(September 15, 1998)

WEAK FUNCTIONAL CONVERGENCE OF QUADRATIC VARIATIONS FOR A LARGE CLASS OF GAUSSIAN PROCESSES WITH APPLICATIONS

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Abstract

We are interested in the weak functional convergence of the process of quadratic variations taken along a regular partition for a large class of Gaussian processes indexed by $[0,1]$, including the standard Wiener process as a particular case. This result is applied to two topics: estimation of a time deformation model for non-stationary Gaussian processes and test on the diffusion coefficient in a diffusion model.

keywords: diffusion, Gaussian process, quadratic variations, time deformation

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60G15,60G17
SECONDARY 60F17,60G12,60J60

1. Introduction.

We are interested in the weak functional convergence of quadratic variation process for a large class of Gaussian processes indexed by $[0,1]$. This convergence result is obtained assuming smoothness of the covariance function outside the diagonal.

The quadratic variations were first introduced by Lévy (1940) who showed that if Z is the standard Wiener process on $[0,1]$, then almost surely ($a.s.$) as $n \to \infty$

$$\sum_{k=1}^{2^n} [Z(k/2^n) - Z((k-1)/2^n)]^2 \longrightarrow 1. \tag{1}$$

Baxter (1956) and further Gladyshev (1961) generalise this result to a large class of Gaussian processes under assumption on the second derivatives of the covariance.

Guyon and León (1989) introduced an important generalisation of these variations for a Gaussian stationary non differentiable process with covariance function $r(u) = 1 - u^\beta L(u)$, where $\beta \in]0,2[$ and L is a slowly varying function at zero. Let H be a real function. The H-variation of process Z indexed by $[0,1]$ is defined by

$$\sum_{k=1}^{n} H\left(\frac{Z(k/n) - Z((k-1)/n)}{(2(r(0) - r(1/n)))^{1/2}}\right).$$

They studied the convergence in distribution of the H-variations, suitably normalised, for non-differentiable Gaussian processes.

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The generalisation of these variations for Gaussian fields is studied in Guyon (1987) and León and Ortega (1989). Another generalisation for non-stationary Gaussian processes and quadratic variations along curves is done in Adler and Pyke (1993).

For Gaussian process Z with stationary increments, Istas and Lang (1997) define general quadratic variations, substituting a general discrete difference operator to the simple difference $Z(k/n) - Z((k-1)/n)$. They give conditions on the discrete difference and on the covariance function of Z that ensure the *a.s.* convergence and the asymptotic normality of these quadratic variations, suitably normalised. Then, they use these quadratic variations to estimate the Hölder index of the process.

For non-stationary Gaussian processes, with stationary increments or not, we give in this paper a general result concerning the functional asymptotic normality of the process of the quadratic variations which corresponds to the linear interpolation of the points $(p/n, V_n(p/n))$ with $V_n(p/n) = \sum_{k=1}^{p}[Z(k/n) - Z((k-1)/n)]^2$, $p = 1, 2, \dots, n$.

Where do quadratic variations appear? We point out two cases which involve quadratic variations.

The first case is the estimation of time deformation for non-stationary models of the form

$$r(x,y) = R(\Phi(y) - \Phi(x)) \tag{2}$$

where r is a correlation function, Φ a bijective, and in general continuous, deformation of the index set and R a stationary correlation function. Model (2) appeared first in Sampson and Guttorp (1992) to give a class of non-stationary correlation functions for random fields. In the one-dimensional case, Perrin (1997) gives different methods for estimating Φ. Perrin and Senoussi (1998, (a) and (b)) exhibit under smoothness assumptions a characterisation for correlation functions r satisfying (2). When the process Z under study is Gaussian, we show it is possible to construct a non-parametric estimator of Φ from one realisation of Z observed at discrete times k/n, $k = 0, 1, \dots, n$ and give the asymptotic normality of this estimator as the number of observations n grows to ∞. Testing the stationarity of Z, *i.e.* testing if Φ is the identity or not, is also considered.

The second case concerns a test on the diffusion coefficient $\sigma(x)$ in the diffusion model $dY(x) = m(x, Y(x))dx + \sigma(x)dW(x)$, $Y(0) = y_0$, $x \in [0,1]$, where $W(x)$ is the standard Wiener process, y_0 is a known real parameter and the functions $m(x,u)$ and $\sigma(x)$ are two deterministic unknown functions. A problem is to estimate the diffusion coefficient $\sigma(x)$ from discrete observations of Y at points k/n, $k = 0, 1, \dots, n$. This problem is of major interest in financial markets and has recently been investigated by Brugière (1991), Soulier (1991), Genon-Catalot *et al.* (1992) and Istas (1996) where quadratic variations (smoothed by convenient kernels) are applied to this estimation. This diffusion process $Y(x)$ is generally non-Gaussian. Nevertheless, under smoothness assumptions on m, Istas (1996) proves that the estimation of the diffusion coefficient $\sigma(x)$ reduces to the estimation of the singularity function of the process $Z(x) = \int_0^x \sigma(u)dW(u)$. We use this to test if $\sigma(x)$ is constant or not.

The paper is structured as follows. The following section sets up notations, assumptions and definitions, describes quadratic variation process more carefully and gives preliminary results needed to prove in section 4 our main result, theorem 4.2, dealing with the weak functional convergence of the process of the quadratic variations. Then we apply the asymptotic result of theorem 4.2 to the related problems: estimation of time deformation and testing if the diffusion coefficient is constant or not.

2. The process of quadratic variations

Let $Z = \{Z(x), x \in [0,1]\}$ be a centred Gaussian process of real-valued random variables with covariance function $r(x, y)$. Assume that

(A1) r is continuous in $[0,1]^2$ and its second derivatives are uniformly bounded for $x \neq y$.

Assumption **(A1)** is satisfied for a large class of processes including: *(i)* standard Wiener process; *(ii)* processes with independent increments with function $x \longmapsto r(x,x)$ of class C^1; *(iii)* stationary processes with rational spectral densities.

