On Hopf-Frobenius algebras

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1 Introduction

A classical notion in the theory of induced representations is that of a *Frobenius extension*. Let $S \to A$ be a homomorphism of unital associative rings, and call this a *ring extension* A/S. A Frobenius extension of rings [12] requires that A be isomorphic to its dual as S-A-bimodules,
${}_S A_A \cong {}_S\mathrm{Hom_S(A_S, S_S)_A}$, and A_S be a finitely generated projective right S-module.[1] The bimodules ${}_S A_A$, ${}_A A_S$, and ${}_S S_S$ are the natural ones, except that ${}_S S_S$ may be replaced by a left twisted bimodule ${}_\beta S_S$, where β is a ring automorphism on S and the left module structure is indicated by $s_1 \cdot s_2 := \beta(s_1)s_2$. This replacement by ${}_\beta S_S$ defines more generally a β-Frobenius extension [18]. A theorem in [20] characterizes a β-Frobenius extension A/S by the existence of a *Frobenius* (bimodule) *homomorphism* $E : {}_S A_S \to {}_\beta S_S$ and a (finite) *dual base* $x_i, y_i \in A$ for the ring extension such that $\sum \beta^{-1}(E(ax_i))y_i = a = \sum_i x_i E(y_i a)$ for every $a \in A$: call (E, x_i, y_i) a *Frobenius system*. As a corollary, a β-Frobenius extension is a separable extension [9] if and only if (iff) there is a $d \in A$ such that $\sum_i x_i d y_i = 1$ and $\beta(s)d = ds$ for every $s \in S$ [11].

For example, a Hopf subalgebra S in a finite dimensional Hopf algebra A over a field is a free β-Frobenius extension, where β is a relative version of the Nakayama automorphism of A and S defined below. A Frobenius system is given in [8]. A_S and ${}_S A$ are free by a theorem in [19].

As another example, a subfactor of type II_1 and finite Jones index is a separable Frobenius extension $S \subset A$ with special Frobenius system (E, x_i, y_i) such that $E(1) = 1$ and $\sum_i x_i y_i = [A : S]$, the trace of the Hattori-Stalling rank of A_S or ${}_S A$ [10]. By an endomorphism ring theorem in [10], the ring

[1] This is equivalent to assuming that ${}_S A$ is finite projective and ${}_A A_S \cong {}_A\mathrm{Hom_S({}_S A, {}_S S)_S}$.

$\mathcal{E} := \mathrm{End_S(A_S)}$ is a separable Frobenius extensions of A with the same properties and index: $[A_1 : A] = [A : S]$. Note that E is an idempotent in $\mathcal{E}$. Iterating the $(\mathcal{E}, E)$ construction for $\mathcal{E}/A$, ..., builds a tower of separable Frobenius extensions with a family of idempotents satisfying the braid-like relations of Jones [10].

As a third example, let A/S be an algebra, i.e., S is a commutative ring k and $k \to A$ has image in the center of A. Thus Frobenius extension reduces to a Frobenius algebra as defined in [6]. If (ϕ, x_i, y_i) is a Frobenius system[2] an algebra automorphism η of A, called the *Nakayama automorphism*, is defined by $\eta(a) = \sum_i \phi(x_i a) y_i$ for all $a \in A$. Making use of η, [2] has shown that the *Frobenius* (or Casimir) *element* $R = \sum_i x_i \otimes y_i$ in $A \otimes A$ satisfies the Yang-Baxter Equation given by $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$ in $A \otimes A \otimes A$. Moreover, R is invertible in $A \otimes A$ iff A is a central separable k-algebra [2]. If A/k is a commutative Frobenius algebra, [1] shows how to interpret this precisely as a two-dimensional topological quantum field theory, based on the coalgebra A with comultiplication given by $a \mapsto Ra = aR$ (the equality follows from the equations above for a dual base) and counit by ϕ.

Now it has been known since [14] and [21] that a finite rank Hopf algebra over a p.i.d. and more generally a finite projective Hopf algebra H over a commutative ring k of trivial Picard group, respectively, is a Frobenius algebra. With f a right norm in H^* and t a right norm in H such that ft equals the counit ϵ, a Frobenius system was given recently by $(f, S^{-1}(t_2), t_1)$ in [3] and in [8] for k a field (the notation we use is borrowed from [8]). [3] proves that the Nakayama automorphism η is the square of the antipode S, if H is unimodular. If k is a field, [8] has an improved formula for η. [3] shows that η has order at most 2 if H and H^* are unimodular, while [8] shows that S and η have order dividing $4 \dim H$ and $2 \dim H$ respectively, if k is a field.

The purpose in this paper is to apply Frobenius systems to Hopf-Frobenius algebras, which is a notion implicit in [21] that is more general than a finite projective Hopf algebra over a ring with trivial Picard group. We will prove that if H is a Hopf-Frobenius algebra, then H^* and the Drinfel'd quantum double $D(H)$ are Hopf-Frobenius algebras as well. We will derive two of Radford's formulas in [23, 25], including the formula for S^4, for Hopf-Frobenius algebras. Then we will show that $D(H)$ is a unimodular, symmetric algebra generalizing [24, 3]. In the last section, we prove that a group-like in a

[2] β is necessarily the identity [18].

general finite projective Hopf algebra has finite order, from which it follows from the formula for S^4 that S and η have finite order.

2 Augmented Frobenius algebras

In this section k denotes a commutative ring. Given an associative, unital k-algebra A, A^* denotes the dual module $\mathrm{Hom_k(A, k)}$, which is an A-A bimodule as follows: given $f \in A^*$ and $a \in A$, af is defined by $(af)(b) = f(ba)$ for every $b \in A$, while fa is defined by $(fa)(b) = f(ab)$.

