Mathematical properties of conically self-similar free-vortex solutions to the Navier–Stokes equations. Part 1. Existence and non-existence

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In this paper a mathematical proof of existence and non-existence of conically self-similar free-vortex solutions to the Navier–Stokes equations is presented. This proof clearly establishes that these solutions do not have any kind of singularity at the symmetry axis. This analysis gives considerably improved existence and non-existence bounds and it is shown that these bounds are close to optimal in the low-swirling limit. This approach links the questions of existence/non-existence for the swirling case and for the non-swirling case. The proof is based on Schauder's Fixed Point Theorem and is therefore non-constructive. Therefore the paper ends with a brief discussion of the question of how to compute the conically self-similar free-vortex solutions to the Navier–Stokes equations.

1. Introduction

The swirling jet is a flow phenomenon of great importance both in nature and in technology. Indeed, tornados, air motion above whirlpools and the flow inside a combustion chamber are all examples of flows characterised by the strong presence of a swirling jet. The study of the swirling jet has also led to the emanation of numerous fundamental flow problems, such as vortex breakdown. Another fundamental flow problem which has emerged from the study of swirling jets is concerned with the properties of swirling conically self-similar solutions to the Navier–Stokes equations. It is still a matter of dispute, which, if any, flow problems can be reasonably modelled by such solutions, nevertheless swirling conically self-similar solutions form one of the most intriguing classes of exact (in the far-field sense) solutions to the Navier–Stokes equations, and hence they pose a fundamental flow problem.

Conically self-similar solutions to the Navier–Stokes equations can be defined as such solutions for which the velocity components decay are $O\left(R^{-1}\right)$ as $R \to \infty$ and for which the quotient of any two velocity components depends only on the azimuthal angle of a spherical coordinate system.The first conically self-similar solution was the Schlichting (1933) profile for a non-swirling round jet, using the boundary layer equation. (Slezkin 1934) derived an equation for the streamfunction of a general non-swirling conically self-similar solution to the Navier–Stokes equation. The first solution to the Slezkin equation was given by Landau (1944), who provided an exact far-field solution to the Navier–Stokes equations in the whole space. His profile, which generalised that of Schichtling, was independently rederived by Squire (1951). A different solution was found in (Squire 1952), which considered a jet emerging over a plane streamsurface, and this was the first time when loss of existence for some regions in parameter space was encountered.

A new interpretation of this solution was given by Wang (1971), who also generalised it somewhat. Finally, the general solution to the Slezkin equation with the fluid bounded by a conical streamsurface was given by Potsch (1981).

Early attempts to construct a solution for swirling jets, using the conservation of angular momentum, was made by Loitsyanskii (1953) for the boundary-layer equations, and independently by Görtler (1954), who obtained an equivalent solution by a quite different procedure. For the full Navier–Stokes equations Tsukker (1955) was the first to construct a solution for a swirling jet, which conserves angular and axial momentum. All of these solutions have the property that the swirling (azimuthal) velocity decays faster than the axial and radial velocities, and none of them allow a central recirculation zone, and hence their value as models for swirling flow is rather limited.

The study of swirling conically self-similar solution trace back to Long (1958) and Long (1961) for the boundary-layer equations and independently to Goldshtik (1960) for the Navier–Stokes equations. Already in this articles the intriguing existence/non-existence properties of these solutions were highlighted. Long's vortex has since been considered as a fundamental flow problem, and as such it has attracted much attention, see for example (Burggraf & Foster 1977; Foster & Smith 1989; Drazin, Banks & Zaturska 1995). The problem considered by Goldshtik was subsequently generalised and given a full mathematical treatment by Serrin (1972). In the final form it describes the flow in a domain bounded by a plane and driven by a vortex half-line. It should be stressed that this solution satisfies the no-slip conditions on the walls, but has a logarithmic singularity on the symmetry axis. In (Goldshtik 1979) swirling conically self-similar solutions driven by a half-line vortex was conisidered in the entire space. In principle, this solution is an extension of Serrin's solution to the entire space, but where some boundary conditions, such as the no-slip condition, have to be relaxed in order to ensure that the solution is regular on the free half-axis. The next development was due to Yih *et al.* (1982), who replaced the half-line vortex in (Goldshtik 1979) by a cone, and hence they obtained a fully regular free-vortex flow inside the domain. What Yih *et al.* (1982) thus achieved was a direct swirling extension of the non-swirling solutions found in (Squire 1952) and (Potsch 1981). Consequently, the solution found in (Yih *et al.* 1982) exhibits the same inability to satisfy the no-slip condition on the conical walls as the non-swirling counterparts do. In fact, what drives these free-vortex solutions is the radial velocity distribution on the bounding conical streamsurface.

Yih *et al.* (1982) found that the their swirling conically self-similar solution allowed both one-cell and two-cell flows, and for this and other reasons this solution has subsequently been studied on several occasions. In the context of conservation laws it was analysed by Paull & Pillow (1985) some asymptotic properties were derived in (Goldshtik & Shtern 1989) and recently extensive asymptotic and numerical studies have been conducted in a half-space (Shtern & Hussain 1993) as well as in cones bounded by streamsurfaces (Shtern & Hussain 1996). In (Shtern & Hussain 1996) it was also suggested that the solution found in (Goldshtik 1979) is a very suitable generalisation of Long's vortex to the full Navier–Stokes equations and the entire space. In short, the solution found in (Yih *et al.* 1982) is perhaps the swirling conically self-similar solution, which has attracted the most recent research effort, yet the state of the mathematical theory for it is much worse than say for Serrin's vortex.

Yih *et al.* (1982) claim to have found their solution is not supported by a formal existence proof. Most of the article is concerned with the important question of properties of solution, if any exists, but the very question of existence is not treated. Existence of their solution is a consequence of the convergence of the numerical method they propose, but although this method indeed seems to converge in the proper region of parameter space, their proof for this convergence is not convincing as we will argue in § 5. Fortunately, their conclusion about the existence of a solution to their problem is correct as we shall demonstrate in the present paper. In order to do this, we will use a technique from (Serrin 1972) as well as the properties of the general non-swirling Squire–Potsch solutions. In doing so we reveal the underlying relationship between existence in the swirling and in the non-swirling case. Furthermore, this procedure will yield substantially improved bounds for existence and non-existence of solutions to the problem.

Another characteristic property of the free-vortex solution found by Yih *et al.* (1982) is the regularity of the solution on the symmetry axis, as opposed to the logarithmic singularity found for the solution to Serrin's problem. However, the absence of a logarithmic singularity at the symmetry axis, would not have followed from the convergence of the numerical method proposed by Yih *et al.* (1982). On the other hand, in this paper, we will give a formal proof of the regularity of the solution at the symmetry axis. In the sequel to this article we will improve this regularity result further, and in addition prove that the solution is uniquely determined by its behaviour on the symmetry axis.

1.1. *Possible applications and interpretations*

The solution found in (Squire 1952) proved difficult to interpret, since it is a solution to the Navier–Stokes equations, which does not satisfy the no-slip condition at the wall. This difficulty has been inherited by the Potsch solutions and the solution found by Yih *et al.* (1982). Squire interpreted his solution as the solution for a jet above a plane wall. An experimental study (Schneider, Zauder & Bohm 1987) shows that a jet emerging from the apex of a cone, and bounded by conical walls is not described by a similarity solution. As pointed out in (Shtern & Hussain 1996) this is easily seen, since the total axial flow force decays at the walls, but is required to be constant by Squire's solution. A further difficulty is that the cone apex need not *a priori* coincide with the mathematical cone apex of the theoretical profile, and if it does not then the experimental geometry is not compatible with that of the theoretical profile. However, for the study of phenomena in the vicinity of the symmetry axis, such as vortex breakdown, the boundary conditions at the walls are commonly believed to be of little importance. Indeed, some numerical simulations of the Navier–Stokes equations aimed at investigating vortex breakdown, such as (Beran & Culick 1992), has neglected the no-slip condition at the walls.

A different approach has recently been presented in (Shtern, Borissov & Hussain 1997), where the solution found by Yih *et al.* (1982) was used as the inner solution, which was then forced to match a generalised vortex sink solution, which satisfies all boundary conditions.

An entirely different interpretation of the Squire solution was presented by Wang (1971), who used this solution to model the spreading of oil from a sinking ship on the ocean. This interpretation inspired Goldshtik & Shtern (1989) to reinterpret the solution found in Yih *et al.* (1982) as a model of the air flow above a whirlpool.