We denote by $r^{(m,m')}$ the m, m'-partial derivative of r with respect to x and y and set

$$\begin{aligned} r^{(0,1)}(x, x^-) &= \lim_{y \nearrow x} r^{(0,1)}(x, y), \quad x \in]0,1], \\ r^{(0,1)}(x, x^+) &= \lim_{y \searrow x} r^{(0,1)}(x, y), \quad x \in [0,1[. \end{aligned}$$

Those limits exist because the second order derivatives of r are uniformly bounded. So, let us define the two following functions

$$\begin{aligned} D^-(x) &= r^{(0,1)}(x, x^-), \quad x \in]0,1], \\ D^+(x) &= r^{(0,1)}(x, x^+), \quad x \in [0,1[. \end{aligned}$$

Then we have the following result whose the obvious proof is given in Appendix A.

***Lemma** 2.1 Assume **(A1)**. Then D^- and D^+ are continuous in* $[0,1]$.

Let us now introduce the singularity function α of Z.

$$\alpha(x) = D^-(x) - D^+(x), \quad x \in [0,1].$$

It follows directly from lemma 2.1 that α is uniformly continuous in $[0,1]$. Note that the existence of the first derivative of $r(x,y)$ at $x = y$ is not assumed. Indeed, the existence of this derivative would make $\alpha(x) = 0$ for all $x \in [0,1]$.

Let n be a positive integer. We set for $k = 1, 2, \ldots, n$

$$\Delta Z_k = Z(k/n) - Z((k-1)/n).$$

We will now define the process of the quadratic variations of Z. Let $\Pi_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\right\}$ be the regular partition of $[0,1]$ at constant scale $1/n$. Then we define the quadratic variations $V_n(x)$ of Z along Π_n truncated by $x \in [0,1]$ as follows

$$V_n(x) = \sum_{k=1}^{\lfloor nx \rfloor} (\Delta Z_k)^2$$

where $\lfloor nx \rfloor$ denotes the greatest integer smaller than or equal to nx. We take the convention $\sum_{k=1}^{0} (\Delta Z_k)^2 = 0$.

Definition *2.1 The process of the quadratic variations of Z, $vq_n = \{vq_n(x), x \in [0,1]\}$, is defined by*

$$\begin{cases} vq_n(x) &= V_n(x) + (nx - \lfloor nx \rfloor)\left(\Delta Z_{\lfloor nx \rfloor + 1}\right)^2, \quad x \in [0,1[, \\ vq_n(1) &= V_n(1). \end{cases}$$

Thus, $vq_n(x)$ is a continuous version of $V_n(x)$ and corresponds to the stochastic linear spline with mesh Π_n that interpolates points $(p/n, V_n(p/n))$, $p = 1, 2, \dots, n$. From now, we do not distinguish anymore the case $x \in [0,1[$ from the case $x = 1$ in the definition of $vq_n(x)$.

3. Related problems

Let us mention two statistical problems related to quadratic variations.

3.1. *Estimation of a deformation model* Let Z be a centred Gaussian process with correlation function satisfying **(A1)**. Consider the problem of estimating the bijective and continuous deformation $\Phi : [0,1] \longmapsto \mathbb{R}$ from one realisation of Z observed at discrete times k/n, $k = 0, 1, \dots, n$, in the model

$$r(x,y) = R(\Phi(y) - \Phi(x)) \tag{3}$$

where R is a stationary correlation function. We assume that the deformation Φ satisfies the following assumption

(B) Φ is bijective and continuously differentiable in [0,1], as is its inverse,

Note that if (Φ, R) is a solution to (3), then for any $b > 0$ and $c \in \mathbb{R}$, $(\tilde{\Phi}, \tilde{R})$ with $\tilde{\Phi}(t) = b\Phi(t) + c$ and $\tilde{R}(u) = R(u/b)$ is a solution as well. Thus, without loss of generality we may impose the restriction that

$$\Phi(0) = 0 \ \text{ and } \ \Phi(1) = 1 \tag{4}$$

Consequently, the stationary correlation function R is uniquely determined as

$$R(u) = r(0, \Phi^{-1}(u)) \ \text{ and } \ R(-u) = R(u). \tag{5}$$

We note Φ' the derivative of Φ and R' the derivative of R. It follows from **(A1)** and **(B)** that the stationary correlation function $R(u)$ is continuous and differentiable for

u different from 0, and that its derivative to the left and its derivative to the right at 0 exist and satisfy

$$\begin{aligned} R'(0^-) &= D^-(x)/\Phi'(x), \\ R'(0^+) &= D^+(x)/\Phi'(x). \end{aligned}$$

Thus, the singularity function α satisfies the following relation

$$\alpha(x) = 2R'(0^-)\Phi'(x).$$

Finally, under Conditions (4), we get for all $x \in [0,1]$

$$\Phi(x) = \frac{\displaystyle\int_0^x \alpha(u)du}{\displaystyle\int_0^1 \alpha(u)du}.$$

Therefore, the estimation of Φ requires an estimation of the primitive of α: $x \longmapsto \displaystyle\int_0^x \alpha(u)du$. Once an estimator of Φ will be built, we will give the asymptotic normality of this estimator suitably normalised as the number of observations n grows to ∞.

3.2. *Testing the diffusion coefficient of a diffusion process* Consider the following stochastic differential equation

$$\left\{ \begin{aligned} dY(x) &= m(x, Y(x))dx + \sigma(x)dW(x), \\ Y(0) &= y_0, \end{aligned} \right.$$

where $W(x)$ is the standard Wiener process, y_0 is a known real number, and the functions $m(x,u)$ and $\sigma(x)$ are unknown. Set $Z(x) = \displaystyle\int_0^x \sigma(u)dW(u)$. The problem is to test if the function $\sigma : x \longmapsto \sigma(x)$ is constant or not. The covariance of the process Z is given by $r(x,y) = \displaystyle\int_0^{x\wedge y} \sigma^2(u)du$. If σ have a bounded first derivative in $[0,1]$ then Z satisfies **(A1)** and $\alpha(x) = \sigma^2(x)$. The diffusion process Y is generally non-Gaussian and cannot satisfy our framework. Nevertheless, under smoothness conditions on the function $m(x,u)$, it is well known (*e.g.* Genon-Catalot *et al.* (1992) and Istas (1996)) that the quadratic variations of Y are asymptotically equivalent to those of Z. We use this result to build our test.