We first consider some preliminaries on Frobenius algebras over commutative rings. A *Frobenius algebra* A/k is a k-algebra where the natural module A_k is *finite projective*, and

$$A_A \cong A^*_A. \tag{1}$$

Suppose $f_i \in A^*$, $x_i \in A$ form a finite projective base, or dual base, of A over k: i.e., for every $a \in A$, $\sum_i x_i f_i(a) = a$. Then there are $y_i \in A$ and a cyclic generator $\phi \in A^*$ such that the A-module isomorphism is given by $a \mapsto \phi a$, and

$$\sum_i x_i \phi(y_i a) = a = \sum \phi(a x_i) y_i, \tag{2}$$

for all $a \in A$. It follows that ϕ is nondegenerate in the following sense: a linear functional ϕ on an algebra A is *nondegenerate* if $a, b \in A$ such that $a\phi = b\phi$ or $\phi a = \phi b$ implies $a = b$.

We refer to ϕ as a *Frobenius homomorphism*, (x_i, y_i) as a *dual base*, and (ϕ, x_i, y_i) as a *Frobenius system.*

It is equivalent to define a k-algebra A Frobenius if A_k is finite projective and ${}_AA \cong {}_AA^*$. In fact, with ϕ defined above, the mapping $a \mapsto a\phi$ is such an isomorphism, by an application of Equations 2.

Note that the bilinear form on A defined by $\langle a, b \rangle := \phi(ab)$ is an inner product in the sense of [16], which is associative: $\langle ab, c \rangle = \langle a, bc \rangle$ for every $a, b, c \in A$.

The Frobenius homomorphism is unique up to an invertible element in A. If ϕ and ψ are Frobenius homomorphisms for A, then $\psi = d\phi$ for some $d \in A$. Similarly, $\phi = d'\psi$ for some $d' \in A$, from which it follows that $dd' = 1$. The element d is referred to as the (left) *derivative* of ψ with respect to ϕ following [27]. Right derivatives in the group of units A° of A are similarly defined.

If (ϕ, x_i, y_i) is a Frobenius system for A, then $e := \sum_i x_i \otimes y_i$ is an invariant element in the tensor-square $A \otimes_k A$, called the *Frobenius element*, depending only on ϕ and not on the dual base. We consider the natural A-bimodule $A \otimes A$ given by $a(b \otimes c) = ab \otimes c$ and $(a \otimes b)c := a \otimes bc$ for every $a, b, c \in A$. By a computation involving Equations 2, e is a Casimir element satisfying $ae = ea$ for every $a \in A$, whence $\sum_i x_i y_i$ is in the center of A. It follows that A is k-separable if and only if there is a $a \in A$ such that $\sum_i x_i a y_i = 1$. [3]

For each $d \in A^\circ$, we easily check that $(\phi d, x_i, d^{-1} y_i)$ and $(d\phi, x_i d^{-1}, y_i)$ are the other Frobenius systems in a one-to-one correspondence. It follows that a Frobenius element is also unique, up to a unit in $A \otimes A$ (either $1 \otimes d^{\pm 1}$ or $d^{\pm 1} \otimes 1$).

A *symmetric algebra* is a Frobenius algebra A/k which satisfies the stronger condition:

$$ {}_A A_A \cong {}_A (A^*)_A . \tag{3} $$

Choosing an isomorphism Φ, the linear functional $\phi := \Phi(1)$ is a Frobenius homomorphism satisfying $\phi(ab) = \phi(ba)$ for every $a, b \in A$: i.e., ϕ is an trace on A. An algebra A *over a field* has a symmetric associative inner product iff A is a symmetric algebra.

A k-algebra A with $\phi \in A^*$ and $x_i, y_i \in A$ satisfying either $\sum_i x_i \phi(y_i a) = a$ for every $a \in A$ or $\sum_i \phi(a x_i) y_i = a$ for every $a \in A$ is automatically Frobenius. As a corollary, one of the dual base equations implies the other. For if $\sum_{i=1}^n (x_i \phi) y_i = \mathrm{Id_A}$, then A is explicitly finite projective over k, and it follows that A^* is finite projective too. The homomorphism ${}_A A \to {}_A A^*$ defined by $a \mapsto a\phi$ for all $a \in A$ is surjective, since given $f \in A^*$, we note that $f = (\sum_i f(y_i) x_i)\phi$. Since A and A^* have the same P-rank, for each prime ideal P in k, the epimorphism $a \mapsto a\phi$ is bijective [26], whence ${}_A A \cong {}_A A^*$. Starting with the other equation in the hypothesis, we similarly prove that $a \mapsto \phi a$ is an isomorphism $A_A \cong A^*_A$.

The Nakayama automorphism of a Frobenius algebra A is an algebra automorphism $\alpha : A \to A$ defined by

$$ \phi \alpha(a) = a \phi \tag{4} $$

for every $a \in A$. In terms of the associative inner product, $\langle x, a \rangle = \langle \alpha(a), x \rangle$ for every $a, x \in A$. α is an inner automorphism iff A is a symmetric algebra.

[3] e is the transpose of the element Q in [3].

The Nakayama automorphism η of another Frobenius homomorphism $\psi = \phi d$, where $d \in A^\circ$, is given by

$$\eta(x) = \sum_i \phi(dx_i x)d^{-1}y_i = \sum_i d^{-1}\phi(\alpha(x)dx_i)y_i = d^{-1}\alpha(x)d, \qquad (5)$$

so that $\alpha\eta^{-1}(x) = dxd^{-1}$. Thus the Nakayama automorphism is unique up to an inner automorphism. A Frobenius algebra A is a symmetric algebra if and only if its Nakayama automorphism is inner.

The left and right derivatives of a pair of Frobenius homomorphisms differ by an application of the Nakayama automorphism.

A k-algebra A is said to be an *augmented algebra* if there is an algebra homomorphism $\epsilon : A \to k$, called an *augmentation.* An element $t \in A$ satisfying $ta = \epsilon(a)t$, $\forall\, \mathrm{a} \in \mathrm{A}$, is called a *right integral* of A. It is clear that the set of right integrals, denoted by $\int_A^r$, is a two-sided ideal of A, since for each $a \in A$, the element at is also a right integral. Similarly for the space of left integrals, denoted by $\int_A^\ell$.

