{J}_n\mathbf{v}\|_{I_n} \le \min\Big\{\int_{I_n}\|\mathbf{v}_t\|\,ds, 2\|\mathbf{v}\|_{I_n}\Big\}.$$

□

Proof of Lemma 5.5. By (46)–(47), we have

$$\begin{aligned}&r_{1,n}(\mathcal{J}_n v - \mathcal{J}_n\Pi_n v) \\ &\quad= \eta\int_{I_n}\big(\psi_{\mathcal{D},t}, \mathcal{J}_n(I-\Pi_n)v\big)\,dt + \eta\big([\psi_{\mathcal{D}}]_{n-1}, \big(\mathcal{J}_n(I-\Pi_n)v\big)_{n-1}^{+}\big) \\ &\qquad+ \int_{I_n}\Big(\frac{i}{\kappa}\nabla\psi_{\mathcal{D}} + \mathbf{A}_{\mathcal{D}}\psi_{\mathcal{D}}, \frac{i}{\kappa}\nabla\big(\mathcal{J}_n(I-\Pi_n)v\big) + \mathbf{A}_{\mathcal{D}}\big(\mathcal{J}_n(I-\Pi_n)v\big)\Big)\,dt \\ &\qquad+ \int_{I_n}\big(-i\eta\kappa(\nabla\cdot\mathbf{A}_{\mathcal{D}})\psi_{\mathcal{D}} + (|\psi_{\mathcal{D}}|^2-1)\psi_{\mathcal{D}}, \mathcal{J}_n(I-\Pi_n)v\big)\,dt,\end{aligned}$$

and

$$\begin{aligned}&r_{2,n}(\mathcal{J}_n\mathbf{v} - \mathcal{J}_n\mathbf{\Pi}_n\mathbf{v}) \\ &\quad= \int_{I_n}\big(\mathbf{A}_{\mathcal{D},t}, \mathcal{J}_n(I-\mathbf{\Pi}_n)\mathbf{v}\big)\,dt + \big([\mathbf{A}_{\mathcal{D}}]_{n-1}, \big(\mathcal{J}_n(I-\mathbf{\Pi}_n)\mathbf{v}\big)_{n-1}^{+}\big) \\ &\qquad+ \int_{I_n}\big\{\big(\nabla\times\mathbf{A}_{\mathcal{D}}, \nabla\times\big(\mathcal{J}_n(I-\mathbf{\Pi}_n)\mathbf{v}\big)\big) + \big(\nabla\cdot\mathbf{A}_{\mathcal{D}}, \nabla\cdot\big(\mathcal{J}_n(I-\mathbf{\Pi}_n)\mathbf{v}\big)\big)\big\}\,dt \\ &\qquad+ \int_{I_n}\Big(\mathrm{Re}\Big\{\frac{i}{\kappa}\overline{\psi}_{\mathcal{D}}\nabla\psi_{\mathcal{D}}\Big\} + |\psi_{\mathcal{D}}|^2\mathbf{A}_{\mathcal{D}}, \mathcal{J}_n(I-\mathbf{\Pi}_n)\mathbf{v}\Big)\,dt \\ &\qquad- \int_{I_n}\big(H, \nabla\times\mathcal{J}_n(I-\mathbf{\Pi}_n)\mathbf{v}\big)\,dt,\end{aligned}$$

so observing that $\mathcal{J}_n v = \frac{1}{k_n}\int_{I_n} v\,ds$, and that $\psi_{\mathcal{D}}$ and $\mathbf{A}_{\mathcal{D}}$ are constant in time, and thus $\psi_{\mathcal{D},t}=0$, $\mathbf{A}_{\mathcal{D},t}=\mathbf{0}$, on each I_n, we obtain

$$\begin{aligned}&r_{1,n}(\mathcal{J}_n v - \mathcal{J}_n\Pi_n v) \\ &\quad= \big(\eta[\psi_{\mathcal{D}}]_{n-1}/k_n - i\eta\kappa(\nabla\cdot\mathbf{A}_{\mathcal{D},n})\psi_{\mathcal{D},n} + (|\psi_{\mathcal{D},n}|^2-1)\psi_{\mathcal{D},n}, \theta_n - \Pi_n\theta_n\big) \\ &\qquad+ \Big(\frac{i}{\kappa}\nabla\psi_{\mathcal{D},n} + \mathbf{A}_{\mathcal{D},n}\psi_{\mathcal{D},n}, \frac{i}{\kappa}\nabla(\theta_n-\Pi_n\theta_n) + \mathbf{A}_{\mathcal{D},n}(\theta_n-\Pi_n\theta_n)\Big) \equiv Q_{1,n},\end{aligned}$$