2. Formulation of the problem

In basic textbooks in fluid mechanics, see for example (Panton 1984), axisymmetric non-swirling solutions to the Navier–Stokes equations are normally found by using a Stokes' streamfunction and conical self-similarity. This approach was extended to the swirling case by Long (1958). The idea is then to seek conically self-similar solutions to the Navier–Stokes equations which are characterised by a streamfunction as well as by a circulation function, *i.e.* to seek solutions on the form:

$$\left.\begin{aligned} u_R = -\frac{\nu\psi'(x)}{R}, \quad u_\theta = -\frac{\nu\psi(x)}{R\sin\theta}, \quad u_\phi = \frac{\nu\Gamma(x)}{R\sin\theta}, \\ p - p_\infty = \frac{\rho\nu^2 q(x)}{R^2}, \quad \Psi = \nu R\psi(x), \quad x = \cos\theta, \end{aligned}\right\} \tag{2.1}$$

where (R, θ, ϕ) are spherical co-ordinates, (u_R, u_θ, u_ϕ) the corresponding velocity components, p the pressure, p_∞ the atmospheric pressure and Ψ a streamfunction. We have also let a prime denote differentiation with respect to x. It may appear that (2.1) is not the most general form of a conically self-similar solution, defined as a solution which is $O\left(R^{-1}\right)$ and where any quotient of velocity components depend only on the azimuthal coordinate. However, if we consider only such forms of conically self-similar solutions which satisfy the continuity equation we find that (2.1) indeed is the most general form of such solutions.

When (2.1) is substituted into the Navier–Stokes equations and terms if lowest order are identified we obtain after some manipulations the following system of ODEs (Serrin 1972) and (Shtern & Hussain 1996):

$$\left(1 - x^2\right)\psi' + 2x\psi - \frac{1}{2}\psi^2 = F, \tag{2.2a}$$

$$\left(1 - x^2\right)F'' + 2xF' - 2F = \Gamma^2, \tag{2.2b}$$

$$\left(1 - x^2\right)\Gamma'' = \psi\Gamma'. \tag{2.2c}$$

2.1. *Boundary conditions*

The boundary conditions we use in this paper are obtained if we assume that we do not have any flow sources on the axis, except at the origin, which implies that

$$\Gamma(1) = \psi(1) = 0. \tag{2.3}$$

For the radial velocity to be bounded outside a neighbourhood of the origin we require that

$$\left|\psi'(1)\right| < \infty. \tag{2.4}$$

If we substitute these conditions into (2.2a) and its derivative, we obtain boundary conditions for F,

$$F(1) = F'(1) = 0. \tag{2.5}$$

We furthermore assume that the swirling flow is driven by a constant circulation along some fixed conical streamsurface $x = x_c$ which implies that

$$\psi(x_c) = 0, \quad \Gamma(x_c) = \Gamma_c. \tag{2.6a, b}$$

Since the system of equations (2.2) is symmetric with respect to the sign of Γ, the sign is immaterial, and we will henceforth assume that $\Gamma_c \geq 0$. The last condition to be specified is the total axial flow force J_z.

Throughout the paper we will make frequent reference to Serrin's problem and therefore we will briefly state the boundary conditions used in this problem. Here we reinforce the no-slip condition at the conical walls

$$\psi(x_c) = \psi'(x_c) = \Gamma(x_c) = 0. \tag{2.7}$$

At the symmetry axis we assume that the circulation is constant, and that we have at most a logarithmic singularity for streamfunction at the symmetry axis, *i.e.* that

$$\lim_{x\to 1^-} \mathit{\Gamma}(x) = C, \qquad \lim_{x\to 1^-} \psi(x) = 0 \ . \tag{2.8}$$

With these conditions equation (2.2*b*) is no longer correct, and instead one must use

$$\left(1-x^2\right) F''' = 2\mathit{\Gamma}\mathit{\Gamma}' \ . \tag{2.9}$$

Indeed this was the definition of the auxiliary function originally introduced by Goldshtik (1960). Under the conditions (2.3)-(2.4) it was integrated by Sozou (1992) to yield (2.2*b*).

It is easily seen that in Serrin's case the boundary conditions for F are

$$F(x_c) = F(1) = 0 \ . \tag{2.10}$$

3. The formulation of the theorem of existence and non-existence

On physical grounds the total axial flow force is perhaps the most suitable parameter to use to characterise the strength of the axial flow. However, it is very difficult to compute and therefore to prove a existence/non-existence results for the system (2.2) we must replace it with a different parameter of lesser physical, but greater mathematical importance. To this end, notice that equation (2.2*b*) is linear and hence we can follow Serrin (1972) and Yih *et al.* (1982) and solve it to obtain

$$F(x) = -(1-x)^2 \int_{x_c}^{x} \frac{t\mathit{\Gamma}^2 \mathrm{d}t}{\left(1-t^2\right)^2} - x\int_x^1 \frac{\mathit{\Gamma}^2 \mathrm{d}t}{(1+t)^2} - \frac{T\mathit{\Gamma}_c^2}{2}(1-x)^2 \ , \tag{3.1}$$

where we have used the boundary conditions (2.5). T is an arbitrary parameter, which will replace J_z in the subsequent analysis. The physical meaning of T is most easily seen for the case $x_c = 0$, where

$$T = -\frac{2F(0)}{\mathit{\Gamma}_c^2} = -\frac{2\psi'(0)}{\mathit{\Gamma}_c^2} \tag{3.2}$$

To see the mathematical importance of T set $\mathit{\Gamma} \equiv 0$, which reduces the system (2.2) to the Slezkin equation:

$$\left(1-x^2\right)\psi' + 2x\psi - \frac{1}{2}\psi^2 = -\frac{T\mathit{\Gamma}_c^2}{2}(1-x)^2 \ , \tag{3.3}$$

and for this equation with $x_c = 0$ Squire (1952) found that the solution ceases to exist if T is sufficiently small. This fact is one of the key elements in our analysis, and therefore we will make a short digression to study the properties of the solutions of (3.3).

3.1. *Analytical solutions in the non-swirling case*

Already in 1934 Slezkin discovered that any non-swirling conically self-similar solution to the Navier–Stokes equations, which satisfies the boundary conditions we have described, must be a solution to

$$\left(1-x^2\right)\psi' + 2x\psi - \frac{1}{2}\psi^2 = C(1-x)^2 \ , \tag{3.4}$$

for some constant $C = \psi'(x_c)(1+x_c)/(1-x_c)$. For the case $x_c = -1$ this was solved by Landau (1944) and independently by Squire (1951), but it turns out that this is a rather special case. In (Squire 1952) solved the (3.4) for $x_c = 0$, and this solution was subsequently generalised to all $x_c \in (-1, 1)$ by Potsch (1981). To calculate the general

Potsch solution one makes the substitution $\psi = -2\left(1-x^2\right) U^{'}/U$ to obtain

$$\frac{\mathrm{d}^2 U}{\mathrm{d}x^2} + \frac{C}{2\left(1+x\right)^2} U = 0 \ . \tag{3.5}$$

Here one can either follow Squire and Potsch and use the general solutions for a hypergeometric equation of this type, or we may change the independent variable to $y = \ln\left(1+x\right)$, which will transform (3.5) to a second order differential equation with constant coefficients.

When $x_c > -1$ the Potsch solutions are given by:

- For the case $C < \frac{1}{2}$

$$\psi_C^P = -C\left(1-x\right) \frac{A_{x_c}\left(x\right)^{\alpha} - 1}{r_2 A_{x_c}\left(x\right)^{\alpha} - r_1}, \tag{3.6}$$

where $A_{x_c}\left(x\right) = \left(1+x\right)/\left(1+x_c\right)$, $\alpha = \left(1-2C\right)^{\frac{1}{2}}$, $r_1 = \frac{1}{2}\left(1+\alpha\right)$ and $r_2 = \frac{1}{2}\left(1-\alpha\right)$. Note that if $C = 0$ we have the trivial solution, which shows that the Landau solution is the only interesting solution in this case.

- For the case $C = \frac{1}{2}$

$$\psi_{\frac{1}{2}}^P = -\left(1-x\right) \frac{\ln A_{x_c}\left(x\right)}{\ln A_{x_c}\left(x\right) - 2} \ . \tag{3.7}$$

- For the case $C > \frac{1}{2}$

$$\psi_C^P = -\left(2C\right)^{\frac{1}{2}}\left(1-x\right) \frac{\sin\left(\beta \ln A_{x_c}\left(x\right)\right)}{\sin\left(\beta \ln A_{x_c}\left(x\right) - \gamma\right)} \ , \tag{3.8}$$

where $\gamma = \arctan 2\beta$ and $\beta = \frac{1}{2}\left(2C-1\right)^{\frac{1}{2}}$.