Now suppose that A is a Frobenius algebra with augmentation ϵ. We claim that a nontrivial right integral exists in A. Since $A^* \cong A$ as right A-modules, an element $n \in A$ exists such that $\phi n = \epsilon$ where ϕ is a Frobenius homomorphism. Call n the *right norm* in A with respect to ϕ. Given $a \in A$, we compute:

$$\phi n a = (\phi n)a = \epsilon a = \epsilon(a)\epsilon = \phi n \epsilon(a).$$

By nondegeneracy of ϕ, n satisfies $na = n\epsilon(a)$ for every $a \in A$.

Proposition 2.1 *If A is an augmented Frobenius algebra, then the set $\int_A^r$ of right integrals is a two-sided ideal which is free cyclic k-summand of A generated by a right norm.*

Proof. Let $\phi \in A^*$ be a Frobenius homomorphism, and $n \in A$ satisfy $\phi n = \epsilon$, the augmentation. Given a right integral $t \neq 0$, we note that

$$\phi t = \phi(t)\epsilon = \phi(t)\phi n = \phi n \phi(t),$$

whence

$$t = \phi(t)n. \qquad (6)$$

Then $\langle n \rangle := \{\rho n |\, \rho \in k\}$ coincides with the set of all right integrals.

Given $\lambda \in k$ such that $\lambda n = 0$, it follows that

$$\phi(n)\lambda = \epsilon(1)\lambda = \lambda = 0,$$

whence $\langle n \rangle$ is a free k-module. Moreover, $\langle n \rangle$ is a direct k-summand in A since $a \mapsto \phi(a)n$ defines a k-linear projection of A onto $\langle n \rangle$. $\square$

In particular, $\int_A^r$ is free of rank 1 in an augmented Frobenius algebra A. The right norm in A is unique up to a unit in k. For, in the notation of the proof above, if t is another right norm in A, then there exist $\lambda_1, \lambda_2 \in k$ such that $t = \lambda_1 n$ and $n = \lambda_2 t$. Then $\lambda_2\lambda_1 n = n$, so that $\lambda_1\lambda_2 = 1_k$ by freeness. Note that right norms are precisely the right integrals that generate $\int_A^r$: the two notions only coincide if k is a field.

Similarly the space $\int_A^\ell$ of left integrals is a rank one free summand in A, generated by any left norm. If $\int_A^r = \int_A^\ell$, A is said to be *unimodular*. If the spaces of right and left integrals do not coincide we define an augmentation on A which measures the deviation from unimodularity. In the notation of the proposition and its proof, for every $a \in A$, the element an is a right integral since the right norm n is. From Equation 6 one concludes that $an = \phi(an)n = (n\phi)(a)n$. The function

$$m := n\phi : A \to k \tag{7}$$

is called the *right modular function*, an augmentation since $\forall\, \mathrm{a}, \mathrm{b} \in \mathrm{A}$, we have $(ab)n = m(ab)n = a(bn) = m(a)m(b)n$ and n freely generates. If η denotes the Nakayama automorphism of ϕ, then A is unimodular if and only if $m = \epsilon = \phi n$ if and only if $\eta(n) = n$.

Proposition 2.2 *Suppose A is an augmented Frobenius algebra with augmentation ϵ, Frobenius homomorphism ϕ, and Nakayama automorphism η. Then A is unimodular if and only if $\epsilon\eta = \epsilon$.*

Proof. Let n denote the right norm such that $\phi n = \epsilon$.

($\Rightarrow$) If A is unimodular, the right modular function $m = \epsilon$. Then $\forall\, \mathrm{a} \in \mathrm{A}$

$$\epsilon(a) = m(a) = (n\phi)(a) = \phi(an) = \phi(n\eta^{-1}(a)) = \epsilon \circ \eta^{-1}(a).$$

Equivalently, $\epsilon = \epsilon\eta$.

($\Leftarrow$) If $\epsilon\eta^{\pm 1} = \epsilon$, then

$$\epsilon(a) = \epsilon\eta^{-1}(a) = \phi(n\eta^{-1}(a)) = \phi(an) = m(a),$$

whence $m = \epsilon$ and A is unimodular. □

We will use several general principles repeatedly in the next section. First, if α is an k-algebra automorphism of the augmented Frobenius algebra (A, ϵ) (satisfying *ϵ-invariance*: $\epsilon \circ \alpha = \epsilon$), then α send integrals to integrals and norms to norms, respecting chirality. If β is an anti-automorphism of (A, ϵ), it sends left norms to right norms, etc. Secondly, with no assumption of an augmentation, a Frobenius system (ϕ, x_i, y_i) of A is transformed by an automorphism α into a Frobenius system $(\phi \circ \alpha^{-1}, \alpha(x_i), \alpha(y_i))$, while an anti-automorphism $\beta : A \to A$ transforms it into the Frobenius system $(\phi \circ \beta^{-1}, \beta(y_i), \beta(x_i))$. Third, if A and B are Frobenius k-algebras with Frobenius homomorphism ϕ_A and ϕ_B, then $A \otimes B$ is a Frobenius algebra with Frobenius homomorphism $\phi_A \otimes \phi_B : A \otimes B \to k$.

In closing this section, we note that Frobenius algebras over commutative rings have been studied in several papers including [6, 2, 11]. Augmented algebras have been treated in [21, 7, 11].

3 Hopf-Frobenius algebras

Let k denote a commutative ring. We review the basics of a Hopf algebra H which is finite projective over k [21, 22]. A bialgebra H is an algebra and coalgebra where the comultiplication and the counit are algebra homomorphisms. We use the *reduced* Sweedler notation $\Delta(a) := \sum a_1 \otimes a_2$ for the values of the comultiplication homomorphism $H \to H \otimes_k H$. The counit is the k-algebra homomorphism $\epsilon : H \to k$ and satisfies $\sum_i \epsilon(a_1)a_2 = \sum a_1 \epsilon(a_2) = a$ for every $a \in H$.

A Hopf algebra H is a bialgebra with antipode. The antipode $S : H \to H$ is an anti-homomorphism of algebras and coalgebras satisfying $\sum S(a_1)a_2 = \epsilon(a)1 = \sum a_1 S(a_2)$ for every $a \in H$.