and

$$\begin{aligned}
&r_{2,n}(\mathcal{J}_n\mathbf{v} - \mathcal{J}_n\mathbf{\Pi}_n\mathbf{v})\\
&\qquad = \Big([\mathbf{A}_{\mathcal{D}}]_{n-1}/k_n + \mathrm{Re}\Big\{\frac{i}{\kappa}\overline{\psi}_{\mathcal{D},n}\nabla\psi_{\mathcal{D},n}\Big\} + |\psi_{\mathcal{D},n}|^2\mathbf{A}_{\mathcal{D},n}, \boldsymbol{\theta}_n - \mathbf{\Pi}_n\boldsymbol{\theta}_n\Big)\\
&\qquad\quad + \big(\nabla\times\mathbf{A}_{\mathcal{D},n}, \nabla\times(\boldsymbol{\theta}_n - \mathbf{\Pi}_n\boldsymbol{\theta}_n)\big) + \big(\nabla\cdot\mathbf{A}_{\mathcal{D},n}, \nabla\cdot(\boldsymbol{\theta}_n - \mathbf{\Pi}_n\boldsymbol{\theta}_n)\big)\\
&\qquad\quad - \big(\mathcal{J}_nH, \nabla\times(\boldsymbol{\theta}_n - \mathbf{\Pi}_n\boldsymbol{\theta}_n)\big) \equiv Q_{2,n},
\end{aligned}$$

where $\theta_n = \int_{I_n} v\,ds$ and $\boldsymbol{\theta}_n = \int_{I_n}\mathbf{v}\,ds$. Now, using Green's formula on each triangle $K\in\mathcal{T}_n$, observing that $\mathbf{A}_{\mathcal{D},n}\cdot\mathbf{n} = 0$ and $(\mathbf{\Pi}_n\boldsymbol{\theta}_n - \boldsymbol{\theta}_n)\cdot\mathbf{n} = 0$ on $\partial\Omega$, we find

$$Q_{1,n} = \sum_{K\in\mathcal{T}_n}\int_{\partial K}\sigma_n\overline{\rho}_n\,ds + \int_\Omega R_{1,i,n}\overline{\rho}_n\,dx = Q_{1,n,e} + Q_{1,n,i},$$

and

$$Q_{2,n} = \sum_{K\in\mathcal{T}_n}\int_{\partial K}\boldsymbol{\sigma}_n\cdot\boldsymbol{\rho}_n\,ds + \int_\Omega \mathbf{R}_{2,i,n}\cdot\boldsymbol{\rho}_n\,dx = Q_{2,n,e} + Q_{2,n,i},$$

where $\rho_n = \theta_n - \Pi_n\theta_n$, $\boldsymbol{\rho}_n = \boldsymbol{\theta}_n - \mathbf{\Pi}_n\boldsymbol{\theta}_n$, and

$$\begin{aligned}
\sigma_n &= \frac{1}{\kappa^2}\Big(\frac{1}{2}\Big\lceil\frac{\partial\psi_{\mathcal{D},n}}{\partial n}\Big\rceil\chi_{\partial K\setminus\partial\Omega} + \frac{\partial\psi_{\mathcal{D},n}}{\partial n}\chi_{\partial K\cap\partial\Omega}\Big),\\
\boldsymbol{\sigma}_n &= \tfrac{1}{2}\big(\lceil\nabla\cdot\mathbf{A}_{\mathcal{D},n}\rceil\mathbf{n} + \lceil\nabla\times\mathbf{A}_{\mathcal{D},n}\rceil\mathbf{t}\big)\chi_{\partial K\setminus\partial\Omega} + (\nabla\times\mathbf{A}_{\mathcal{D},n} - \mathcal{J}_nH)\mathbf{t}\chi_{\partial K\cap\partial\Omega},
\end{aligned}$$