4. Weak functional convergence

Consider the centred Gaussian vector

$$W_{\lfloor nx \rfloor} = \left(Z\left(\frac{0}{n}\right), Z\left(\frac{1}{n}\right), \ldots, Z\left(\frac{\lfloor nx \rfloor}{n}\right) \right)^T.$$

Its covariance matrix is

$$\Sigma_{\lfloor nx\rfloor} = \begin{pmatrix} r(0/n,0/n) & r(0/n,1/n) & \cdots & r(0/n,\lfloor nx\rfloor/n) \\ & r(1/n,1/n) & \cdots & r(1/n,\lfloor nx\rfloor/n) \\ & & \ddots & \vdots \\ & & & r(\lfloor nx\rfloor/n,\lfloor nx\rfloor/n) \end{pmatrix}.$$

Let $L_{\lfloor nx\rfloor}$ be a matrix with $\lfloor nx\rfloor$ rows and $\lfloor nx\rfloor+1$ columns defined as follows

$$L_{\lfloor nx\rfloor} = \begin{pmatrix} -1 & +1 & 0 & \cdots & 0 \\ 0 & -1 & +1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -1 & +1 \end{pmatrix}.$$

Having $V_n(x) = W_{\lfloor nx\rfloor}^T L_{\lfloor nx\rfloor}^T L_{\lfloor nx\rfloor} W_{\lfloor nx\rfloor}$, we deduce (*e.g.* Johnson et Kotz (1970), pp. 149-151).

Proposition 4.1 *For all* $x \in [0,1]$

$$V_n(x) \overset{\mathcal{L}}{\sim} \sum_{k=1}^{\lfloor nx\rfloor} \lambda_{k,\lfloor nx\rfloor} \chi^2_{k,\lfloor nx\rfloor}$$

where $\lambda_{1,\lfloor nx\rfloor} \geq \lambda_{2,\lfloor nx\rfloor} \geq \cdots \geq \lambda_{\lfloor nx\rfloor,\lfloor nx\rfloor} \geq 0$ *are the eigenvalues of the covariance matrix* $L_{\lfloor nx\rfloor}\Sigma_{\lfloor nx\rfloor}L^T_{\lfloor nx\rfloor}$ *and the* $\chi^2_{k,\lfloor nx\rfloor}$ *are independent chi-square variables with one degree of freedom.*

The following theorem gives an uniform upper bound for the largest eigenvalue $\lambda_{1,\lfloor nx\rfloor}$.

Theorem 4.1 *Assume **(A1)**. Then*

$$\sup_{x\in[0,1]} \lambda_{1,\lfloor nx\rfloor} = O(1/n).$$

Proof. For $(j,k) \in [1,\ldots,n]^2$, let

$$a_{j,k} = r\left(\frac{j}{n},\frac{k}{n}\right) + r\left(\frac{j-1}{n},\frac{k-1}{n}\right) - r\left(\frac{j}{n},\frac{k-1}{n}\right) - r\left(\frac{j-1}{n},\frac{k}{n}\right). \tag{6}$$

When $j=k$, a Taylor series expansion of order one with Young remainder gives

$$a_{k,k} = \frac{1}{n}\left(D^-\left(\frac{k}{n}\right) - D^+\left(\frac{k-1}{n}\right)\right) + o(1/n), \quad k=1,2,\ldots,n, \tag{7}$$

where $o(1/n)$ is independent of k. According to lemma 2.1, D^+ is uniformly continuous in $[0,1]$. Thus

$$D^+\left(\frac{k-1}{n}\right) = D^+\left(\frac{k}{n}\right) + o(1)$$

where $o(1)$ is independent of k. Therefore

$$a_{k,k} = \frac{1}{n}\alpha\left(\frac{k}{n}\right) + o(1/n), \quad k = 1, 2, \ldots, n, \tag{8}$$

where $o(1/n)$ is independent of k. α being uniformly bounded in $[0,1]$, we have

$$|a_{k,k}| = O(1/n), \quad k = 1, 2, \ldots, n$$

where $O(1/n)$ is independent of k.
When $j \neq k$, a Taylor series expansion of order one with Lagrange remainder gives

$$|a_{j,k}| = O(1/n^2) \tag{9}$$

where $O(1/n^2)$ is independent of (j,k).
To give an upper bound for the largest eigenvalue $\lambda_{1,\lfloor nx \rfloor}$, it is sufficient to use the following inequality (*e.g.* Horn and Johnson, p. 33).

$$\lambda_{1,\lfloor nx \rfloor} \leq \max_{1 \leq k \leq \lfloor nx \rfloor} \sum_{j=1}^{\lfloor nx \rfloor} |a_{j,k}| \leq \max_{1 \leq k \leq n} \sum_{j=1}^{n} |a_{j,k}| = O(1/n).$$

□

We assume the following assumption for the singularity function α.

(A2) α has a bounded first derivative in $[0,1]$.

For instance, assumption **(A2)** is satisfied: *(i)* for processes with independent increments if the function $x \longmapsto r(x,x)$ is C^2; *(ii)* for stationary processes with rational spectral densities.
The following lemma gives a property of the trajectories of Z.

***Lemma** 4.1 Assume **(A2)**. Then for any constant* $\gamma \in]0, 1/2[$

$$\lim_{h \to 0} h^{-\gamma} \sup_{|y-x| \leq h} |Z(y) - Z(x)| = 0 \ \ a.s.$$

It follows that Z is continuous (in the sense that a.s. Z has continuous trajectories).