A *group-like element* in H is defined to be a $g \in H$ such that $\Delta(g) = g \otimes g$ *and* $\epsilon(g) = 1$. It follows that $g \in H^\circ$ and $S(g) = g^{-1}$.

Finite projective Hopf algebras enjoy the duality properties of finite dimensional Hopf algebras. H^* is a Hopf algebra with convolution product $(fg)(x) := \sum f(x_1)g(x_2)$. The counit is given by $f \mapsto f(1)$. The unit of H^* is the counit of H. The comultiplication on H^* is given by $\sum f_1 \otimes f_2(a \otimes b) = f(ab)$ for every $f \in H^*$, $a, b \in H$. The antipode is the dual of S, a mapping of H^* into H^*, denoted again by S when the context is clear. Note that an augmentation f in H^* is a group-like element in H^*, and *vice versa*, with inverse given by $Sf = f \circ S$.

As Hopf algebras, $H \cong H^{**}$, the isomorphism being given by $x \mapsto \mathrm{ev}_x$, the evaluation map at x: we fix this isomorphism as an identification of H with H^{**}. The usual left and right action of an algebra on its dual specialize to the left action of H^* on $H^{**} \cong H$ given by $g \rightharpoonup a := \sum a_1 g(a_2)$, and the right action given by $a \leftharpoonup g := \sum g(a_1)a_2$.

Definition 3.1 *A k-algebra H is Hopf-Frobenius if H is a bialgebra and a Frobenius algebra with Frobenius homomorphism f a right integral in H^*. Call f the Hopf-Frobenius homomorphism.*

The condition that $f \in \int_{H^*}^r$ is equivalent to

$$\sum f(a_1)a_2 = f(a)1 \tag{8}$$

for every $a \in H$. Note that H is an augmented Frobenius algebra with augmentation ϵ. Let $t \in H$ be a right norm such that $ft = \epsilon$. Note that $f(t) = 1$. Fix the notation f and t for a Hopf-Frobenius algebra. We show below that a Hopf-Frobenius homomorphism is unique up an invertible scalar in k. If H is a Hopf-Frobenius algebra and a symmetric algebra, we say that H is a *symmetric Hopf-Frobenius algebra*.

It follows from [21, Theorem 2] that a Hopf-Frobenius algebra H automatically has an antipode. With f its Hopf-Frobenius homomorphism and t a right norm, define $S : H \to H$ by

$$S(a) = \sum f(t_1 a)t_2. \tag{9}$$

Then for every $a \in H$

$$\sum S(a_1)a_2 = \sum f(t_1 a_1)t_2 a_2 = f(ta)1 = \epsilon(a)1.$$

Now in the convolution algebra structure on $\mathrm{End_k(H)}$, this shows S has $\mathrm{Id_H}$ as right inverse. Since $\mathrm{End_k(H)}$ is finite projective over k, it follows that $\mathrm{Id_H}$ is also a left inverse of S; whence S is the unique antipode.

The Pareigis Theorem [21] generalizing the Larson-Sweedler Theorem [14] shows that a finite projective Hopf algebra H over a ground ring k with trivial Picard group is a Hopf-Frobenius algebra. In detail, the theorem proves the following in the order given. The first two items are proven without the hypothesis on the Picard group of k. The last two items require only that $\int_{H^*}^\ell$ be free of rank 1.

1. There is a right Hopf H-module structure on H^*. Since all Hopf modules are trivial, $H^* \cong P(H^*) \otimes H$, for the coinvariants $P(H^*) = \int_{H^*}^{\ell}$.

2. The antipode S is bijective.

3. There exists a left integral f in H^* such that the mapping $\Theta : H \to H^*$ defined by
$$\Theta(x)(y) = f(yS(x)) \tag{10}$$
is a right Hopf module isomorphism.

4. H is a Frobenius algebra with Frobenius homomorphism f.

It follows from 2. above that a Hopf-Frobenius algebra H possesses an ϵ-invariant anti-automorphism S. If $f \in H^*$ is a Hopf-Frobenius homomorphism, then Sf is a Frobenius homomorphism and *left integral* in H^*. It is therefore equivalent to replace right with left in Definition 3.1.

Let $m : H \to k$ be the right modular function of H. Since m is an algebra homomorphism, it is group-like in H^*, whence m at times is called the *right distiguished group-like element* in H^*.

Proposition 3.1 *Let H be a Hopf-Frobenius algebra with Hopf-Frobenius homomorphism f and right norm t. Then $(f, S^{-1}t_2, t_1)$ is a Frobenius system for H.*

Proof. Applying S^{-1} to both sides of Equation 9 yields
$$\sum S^{-1}(t_2) f(t_1 a) = a, \tag{11}$$
for every $a \in H$. It follows from the finite projectivity assumption on H that $(f, S^{-1}(t_2), t_1)$ is a Frobenius system. $\square$

It follows from the proposition that $t \leftharpoondown f = 1$. Together with the corollary below this implies that f is a right norm in H^*, since 1 is the counit for H^*. It follows that g is another Hopf-Frobenius homomorphism for H iff $g = f\lambda$ for some $\lambda \in k^\circ$.

Proposition 3.2 *H is a Hopf-Frobenius algebra if and only if H^* is a Hopf-Frobenius algebra.*

Proof. It suffices by duality to establish the forward implication. Suppose f is a Hopf-Frobenius homomorphism for H and t a right norm. Note that $\sum g_1(t)g_2 = g(t)\epsilon$ and $(fg)(t) = \sum f(t_1)g(t_2) = g(1)$ for every $g \in H^*$. Then by Equation 9 and the argument after it,

$$S(g) = \sum (f_1 g)(t) f_2 \tag{12}$$

is an equation for the antipode in H^*.

Then $(t, S^{-1}f_2, f_1)$ is a Frobenius system for H^* by taking S^{-1} of both sides. Whence t is a Hopf-Frobenius homomorphism for H^* with right norm f. □

It follows that H^* is also an augmented Frobenius algebra. Let $b \in H$ be the right *distiguished group-like element* satisfying

$$gf = g(b)f \tag{13}$$

for every $g \in H^*$.