where χ_E denotes the characteristic function of E, and $\mathbf{t} = (-n_2, n_1)$, if $\mathbf{n} = (n_1, n_2)$. We now use a trace inequality, see [1],

$$\|v\|^2_{L^2(\partial K)} \le c\|v\|_{L^2(K)}|v|_{H^1(K)},\quad \forall v\in\mathcal{H}^1(K), \tag{48}$$

to obtain

$$h_K^{-2l+1}\|\rho_n\|^2_{L^2(\partial K)} \le c\big(h_K^{-2l}\|\rho_n\|^2_{L^2(K)} + h_K^{-2(l-1)}|\rho_n|^2_{H^1(K)}\big).$$

Using this, the Cauchy-Schwarz inequality, and Theorem 4.2 with $m = 0$ and $m = 1$, we have

$$\begin{aligned}
|Q_{1,n,e}| &\le \sum_{K\in\mathcal{T}_n} h_K^{l-\frac{1}{2}}\|\sigma_n\|_{L^2(\partial K)}h_K^{-l+\frac{1}{2}}\|\rho_n\|_{L^2(\partial K)}\\
&\le c\bigg(\sum_{K\in\mathcal{T}_h^n} h_K^{2l-1}\frac{1}{\kappa^4}\Big(\Big\|\frac{1}{2}\Big\lceil\frac{\partial\psi_{\mathcal{D},n}}{\partial n}\Big\rceil\Big\|^2_{L^2(\partial K\setminus\partial\Omega)} + \Big\|\frac{\partial\psi_{\mathcal{D},n}}{\partial n}\Big\|^2_{L^2(\partial K\cap\partial\Omega)}\Big)\bigg)^{\frac{1}{2}}\\
&\quad\times\bigg(\sum_{K\in\mathcal{T}_h^n}\big(h_K^{-l}\|\theta_n - \Pi_n\theta_n\|_{L^2(K)}\big)^2 + \big(h_K^{-(l-1)}|\theta_n - \Pi_n\theta_n|_{H^1(K)}\big)^2\bigg)^{\frac{1}{2}}\\
&\le c\|h^l_{\mathcal{T}_n}R_{1,e,n}\|\,|\theta_n|_{H^l(\Omega)},
\end{aligned}$$

where we also used the definition of $R_{1,e}$ in the last step. By using Theorem 4.2 once more with $m = 0$, we find

$$|Q_{1,n,i}| \le \|h^l_{\mathcal{T}_n}R_{1,i,n}\|\|h^{-l}_{\mathcal{T}_n}(\theta_n - \Pi_n\theta_n)\| \le c\|h^l_{\mathcal{T}_n}R_{1,i,n}\|\,|\theta_n|_{H^l(\Omega)},$$

which concludes the proof of the first inequality in Lemma 5.5. The proof of the second one is similar, this time using Theorem 4.5, and observing that the trace inequality (48) is valid for vector-valued functions, and that

$$\begin{aligned}
\|\boldsymbol{\sigma}_n\|^2_{L^2(\partial K)} &= \big\|\tfrac{1}{2}\lceil\nabla\cdot\mathbf{A}_{\mathcal{D},n}\rceil\big\|^2_{L^2(\partial K\setminus\partial\Omega)} + \big\|\tfrac{1}{2}\lceil\nabla\times\mathbf{A}_{\mathcal{D},n}\rceil\big\|^2_{L^2(\partial K\setminus\partial\Omega)}\\
&\quad + \|\nabla\times\mathbf{A}_{\mathcal{D},n} - \mathcal{J}_nH\|^2_{L^2(\partial K\cap\partial\Omega)}.
\end{aligned}$$

□

5.3. **Stability Estimates.** The following lemma estimates the quantities of test functions that appear in Proposition 5.3.

