Reducing non-stationary processes to stationarity by a time deformation

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Abstract

A necessary and sufficient condition is given to reduce a non-stationary random process $\{Z(t) : t \in T \subseteq \mathbb{R}\}$ to stationarity *via* a bijective differentiable time deformation Φ so that its correlation function $r(t, t')$ depends only on the difference $\Phi(t') - \Phi(t)$ through a stationary correlation function R: $r(t, t') = R(\Phi(t') - \Phi(t))$.

keywords: bijective time deformation; characterisation; correlation function; non-stationarity

1 Introduction

In most applications dealing with non-stationary processes, the first step in classical approaches consists in removing expectation, dividing the residuals by standard deviation and modelling the residuals as stationary processes (AR, MA, $ARMA$, ...). The process $Z = \{Z(t) : t \in T \subseteq \mathbb{R}\}$ under study is of the form

$$Z(t) = \mu(t) + \sqrt{\sigma(t)}\epsilon(t),$$

where $\mu(t) = EZ(t)$, $\sigma(t) = E(Z(t) - \mu(t))^2$ and $\epsilon(t)$ a centred and standardised weakly (or strongly) stationary process. The non-stationarity of Z is then understood as non-stationarity of both the first order moment $\mu(t)$ and the variance $\sigma(t)$.

Modelling $\epsilon(t)$ as

$$\epsilon(t) = \delta(\Phi(t)),$$

where δ is weakly stationary and Φ is a bijective deformation, or equivalently modelling the correlation function $r(t, t')$ of the process Z as

$$r(t, t') = R(\Phi(t') - \Phi(t)), \tag{1}$$

where R is a stationary correlation function, is a way to introduce further non-stationarity. This is discussed by Sampson and Guttorp (1992) and is later developed in Meiring (1995) and Perrin (1997).

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When a correlation function satisfies (1) we call it a stationary reducible correlation function. Unlike the classical approaches, non-stationarity through second order moments is thus taken into account and (1) gives opportunity to enlarge the class of models for studying non-stationary processes.

According to proposition 7.1.4 in Samorodnitsky and Taqqu (1994) the correlation function of any H-self similar process with parameter $H > 0$ indexed by $T = \{t : t > 0\}$ satisfies model (1) with $\Phi(t) = \ln(t)$. However not every correlation function can be reduced to a stationarity one.

This work appears closely related to the following question of Krivine (in Assouad (1980))

"On a locally compact Abelian group $(\mathbb{T}, +)$*, how can the positive Hermitian kernels* r *which satisfy* $r(t, t') = R(\Phi(t') - \Phi(t))$ *be characterised, where* Φ *is a transformation of the space* $\mathbb{T}$ *and* R *a positive definite function?"*

In this paper we characterise stationary reducible correlation functions under smoothness assumptions. The paper is organised as follows. In Section 2 we give some properties of the model (1) and propose a characterisation for smooth stationary reducible correlation functions in the form of a differential equation. Further, we prove uniqueness of the deformation Φ up to a translation and a scale change. Section 3 contains two examples of stationary reducible correlations and two counterexamples.

2 Properties, characterisation and uniqueness

2.1 Properties of stationary reducibility

So far we have considered correlation functions. However the model (1) can be applied to covariance functions provided that the variance of the process is constant. Moreover, when the mean of the process is also constant, the model (1) generalises the notion of weak stationarity (or strong stationarity for Gaussian processes); indeed, when the deformation Φ is the identity function stationarity and reducible stationarity agree.

2.2 Characterisation and uniqueness

Let the index space T be either $\mathbb{R}$, $\{t : t \geq 0\}$, $\{t : t > 0\}$ or an interval $[a, b] \subset \mathbb{R}$ and consider for the bijective deformation Φ the following assumption

(A1) Φ is continuous and differentiable in T as is its inverse.