The convolution product inverse of m is $m^{-1} = m \circ S$. Given a left norm $v \in H$, we claim that

$$va = vm^{-1}(a).$$

Since t is a right norm, S an anti-automorphism and ϵ-invariant, it follows that St is a left norm. Then we may assume $v = St$. Then $S(at) = StSa = m(a)St$, whence $vx = vmS^{-1}(x)$ for every $a, x \in H$. The claim will follow from $m \circ S^2 = m$, since this implies that $m \circ S^{-1} = m^{-1}$. Since S^2 is an ϵ-invariant k-automorphism of H, $S^2 t$ is also a right norm and differs by a unit from t. It follows readily that $m \circ S^2 = m$.

Lemma 3.1 *Given a Hopf-Frobenius algebra H with right norm $f \in H^*$ and right norm $t \in H$ such that $f(t) = 1$, the Nakayama automorphism, relative to f, and its inverse are given by:*

$$\eta(a) = S^2(a \leftharpoonup m^{-1}) = (S^2 a) \leftharpoonup m^{-1}, \tag{14}$$

$$\eta^{-1}(a) = S^{-2}(a \leftharpoonup m) = (S^{-2} a) \leftharpoonup m.$$

Proof. Using the Frobenius system $(f, S^{-1}t_2, t_1)$, we note that

$$\eta^{-1}(a) = \sum S^{-1}(t_2)f(t_1\eta^{-1}(a)) = \sum S^{-1}(t_2)f(at_1).$$

We compute:

$$\begin{aligned} S^2(\eta^{-1}(a)) &= \sum f(at_1)St_2 \\ &= \sum f(a_1t_1)a_2t_2St_3 \\ &= \sum f(a_1t)a_2 \\ &= a \leftharpoonup m \end{aligned}$$

since $a \leftharpoonup f = f(a)1$, $at = m(a)t$ for every $a \in H$ and $f(t) = 1$. Whence $\eta^{-1}(a) = S^{-2}(a \leftharpoonup m)$. Since $mS^{-2} = m$ it follows that $\eta^{-1}(a) = (S^{-2}a) \leftharpoonup m$.

It follows that $a = (S^{-2}\eta a) \leftharpoonup m$, so let the convolution inverse m^{-1} act on both sides: $(a \leftharpoonup m^{-1}) = S^{-2}\eta(a)$. Whence $\eta(a) = S^2(a \leftharpoonup m^{-1}) = (S^2a) \leftharpoonup m^{-1}$, since $m^{-1}S^2 = m^{-1}$. $\square$

As a corollary, we obtain [3, Proposition 3.8]: if H is a unimodular Hopf-Frobenius algebra, then the Nakayama automorphism is the square of the antipode.

Proposition 3.3 *If H is a Hopf-Frobenius algebra with Hopf-Frobenius homomorphism f and right norm t, then*

$$\sum t_2 \otimes t_1 = \sum b^{-1}S^2t_1 \otimes t_2. \tag{15}$$

Proof. On the one hand, we have seen that $(f, S^{-1}t_2, t_1)$ is a Frobenius system for H. On the other hand, the equation $f \rightharpoonup x = bf(x)$ for every $x \in H$ follows from Equation 13 and gives

$$\begin{aligned} \sum S(t_1)bf(t_2a) &= \sum S(t_1)t_2a_1f(t_3a_2) \\ &= \sum a_1f(ta_2) \\ &= \sum a_1\epsilon(a_2)f(t) = a. \end{aligned}$$

Then $(f, S(t_1)b, t_2)$ is another Frobenius system for H.

Since $(S^{-1}(t_2), t_1)$ and $(S(t_1)b, t_2)$ are both dual bases to f, it follows that $\sum S^{-1}t_2 \otimes t_1 = \sum S(t_1)b \otimes t_2$. Equation 15 follows from applying $S \otimes 1$ to both sides. $\Box$

Proposition 3.1 with $a = S^{-1}t$ gives

$$\sum S^{-1}t_2 f(t_1 S^{-1}t) = S^{-1}tf(S^{-1}t) = S^{-1}t.$$

Since $S^{-1}t$ is a left norm, it follows that

$$f(S^{-1}t) = 1. \tag{16}$$

Proposition 3.4 *Given a Hopf-Frobenius algebra H with Hopf-Frobenius homomorphism f, the right distinguished group-like element b is equal to the derivative d of the left integral Frobenius homomorphism $S^{-1}f$ with respect to f.*

Proof. Another Frobenius system for H is given by $(S^{-1}f, St_1, t_2)$, since S is an anti-automorphism. Then there exists a (derivative) $d \in H^\circ$ such that

$$df = S^{-1}f. \tag{17}$$

$S^{-1}f$ is a left norm in H^* since S^{-1} is an ϵ-invariant anti-automorphism. Also bf is a left integral in H^* by the following argument. For any $g, g' \in H^*$, we have $b(gg') = (bg)(bg')$ as b is group-like. Then for every $h \in H^*$

$$\begin{aligned} h(bf) &= b[(b^{-1}h)f] \\ &= b[(b^{-1}h)(b)f] \\ &= h(1)(bf). \end{aligned}$$

Now both $S^{-1}f(t)$ and $bf(t)$ equal 1, since $f(S^{-1}t) = 1$, $f(tb) = \epsilon(b)f(t) = 1$ and b is group-like. Since bf is a scalar multiple of the norm $S^{-1}f$, it follows that

$$S^{-1}f = bf. \tag{18}$$

Finally, $d = b$ since $df = bf$ from Equations 17 and 18, and f is nondegenerate. $\Box$

We next derive a formula for the fourth power of the antipode of a Hopf-Frobenius algebra by noting that the Nakayama automorphisms associated with the two Frobenius homomorphisms $S^{-1}f$ and f differ by an inner automorphism determined by the derivative in Proposition 3.4.