Lemma 5.6. *We have*

$$\sum_{n=1}^{N}\Big|\int_{I_n} v\,ds\Big|_{H^l(\Omega)} \le G_{l,N}\begin{cases}\Big(\int_{J_N}(t_N-s)^{l-1}|v|^2_{H^l(\Omega)}\,ds\Big)^{\frac12}, & 1\le l<2,\\ \Big(\int_{J_{N-1}}(t_N-s)|v|^2_{H^2(\Omega)}\,ds\Big)^{\frac12}+\Big|\int_{I_N} v\,ds\Big|_{H^2(\Omega)}, & l=2,\end{cases}$$

and

$$\sum_{n=1}^{N}\min\Big\{\int_{I_n}\|v_t\|\,ds, 2\|v\|_{I_n}\Big\} \le G_{2,N}\bigg(\Big(\int_{J_{N-1}}(t_N-s)\|v_t\|^2\,ds\Big)^{\frac12}+2\|v\|_{I_N}\bigg),$$

where $G_{l,N}$ is defined in (29).

Proof. We have, by the Cauchy-Schwarz inequality, for $1\le l<2$,

$$\int_{J_N}|g|\,ds \le G_{l,N}\Big(\int_{J_N}(t_N-s)^{l-1}|g|^2\,ds\Big)^{\frac12}, \tag{49}$$

and, for $l=2$,

$$\int_{J_{N-1}}|g|\,ds \le (G_{2,N}-1)\Big(\int_{J_{N-1}}(t_N-s)|g|^2\,ds\Big)^{\frac12}. \tag{50}$$

We now use Minkowski's inequality for integrals (see [16]), to find that

$$\Big|\int_{I_n} v\,ds\Big|_{H^l(\Omega)} \le \int_{I_n}|v|_{H^l(\Omega)}\,ds, \tag{51}$$

which together with (49)–(50) proves the first inequality. Further,

$$\sum_{n=1}^{N}\min\Big\{\int_{I_n}\|v_t\|\,ds, 2\|v\|_{I_n}\Big\} \le \int_{J_{N-1}}\|v_t\|\,ds+2\|v\|_{I_N}, \tag{52}$$

so by using (50), we obtain the second inequality, which completes the proof of the lemma. ☐

Thus, in view of the above lemma, our basic stability assumption will be that for some l,l', $1\le l,l'\le 2$, there exist constants $S^{\text{space}}_{l,1,N}$, $S^{\text{space}}_{l',2,N}$ and $S^{\text{time}}_{1,N}$, $S^{\text{time}}_{2,N}$ such that for the solutions z, $\mathbf{z}$ to the dual problem (42)–(43), we have

$$\left.\begin{array}{ll}\Big(\int_{J_N}(t_N-s)^{l-1}|z|^2_{H^l(\Omega)}\,ds\Big)^{\frac12}, & 1\le l<2\\ \Big(\int_{J_{N-1}}(t_N-s)|z|^2_{H^2(\Omega)}\,ds\Big)^{\frac12}+\Big|\int_{I_N} z\,ds\Big|_{H^2(\Omega)}, & l=2\end{array}\right\}\le S^{\text{space}}_{l,1,N}\|\tilde e\|,$$
$$\left.\begin{array}{ll}\Big(\int_{J_N}(t_N-s)^{l'-1}|\mathbf{z}|^2_{H^{l'}(\Omega)}\,ds\Big)^{\frac12}, & 1\le l'<2\\ \Big(\int_{J_{N-1}}(t_N-s)|\mathbf{z}|^2_{H^2(\Omega)}\,ds\Big)^{\frac12}+\Big|\int_{I_N} z\,ds\Big|_{H^2(\Omega)}, & l'=2\end{array}\right\}\le S^{\text{space}}_{l',2,N}\|\tilde e\|, \tag{53}$$

and

$$\begin{aligned}\Big(\int_{J_{N-1}}(t_N-s)\|z_t\|^2\,ds\Big)^{\frac12}+\|z\|_{I_N}&\le S^{\text{time}}_{1,N}\|\tilde e\|,\\ \Big(\int_{J_{N-1}}(t_N-s)\|\mathbf{z}_t\|^2\,ds\Big)^{\frac12}+\|\mathbf{z}\|_{I_N}&\le S^{\text{time}}_{2,N}\|\tilde e\|,\end{aligned} \tag{54}$$

where we used the abbreviation $\|\tilde e\|=(\eta\|e_N^-\|^2+\|\mathbf{e}_N^-\|^2)^{\frac12}$.