Proof. A Taylor series expansion of order one with Young remainder gives for all $(x,y) \in [0,1]^2$

$$E\left(Z(y) - Z(x)\right)^2 = r(x,x) + r(y,y) - 2r(x,y) = O(|y-x|).$$

Therefore, for any constant $\gamma \in]0, 1/2[$

$$\lim_{h \to 0} h^{-\gamma} \sup_{|y-x| \leq h} |Z(y) - Z(x)| = 0 \ \ a.s.$$

(*e.g.* Neveu (1980), p. 93).

□

We now from Baxter (1956) that $V_n(x)$ is a consistent estimator of $\int_0^x \alpha(u)du$. The following lemma gives an upper bound for the bias of $V_n(x)$.

Lemma *4.2 Assume **(A1)**-**(A2)**. Then, the following holds*

$$\sup_{x\in[0,1]} \left| E(V_n(x)) - \int_0^x \alpha(u)du \right| = O(1/n).$$

Proof. Observe from proposition 4.1 that

$$E(V_n(x)) = \sum_{k=1}^{\lfloor nx \rfloor} a_{k,k}. \tag{10}$$

Recall (8)

$$\sum_{k=1}^{\lfloor nx \rfloor} a_{k,k} = \frac{1}{n} \sum_{k=1}^{\lfloor nx \rfloor} \alpha\left(\frac{k}{n}\right) + o(1/n).$$

We have

$$\left| \frac{1}{n} \sum_{k=1}^{\lfloor nx \rfloor} \alpha\left(\frac{k}{n}\right) - \int_0^x \alpha(u)du \right| \leq \sum_{k=1}^{\lfloor nx \rfloor} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left| \alpha\left(\frac{k}{n}\right) - \alpha(u) \right| du + \int_{\frac{\lfloor nx \rfloor}{n}}^{x} |\alpha(u)|du.$$

Since α is continuous (direct consequence of lemma 2.1) and has a bounded first derivative in $[0, 1]$ (assumption **(A2)**)

$$\sup_{x\in[0,1]} \int_{\frac{\lfloor nx \rfloor}{n}}^{x} |\alpha(u)|du = O(1/n)$$

and

$$\sup_{x\in[0,1]} \sum_{k=1}^{\lfloor nx \rfloor} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left| \alpha\left(\frac{k}{n}\right) - \alpha(u) \right| du = O(1/n).$$

□

Remark *4.1 When only assuming **(A1)** we have*

$$\sup_{x\in[0,1]} \left| E(V_n(x)) - \int_0^x \alpha(u)du \right| = o(1).$$

We set for $k = 1, 2, \ldots, n$

$$\xi_{k,n}(x) = \sqrt{n}\lambda_{k,\lfloor nx \rfloor}(\chi^2_{k,\lfloor nx \rfloor} - 1), \quad x \in [0,1],$$

and we define the corresponding truncated variables $\xi^{\epsilon}_{k,n}(x)$ as follows

$$\xi^{\epsilon}_{k,n}(x) = \xi_{k,n}(x)\mathbb{1}_{|\xi_{k,n}(x)| \leq \epsilon} + \epsilon\mathbb{1}_{\xi_{k,n}(x) > \epsilon} - \epsilon\mathbb{1}_{\xi_{k,n}(x) < -\epsilon}, \quad x \in [0,1] \text{ and } \epsilon > 0.$$

Before giving our main theorem, we need first to establish the following result whose the technical and fairly arduous proof is given in Appendix B.

***Lemma** 4.3 Assume **(A1)**-**(A2)**. Then, for all $x \in [0,1]$ as n becomes infinite*

$$(i) \quad \sum_{k=1}^{\lfloor nx \rfloor} P(|\xi_{k,n}(x)| > \epsilon) \longrightarrow 0, \quad \forall \epsilon > 0.$$

and for some $\epsilon > 0$

$$(ii) \quad \sum_{k=1}^{\lfloor nx \rfloor} E(\xi^{\epsilon}_{k,n}(x)) \longrightarrow 0,$$

$$(iii) \quad \sum_{k=1}^{\lfloor nx \rfloor} E^2(\xi^{\epsilon}_{k,n}(x)) \longrightarrow 0,$$

$$(iv) \quad \sum_{k=1}^{\lfloor nx \rfloor} E((\xi^{\epsilon}_{k,n}(x))^2) \longrightarrow 2\int_0^x \alpha^2(u)du.$$

Here is our main theorem.

***Theorem** 4.2 Assume **(A1)**-**(A2)**. Then $\left\{\sqrt{n}(vq_n(x) - \int_0^x \alpha(u)du), x \in [0,1]\right\}$ converges weakly in $C([0,1])$ to the centred Gaussian process $\left\{\int_0^x \sqrt{2}\alpha(u)dW(u), x \in [0,1]\right\}$ as $n \to \infty$.*

Proof. For any $x \in [0,1]$ we have the decomposition

$$\sqrt{n}\left(vq_n(x) - \int_0^x \alpha(u)du\right) = \sqrt{n}\left(vq_n(x) - E(vq_n(x))\right) + \sqrt{n}\left(E(vq_n(x)) - \int_0^x \alpha(u)du\right). \quad (11)$$

The second term in the right-hand side of (11) can be decomposed as follows

$$\sqrt{n}\left(E(vq_n(x)) - \int_0^x \alpha(u)du\right) = \sqrt{n}\left(E(V_n(x)) - \int_0^x \alpha(u)du\right) + \sqrt{n}\left(nx - \lfloor nx \rfloor\right) E\left(\Delta Z_{\lfloor nx \rfloor + 1}\right)^2. \quad (12)$$

According to lemma 4.2, the first term in the right-hand side of (12) converges uniformly in $[0,1]$ to 0. We have $\sup_{x\in[0,1]} |nx - \lfloor nx \rfloor| \leq 1$ with $|nx - \lfloor nx \rfloor| = 0$ for $x = 1$, and for $x \in [0,1[$ there is one $k \in [1,2,\dots,n]$ such that $\lfloor nx \rfloor + 1 = k$ and $E\left(\Delta Z_{\lfloor nx \rfloor+1}\right)^2 = a_{k,k}$ as defined by (6). As we have seen that $a_{k,k} = O(1/n)$ uniformly in k (theorem 4.1), the second term in the right hand side of (12) converges to 0 uniformly in $[0,1]$.