Theorem 3.1 *Given a Hopf-Frobenius algebra H with right distinguished group-like elements $m \in H^*$ and $b \in H$, the fourth power of the antipode is given by*

$$S^4(a) = b(m^{-1} \rightharpoonup a \leftharpoonup m)b^{-1} \tag{19}$$

for every $a \in H$.

Proof. Let $g := S^{-1}f$ and denote the left norm St by Λ. Note that $g(\Lambda) = 1 = g(S^{-1}\Lambda)$ since $f(t) = 1 = f(S^{-1}t)$. We note that $(g, \Lambda_2, S^{-1}\Lambda_1)$ is a Frobenius system for H, since S is an anti-automorphism of H

Then the Nakayama automorphism α associated with g has inverse satisfying

$$\alpha^{-1}(a) = \sum \Lambda_2 g(aS^{-1}\Lambda_1)$$

whence

$$\begin{aligned} S^{-1}\alpha^{-1}(a) &= \sum S^{-1}g(\Lambda_1 Sa)S^{-1}(\Lambda_2) \\ &= \sum S^{-1}(\Lambda_3)S^{-1}g(\Lambda_1 Sa_2)\Lambda_2 Sa_1 \\ &= \sum S^{-1}g(\Lambda Sa_2)Sa_1 \\ &= g(S^{-1}\Lambda)\sum m^{-1}(Sa_2)Sa_1 = S(m \rightharpoonup a), \end{aligned}$$

since $Sm^{-1} = m$. It follows that

$$\alpha^{-1}(a) = S^2(m \rightharpoonup a) = m \rightharpoonup S^2 a \tag{20}$$

$$\alpha(a) = m^{-1} \rightharpoonup S^{-2}a = S^{-2}(m^{-1} \rightharpoonup a). \tag{21}$$

Recall from Proposition 3.4 that $g = bf = f\eta^{-1}(b)$, where η is the Nakayama automorphism of f. By Equation 5 and Lemma 3.1,

$$\begin{aligned} m^{-1} \rightharpoonup S^{-2}a &= \alpha(a) \\ &= \eta(b^{-1})\eta(a)\eta(b) \\ &= m^{-1}(b^{-1})b^{-1}(S^2(a) \leftharpoonup m^{-1})bm^{-1}(b) \\ &= b^{-1}S^2(a)b \leftharpoonup m^{-1}, \end{aligned}$$

since b and m are group-likes and S^2 leaves m and b fixed. It follows that

$$a = m \rightharpoonup b^{-1}S^4(a)b \leftharpoonup m^{-1},$$

for every $a \in H$. Equation 19 follows. $\Box$

The theorem implies [3, Corollary 3.9], which states that $S^4 = \mathrm{Id_H}$, if H and H^* are unimodular finite projective Hopf algebra over k. For localizing with respect to any maximal ideal $\mathcal{M}$, we obtain unimodular Hopf-Frobenius algebras $H \otimes k_{\mathcal{M}}$ and its dual, since $k_{\mathcal{M}}$ has trivial Picard group. By Theorem 3.1, the localized antipode satisfies $(S_{\mathcal{M}})^4 = \mathrm{Id}$ for every maximal ideal $\mathcal{M}$ in k; whence $S^4 = \mathrm{Id_H}$ [26].

In closing this section, we note that relationships among the antipode, integrals, the distinguished group-likes and Nakayama automorphism for Hopf algebras over fields were investigated in [3, 8, 23, 25].

4 The quantum double

Let k be a commutative ring. We note that the *quantum double* $D(H)$, due to Drinfel'd [5], is definable for a finite projective Hopf algebra H over k: at the level of coalgebras it is given by

$$D(H) := H^{*\,\mathrm{cop}} \otimes_k H,$$

where $H^{*\,\mathrm{cop}}$ is the co-opposite of H^*, the coproduct being Δ^{op}.

The multiplication on $D(H)$ is described in two equivalent ways as follows [17, Lemma 10.3.11]. In terms of the notation gx replacing $g \otimes x$ for every $g \in H^*, x \in H$, both H and H^* are subalgebras of $D(H)$, and for each $g \in H^*$ and $x \in H$,

$$xg := \sum (x_1 g S^{-1} x_3) x_2 = \sum g_2 (S^{-1} g_1 \rightharpoonup x \leftharpoonup g_3). \tag{22}$$

The algebra $D(H)$ is a Hopf algebra with antipode $S'(gx) := SxSg$, the proof proceeding as in [13]. A Hopf algebra H' is *almost cocommutative*, if there exists $R \in H' \otimes H'$, called the *universal R-matrix*, such that $R\Delta(a)R^{-1} = \Delta^{\mathrm{op}}(a)$ for every $a \in H'$. A *quasi-triangular Hopf algebra H'* is almost cocommutative with universal R-matrix satisfying the two equations,

$$(\Delta \otimes \mathrm{Id})\mathrm{R} \;=\; R_{13}R_{23} \tag{23}$$

$$(\mathrm{Id} \otimes \Delta)\mathrm{R} \;=\; R_{13}R_{12}. \tag{24}$$

By a proof like that in [13, Theorem IX.4.4], $D(H)$ is a quasi-triangular Hopf algebra with universal R-matrix

$$R = \sum_i e_i \otimes e^i \in D(H) \otimes D(H), \tag{25}$$

where (e_i, e^i) is a finite projective base of H [5].

Theorem 4.1 *If H is a Hopf-Frobenius algebra, then the quantum double $D(H)$ is a unimodular, symmetric Hopf-Frobenius algebra.*

Proof. We first show that $D(H)$ is a unimodular Hopf-Frobenius algebra. Let f be a Hopf-Frobenius homomorphism with t a right norm. Then $T := S^{-1}f$ is a left norm in H^*, and b^{-1} is the left distinguished group-like element in H satisfying $Tg = g(b^{-1})T$ for every $g \in H^*$. Moreover, note that $\ell := S^{-1}(t)$ be a left norm in H.

In this proof we denote elements of $D(H)$ as tensors in $H^* \otimes H$. We claim that $T \otimes t$ is a left and right integral in $D(H)$. We first show that it is a right integral.