The proposition below indicates that such estimates are possible, at least under strong boundedness assumptions on the coefficients, and with exponential growth of the stability factors. However, these crude estimates may not reflect the actual stability properties of the dual problem. We believe that, in an adaptive algorithm based on the a posteriori error analysis in Theorem 5.1, one should solve the dual problem numerically, and insert z, $\mathbf{z}$ as weights in the error estimates, rather than as stability factors. We will address this topic, together with other implementation issues, in future work.

Proposition 5.7. *Assume that* $\|\tilde{\psi}\|_{L^\infty(J_N;L^\infty(\Omega))} < \infty$, $\|\tilde{\mathbf{A}}\|_{L^\infty(J_N;L^\infty(\Omega))} < \infty$. *Then, for the solutions* z, $\mathbf{z}$ *to the dual problem (42)–(43), we have*

$$\eta\|z\|^2_{J_N} + \|\mathbf{z}\|^2_{J_N} + \int_{J_N}\Big\{\frac{1}{\kappa^2}|z|^2_{H^1(\Omega)} + |\mathbf{z}|^2_{H^1(\Omega)}\Big\}\,dt \le C_1(t_N)\big(\eta\|e_N^-\|^2 + \|\mathbf{e}_N^-\|^2\big). \tag{55}$$

If in addition, $\|\nabla\tilde{\psi}\|_{L^\infty(J_N;L^\infty(\Omega))} < \infty$, $\|\nabla\cdot\tilde{\mathbf{A}}\|_{L^\infty(J_N;L^\infty(\Omega))} < \infty$, *then we also have*

$$\int_{J_N}(t_N - t)\big\{\eta\|z_t\|^2 + \|\mathbf{z}_t\|^2\big\}\,dt \le C_2(t_N)\big(\eta\|e_N^-\|^2 + \|\mathbf{e}_N^-\|^2\big). \tag{56}$$

Proof. Let $t \in J_N$ and $\tilde{J} = (t, t_N)$. To prove (55), we set $w = z\chi_{\Omega\times\tilde{J}}$ in (42) and $\mathbf{w} = \mathbf{z}\chi_{\Omega\times\tilde{J}}$ in (43), to find

$$\eta\int_{\tilde{J}}(z, -z_t)\,dt + \int_{\tilde{J}}\Big\|\frac{i}{\kappa}\nabla z + \tilde{\mathbf{A}}z\Big\|^2\,dt + \int_{\tilde{J}}(z, \gamma_{11}z + \tilde{\gamma}_{11}\overline{z} + \gamma_{12}\mathbf{z})\,dt = 0, \tag{57}$$

and

$$\int_{\tilde{J}}(\mathbf{z}, -\mathbf{z}_t)\,dt + \int_{\tilde{J}}\big\{\|\nabla\times\mathbf{z}\|^2 + \|\nabla\cdot\mathbf{z}\|^2\big\}\,dt + \int_{\tilde{J}}(\mathbf{z}, \tilde{\tilde{\psi}}\mathbf{z} + \gamma_{21}z)\,dt = 0. \tag{58}$$

Recall that γ_{11}, $\tilde{\gamma}_{11}$ were defined in (37), and γ_{12}, γ_{21} in (40). Simple calculations, with $\hat{\psi} = |\psi| - |\psi_{\mathcal{D}}|$, yield

$$\begin{aligned}
\Big\|\frac{i}{\kappa}\nabla z + \tilde{\mathbf{A}}z\Big\|^2 &= \frac{1}{\kappa^2}\|\nabla z\|^2 + \mathrm{Re}\,\Big\{\frac{2i}{\kappa}(\nabla z, \tilde{\mathbf{A}}z)\Big\} + \|\tilde{\mathbf{A}}z\|^2,\\
(z, \gamma_{11}z + \tilde{\gamma}_{11}\overline{z}) &= \|\hat{\mathbf{A}}z\|^2 - i\eta\kappa\big((\nabla\cdot\tilde{\mathbf{A}})z, z\big) + \|\hat{\psi}z\|^2 - \|z\|^2\\
&\quad + (|\psi||\psi_{\mathcal{D}}|z, z) + (z, |\psi\psi_{\mathcal{D}}|z + \psi\psi_{\mathcal{D}}\overline{z}),\\
(z, \gamma_{12}\mathbf{z}) &= \frac{2i}{\kappa}\big(\overline{\tilde{\psi}}\nabla z, \mathbf{z}\big) + \frac{i}{\kappa}\big(\overline{\tilde{\psi}}z, \nabla\cdot\mathbf{z}\big) + 2\big(\overline{\tilde{\psi}}z, \tilde{\mathbf{A}}\cdot\mathbf{z}\big),
\end{aligned}$$