Of even greater importance to us in this paper are the modified Potsch solutions defined by

$$f_C^P = -\frac{\psi_C^P}{2\left(1-x^2\right)} \ . \tag{3.9}$$

We notice that f_C^P is positive if C is negative and vice versa. It is easily seen that f_C^P is monotone: increasing if $C \leq 0$ and decreasing if $C \geq 0$. We also mote that f_C^P satisfies the equation

$$f^{'} + f^2 = -\frac{C}{2\left(1+x\right)^2} \tag{3.10}$$

as well as the condition $f_C^P\left(x_c\right) = 0$.

The most interesting property of the Potsch solutions is the fact that they no longer remain bounded in $[x_c, 1]$ if C is decreased past a certain negative value, which depends only on x_c. This was discovered by Squire (1952) but was not mentioned in (Potsch 1981). This property is a key element in our analysis and therefore we will explicitly state it here. In fact we need a more refined concept:

Definition 1. *For any $x_1 \in (x_c, 1]$, let $C_{x_1}^*$ be the unique value of C such that f_C^P is bounded in $[x_c, x_1]$ if and only if $C < C_{x_1}^*$. The case $x_1 = 1$ is of special importance and we will therefore use the notation $C^* \equiv C_1^*$.*

Remark. This definition is stronger than if we had used ψ_C^P instead of f_C^P. The two definitions would coincide for $x_1 \in (x_c, 1)$, but for $x_1 = 1$ it requires also that $\psi^{'}(1)$ is bounded.

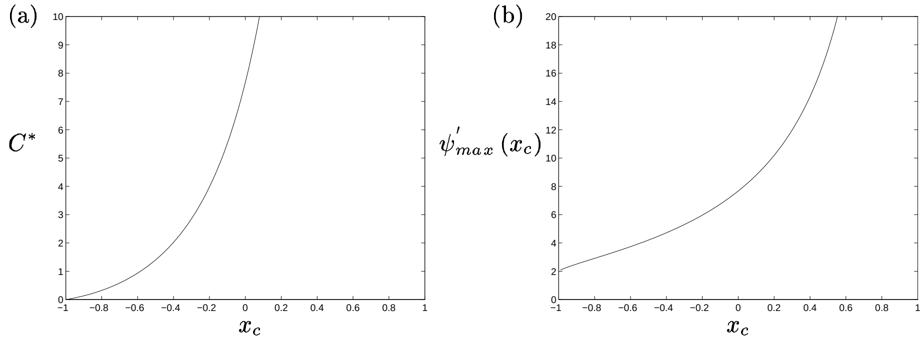

FIGURE 1. Plots of (a) C^* and (b) the maximum obtainable value of $\psi'(x_c)$ as functions of x_c.

To see that there exists a unique $C^*_{x_1}$ for any x_c and x_1, consider the denominators in (3.6)-(3.8). In the vicinity of x_c they are all negative, but if we set $x = x_1$ and increase C there is exactly one value of C where they will vanish. This value is precisely $C^*_{x_1}$. To calculate $C^*_{x_1}$ we thus simply look for the zeros of the denominators in (3.6)-(3.8) and finally we have to see which of the conditions is the critical one. If this is done we obtain the following:

- When $x_c < \left(1 + x_1 - e^2\right)/e^2$ we get that $C^*_{x_1}$ is the positive solution to

$$\frac{1}{\alpha}\ln\left(\frac{1+\alpha}{1-\alpha}\right) = \ln\left(\frac{1+x_1}{1+x_c}\right). \tag{3.11}$$

- When $x_c = \left(1 + x_1 - e^2\right)/e^2$ we have that

$$C^*_{x_1} = \frac{1}{2} \tag{3.12}$$

- When $x_c > \left(1 + x_1 - e^2\right)/e^2$ we find that

$$C^*_{x_1} = \frac{1}{2}\left(1 + 4\left(\beta^*\right)^2\right) \tag{3.13}$$

where in turn β^* is the positive solution to

$$\beta\ln\left(\frac{1+x_1}{1+x_c}\right) = \arctan 2\beta\ . \tag{3.14}$$

In figure 1(a) we have plotted C^* versus x_c. In figure 1(b) we have shown the maximum value of $\psi'(x_c) = C\left(1 - x_c\right)/\left(1 + x_c\right)$ versus x_c, and for this graph we see that all solutions with $\psi'(x_c) < 2$ are obtainable. We also see that as $x_c \to 1^-$ we have that the maximum possible value of $\psi'(x_c) \to 2^+$, and this is consistent with the Landau solution for $x_c = -1$, for which it is easy to verify that $\psi'(-1) < 2$.

3.2. *The main result*

Unlike the argument produced by Yih *et al.* (1982) our argument will emphasize that the question of existence/non-existence of a solution to the swirling problem is almost the same as that for the non-swirling one, which has been completely resolved. From this it is evident that T defined in (3.1) will be a parameter highly relevant for the question of existence and non-existence. Indeed, we will spend most of this paper proving the following theorem:

THEOREM 1. *(Existence/Non-existence) For any $x_c \in (-1, 1)$ and any Γ_c there exist numbers $T_* < T^*$ such that*

(a) If $T > T^*$ then there exists a solution to (2.2) which satisfies the boundary conditions (2.3)-(2.6). Furthermore $\psi \in C^1([x_c, 1])$, $F \in C^2([x_c, 1])$ and $\Gamma \in C^2([x_c, 1])$.
(b) If $T \leq T_*$ there exists no solution to (2.2) which satisfies the boundary conditions (2.3)-(2.6).

Remark. 1. The theorem does not say anything about uniqueness. However, in the sequel to this article we will prove a much weaker uniqueness result, namely that the solution is uniquely determined by its behaviour on the symmetry axis.

Remark. 2. The bounds T_* and T^* depend only on x_c and Γ_c and they are given by the following expressions:

$$T^* = -\frac{2C^*}{\Gamma_c^2} - \frac{x_c H^\dagger(-x_c)}{1-x_c} + \frac{x_c^2 H^\dagger(-x_c)}{1-x_c^2}, \tag{3.15}$$

$$T_* = -\min_{x_1}\left\{\frac{2C^*_{x_1}}{\Gamma_c^2} + \frac{x_1 H^\dagger(x_1)}{1-x_1}\right\} + \frac{x_c^2 H^\dagger(x_c)}{1-x_c^2}, \tag{3.16}$$

where $H^\dagger$ is a Heaviside function taking the value zero at zero.

In figure 2 our bounds are compared with those obtained in (Yih *et al.* 1982). It is seen that we have improved the bounds substantially, and the improvement is greater the smaller the value of x_c and the smaller the value of Γ_c. When $x_c = 0$ we can use (3.2) to illustrate these bounds in terms of the renormalised tangential velocities in the plane $(-\psi'(0))$ and Γ_c. This has been done in figure 3 and here we see that as $\Gamma_c \to 0$ the critical velocity approaches the value 7.6727, which was found by Squire (1952) for the non-swirling case. It is also evident that our estimates are very good in the low-Γ_c limit, but worse in the high-Γ_c limit. Using an asymptotic analysis technique Goldshtik & Shtern (1989) has been able to obtain better limits in the high-Γ_c limit, however, they have not formally established that the asymptotic behaviour accurately describes the situation for finite values of Γ_c. Our analysis will in no way depend on any asymptotic approximation of (2.2) and is therefore directly seen to be valid for all values of Γ_c.

4. Proof of Theorem 1

4.1. *An outline of the proof*

The ideas we use to prove Theorem 1 were originally developed by Serrin (1972), who applied them to a similar, yet different, problem. Yih *et al.* (1982) applied some of Serrin's ideas to the present problem and their effort has strongly inspired the present effort. However, (Yih *et al.* 1982) is primarily concerned with the properties of solutions and they do not present a complete mathematical proof of a statement similar to Theorem 1. In this paper we will use different ideas of Serrin's to make a rigorous treatment of the question of existence of conically self-similar free-vortex solutions. Furthermore, as already pointed out this will yield sharper bounds in parameter space for the existence and non-existence of conically self-similar solutions.