It remains to study the weak convergence of $\sqrt{n}\,(vq_n(x) - E(vq_n(x)))$. We have

$$\begin{aligned}\sqrt{n}\,(vq_n(x) - E(vq_n(x))) &= \sqrt{n}\,(V_n(x) - E(V_n(x))) \\ &+ \sqrt{n}\,(nx - \lfloor nx \rfloor)\left(\Delta Z_{\lfloor nx \rfloor+1}\right)^2 \\ &- \sqrt{n}\,(nx - \lfloor nx \rfloor)\,E\left(\Delta Z_{\lfloor nx \rfloor+1}\right)^2. \end{aligned} \tag{13}$$

As previously, the third term in the right hand side of (13) converges to 0 uniformly in $[0,1]$. From lemma 4.1 it follows that the second term in the right hand side of (13) converges to 0 uniformly $a.s.$ in $[0,1]$. Finally, it follows from lemma 4.3 and theorem 7.4.28 in Dacunha-Castelle and Duflo (1986) that $\left\{\sqrt{n}\,(V_n(x) - E(V_n(x)))\,, x \in [0,1]\right\}$ converges weakly to the centred Gaussian process with independent increments $\left\{\int_0^x \sqrt{2}\alpha(u)dW(u), x \in [0,1]\right\}$.

□

5. Applications

We come back to the statistical problems related to the quadratic variations.

5.1. *Estimation of a deformation model* We want to estimate the deformation Φ in the model

$$r(x,y) = R(\Phi(y) - \Phi(x)) \tag{14}$$

where R is a stationary correlation function. Under assumption on r (**(A1)**) and on Φ (**(B)**) we showed in section 3.1 that for all $x \in [0,1]$

$$\Phi(x) = \frac{\int_0^x \alpha(u)du}{\int_0^1 \alpha(u)du}.$$

An estimator of Φ is

$$\hat{\Phi}_n(x) = \frac{vq_n(x)}{vq_n(1)}. \tag{15}$$

Perrin (1997) showed the following result.

Theorem *5.1 Assume **(A1)** and **(B)**. Then*

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |\hat{\Phi}_n(x) - \Phi(x)| = 0 \quad a.s.$$

Proof. Since $\left(\hat{\Phi}_n\right)_{n\geq 1}$ is a sequence of increasing functions in $C([0,1])$, it suffices to show the pointwise $a.s.$ convergence instead of the uniform $a.s.$ convergence. Thus, by definition (15) of $\hat{\Phi}_n$, we must show that for all $x \in [0,1]$

$$\lim_{n\to\infty} vq_n(x) = \int_0^x \alpha(u)du \quad a.s.$$

For all $x \in [0,1]$ we have

$$vq_n(x) = V_n(x) + (nx - \lfloor nx \rfloor)\left(\Delta Z_{\lfloor nx \rfloor+1}\right)^2. \tag{16}$$

It follows from lemma 4.1 that the second term in the right-hand side of (16) converges $a.s.$ to 0. It remains to study

$$V_n(x) = EV_n(x) + V_n(x) - EV_n(x). \tag{17}$$

According to proposition 4.1

$$V_n(x) - EV_n(x) = \sum_{k=1}^{\lfloor nx \rfloor} \lambda_{k,\lfloor nx \rfloor}(\chi^2_{k,\lfloor nx \rfloor} - 1).$$

If we set $M_k = E(\chi^2_{k,\lfloor nx \rfloor} - 1)^k$, then

$$\begin{aligned} E(V_n(x) - EV_n(x))^4 &= M_4 \sum_{k=1}^{\lfloor nx \rfloor} \lambda^4_{k,\lfloor nx \rfloor} + M_2^2 \sum_{k=1}^{\lfloor nx \rfloor} \sum_{k'=1}^{\lfloor nx \rfloor} \lambda^2_{k,\lfloor nx \rfloor} \lambda^2_{k',\lfloor nx \rfloor} \\ &\leq M_4 \left(\sum_{k=1}^{\lfloor nx \rfloor} \lambda^2_{k,\lfloor nx \rfloor} \right)^2. \end{aligned}$$

Thus, it follows from theorem 4.1 that $E(V_n(x)-EV_n(x))^4 = O(1/n^2)$. Using Markov inequality and Borel-Cantelli lemma, we obtain that $V_n(x) - EV_n(x)$ converges $a.s.$ to 0. As remark 4.1 shows that $EV_n(x)$ converges to $\int_0^x \alpha(u)du$, the left-hand side of (17) converges to $\int_0^x \alpha(u)du$ $a.s.$

□

Hereafter, we prove the weak functional convergence.

***Corollary** 5.1 Assume **(A1)-(A2)** and **(B)**. Then*

$$\left\{\sqrt{n}(\hat{\Phi}_n(x)-\Phi(x)), x\in[0,1]\right\}$$

converges weakly in $C([0,1])$ to the centred Gaussian process

$$\left\{\frac{\int_0^x \sqrt{2}\alpha(u)dW(u)}{\int_0^1 \alpha(u)du} - \Phi(x)\frac{\int_0^1 \sqrt{2}\alpha(u)dW(u)}{\int_0^1 \alpha(u)du}, x\in[0,1]\right\}$$

as $n\to\infty$.

Proof. For all $x\in[0,1]$ we have the following decomposition

$$\hat{\Phi}_n(x)-\Phi(x) = \frac{vq_n(x)-\int_0^x \alpha(u)du}{\int_0^1 \alpha(u)du} - \hat{\Phi}_n(x)\frac{vq_n(1)-\int_0^1 \alpha(u)du}{\int_0^1 \alpha(u)du}.$$

The result follows directly from theorems 4.2 and 5.1.