and

$$(\mathbf{z}, \gamma_{21}z) = \mathrm{Re}\,\Big\{\frac{i}{\kappa}\big(\overline{\tilde{\psi}}, \nabla\cdot(\overline{z}\mathbf{z})\big) + \frac{i}{\kappa}\big(\overline{\tilde{\psi}}\nabla z, \mathbf{z}\big) + 2\big(\overline{\tilde{\psi}}z, \tilde{\mathbf{A}}\cdot\mathbf{z}\big) + i\eta\kappa\big(\overline{\tilde{\psi}}z, \nabla\cdot\mathbf{z}\big)\Big\}.$$

Thus, adding the real parts of (57) to (58), we obtain

$$\begin{aligned}
&\frac{\eta}{2}\|z(\cdot,t)\|^2 + \frac{1}{2}\|\mathbf{z}(\cdot,t)\|^2 + \int_{\tilde{J}}\Big\{\frac{1}{\kappa^2}\|\nabla z\|^2 + \|\nabla\times\mathbf{z}\|^2 + \|\nabla\cdot\mathbf{z}\|^2\Big\}\,ds\\
&\quad + \int_{\tilde{J}}\big\{\|\tilde{\mathbf{A}}z\|^2 + \|\hat{\mathbf{A}}z\|^2 + \|\hat{\psi}z\|^2\big\}\,ds\\
&\quad + \int_{\tilde{J}}\big\{\mathrm{Re}\,(z, |\psi\psi_{\mathcal{D}}|z + \psi\psi_{\mathcal{D}}\overline{z}) + (|\psi||\psi_{\mathcal{D}}|z, z) + (\tilde{\tilde{\psi}}\mathbf{z}, \mathbf{z})\big\}\,ds\\
&= \frac{\eta}{2}\|z(\cdot,t_N)\|^2 + \frac{1}{2}\|\mathbf{z}(\cdot,t_N)\|^2 + \int_{\tilde{J}}\Big\{\|z\|^2 - \mathrm{Re}\,\Big\{\frac{2i}{\kappa}(\nabla z, \tilde{\mathbf{A}}z)\Big\}\Big\}\,ds\\
&\quad - \mathrm{Re}\int_{\tilde{J}}\Big\{\frac{4i}{\kappa}\big(\overline{\tilde{\psi}}\nabla z, \mathbf{z}\big) + \frac{2i}{\kappa}\big(\overline{\tilde{\psi}}z, \nabla\cdot\mathbf{z}\big) + 4\big(\overline{\tilde{\psi}}z, \tilde{\mathbf{A}}\cdot\mathbf{z}\big) + i\eta\kappa\big(\overline{\tilde{\psi}}z, \nabla\cdot\mathbf{z}\big)\Big\}\,ds.
\end{aligned}$$

But, $z(\cdot,t_N)=e_N^-$, $\mathbf{z}(\cdot,t_N)=\mathbf{e}_N^-$, and

$$\begin{aligned}&\Big|\frac{2i}{\kappa}(\nabla z,\tilde{\mathbf{A}}z)+\frac{4i}{\kappa}(\overline{\tilde\psi}\nabla z,\mathbf{z})+\frac{2i}{\kappa}(\overline{\tilde\psi}z,\nabla\cdot\mathbf{z})+4(\overline{\tilde\psi}z,\tilde{\mathbf{A}}\cdot\mathbf{z})+i\eta\kappa(\overline{\tilde\psi}z,\nabla\cdot\mathbf{z})\Big|\\&\quad=\Big|\frac{2i}{\kappa}(\nabla z,2\overline{\tilde\psi}\mathbf{z}+\tilde{\mathbf{A}}z)+\Big(\Big(\frac{2i}{\kappa}+i\eta\kappa\Big)z,\nabla\cdot\mathbf{z}\Big)+4(\overline{\tilde\psi}z,\tilde{\mathbf{A}}\cdot\mathbf{z})\Big|\\&\quad\le\frac{1}{2\kappa^2}\|\nabla z\|^2+\frac12\|\nabla\cdot\mathbf{z}\|^2+\|\tilde{\mathbf{A}}z\|^2\\&\qquad+2\|2\overline{\tilde\psi}\mathbf{z}+\tilde{\mathbf{A}}z\|^2+\frac12\Big\|\Big(\frac{2i}{\kappa}+i\eta\kappa\Big)z\Big\|^2+4\|\overline{\tilde\psi}\mathbf{z}\|^2.\end{aligned}$$