The pivot of our proof is the Schauder Fixed Point Theorem, and most of the proof is concerned with finding a suitable set to which it can be applied. We will present the basic structure of the proof in more detail below, but first of all, let us transform our problem into the form used by Serrin (1972) and Yih *et al.* (1982). To this end let us make the substitutions

$$f = -\frac{\psi}{2(1-x^2)}, \quad \Omega = \frac{\Gamma}{\Gamma_c}, \quad G = -\frac{2}{\Gamma_c^2}F, \tag{4.1$a-c$}$$

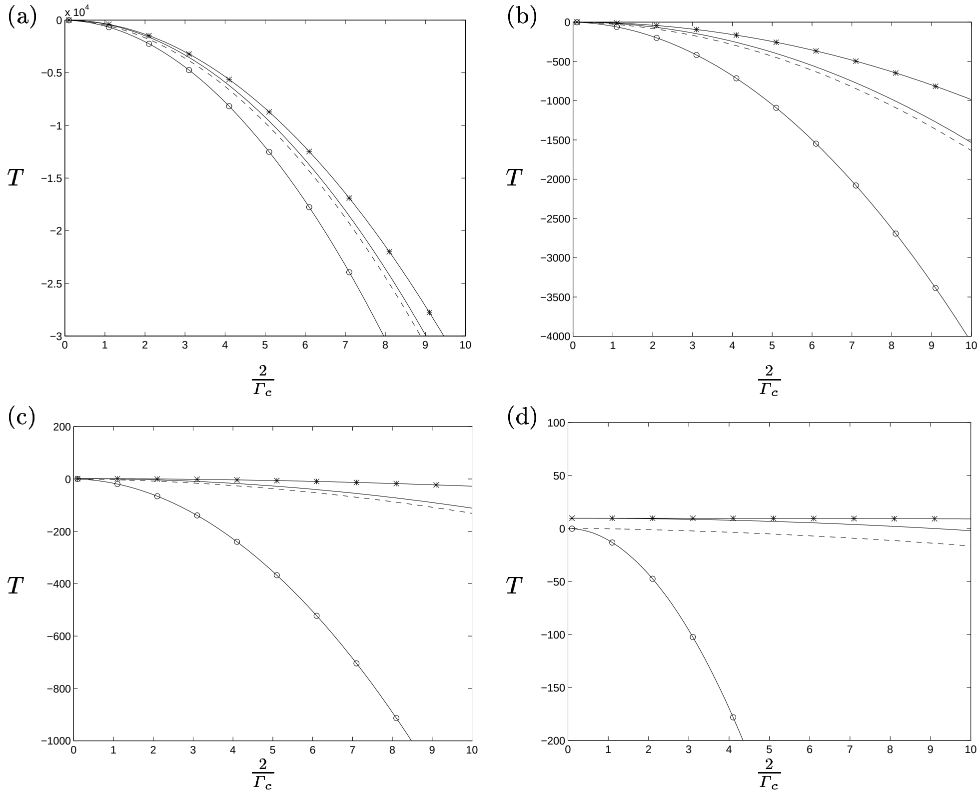

FIGURE 2. Plots of the regions of existence and non-existence for (a) $x_c = 2^{-\frac{1}{2}}$, (b) $x_c = 0$, (c) $x_c = -2^{-\frac{1}{2}}$ and (d) $x_c = -0.95$. The solid line shows T^* and solutions exist above it. The dashed line illustrates T_* so on and below this line no solution exists. The lines marked with stars and circles show the existence and non-existence bounds from (Yih *et al.* 1982).

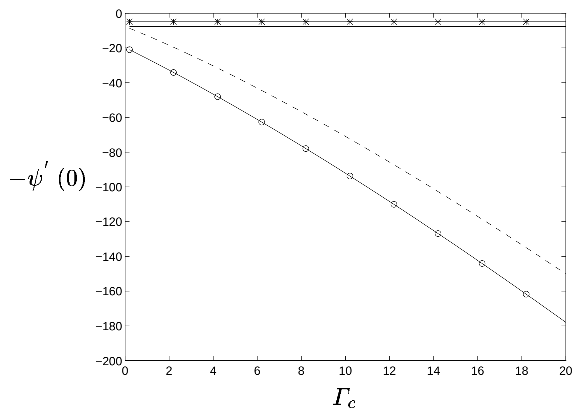

FIGURE 3. Plots of the regions of existence and non-existence for $x_c = 0$ for various tangential velocities in the plane and various Γ_c. Solutions exist above the solid line, and do not exist below the dashed line. The lines marked with stars and circles show the corresponding existence and non-existence bounds from (Yih *et al.* 1982).

and then we obtain the system

$$f' + f^2 = \left(\frac{\Gamma_c}{2}\right)^2 \frac{G(x)}{(1-x^2)^2} \tag{4.2a}$$

$$\Omega'' + 2f\Omega' = 0 \ , \tag{4.2b}$$

where

$$G(x) = 2(1-x)^2 \int_{x_c}^{x} \frac{t\Omega^2 \mathrm{d}t}{(1-t^2)^2} + 2x \int_x^1 \frac{\Omega^2 \mathrm{d}t}{(1+t)^2} + T(1-x)^2 \ . \tag{4.2c}$$

To satisfy the boundary conditions we must have that

$$f(x_c) = 0 \ , \quad |f(1)| < \infty \ , \tag{4.3a,b}$$

$$\Omega(x_c) = 1 \ , \quad \Omega(1) = 0 \ . \tag{4.3c,d}$$

To begin our analysis, let us fix the values of T, Γ_c and x_c and assume that we have an arbitrary $f \in C([x_c, 1])$. Let $\mathcal{S}_\Omega$ be the mapping which takes this function f to the solution Ω which is the solution to (4.2*b*) with the boundary conditions (4.3*c,d*). Now $\Omega = \mathcal{S}_\Omega f$ enables us to calculate G using (4.1), and we denote this mapping $\mathcal{S}_{G,\Omega}$. Hence

$$G = \mathcal{S}_{G,\Omega}\Omega = \mathcal{S}_{G,\Omega} \circ \mathcal{S}_\Omega f \equiv \mathcal{S}_G f \tag{4.4}$$

and we let this serve as the definition of $\mathcal{S}_G$. Finally this value of G can be used to find a new value of f, $\widetilde{f}$ as the solution to (4.2*a*) which satisfies the boundary conditions (4.3*a,b*), and this mapping we denote $\mathcal{S}_{f,G}$. To summarise, we have that

$$\widetilde{f} = \mathcal{S}_{f,G}G = \mathcal{S}_{f,G} \circ \mathcal{S}_{G,\Omega}\Omega \equiv \mathcal{S}_{f,\Omega}\Omega \tag{4.5}$$

$$= \mathcal{S}_{f,G} \circ \mathcal{S}_{G,\Omega} \circ \mathcal{S}_\Omega f \equiv \mathcal{S}f \tag{4.6}$$

and this defines both $\mathcal{S}_{f,\Omega}$ and $\mathcal{S}f$. At this stage we do not know if these mappings are well defined in all of $C([x_c, 1])$, but we will henceforth adopt the convention that $\mathcal{S}C([x_c, 1])$, is the image of the largest subset of $C([x_c, 1])$ for which it is well-defined.

Note that if $\mathcal{S}f = f$, f is sufficiently regular and $f(1)$ is bounded then $(f, \mathcal{S}_\Omega f)$ is a solution to (4.2) satisfying all the boundary conditions (4.3), *i.e.* the existence (non-existence) of solutions to our problem is primarily reduced to the question of the presence (absence) of fixed points of $\mathcal{S}$. If we had followed Yih *et al.* (1982) and had not distinguished between f and $\mathcal{S}f$, we will only have been able to study properties of solutions to (4.2*b*)-(4.1), if any exists. However, the fundamental question of existence of a solution would have been impractical for us to treat in that framework.

Consider now the space $C([x_c, 1])$ equipped with the uniform (supremum) norm. This is a well-known example of a Banach space. Suppose now that for every $T > T^*$ we can find a non-empty, closed, bounded and convex subset $X \subset C([x_c, 1])$ such that

(1) $\mathcal{S}X \subset X$,
(2) $\mathcal{S}$ is continuous on X,
(3) $\mathcal{S}X$ is conditionally compact in X,
(4) $\mathcal{S}X \subset C^1([x_c, 1))$,
(5) $\mathcal{S}_\Omega X \subset C^2([x_c, 1])$,
(6) $\mathcal{S}_G X \subset C^2([x_c, 1])$.

From conditions (1)-(3) the existence of a solution to (4.2) satisfying all the boundary conditions (4.3) follows from the Schauder Fixed Point Theorem, here presented in the form which appears in (Deimling 1985).

THEOREM 2. *(Schauder) Let X be a real Banach space, $C \subset X$ nonempty, closed, bounded and convex, and $F : C \to C$ compact. Then F has a fixed point in C.*

Conditions (4)-(6) imply that our solution has sufficient regularity, and therefore it is clear that to prove Theorem 1 it is enough to find a non-empty, closed, bounded and convex set $X \subset C([x_c,1])$ such that (1)-(6) hold. In the next subsection we will find a candidate for X and show that it satisfies (1) and (4)-(6), and in the following subsection we will complete the proof.

To prove the non-existence statement, it suffices to prove that $\mathcal{S}C([x_c,1]) \cap C([x_c,1]) = \emptyset$ when $T \leq T_*$.

4.2. *Properties of the mapping $\mathcal{S}$*

In this subsection our primary aim is to find a non-empty, closed, bounded and convex set $X \subset C([x_c,1])$ such that $\mathcal{S}X \subset X$, and to this end we will state two comparison results which appear in (Serrin 1972) and in (Yih *et al.* 1982) respectively.