Note that if (Φ, R) is a solution to (1), then for any $\alpha > 0$ and $\beta \in \mathbb{R}$, $(\tilde{\Phi}, \tilde{R})$ with $\tilde{\Phi}(x) = \alpha\Phi(x) + \beta$ and $\tilde{R}(u) = R(u/\alpha)$ is a solution as well. Thus, without loss of generality we may impose the restriction that for some fixed point x_0 in T

$$\Phi(t_0) = 0 \quad \text{and} \quad \Phi'(t_0) = 1 \tag{2}$$

where Φ' is the derivative of Φ. Consequently, when the non-stationary correlation $r(t,t')$ satisfying (1) and the deformation Φ are given, the stationary correlation function R is uniquely determined as

$$R(u) = r(t_0, \Phi^{-1}(u)),\ \forall u \in \{t'-t : (t,t') \in \Phi(T) \times \Phi(T)\}. \tag{3}$$

We denote by $\partial_i r(t,t')$, $i = 1,2$, the first partial derivatives of $r(t,t')$. We consider correlation functions $r(t,t')$ such that

(A2) $r(t,t')$ is continuous and differentiable for $t \neq t'$.

(A3) the function $u \longmapsto \dfrac{\partial_1 r(u,t_0)}{\partial_2 r(u,t_0)}$ is locally Lebesgue integrable in T.

Assumptions **(A2)** and **(A3)** are satisfied by a large class of processes including mth order iterated integrals of a Wiener process, non-degenerate fractional Brownian motions (which are a particular case of H-self similar processes) with parameter $H \in]0,1]$ and more generally processes with independent increments provided the variance is continuously differentiable.

It follows from **(A1)** and **(A2)** that the stationary correlation function $R(u)$ is continuous and differentiable for u different from 0.

Here is the necessary and sufficient condition for stationary reducibility *via* bijective deformation.

***Theorem* 2.1** *Assume (A1), (A2) and (A3). A correlation function $r(t,t')$ satisfies (1) if and only if almost everywhere for $t \neq t'$*

$$\partial_1 r(t,t') \frac{\partial_1 r(t',t_0)}{\partial_2 r(t',t_0)} + \partial_2 r(t,t') \frac{\partial_1 r(t,t_0)}{\partial_2 r(t,t_0)} = 0. \tag{4}$$

The pair (Φ, R) in (1) is uniquely determined with Φ given by

$$\Phi(t) = -\int_{t_0}^{t} \frac{\partial_1 r(u,t_0)}{\partial_2 r(u,t_0)} du \tag{5}$$

and R given by (3) and (5).

Proof. First we show that (1) is equivalent to

$$\partial_1 r(t,t')\Phi'(t') + \partial_2 r(t,t')\Phi'(t) = 0,\ \ \forall t \neq t'. \tag{6}$$

Indeed if (1) holds for $t \neq t'$

$$\left\{ \begin{array}{lcl} \partial_1 r(t,t') & = & -\Phi'(t) R' \left(\Phi(t') - \Phi(t)\right) \\ \partial_2 r(t,y') & = & \Phi'(t') R' \left(\Phi(t') - \Phi(t)\right), \end{array} \right.$$

so that (6) is satisfied. Conversely assume (6) holds, consider the bijective coordinate change

$$\left\{ \begin{array}{lcl} u & = & \Phi(t') - \Phi(t) \\ u' & = & \Phi(t') + \Phi(t) \end{array} \right.$$

and set $\Gamma(u,u') = r\left(t(u,u'), t'(u,u')\right)$. Equation (6) then leads to

$$\partial_2 \Gamma(u,u') = 0.$$

Thus Γ does not depend on the second coordinate. Therefore we may define $R(u) = \Gamma(u,u')$ and (1) follows. Further, under Conditions (2), we infer from (6) that (5) holds and (4) now follows from (5) and (6). Conversely, if (4) holds, we may define Φ through (5), and this Φ satisfies (6). Clearly, Φ is uniquely determined by (5) and R by (3).