We also observe that

$$\operatorname{Re}(z,|\psi\psi_{\mathcal{D}}|z+\psi\psi_{\mathcal{D}}\overline{z})+(|\psi||\psi_{\mathcal{D}}|z,z)+(\tilde{\tilde\psi}\mathbf{z},\mathbf{z})\ge 0.$$

Thus, using the assumption that $\tilde\psi$ and $\tilde{\mathbf{A}}$ are bounded, we find

$$\begin{aligned}&\eta\|z(\cdot,t)\|^2+\|\mathbf{z}(\cdot,t)\|^2+\int_{\tilde J}\Big\{\frac{1}{\kappa^2}\|\nabla z\|^2+\|\nabla\times\mathbf{z}\|^2+\|\nabla\cdot\mathbf{z}\|^2\Big\}\,ds\\&\qquad\le\eta\|e_N^-\|^2+\|\mathbf{e}_N^-\|^2+C\int_{\tilde J}\{\eta\|z\|^2+\|\mathbf{z}\|^2\}\,ds.\end{aligned}$$

Invoking Grönwall's lemma, we obtain

$$\begin{aligned}&\eta\|z\|^2_{J_N}+\|\mathbf{z}\|^2_{J_N}+\int_{J_N}\Big\{\frac{1}{\kappa^2}\|\nabla z\|^2+\|\nabla\times\mathbf{z}\|^2+\|\nabla\cdot\mathbf{z}\|^2\Big\}\,ds\\&\qquad\le C(t_N)\big(\eta\|e_N^-\|^2+\|\mathbf{e}_N^-\|^2\big),\end{aligned}\tag{59}$$

which together with (2) proves (55).

To prove (56), we now set $w=-(t_N-t)z_t$ in (42) and $\mathbf{w}=-(t_N-t)\mathbf{z}_t$ in (43), to find

$$\begin{aligned}&\eta\int_{J_N}(t_N-t)(-z_t,-z_t)\,dt-\int_{J_N}(t_N-t)\Big(\frac{i}{\kappa}\nabla z_t+\tilde{\mathbf{A}}z_t,\frac{i}{\kappa}\nabla z+\tilde{\mathbf{A}}z\Big)\,dt\\&\quad-\int_{J_N}(t_N-t)(z_t,\gamma_{11}z+\tilde\gamma_{11}\overline{z}+\gamma_{12}\mathbf{z})\,dt=0,\end{aligned}$$

and

$$\begin{aligned}&\int_{J_N}(t_N-t)(-\mathbf{z}_t,-\mathbf{z}_t)\,dt-\int_{J_N}(t_N-t)\big\{(\nabla\times\mathbf{z}_t,\nabla\times\mathbf{z})+(\nabla\cdot\mathbf{z}_t,\nabla\cdot\mathbf{z})\big\}\,dt\\&\quad-\int_{J_N}(t_N-t)(\mathbf{z}_t,\tilde{\tilde\psi}\mathbf{z}+\gamma_{21}z)\,dt=0.\end{aligned}$$