PROPOSITION 1. *(a) Let x_1 be an arbitrary point in $[x_c,1]$. Suppose that $G_{1,2} \in C([x_c,x_1])$ and that $G_1 \geq G_2$, then $\mathcal{S}_{f,G}G_1 \geq \mathcal{S}_{f,G}G_2$ in $[x_c,x_1]$, as long as both of the expressions are finite.*
(b) Suppose that $f_{1,2}$ are integrable in $[x_c,1]$ and that $f_1 \geq f_2$ then $\mathcal{S}_\Omega f_1 \leq \mathcal{S}_\Omega f_2$.

Remark. A direct consequence of (a) is that $\psi^P_{C_1} > \psi^P_{C_2}$ if $C_1 > C_2$, which is not obvious from the explicit formula for ψ^P_C.

The proof of (a) appears in (Serrin 1972) but by a Riccati transform $f = U'/U$ it can also be deduced directly from one of Sturm's comparison theorems, see for example (Ince 1956, p. 229). The proof of (b) can be found in (Yih *et al.* 1982), and it uses the following auxiliary result, which is of interest in its own right.

LEMMA 1. *If $f \in C([x_c,1)) \cap L^1([x_c,1])$ then $\mathcal{S}_\Omega f \in C^1([x_c,1])$, it is decreasing and it satisfies $0 \leq \mathcal{S}_\Omega f \leq 1$. Furthermore, let $Y \subset [x_c,1]$ be any interval and suppose that $f \in C(Y)$ then $\mathcal{S}_\Omega f \in C^2(Y) \cap C^1([x_c,1])$.*

Proof. Let us denote $\mathcal{S}_\Omega f$ by Ω. If f is integrable we can integrate (4.2*b*) to obtain

$$\Omega'(x) = \Omega'(x_c)\exp\left(-2\int_{x_c}^{x} f\,\mathrm{d}z\right). \tag{4.7}$$

From this it is clear that Ω is monotone, and if it is to satisfy the boundary conditions: $\Omega(x_c) = 1$ and $\Omega(1) = 0$, it must be decreasing. If we integrate (4.7) once more and enforce the boundary conditions we obtain

$$\Omega(x) = \frac{\int_x^1 \exp\left(-2\int_{x_c}^{y} f\,\mathrm{d}z\right)\mathrm{d}y}{\int_{x_c}^1 \exp\left(-2\int_{x_c}^{y} f\,\mathrm{d}z\right)\mathrm{d}y}. \tag{4.8}$$

From the integrability of f it thus follows that $\Omega \in C^1([x_c,1])$. Finally, if $f \in C(Y)$ then it follows from the last statement and (4.2*b*) that $\Omega \in C^2(Y)$. □

We will use this lemma to prove an almost universal estimate of $\mathcal{S}_G$:

PROPOSITION 2. *If f is integrable in $[x_c,1]$ then $\mathcal{S}_G f$ satisfies the estimate*

$$(1-x)^2\left\{T + \frac{xH^\dagger(-x)H^\dagger(-x_c)}{1-x} - \frac{x_c^2 H^\dagger(-x_c)}{1-x_c^2}\right\} \leq$$
$$\leq \mathcal{S}_G f \leq (1-x)^2\left\{T + \frac{xH^\dagger(x)}{1-x} - \frac{x_c^2 H^\dagger(x_c)}{1-x_c^2}\right\}, \tag{4.9}$$

where $H^{\dagger}$ is a modified Heaviside function, i.e.

$$H^{\dagger}(x) = \begin{cases} 1, & \text{for} \quad x > 0 \\ 0, & \text{for} \quad x \leq 0 \end{cases} . \tag{4.10}$$

Furthermore, $\mathcal{S}_G f \in C^2([x_c, 1])$.

Proof. Just like in (Yih *et al.* 1982) we first notice that Lemma 1 implies that $0 \leq \Omega \leq 1$, and then (4.9) follows directly from (4.1) by setting $\Omega = 1$ or $\Omega = 0$ depending on whether the contributions are positive or negative. The second statement follows from the second statement in Lemma 1. Some caution must be taken at $x = 1$ since the denominator of the first integral in (4.1) vanishes there. However, from Lemma 1 we have that $\Omega \in C^1([x_c, 1])$ thus we have that $\Omega = O(1 - x)$, and hence the first integral is finite as $x \to 1$. □

Thus far we have only restated some of the results in (Serrin 1972) and (Yih *et al.* 1982), but at this point when we will discuss the properties of $S_{f,G}$ our arguments will deviate. In fact, the results for Serrin's problem are no longer applicable to our problem and our use of the Potsch solutions as comparison functions is considerably sharper than the corresponding estimates in (Yih *et al.* 1982).

PROPOSITION 3. *Suppose that G satisfies the left inequality in (4.9) and that $T > T^*$, where T^* is defined in (3.15), then there exists a constant C depending only on Γ_c, T and x_c such that $\mathcal{S}_{f,G} G \geq f_C^P > -\infty$.*

Proof. We have that

$$G \geq (1 - x)^2 \left\{ T + \frac{x H^{\dagger}(-x) H^{\dagger}(-x_c)}{1 - x} - \frac{x_c^2 H^{\dagger}(-x_c)}{1 - x_c^2} \right\} \tag{4.11}$$

$$\geq (1 - x)^2 \left\{ T + \frac{x_c H^{\dagger}(-x_c)}{1 - x_c} - \frac{x_c^2 H^{\dagger}(-x_c)}{1 - x_c^2} \right\} \tag{4.12}$$

$$\equiv (1 - x)^2 G^* \tag{4.13}$$

By Proposition 1(a) we have that $\mathcal{S}_{f,G} G \geq \tilde{f}$ where $\tilde{f}$ satisfies

$$\tilde{f}' + \tilde{f}^2 = \left(\frac{\Gamma_c}{2} \right)^2 \frac{G^*}{(1 + x)^2} \tag{4.14}$$

as well as the condition $\tilde{f}(x_c) = 0$. However, this we recognise as (3.10) with $C = -\Gamma_c^2 G^*/2$. Consequently, the solution to (4.14) is given by f_C^P as defined in (3.9) with this value of C. However, from (3.11)-(3.13) we know that f_C^P is bounded in $[x_c, 1]$ if and only if $C < C^*$. Consequently, $\mathcal{S}_{f,G} G$ is bounded from below if $\Gamma_c^2 G^*/2 > -C^*$. This concludes the proof of the proposition. □

Next we would like to show that if $T < T_*$ then $\mathcal{S}_{f,G} G \to -\infty$ somewhere in $[x_c, 1]$, *i.e.* we would like to prove the following proposition

PROPOSITION 4. *Suppose that G satisfies the right inequality in (4.9) and that $T \leq T_*$, where T_* is defined in (3.16) then $\mathcal{S}_{f,G} G \to -\infty$ somewhere in $[x_c, 1]$.*

Proof. Pick any x_1 such that $x_c < x_1 < 1$, and consider the interval $[x_c, x_1]$. In this interval we have that

$$G \leq (1-x)^2 \left\{ T + \frac{xH^\dagger(x)}{1-x} - \frac{x_c^2 H^\dagger(x_c)}{1-x_c^2} \right\} \tag{4.15}$$

$$\leq (1-x)^2 \left\{ T + \frac{x_1 H^\dagger(x_1)}{1-x_1} - \frac{x_c^2 H^\dagger(x_c)}{1-x_c^2} \right\} \tag{4.16}$$

$$\equiv (1-x)^2 G_{*,x_1} \tag{4.17}$$

By Proposition 1(a) we have that $\mathcal{S}_{f,G}G \leq \widetilde{f}$ in $[x_c, x_1]$, where $\widetilde{f}$ satisfies

$$\widetilde{f}' + \widetilde{f}^2 = \left(\frac{\Gamma_c}{2}\right)^2 \frac{G_{*,x_1}}{(1+x)^2} \tag{4.18}$$

as well as the condition $\widetilde{f}(x_c) = 0$. Once again, we recognise this as (3.10) with $C = -\Gamma_c^2 G_{*,x_1}/2$. Consequently, the solution to (4.18) is given by f_C^P (as defined in (3.9) with this value of C. However, from the definition of $C_{x_1}^*$ we know that $f_C^P \to -\infty$ in $[x_c, x_1]$ if and only if $C \geq C_{x_1}^*$. Consequently, $\mathcal{S}_{f,G}G \to -\infty$ in $[x_c, x_1]$ if $\Gamma_c^2 G^*/2 > -C_{x_1}^*$. If $\mathcal{S}_{f,G}G \to -\infty$ in $[x_c, x_1]$ for any $x_1 \in (x_c, 1)$ then it is clear that $\mathcal{S}_{f,G}G \to -\infty$ in $[x_c, 1]$. Consequently, we have that $\mathcal{S}_{f,G}G \to -\infty$ in $[x_c, 1]$ if $T \leq T_*$ where T_* is given by (3.16). This concludes the proof of the proposition. □

Having now found a condition which ensures that $\mathcal{S}_{f,G}G$ is bounded from below, we must now obtain an upper bound. To begin with we will prove the following proposition, which is a modification of a result in Serrin (1972).