The transpose of Formula 15 in Proposition 3.3 is $\sum t_1 \otimes t_2 = \sum t_2 \otimes b^{-1}S^2 t_1$. Applying $\Delta \otimes S^{-1}$ to both sides yields $\sum t_1 \otimes t_2 \otimes S^{-1}t_3 = \sum t_2 \otimes t_3 \otimes (St_1)b$. It follows easily that

$$\sum S^{-1}t_3 b^{-1} t_1 \otimes t_2 = 1 \otimes t. \tag{26}$$

Given a simple tensor $g \otimes x \in D(H)$, note that in the second line below we use $Tg = g(b^{-1})T$ for each $g \in H^*$, and in the third line we use Equation 26:

$$\begin{aligned}(T \otimes t)(g \otimes x) &= \sum Tg(S^{-1}t_3(-)t_1) \otimes t_2 x \\ &= Tg(S^{-1}t_3 b^{-1} t_1) \otimes t_2 x \\ &= g(1)T \otimes tx \\ &= g(1)\epsilon(x) T \otimes t\end{aligned}$$

In order to show that $T \otimes t$ is also a left integral, we note that Formula 15 applied to the right norm $T' = S^{-1}T$ in H^* is $\sum T'_1 \otimes T'_2 = \sum T'_2 \otimes m^{-1}S^2 T'_1$. Apply $S \otimes S$ to obtain

$$\sum T_2 \otimes T_1 = \sum T_1 \otimes S^2 T_2 m. \tag{27}$$

Applying $\Delta \otimes S^{-1}$ to both sides yields $\sum T_2 \otimes T_3 \otimes mS^{-1}T_1 = \sum T_1 \otimes T_2 \otimes ST_3$. Whence

$$\begin{aligned}\sum T_2 \otimes T_3 m S^{-1} T_1 &= \sum T_1 \otimes T_2 S T_3 \\ &= T \otimes 1. \end{aligned}\tag{28}$$

Then

$$
\begin{aligned}
(g \otimes x)(T \otimes t) &= \sum gT_2 \otimes (S^{-1}T_1 \rightharpoonup x \leftharpoonup T_3)t \\
&= \sum gT_2 \otimes S^{-1}T_1(x_3)T_3(x_1)x_2 t \\
&= \sum gT_2 \otimes [T_3 m S^{-1}T_1](x)t \\
&= gT \otimes \epsilon(x)t = g(1)\epsilon(x)T \otimes t
\end{aligned}
$$

Thus $T \otimes t$ is also a left integral.

Next we note that $T \otimes t$ is a Hopf-Frobenius homomorphism for $D(H)^*$, since $D(H)^* \cong H^{\rm op} \otimes H^*$, the ordinary tensor product of algebras (recall that $D(H)$ is the ordinary tensor product of coalgebras $(H^{\rm op})^* \otimes H$). This follows from $T \otimes t$ being a right integral in $D(H)$ on the one hand, while, on the other hand, $H^{\rm op}$ and H^* are Hopf-Frobenius algebras with Hopf-Frobenius homomorphisms $T = S^{-1}f$ and t.

Since $T \otimes t$ is a Hopf-Frobenius homomorphism for $D(H)^*$, it follows that $T \otimes t$ is a right norm in $D(H)$. Since $T \otimes t$ is a left integral in $D(H)$, it follows that it is a left norm too. Hence, $D(H)$ is unimodular.

We finally prove that $D(H)$ is a symmetric algebra. Since $D(H)$ is unimodular, Lemma 3.1 shows that $D(H)$ has Nakayama automorphism S^2. Now $D(H)$ is almost cocommutative. The computation in [17, Proposition 10.1.4] shows that S^2 of an almost commutative Hopf algebra H is an inner automorphism as follows: recalling the universal R-matrix in Equation 25, $R = \sum_i e_i \otimes e^i$, then $S^2(a) = uau^{-1}$ where $u = \sum_i (Se^i)e_i$. Since the Nakayama automorphism is inner, $D(H)$ is a symmetric algebra. □

Corollary 4.1 *$S(t) \otimes f$ is a Hopf-Frobenius homomorphism for $D(H)$.*

Proof. Note that $S(t) \otimes f$ is a right integral in $D(H)^* \cong H^{\rm op} \otimes H^*$, since $S(t)$ and f are right integrals in $H^{\rm op}$ and H^*, respectively. Then

$$
(T \otimes t)(S(t) \otimes f) = \epsilon_{D(H)^*} T(S(t)) f(t) = \epsilon_{D(H)^*}. \tag{29}
$$

so that $S(t) \otimes f$ is a right norm in $D(H)^*$. By Proposition 3.2, $D(H)$ is a Hopf-Frobenius algebra with Hopf-Frobenius homomorphism $S(t) \otimes f$. □

Theorem 4.1 is a generalization of the theorem that $D(H)$ is unimodular in [24] and the theorem that $D(H)$ is a symmetric algebra [3, Corollary 3.12], both for a finite dimensional Hopf algebra H over a field.

5 Finite order elements

Let H be a finite projective Hopf algebra over a commutative ring k and $d \in H$ be a group-like element. Our aim in this section is to prove that $d^N = 1$ for some integer N. Then we will prove as corollaries of Theorem 3.1 that the antipode S and Nakayama automorphism η have finite order.

Let $k[d, d^{-1}]$ denote the subalgebra of H generated over k by 1 and the negative and positive powers of d. Let $k[d]$ denote only the k-span of 1 and the positive powers of d. Clearly $k[d, d^{-1}]$ is Hopf subalgebra of H. d has a *minimal polynomial* $p(x) \in k[x]$ if $p(x)$ is a polynomial of least degree such that $p(d) = 0$ and the gcd of all the coefficients is 1. We first consider the case where k is a domain.

Lemma 5.1 *If k is a domain, each group-like $d \in H$ has a minimal polynomial $p(x) = x^s - 1$ for some integer s.*

Proof. Let $\overline{k}$ denote the field of fractions of k. We work at first in the Hopf algebra $H \otimes_k \overline{k}$ in which H is embedded. Since $\overline{k}[d, d^{-1}]$ is a finite dimensional Hopf algebra, there is a unique minimal polynomial of d, given by $\overline{p}(x) = x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_0 1$. Since d is invertible, $\lambda_0 \neq 0$ and $\overline{k}[d, d^{-1}] = k[d]$.