But,

$$\begin{aligned}&\operatorname{Re}\int_{J_N}(t_N-t)\Big(\frac{i}{\kappa}\nabla z_t+\tilde{\mathbf{A}}z_t,\frac{i}{\kappa}\nabla z+\tilde{\mathbf{A}}z\Big)\,dt=-\frac{1}{2\kappa^2}t_N\|\nabla z(0)\|^2\\&\quad+\frac{1}{2\kappa^2}\int_{J_N}\|\nabla z\|^2\,dt+\operatorname{Re}\int_{J_N}(t_N-t)\Big(z_t,\frac{i}{\kappa}\nabla\cdot(\tilde{\mathbf{A}}z)+\frac{i}{\kappa}\tilde{\mathbf{A}}\cdot\nabla z+\tilde{\mathbf{A}}^2z\Big)\,dt,\end{aligned}$$

and

$$\begin{aligned}&\int_{J_N}(t_N-t)\big\{(\nabla\times\mathbf{z}_t,\nabla\times\mathbf{z})+(\nabla\cdot\mathbf{z}_t,\nabla\cdot\mathbf{z})\big\}\,dt\\&\quad=-\frac12 t_N\big(\|\nabla\times\mathbf{z}(0)\|^2+\|\nabla\cdot\mathbf{z}(0)\|^2\big)+\frac12\int_{J_N}\{\|\nabla\times\mathbf{z}\|^2+\|\nabla\cdot\mathbf{z}\|^2\}\,dt.\end{aligned}$$

Thus, using the boundedness assumptions on $\tilde{\psi}$ and $\tilde{\mathbf{A}}$, we obtain

$$\eta \int_{J_N} (t_N - t)\|z_t\|^2\,dt + \frac{1}{2\kappa^2} t_N \|\nabla z(0)\|^2 = \frac{1}{2\kappa^2}\int_{J_N} \|\nabla z\|^2\,dt$$
$$+ \operatorname{Re}\int_{J_N} (t_N - t)\Big(z_t, \frac{i}{\kappa}\nabla\cdot(\tilde{\mathbf{A}} z) + \frac{i}{\kappa}\tilde{\mathbf{A}}\cdot\nabla z + \tilde{\mathbf{A}}^2 z + \gamma_{11} z + \tilde{\gamma}_{11}\overline{z} + \gamma_{12}\mathbf{z}\Big)\,dt$$
$$\le \frac{\eta}{2}\int_{J_N} (t_N - t)\|z_t\|^2\,dt + C\Big(\|z\|_{J_N}^2 + \|\mathbf{z}\|_{J_N}^2 + \int_{J_N} \big\{\|\nabla z\|^2 + \|\nabla\cdot\mathbf{z}\|^2\big\}\,dt\Big),$$

and

$$\int_{J_N} (t_N - t)\|\mathbf{z}_t\|^2\,dt + \frac{1}{2} t_N \big(\|\nabla\times\mathbf{z}(0)\|^2 + \|\nabla\cdot\mathbf{z}(0)\|^2\big)$$
$$= \frac{1}{2}\int_{J_N} \big\{\|\nabla\times\mathbf{z}\|^2 + \|\nabla\cdot\mathbf{z}\|^2\big\}\,dt + \int_{J_N} (t_N - t)\big(\mathbf{z}_t, \tilde{\tilde{\psi}}\mathbf{z} + \gamma_{21} z\big)\,dt$$
$$\le \frac{1}{2}\int_{J_N} (t_N - t)\|\mathbf{z}_t\|^2\,dt$$
$$+ C\Big(\|z\|_{J_N}^2 + \|\mathbf{z}\|_{J_N}^2 + \int_{J_N} \big\{\|\nabla z\|^2 + \|\nabla\times\mathbf{z}\|^2 + \|\nabla\cdot\mathbf{z}\|^2\big\}\,dt\Big),$$

so (56) now follows from (59). □

Acknowledgments. This licentiate thesis was written within the ECMI (European Consortium for Mathematics in Industry) programme in applied mathematics.

I would like to express my gratitude to my advisor Stig Larsson for all the encouragement, discussions, and time, that he has given me during the course of this work.

I am also in debt to several of my colleagues, especially Mats Boman, with whom I have had many inspiring and stimulating discussions about mathematics and everything else.

This work was financially supported by Chalmers Finite Element Center Φ and the Swedish Research Council for Engineering Sciences (TFR).

I also would like to thank the Oxford Center for Industrial and Applied Mathematics (OCIAM) and the Mittag-Leffler Institute, where parts of this work was done, for their hospitality.

Finally, I would like to thank my parents, and my girlfriend Lina, for their support.

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