PROPOSITION 5. *Suppose that G satisfies the right inequality in (4.9) then we have that*
(a) if $T > 0$ and $x_c \leq T/(T+1)$ then

$$\mathcal{S}_{f,G}G \leq f_{UP} \equiv \begin{cases} f_{-T\Gamma_c^2}^P(x) & \text{for} \quad x \leq \dfrac{T}{T+1} \\ f_{-T\Gamma_c^2}^P\left(\dfrac{T}{T+1}\right) + \dfrac{\Gamma_c^2}{8} \ln \dfrac{1}{(T+1)(1-x)} & \text{for} \quad x \geq \dfrac{T}{T+1} \end{cases} \tag{4.19}$$

(b) if $T > 0$ and $x_c > T/(T+1)$ then

$$\mathcal{S}_{f,G}G \leq f_{UP} \equiv \frac{\Gamma_c^2}{8} \ln \frac{1-x_c}{1-x} \tag{4.20}$$

(c) if $T \leq 0$ and $x_c \leq 0$ then

$$\mathcal{S}_{f,G}G \leq f_{UP} \equiv \begin{cases} 0 & \text{for} \quad x \leq 0 \\ \dfrac{\Gamma_c^2}{16} \ln \dfrac{1}{1-x} & \text{for} \quad x \geq 0 \end{cases} \tag{4.21}$$

(d) if $T > 0$ and $x_c > 0$ then

$$\mathcal{S}_{f,G}G \leq f_{UP} \equiv \frac{\Gamma_c^2}{16} \ln \frac{1-x_c}{1-x} \tag{4.22}$$

Remark. 1. If this proposition holds then it is clear that $\mathcal{S}_{f,G}G$ is integrable.

Remark. 2. Note that in all cases we have that $f_{UP} \geq 0$.

Proof. Let us begin with the case (a). Since G satisfies the right inequality in (4.9) we have that

$$G(x) \le \begin{cases} 2T(1-x)^2 & \text{for} \quad x \le \dfrac{T}{T+1} \\ 2x(1-x) & \text{for} \quad x \ge \dfrac{T}{T+1} \end{cases} . \tag{4.23}$$

From Proposition 1(a) we find as in the last two propositions that

$$\mathcal{S}_{f,G}G \le f^P_{-T\Gamma_c^2}(x) \quad \text{if} \quad x \le \frac{T}{T+1} \tag{4.24}$$

For $x \ge T/(T+1)$ we have that $\mathcal{S}_{f,G}G$ is bounded from above by the function f_{UP}, which satisfies that $f_{UP}(T/(T+1)) = f^P_{-T\Gamma_c^2}(T/(T+1))$ and

$$f'_{UP} + f^2_{UP} = \frac{\Gamma_c^2}{2}\frac{x}{(1+x)^2}\frac{1}{1-x} \le \frac{\Gamma_c^2}{8}\frac{1}{1-x} , \tag{4.25}$$

where we have used the basic inequality $4x \le (1+x)^2$. Hence we have that

$$f'_{UP} \le \frac{\Gamma_c^2}{8}\frac{1}{1-x} , \tag{4.26}$$

which we can integrate to find that

$$\mathcal{S}_{f,G}G \le f_{UP} = f^P_{-T\Gamma_c^2}\left(\frac{T}{T+1}\right) + \frac{\Gamma_c^2}{8}\ln\frac{1}{(T+1)(1-x)} \quad \text{for} \quad x \ge \frac{T}{T+1} . \tag{4.27}$$

This concludes the proof of (a). (b) follows if one makes the obvious modifications. To prove (c) replace (4.23) by

$$G(x) \le \begin{cases} 0 & \text{for} \quad x \le 0 \\ x(1-x) & \text{for} \quad x \ge 0 \end{cases} . \tag{4.28}$$

and the rest of the proof is analogous to that used to prove (a). Finally (d) follows if the proof of (c) is modified slightly. □

For given $T > T^*$, *Re* and x_c we now define X_1 to be the closed convex (but unbounded) set

$$X_1 = \left\{ f \in C([x_c, 1)) : f_C^P \le f \le f_{UP} \right\} , \tag{4.29}$$

where $C = -\Gamma_c^2 G_*/2$, G_* is defined as in Proposition 3 and f_{UP} is defined as in Proposition 5. Propositions 3 and 5 imply that $\mathcal{S}X_1 \subset X_1$. One can proceed as in (Serrin 1972) and prove that X_1 contains a fixed point, but then one would obtain a solution which possibly has a logarithmic singularity at $x = 1$ and this we should not have to accept, and hence we will continue our search for a closed convex and bounded subset $X \subset C([x_c, 1])$ such that $\mathcal{S}X \subset X$. In fact, most of the remaining analysis is involved in proving the following proposition.

PROPOSITION 6. *Suppose that $f \in X_1$ then there exists a $\widetilde{C} > 0$ depending only on Γ_c, T and x_c such that $\mathcal{S}f \le f^P_{-\widetilde{C}\Gamma_c^2/2} < \infty$.*

Proof. Since f_{UP} is integrable there exists an $0 \le M < \infty$ such that for all $f \in X_1$ and all $y \in [x_c, 1]$

$$\int_{x_c}^{y} f \mathrm{d}x \le \int_{x_c}^{y} f_{UP}\mathrm{d}x \le \int_{x_c}^{1} f_{UP}\mathrm{d}x \equiv M . \tag{4.30}$$

Furthermore, let C be as in the definition of X_1 and define

$$m = \inf_{x\in[x_c,1]} f_C^P = \begin{cases} 0 & \text{if} \quad C \leq 0 \\ f_C^P(1) & \text{if} \quad C \geq 0 \end{cases} . \tag{4.31}$$

From these definitions and the definitions of C and f_{UP} above it is clear that m and M depend only on T, Γ_c and x_c. Consequently, for all $f \in X_1$ we have the estimate

$$\exp(-2M) \leq \exp\left(-2\int_{x_c}^{x} f\mathrm{d}z\right) \leq \exp(-2m[x - x_c]) . \tag{4.32}$$

If these relations are used to estimate $\mathrm{d}/\mathrm{d}x\,(\mathcal{S}_\Omega f)$ uniformly from (4.7) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mathcal{S}_\Omega f)(x) &\geq -\frac{\exp(2M - 2m[x - x_c])}{1 - x_c} \\ &\geq -\frac{\exp(2M - 2m[1 - x_c])}{1 - x_c} \equiv -N \end{aligned} \tag{4.33}$$

which is a uniform bound for each $f \in X_1$, where N only depends on T, Γ_c and x_c. (Here we have calculated $\Gamma'(x_c)$ from (4.8)). Using (4.33) we obtain the following upper bound for $\mathcal{S}_\Omega f$ which holds for all $f \in X_1$.

$$\mathcal{S}_\Omega f(x) \leq \begin{cases} N(1 - x) & \text{if} \quad \dfrac{N-1}{N} \leq x \leq 1 \\ 1 & \text{if} \quad x \leq \dfrac{N-1}{N} \end{cases} . \tag{4.34}$$

This is well defined since $x_c > (N-1)/N$ cannot occur as follows from

$$\frac{N-1}{N} = x_c + (1 - x_c)(1 - \exp(2m(1 - x_c) - 2M)) \geq x_c , \tag{4.35}$$

where we have used that $2m(1 - x_c) \leq 2M$, which is an immediate consequence of the definitions of m and M.