□

We can now propose a test of stationarity for Gaussian processes satisfying model (14). For this, we simply test Φ is the identity function against Φ is not the identity function. The former is equivalent to $\alpha(x)$ is constant for all $x\in[0,1]$, more precisely $\alpha(x) = 2R'(0^-)$ (*cf.* section 3.1). In this case we have $\left\{\sqrt{n}(\hat{\Phi}_n(x)-x), x\in[0,1]\right\}$ converges weakly in $C([0,1])$ to the Brownian bridge $\left\{\sqrt{2}(W(x)-xW(1)), x\in[0,1]\right\}$ as $n\to\infty$. Thus, $\sqrt{n}\sup_{x\in[0,1]}|\hat{\Phi}_n(x)-x|$ converges weakly to the Kolmogorov distribution (*e.g.* Dacunha-Castelle and Duflo (1986)) $\sqrt{2}\sup_{x\in[0,1]}|W(x)-xW(1)|$. We set $D = \sup_{x\in[0,1]}|W(x)-xW(1)|$ and we recall that $P(D\le y) = 1+\sum_{k=1}^{\infty}\exp(-2k^2y^2)$ for all $y>0$. Therefore, we reject stationary hypothesis at the level of significance a if $\sqrt{n}\sup_{x\in[0,1]}|\hat{\Phi}_n(x)-x| \ge \sqrt{2}Q_{1-a}$ where Q_{1-a} is the quantile of order $1-a$ of D.

5.2. *A test on the diffusion coefficient of a diffusion process* We come back to the diffusion model $dY(x) = m(x,Y(x))dx+\sigma(x)dW(x)$, $Y(0)=y_0$ of section 3.2. Assume the following

(C1) $m(x,u)\in C^1([0,1]\times\mathbb{R})$.
(C2) $\exists\lambda>0, \forall x\in[0,1], \forall u\in\mathbb{R}, \; |m(x,u)|\le\lambda(1+|u|)$.
(C3) σ has a bounded first derivative in $[0,1]$.

Assumptions **(C1)**-**(C2)** ensure the existence and uniqueness of the solution of the diffusion (*e.g.* Ikeda and Watanabe (1989)). We recall that under assumption **(C3)** Z satisfies **(A1)** and $\alpha(x) = \sigma^2(x)$, and under assumptions **(C1)**-**(C2)** the quadratic variations of Y are asymptotically equivalent to those of Z (Genon-Catalot *et al.* (1992) and Istas (1996)). Thus, we can use the same test as before to test if σ is constant or not. More precisely, we reject that σ is constant at the level of significance a if $\sqrt{n} \sup_{x\in[0,1]} \left| \frac{vq_n(x)}{vq_n(1)} - x \right| \geq \sqrt{2} Q_{1-a}$ where Q_{1-a} is the quantile of order $1-a$ of D.

Appendix A. Proof of lemma 2.1

We set $\Delta = \{(x,y) \in [0,1]^2, x \neq y\}$. Firstly, we show that D^- is continuous in $]0,1]$. The decomposition for all $x \in]0,1]$ and all $h > 0$

$$\begin{aligned} |D^-(x+h) - D^-(x)| &\leq |r^{(0,1)}(x+h,(x+h)^-) - r^{(0,1)}(x+h,x^-)| \\ &+ |r^{(0,1)}(x+h,x^-) - r^{(0,1)}(x,x^-)| \end{aligned}$$

leads to the inequality

$$|D^-(x+h) - D^-(x)| \leq h \sup_{(x,y)\in\Delta} |r^{(0,2)}(x,y)| + h \sup_{(x,y)\in\Delta} |r^{(1,1)}(x,y)|$$

from which we deduce the continuity of D^- in $]0,1]$.

Then, it remains to prove that $\lim_{x\searrow 0} D^-(x)$ is finite. For that, we write

$$\begin{aligned} \lim_{x\searrow 0} D^-(x) &= \lim_{x\searrow 0} \lim_{h\searrow 0} r^{(0,1)}(x,x-h) \\ &= \lim_{x\searrow 0} \lim_{h\searrow 0} \left(r^{(0,1)}(x,x-h) - r^{(0,1)}(x,0) + r^{(0,1)}(x,0) \right) . \end{aligned}$$

We have

$$|r^{(0,1)}(x,x-h) - r^{(0,1)}(x,0)| \leq |x-h| \sup_{(x,y)\in\Delta} |r^{(0,2)}(x,y)|.$$

Moreover $r^{(0,1)}(x,0)$ being piecewise continuous in $[0,1]$, $\lim_{x\searrow 0} r^{(0,1)}(x,0) = r^{(0,1)}(0^+,0)$. Therefore

$$\lim_{x\searrow 0} D^-(x) = r^{(0,1)}(0^+,0).$$

A similar treatment gives D^+ continuous in $[0,1[$ and

$$\lim_{x\nearrow 1} D^+(x) = r^{(0,1)}(1^-,1).$$

In conclusion we get continuous extensions of D^- and D^+ by setting

$$\begin{aligned} D^-(0) &= r^{(0,1)}(0^+,0), \\ D^+(1) &= r^{(0,1)}(1^-,1). \end{aligned}$$

□

Appendix B. Proof of lemma 4.3

We recall that $\lambda_{1,\lfloor nx\rfloor} \geq \lambda_{2,\lfloor nx\rfloor} \geq \ldots \geq \lambda_{\lfloor nx\rfloor,\lfloor nx\rfloor} \geq 0$ are the $\lfloor nx\rfloor$ eigenvalues of the covariance matrix $L_{\lfloor nx\rfloor}\Sigma_{\lfloor nx\rfloor}L^T_{\lfloor nx\rfloor}$. Without loss of generality, we suppose hereafter that $\lambda_{1,\lfloor nx\rfloor} > 0$.
(i) We have the decomposition for any $x \in [0,1]$, for any $\epsilon > 0$ and for any $k = 1, 2, \ldots, \lfloor nx\rfloor$

$$\begin{aligned} P(|\xi_{k,n}(x)| > \epsilon) &\leq P(\sqrt{n}\lambda_{1,\lfloor nx\rfloor}|\chi^2_{k,\lfloor nx\rfloor} - 1| > \epsilon) \\ &\leq P(\chi^2_{k,\lfloor nx\rfloor} > \frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx\rfloor}} + 1) + P(\chi^2_{k,\lfloor nx\rfloor} < -\frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx\rfloor}} + 1). \end{aligned}$$