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3 Examples and counterexamples

3.1 Examples

3.1.1 Fractional Brownian motions

As already mentioned, non-degenerate fractional Brownian motions are a particular case of H-self similar processes with parameter $H \in]0,1]$. Their correlation functions are of the form for all t, t' such that $tt' \neq 0$

$$r(t,t') = \frac{\left(t^{2H} + t'^{2H} - |t' - t|^{2H}\right)}{2t^H t'^H}. \tag{7}$$

As pointed out in the introduction, they satisfy (1) with $\Phi(t) = \ln(t)$ (proposition 7.1.4 in Samorodnitsky and Taqqu (1994)). The corresponding stationary correlation function R is

$$R(u) = \cosh(Hu) - 2^{(2H-1)}|\sinh(|u|/2)|^{2H}$$

where cosh and sinh denote hyperbolic cosine and sine functions.

3.1.2 A simple diffusion process

For a measurable function f on $T = \{t : t \geq 0\}$, with $f(t) \neq 0$, locally square integrable with respect to the Lebesgue measure, and W the standard Wiener process, the process $Z(t) = \int_0^t f(s)dW(s)$ is a centred Gaussian process with correlation function defined for all t, t' such that $tt' \neq 0$

$$r(t,t') = \frac{\sqrt{\int_0^{t\wedge t'} f^2(u)du}}{\sqrt{\int_0^{t\vee t'} f^2(u)du}}.$$

Calculation of the partial derivatives of $r(t,t')$ show that (4) is satisfied for $tt' \neq 0$. Thus, it follows from (5) that the deformation Φ which satisfies (2) with $t_0 = 0$ and that makes Z strongly stationary is defined by

$$\Phi(t) = \frac{\int_0^1 f^2(u)du}{f^2(1)} \ln\left(\frac{\int_0^t f^2(u)du}{\int_0^1 f^2(u)du}\right).$$

If $f(u) = u^\beta$ with $\beta > -1/2$, the same deformation $\Phi(t) = \ln(t)$ as before is obtained. The corresponding stationary correlation function R is

$$R(u) = \exp\left(-(\frac{2\beta+1}{2})|u|\right).$$

3.2 Counterexamples

3.2.1 Smooth correlation function

Consider a stationary isotropic Gaussian random field $\tilde{Z}(x)$ indexed by $\mathbb{R}^2$ which has the correlation function $\tilde{r}(x, y) = \exp(-\|y - x\|_2^2)$ where $\|u\|_2$ is Euclidean distance in $\mathbb{R}^2$. Strolling along the parabolic path $t \longmapsto (t, t^2)$, we consider the process $Z(t) = \tilde{Z}(t, t^2)$ indexed by $\mathbb{R}$. Its correlation function is $r(t, t') = \exp\{-(t' - t)^2(1 + (t + t')^2)\}$ and is infinite differentiable. Suppose there is a continuous and differentiable deformation Φ reducing r. Then with $t_0 = 0$ (4) gives

$$(1 + 2t(t + t'))(1 + 2t'^2) = (1 + 2t'(t + t'))(1 + 2t^2), \quad \forall t \neq t'.$$

This equation implies necessarily that $t = t'$. Therefore, $Z(t)$ is not stationary reducible.

3.2.2 Assouad's counterexample

This counterexample constitutes a partial answer of Assouad (1980) to the question of Krivine quoted in the introduction.

We endow $\mathbb{R}^3$ with the canonical scalar product $< .,. >_3$. The counterexample is obtained as follows

1. consider six points $X_1, \ldots, X_6$ on the unit sphere S_2 of $\mathbb{R}^3$ that meet the always satisfied conditions

$$X_1 + X_2 = 2X_6 \cos\alpha, \; X_2 + X_3 = 2X_4 \cos\alpha, \; X_1 + X_3 = 2X_5 \cos\alpha$$

   for any $\alpha \in \left]0, \frac{\pi}{3}\right[$;

2. build the Gram matrix (symmetric and positive definite)

$$(r_{i,j}) = (< X_i, X_j >_3)_{1 \leq i,j \leq 6};$$

3. then construct a continuous (in the mean square sense) Gaussian process $Z = \{Z(t), t \in [0, 1]\}$ with correlation $r(t, t')$ that satisfies for $1 \leq i, j \leq 6$

$$r((i - 1)/5, (j - 1)/5) = r_{i,j};$$

4. finally Perrin (1997) proves that there is no bijective continuous deformation Φ of $\mathbb{R}$ that makes Z stationary.

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