Our next task is to use (4.34) to estimate $\mathcal{S}_G f$ for all $f \in X_1$. By straightforward calculations one easily establishes that for $(N-1)/N \leq x \leq 1$ we have that

$$\mathcal{S}_G f(x) \leq (1 - x)^2 \left\{ T + 2I_1\left(\frac{N-1}{N}, x_c^+\right) + 2N^2 I_2(x) + 2N^2 \frac{xI_3(x)}{(1 - x)^2} \right\} \tag{4.36}$$

where $x_c^+ = \max\{x_c, 0\}$ and

$$I_1(y,z) = \int_z^y \frac{t\mathrm{d}t}{(1 - t^2)^2} = \frac{y^2 - z^2}{2(1 - y^2)(1 - z^2)} \tag{4.37}$$

$$I_2(x) = \int_{(N-1)/N}^{x} \frac{t\mathrm{d}t}{(1 + t)^2} = \frac{1}{1 + x} - \frac{N}{2N - 1} + \ln\left(\frac{(2N-1)(1 + x)}{N}\right) \tag{4.38}$$

$$I_3(x) = \int_x^1 \frac{(1 - t)^2}{(1 + t)^2}\mathrm{d}t = -(x + 1) + 4\ln\frac{x + 1}{2} + \frac{4}{x + 1} \tag{4.39}$$

From (4.37)-(4.39) and the definition of N it follows immediately that the right-hand side in (4.36) depends only on T, Γ_c and x_c. From (4.37)-(4.38) it follows that $I_1((N-1)/N, x_c^+)$ is bounded and that $I_2(x) \leq I_2(1) < \infty$. Let us now consider I_3: it is evident that $I_3 \in C^\infty([x_c, 1])$ and that $I_3(1) = 0$, and since the integrand in (4.39) has a double zero at $t = 1$ we have that $I_3'(1) = I_3''(1) = 0$. Hence, by Taylor's theorem it follows that there exists a bounded function R, such that

$$I_3(x) = (1-x)^2 R(x) \tag{4.40}$$

If we now define $\mathcal{R} \equiv \sup_{x\in[(N-1)/N,1]} R(x)$ we can express (4.36) as

$$\mathcal{S}_G f(x) \leq (1-x)^2 \left\{T + 2I_1\left(\frac{N-1}{N}, x_c^+\right) + 2N^2 I_2(1) + 2N^2\mathcal{R}\right\} \tag{4.41}$$

$$\equiv C_1 (1-x)^2 \tag{4.42}$$

where C_1 depends only on N and x_c and hence only on T, Γ_c and x_c.

For $x \leq x_c$ we similarly obtain

$$\begin{aligned} \mathcal{S}_G f(x) \leq (1-x)^2 \{T + 2I_1(x^+, x_c^+) + \\ + 2\frac{x^+}{(1-x)^2}\left[N^2 I_3\left(\frac{N-1}{N}\right) + I_4(x^+)\right]\right\} \end{aligned} \tag{4.43}$$

where $x^+ \equiv \max\{x, 0\}$, I_1, I_3 are defined as in (4.37) and (4.39) and I_4 is defined as

$$I_4(x) = \int_x^{(N-1)/N} \frac{\mathrm{d}t}{(1+t)^2} = \frac{1}{1+x} - \frac{N}{2N-1} \ . \tag{4.44}$$

I_4 is clearly a decreasing function of x which depends only on N and x_c. We now note that since $x \leq (N-1)/N$ we have that $x/(1-x)^2 = 2N(N-1)$, and hence we may estimate (4.43) as

$$\begin{aligned} \mathcal{S}_G f(x) \leq (1-x)^2 \{T + 2I_1(x^+, x_c^+) + \\ + 4N(N-1)\left[N^2 I_3\left(\frac{N-1}{N}\right) + I_4(x_c^+)\right]\right\} \end{aligned} \tag{4.45}$$

$$\equiv C_2 (1-x)^2 \tag{4.46}$$

where C_2 depends only on N and x_c and hence only on T, Γ_c and x_c.

To summarise we now let $\widetilde{C} = \max\{C_1, C_2\}$ and we have shown that

$$\mathcal{S}_G f(x) \leq \widetilde{C}(1-x)^2 \ . \tag{4.47}$$

According to Proposition 1(a) we have that $\mathcal{S}f \leq \widetilde{f}$ where $\widetilde{f}$ is the solution to

$$\widetilde{f}' + \widetilde{f}^2 = \left(\frac{\Gamma_c}{2}\right)^2 \frac{\widetilde{C}}{(1+x)^2} \tag{4.48}$$

which satisfies $\widetilde{f}(x_c) = 0$. However, this means that $\widetilde{f} = f^P_{-\widetilde{C}\Gamma_c^2/2}$. Hence we have found a $\widetilde{C}$ such that $\mathcal{S}f \leq f^P_{-\widetilde{C}\Gamma_c^2/2}$ for all $f \in X_1$. □

With Proposition 6 at our disposal it is natural to define the set X as

$$X = \left\{f \in C([x_c, 1]) : f^P_{\widetilde{C}} \leq f \leq \min\left\{f_{UP}, f^P_{-\widetilde{C}\Gamma_c^2/2}\right\}\right\} \ . \tag{4.49}$$

With this definition it is clear that $X \subset X_1$ and that X is a non-empty, closed, convex and bounded subset of $C([x_c, 1])$. Propositions 3, 5 and 6 taken together show that

$$\mathcal{S}X \subset X \ . \tag{4.50}$$

Since $X \subset C([x_c, 1])$ Lemma 1 tells us that

$$\mathcal{S}_\Omega X \subset C^2([x_c, 1]) \ . \tag{4.51}$$

Furthermore, it follows from Proposition 2 that

$$\mathcal{S}_G X \subset C^2\left([x_c,1]\right) \ . \tag{4.52}$$

To conclude this subsection we would like to prove that

$$\mathcal{S} X \subset C^1\left([x_c,1)\right) \ , \tag{4.53}$$

which follows from (4.52) and the following lemma.

LEMMA 2. *Suppose that $G \in C^2\left([x_c,1)\right)$, then $\mathcal{S}_{f,G}G$ is C^1 in any open interval in which it is bounded. Additionally, if $\mathcal{S}_{f,G}G$ is bounded in $[x_c,1]$ then $\mathcal{S}_{f,G}G \in C^1\left([x_c,1)\right) \cap C\left([x_c,1]\right)$.*

Proof. The equation (4.2*a*) is regular in $[x_c,1)$ and therefore it follows from standard ODE theory that the solution is C^1 whenever it is bounded. However, if $f = \mathcal{S}_{f,G}G$ is bounded in $[x_c,1]$ then we can define $f(1) = \lim_{x\to 1^-} f(x)$, which ensures that $\mathcal{S}_{f,G}G \in C\left([x_c,1]\right)$. □

4.3. *End of the proof of Theorem 1*

Proof. of Theorem 1
(a) Since $T > T^*$ we may define X as in (4.49). We then have that X is a non-empty, closed, convex and bounded subset of $C\left([x_c,1]\right)$ such that $\mathcal{S}X \subset X$. To use Theorem 2 we must now only prove that $\mathcal{S} : X \to X$ is compact, i.e. that it is continuous and that $\mathcal{S}X$ is conditionally compact in X.

Let us now prove that $\mathcal{S}$ is continuous in X. To this end, assume that f converges to f_0 in X, this convergence is uniform since $X \subset C\left([x_c,1]\right)$. If we can prove that

$$\left|\Omega f - \mathcal{S}_\Omega f_0\right| \le K_1 \left(1-x\right) \left\|f - f_0\right\|_u \ , \tag{4.54}$$

where $\left\|\cdot\right\|_u$ denotes the uniform (supremum) norm, we can use the estimate (4.34) to establish that $\mathcal{S}_G^T f$ converges to $\mathcal{S}_G^T f_0$ uniformly in such a way that

$$\left|\mathcal{S}_G f - \mathcal{S}_G f_0\right| \le K_2 \left(1-x\right)^2 \left\|f - f_0\right\|_u \tag{4.55}$$

By applying Gronwall's lemma one sees that this implies that $\mathcal{S}f$ converges uniformly to $\mathcal{S}f_0$ in X. Hence to show that $\mathcal{S}$ is continuous on X all that remains is to establish the inequality (4.54). To simply the notations we introduce the following variables

$$E(y) = \exp\left(-2\int_{x_c}^{y} f \mathrm{d}z\right) \tag{4.56}$$

$$E_0(y) = \exp\left(-2\int_{x_c}^{y} f_0 \mathrm{d}z\right) \ . \tag{4.57}$$

If we use (4.8) and (4.32) we find that

$$\left|\mathcal{S}_\Omega f \ - \ \mathcal{S}_\Omega f_0\right| = \frac{\left|\int_x^1 E(y)\,\mathrm{d}y \int_{x_c}^1 E_0(y)\,\mathrm{d}y - \int_{x_c}^1 E(y)\,\mathrm{d}y \int_x^1 E_0(y)\,\mathrm{d}y\right|}{\int_{x_c}^1 E(y)\,\mathrm{d}y \int_{x_c}^1 E_0(y)\,\mathrm{d}y} \tag{4.58}$$

$$\le \frac{\exp(4M)}{\left(1-x_c\right)^2}\left|\int_x^1 E(y)\,\mathrm{d}y \int_{x_c}^x E_0(y)\,\mathrm{d}y - \int_{x_c}^x E(y)\,\mathrm{d}y \int_x^1 E_0(y)\,\mathrm{d}y\right| \tag{4.59}$$