$\overline{k}[d]$ is a Hopf-Frobenius algebra with Hopf-Frobenius homomorphism $f : \overline{k}[d] \to \overline{k}$. Then $f(d^k)d^k = f(d^k)1$ for every integer k, since each d^k is group-like. If $f(d^k) \neq 0$, then $k \geq s$, since otherwise d is root of $x^k - 1$, a polynomial of degree less than s.

Thus, $f(d) = \cdots = f(d^{s-1}) = 0$, but $f(1) \neq 0$ since $f \neq 0$ on $\overline{k}[d]$. Then $f(p(d)) = f(d^s) + \lambda_0 f(1) = 0$, so that $f(d^s) = -\lambda_0 f(1) \neq 0$. Since $f(d^s)d^s = f(d^s)1$, it follows that $r(d^s - 1) = 0$ for some nonzero $r \in k$. Since H is finite projective over an integral domain, it follows that $d^s - 1 = 0$. $\Box$

It follows easily from the proof that if $f(x) \in k[x]$ such that $f(d) = 0$, then $d^s = 1$ for some integer $s \leq \deg f$.

Theorem 5.1 *Let H be a finite projective Hopf algebra over a commutative ring k. If $d \in H$ is a group-like element, then $d^N = 1$ for some integer N.*

Proof. Let $a_1, \ldots, a_n$ be generators of H as a k-module. Then there are $\lambda_{ij} \in k$ such that $da_i = \sum_{j=1}^n \lambda_{ij} a_j$. Let $p(x) = \det(\delta_{ij}x - a_{ij})$, where δ_{ij} is the Kronecker delta. It follows that $p(x)$ is a monic polynomial of degree n such that $p(d) = 0$.

Let $\mathcal{P}$ be a prime ideal in k. Note that $H/\mathcal{P}H \cong H \otimes_k (k/\mathcal{P})$ is a finite projective Hopf algebra over the domain $k/\mathcal{P}$. By Lemma 5.1, there is an integer $s_{\mathcal{P}} \leq n$ such that $d^{s_{\mathcal{P}}} - 1 \in \mathcal{P}H$. Since $x^t - 1$ divides $x^{n!} - 1$ for each integer $t \leq n$, it follows that

$$d^{n!} - 1 \in \mathcal{P}H$$

for each prime ideal $\mathcal{P}$ of k. Since H is a finite projective over k, a standard argument gives $\mathrm{Nil}(k)H = \cap(\mathcal{P}H)$ over all prime ideals, where the nilradical $\mathrm{Nil}(k) = \cap\mathcal{P}$ is equal to the intersection of all prime ideals in k. Thus, $d^{n!} - 1 = \sum r_i a_i$ where $r_i \in \mathrm{Nil}(k)$. Let k_i be integers such that $r_i^{k_i} = 0$. Then

$$(d^{n!} - 1)^{(\sum_{i=1}^{n} k_i)+1} = 0. \tag{30}$$

It is clear that $P(x) := (x^{n!} - 1)^{(\sum_{i=1}^{n} k_i)+1}$ is a monic polynomial with integer coefficients.

Suppose there is a least positive $m \in \mathcal{Z}$ such that $m1 = 0$. Then $\mathcal{Z}_m := \mathcal{Z}/m\mathcal{Z} \subseteq k$. Consider $\mathcal{Z}_m[d, d^{-1}]$ in H. From Equation 30 we see that $\mathcal{Z}_m[d, d^{-1}] = \mathcal{Z}_m[d]$. But $\mathcal{Z}_m[d]$ and so $\mathcal{Z}_m[d]^\circ$ are finite, whence $d^N = 1$ for some N.

Suppose that $m1 \neq 0$ for any integer m. Then $\mathcal{Z} \subseteq k$. Again by Equation 30, $\mathcal{Z}[d, d^{-1}] = \mathcal{Z}[d]$ is a Hopf algebra over $\mathcal{Z}$. It suffices by Lemma 5.1 to prove that $\mathcal{Z}[d]$ is a free module.

Because of Equation 30, there is a polynomial $q(x) = \lambda_s x^s + \cdots + \lambda_0 \in \mathcal{Z}[x]$ of least degree such that $q(d) = 0$. Since $1, d, \ldots, d^{s-1}$ freely generate a free submodule of $\mathcal{Z}[d]$, it follows that $q(x)$ may chosen such that the content $c(q(x)) = 1$.

It suffices to prove $q(x)$ monic, for then $\mathcal{Z}[d]$ is freely generated by $1, d, \ldots, d^{s-1}$. Now the minimal polynomial $q(x)$ divides $P(x)$ in $\mathcal{Q}[x]$, so that $rP(x) = h(x)q(x)$ for some $r \in \mathcal{Z}$ and $h(x) \in \mathcal{Z}[x]$. By the Gauss Lemma, $c(h) = r$ since $c(P) = c(q) = 1$. Then $P(x) = q(x)h_1(x)$ where $h_1(x) \in \mathcal{Z}[x]$. If γ is the leading coefficient of $h_1(x)$, then $\gamma\lambda_s = 1$, so that $\lambda_s = \pm 1$. Hence, $q(x)$ is monic. □

As a consequence of Theorem 3.1, Theorem 5.1 and Equation 14, we obtain the following corollaries.

Corollary 5.1 *Let H be a Hopf-Frobenius algebra. Then $S^{4M} = \eta^{2M} = \mathrm{Id_H}$ for some integer M.*

Proof. Let M be an integer such that $b^M = 1_H$ and $m^M = 1_{H^*}$. □

Corollary 5.2 *Let H be a finite projective Hopf algebra over a commutative ring k. Then $S^{4M} = \mathrm{Id_H}$ for some integer M.*

Proof. Localizing with respect to any maximal ideal of k, we reduce the statement to the Hopf-Frobenius case. □

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