We deduce from theorem 4.1 that $\exists n_1 > 0$ such that if $n \geq n_1$

$$\lambda_{1,\lfloor nx\rfloor} \leq \frac{\epsilon}{2n} \text{ or } -\frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx\rfloor}} + 1 \leq -2\sqrt{n} + 1 < 0.$$

Hence for all $n \geq n_1$

$$P(|\xi_{k,n}(x)| > \epsilon) \leq P(\chi^2_{k,\lfloor nx\rfloor} > \frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx\rfloor}} + 1).$$

Since the density of a chi-square with one degree of freedom is $\frac{exp(-x/2)}{\sqrt{2\pi x}}$ for $x > 0$

$$\begin{aligned} P(\chi^2_{k,\lfloor nx\rfloor} > \frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx\rfloor}} + 1) &\leq \int_{\frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx\rfloor}}}^{\infty} \frac{\exp(-x/2)}{\sqrt{2\pi x}}dx \\ &\leq \frac{\sqrt{2\lambda_{1,\lfloor nx\rfloor}\sqrt{n}}}{\sqrt{\pi\epsilon}} \exp\left(-\frac{\epsilon}{2\lambda_{1,\lfloor nx\rfloor}\sqrt{n}}\right). \end{aligned}$$

Thus, for all $n \geq n_1$

$$P(|\xi_{k,n}(x)| > \epsilon) \leq \frac{1}{\sqrt{\pi}n^{1/4}} \exp(-\sqrt{n}).$$

Therefore, for any $\epsilon > 0$

$$\sum_{k=1}^{\lfloor nx\rfloor} P(|\xi_{k,n}(x)| > \epsilon) \leq \frac{n^{3/4}}{\sqrt{\pi}} \exp(-\sqrt{n}) \tag{18}$$

and $\sum_{k=1}^{\lfloor nx\rfloor} P(|\xi_{k,n}(x)| > \epsilon)$ converges to 0 as n becomes infinite.
(ii) We have for some $\epsilon > 0$

$$E(\xi^{\epsilon}_{k,n}(x)) = E(\xi_{k,n}(x)\mathbb{1}_{|\xi_{k,n}|\leq\epsilon}) + \epsilon(P(\xi_{k,n}(x) > \epsilon) - P(\xi_{k,n}(x) < -\epsilon)). \tag{19}$$

In the proof of *(i)* it has been shown that $P(\xi_{k,n}(x) < -\epsilon)$ is equal to 0 for n sufficiently large and that $\sum_{k=1}^{\lfloor nx \rfloor} P(\xi_{k,n}(x) > \epsilon)$ converges to 0 as $n \to \infty$. Then it remains to prove that

$$\lim_{n\to\infty} \sum_{k=1}^{\lfloor nx \rfloor} E(\xi_{k,n}(x)\mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon}) = 0. \tag{20}$$

We have

$$\begin{aligned}&E(\xi_{k,n}(x)\mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon})\\ &\qquad = \sqrt{n}\lambda_{k,\lfloor nx \rfloor}(E(\chi^2_{k,\lfloor nx \rfloor}\mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon}) - 1) + \sqrt{n}\lambda_{k,\lfloor nx \rfloor}P(|\xi_{k,n}(x)| > \epsilon).\end{aligned}$$

Keeping in mind that $P^2(|\xi_{k,n}(x)| > \epsilon) \leq P(|\xi_{k,n}(x)| > \epsilon)$, Cauchy-Schwarz's inequality gives

$$\sum_{k=1}^{\lfloor nx \rfloor} \sqrt{n}\lambda_{k,\lfloor nx \rfloor}P(|\xi_{k,n}(x)| > \epsilon) \leq \sqrt{\sum_{k=1}^{\lfloor nx \rfloor} n\lambda^2_{k,\lfloor nx \rfloor}}\sqrt{\sum_{k=1}^{\lfloor nx \rfloor} P(|\xi_{k,n}(x)| > \epsilon)}.$$

Theorem 4.1 implies $\sum_{k=1}^{\lfloor nx \rfloor} n\lambda^2_{k,\lfloor nx \rfloor} = O(1)$. Moreover, it follows from the previous point *(i)* that $\sum_{k=1}^{\lfloor nx \rfloor} P(|\xi_{k,n}(x)| > \epsilon)$ converges to 0 as $n \to \infty$. Using again the density of a chi-square, we get the following bound for $n \geq n_1$ with n_1 defined in *(i)*

$$\begin{aligned}|E(\chi^2_{k,\lfloor nx \rfloor}\mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon}) - 1| &\leq \frac{1}{\sqrt{2\pi}}\int_{\frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx \rfloor}}}^{\infty} \sqrt{u}\exp(-u/2)du\\ &\leq \frac{1}{\sqrt{2\pi}}\int_{\frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx \rfloor}}}^{\infty} \exp(-u/4)du \qquad .\\ &\leq \frac{4}{\sqrt{2\pi}}\exp(-\sqrt{n}/2)\end{aligned}$$

The second inequality is due to the fact that $\sqrt{u} \leq \exp(u/4)$ for all $u \geq 0$. Thus

$$|\sum_{k=1}^{\lfloor nx \rfloor} \sqrt{n}\lambda_{k,\lfloor nx \rfloor}(E(\chi^2_{k,\lfloor nx \rfloor}\mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon}) - 1)| \leq \sqrt{\frac{2}{\sqrt{\pi}}}\epsilon\sqrt{n}\exp(-\sqrt{n}/2). \tag{21}$$

Consequently, (20) is proved, as is the point *(ii)*.
(iii) From (19) we get

$$E^2(\xi^\epsilon_{k,n}(x)) \leq 2E^2(\xi_{k,n}(x)\mathbb{1}_{|\xi_{k,n}|\leq\epsilon}) + 2\epsilon^2(P(\xi_{k,n}(x) > \epsilon) - P(\xi_{k,n}(x) < -\epsilon))^2.$$