If we let $F = \min(f, f_0)$ and use the fact that $\exp(-x) \ge 1 - x$ we obtain that

$$\exp\left(-2\int_{x_c}^{y} F \mathrm{d}z\right)\left\{1 - 2\left(1-x_c\right)\left\|f - f_0\right\|_u\right\} \le E(y) \le \exp\left(-2\int_{x_c}^{y} F \mathrm{d}z\right) \tag{4.60}$$

and exactly the same estimate for E_0. Hence (4.59) becomes

$$|\mathcal{S}_\Omega f - \mathcal{S}_\Omega f_0| \leq \frac{\exp(4M)}{(1-x_c)^2} \int_x^1 \exp\left(-2\int_{x_c}^y F \mathrm{d}z\right) \mathrm{d}y \int_{x_c}^x \exp\left(-2\int_{x_c}^y F \mathrm{d}z\right) \mathrm{d}y \tag{4.61}$$

$$\times \left|1 - \left(1 - 2(1-x_c)\|f - f_0\|_u\right)^2\right| \tag{4.62}$$

$$\leq K_1 (1-x) \|f - f_0\|_u \ , \tag{4.63}$$

where K_1 is given by

$$K_1 = \frac{4\exp(4M - 4m(1-x_c))}{1-x_c} (x - x_c)(1 + 2(1-x_c)\|X\|_u) . \tag{4.64}$$

Here we have used (4.32) and here $\|X\|_u$ denotes the supremum of the uniform norm of the elements in X, which is finite since X is a bounded set in this norm. We have thus found a constant K_1 such that (4.54) holds. Hence we have proved that $\mathcal{S}$ is continuous on X.

Since $\mathcal{S}X \subset X$ we know that $\mathcal{S}X$ is pointwise bounded for all $x \in [x_c, 1]$. From (4.47) it is also follows that $(\mathcal{S}_G X) / (1-x^2)^2$ is pointwise bounded as well. However, from (4.2a) we have that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathcal{S}f) = \left(\frac{\Gamma_c}{2}\right)^2 \frac{\mathcal{S}_G f}{(1-x^2)^2} - (\mathcal{S}f)^2 \tag{4.65}$$

for any $f \in X_1$. Hence $\mathcal{S}f$ consists of continuous functions with uniformly bounded gradients. Consequently, $\mathcal{S}X$ is equicontinuous in $[x_c, 1]$. The conditional compactness of $\mathcal{S}X$ thus follows from the Arzela-Ascoli theorem (See for example Folland 1984).

To summarise, we have shown that X and $\mathcal{S}$ satisfy all the conditions of Theorem 2, and hence there exists an $f \in X$ such that $\mathcal{S}f = f$, and thus f, $\Omega = \mathcal{S}_\Omega f$ and $G = \mathcal{S}_G f$ solve our system of equations (4.2) and the boundary conditions (4.3). From (4.51)-(4.53) it follows that

$$f \in C^1([x_c, 1)) \cap C([x_c, 1]) \tag{4.66}$$

$$\Omega \in C^2([x_c, 1]) \tag{4.67}$$

$$G \in C^2([x_c, 1]) \ . \tag{4.68}$$

After back-substitution to our original variables ψ, F and Γ, all that remains is to establish that $\psi \in C^1([x_c, 1])$. However, we have that

$$\psi' = 4xf - 2(1-x^2) f' \tag{4.69}$$

and we know that $\lim_{x\to 1^-} (1-x) f'(x) = 0$ or else we would not have that $|f(1)| < \infty$. This concludes the proof of (a).

Proof of (b)

Let $\widehat{X} = C([x_c, 1])$. All functions $f \in \widehat{X}$ are integrable in $[x_c, 1]$ and hence (4.9) holds. Since $T \leq T_*$, Proposition 4 shows that $\mathcal{S}f \to -\infty$ somewhere in $[x_c, 1]$, for all $f \in \widehat{X}$. On the other hand, there is no $f \in \widehat{X}$ such that $f \to -\infty$ somewhere in $[x_c, 1]$, and therefore we have established that

$$\mathcal{S}\widehat{X} \cap \widehat{X} = \emptyset \tag{4.70}$$

This obviously rules out the possible presence of a fixed point in $\widehat{X}$, and hence for $T \leq T_*$ there can be no continuous solution to (4.2) which satisfies the boundary conditions (4.3).

□

5. Some remarks on the computability of the solutions obtained in Theorem 1

A problem with existence theorems based on the Schauder Fixed Point Theorem is that theorems do not establish that it is possible to compute the solution approximately with a finite number of operations. Suppose that $f \in X$ is a fixed point to $\mathcal{S}$, where X is as defined in (4.49) then it is clear that $f \in \mathcal{S}^n X$ for all n. Inspired by this Yih *et al.* (1982) suggested the following computational approach. Let $\Omega_0 \equiv 1$. Let $f_1 = \mathcal{S}_{f,\Omega}\Omega$ and then define $f_n = \mathcal{S}^{n-1}f_1$ and $\Omega_n = \mathcal{S}_\Omega f_n$. Yih *et al.* (1982) claimed to have proved convergence for this numerical method if $x_c \geq 0$. Although numerical experiments suggest that this method indeed converges, in our opinion the proof presented in (Yih *et al.* 1982) is not complete. To see this we will repeat and comment on their argument.

If $x_c \geq 0$ we may add a further comparison result to those in Proposition 1, namely

LEMMA 3. *If $x_c \geq 0$ and $\Omega_1 \geq \Omega_2$ then $\mathcal{S}_{G,\Omega}\Omega_1 \geq \mathcal{S}_{G,\Omega}\Omega_2$.*

The proof is a trivial consequence of (4.1).

Using this, Proposition 1 and Lemma 1 we can follow Yih *et al.* (1982) and establish that the f_ns and Ω_ns form nested sequences

$$f_1 \geq f_3 \geq \ldots \geq \ldots \geq f_4 \geq f_2 \tag{5.1}$$

$$\Omega_0 \geq \Omega_2 \geq \ldots \geq \ldots \geq \Omega_3 \geq \Omega_1 \tag{5.2}$$

Yih *et al.* (1982) claimed that this is enough to ensure convergence of the numerical method, but in our opinion this is not necessarily true. The first problem is that we cannot establish that $f_1(1) < \infty$. This problem can be remedied with an argument similar to that which led up to Proposition 6. Having done so, it is true that odd sequence entries of f_n approach some limit l from below and that the even sequence entries approach some limit L from above. There is, however, no reason why these limits should necessarily coincide. We claim that without additional knowledge of the iterated sequences it is quite possible that some limit cycle (l, L), such that $\mathcal{S}l = L$ and $\mathcal{S}L = l$, is approached in the limit.

As mentioned in Yih *et al.* (1982) nested sequences of this type have previously been used by Weyl (1942) to prove the existence of similarity solutions for some boundary-layer problems. However, an essential part of Weyl's proof is the establishment that for his problems

$$|f_{n+1} - f_n| \to 0 \tag{5.3}$$

pointwise and uniformly on compacts as $n \to \infty$. This is clearly sufficient to establish convergence of the numerical method outlined above, unfortunately however, a property like (5.3) is difficult to establish for our problem. It is precisely this difficulty in proving a suitable contraction property, which has forced us to invoke the abstract framework of the Schauder Fixed Point Theorem.

Finally, we should say a few words about uniqueness. Using Proposition 1 we easily establish that for $x_c \geq 0$ any solution obtained from the procedure outlined above is unique. However, this does not imply that this is the unique solution to (4.2) which satisfies (4.3) for given values of T, Γ_c and x_c. The reason for this, is that we have started the iterations with $\Omega_0 \equiv 1$. This choice is clearly arbitrary, to obtain all solution we should therefore try all continuous decreasing functions in $[x_c, 1]$ taking values between 0 and 1 as starting functions. This would be an enormous set of initial functions, all of which could in principle yield different solutions. In the sequel to this article we will prove a theorem which shows that all possible solutions to our problem, can be exhaustively indexed by a three-dimensional set, and some evidence indicates that it may in fact be one-dimensional. In doing so we will mathematically justify a numerical procedure described in (Shtern & Hussain 1993) and (Shtern & Hussain 1996).

6. Conclusion

We have mathematically proved sufficient conditions for the existence and non-existence of conically self-similar free-vortex solutions to the Navier–Stokes equations. Specifically, we have proved the absence of a logarithmic singularity at the symmetry axis. In doing so, we have also improved the bounds for solution existence and non-existence, and the resulting bounds are shown to be close to optimal in the low-swirling limit, which emphasizes the relationship between solution existence and non-existence in the swirling case and in the non-swirling case. Finally, we have discussed briefly some mathematical difficulties which occur when we try to compute these solutions.

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