We deduce from the proof of *(i)* that

$$\sum_{k=1}^{\lfloor nx \rfloor} \epsilon^2 (P(\xi_{k,n}(x) > \epsilon) - P(\xi_{k,n}(x) < -\epsilon))^2 \leq \epsilon^2 \sum_{k=1}^{\lfloor nx \rfloor} (P(\xi_{k,n}(x) > \epsilon) + P(\xi_{k,n}(x) < -\epsilon)).$$

converges to 0 as $n \to \infty$ and from the proof of *(ii)* that

$$\begin{aligned}\sum_{k=1}^{\lfloor nx \rfloor} E^2(\xi_{k,n}(x)\mathbb{1}_{|\xi_{k,n}|\leq\epsilon}) &\leq 2\sum_{k=1}^{\lfloor nx \rfloor} n\lambda_{k,\lfloor nx \rfloor}^2 (E(\chi_{k,\lfloor nx \rfloor}^2 \mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon}) - 1)^2 \\ &+ 2\sum_{k=1}^{\lfloor nx \rfloor} n\lambda_{k,\lfloor nx \rfloor}^2 P(|\xi_{k,n}(x)| > \epsilon).\end{aligned}$$

For $n \geq n_1$ we have $n\lambda_{k,\lfloor nx \rfloor}^2 \leq \dfrac{\epsilon^2}{4n}$. Combining this inequality with (18) we obtain

$$\sum_{k=1}^{\lfloor nx \rfloor} n\lambda_{k,\lfloor nx \rfloor}^2 P(|\xi_{k,n}(x)| > \epsilon) \leq \frac{\epsilon^2}{4\sqrt{\pi}n^{1/4}} \exp(-\sqrt{n}).$$

Moreover, due to (21)

$$\begin{aligned}\sum_{k=1}^{\lfloor nx \rfloor} n\lambda_{k,\lfloor nx \rfloor}^2 (E(\chi_{k,\lfloor nx \rfloor}^2 \mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon}) - 1)^2 &\leq \left(\sum_{k=1}^{\lfloor nx \rfloor} \sqrt{n}\lambda_{k,\lfloor nx \rfloor} |E(\chi_{k,\lfloor nx \rfloor}^2 \mathbb{1}_{|\xi_{k,n}(x)|\leq\epsilon}) - 1|\right)^2 \\ &\leq \frac{2\epsilon^2}{\pi} n \exp(-\sqrt{n}).\end{aligned}$$

Therefore $E^2(\xi_{k,n}^\epsilon(x))$ converges to 0 as $n \to \infty$.

(iv) To prove that $\lim_{n\to\infty} \sum_{k=1}^{\lfloor nx \rfloor} E((\xi_{k,n}^\epsilon(x))^2) = 2\int_0^x \alpha^2(x)dx$, we proceed in three stages

- Firstly, we show that $\sum_{k=1}^{\lfloor nx \rfloor} E(\xi_{k,n}^2(x))$ converges to $2\int_0^x \alpha^2(x)dx$ as $n \to \infty$. To prove this result we start from the following equality due to proposition 4.1

$$\begin{aligned}\sum_{k=1}^{\lfloor nx \rfloor} E(\xi_{k,n}^2(x)) = 2n\sum_{k=1}^{\lfloor nx \rfloor} \lambda_{k,\lfloor nx \rfloor}^2 &= nVar(V_n(x)) \\ &= n(E(V_n^2(x)) - (EV_n(x))^2).\end{aligned}$$

  Let us recall the definition of $a_{j,k}$, $(j,k) \in [1,\ldots,n]^2$

$$a_{j,k} = r\left(\frac{j}{n},\frac{k}{n}\right) + r\left(\frac{j-1}{n},\frac{k-1}{n}\right) - r\left(\frac{j}{n},\frac{k-1}{n}\right) - r\left(\frac{j-1}{n},\frac{k}{n}\right).$$

We have

$$
\begin{aligned}
E\left(V_n(x)\right) &= \sum_{k=1}^{\lfloor nx\rfloor} a_{k,k}, \\
E\left(V_n^2(x)\right) &= 3\sum_{k=1}^{\lfloor nx\rfloor} a_{k,k}^2 + 2\sum_{k=1}^{\lfloor nx\rfloor}\sum_{j>k}\left(a_{k,k}a_{j,j} + 2a_{j,k}^2\right).
\end{aligned}
$$

Then

$$
\sum_{k=1}^{\lfloor nx\rfloor} E(\xi_{k,n}^2(x)) = 2n\sum_{k=1}^{\lfloor nx\rfloor}\sum_{j=1}^{\lfloor nx\rfloor} a_{j,k}^2 = 2n\sum_{k=1}^{\lfloor nx\rfloor} a_{k,k}^2 + 4n\sum_{k=1}^{\lfloor nx\rfloor}\sum_{j>k} a_{j,k}^2. \tag{22}
$$

In the proof of theorem 4.1, we obtained (*cf.* (8) and (9))

$$
\left\{
\begin{aligned}
a_{k,k} &= \frac{1}{n}\alpha\left(\frac{k}{n}\right) + o(1/n), \quad k\in[1,\dots,n], \\
|a_{j,k}| &= O(1/n^2), \quad j\neq k.
\end{aligned}
\right.
$$

Thus, the second term in the right-hand side of (22) converges to 0 as $n\to\infty$ and

$$
2n\sum_{k=1}^{\lfloor nx\rfloor} a_{k,k}^2 = \frac{2}{n}\sum_{k=1}^{\lfloor nx\rfloor}\alpha_0^2\left(\frac{k}{n}\right) + o(1).
$$

Since α is Riemann integrable in $[0,1]$, $2n\sum_{k=1}^{\lfloor nx\rfloor} a_{k,k}^2$ converges to $2\int_0^x \alpha^2(x)dx$ as $n\to\infty$.

- Then we show that $\sum_{k=1}^{\lfloor nx\rfloor} E(\xi_{k,n}^2(x)\mathbb{1}_{|\xi_{k,n}(x)|\le\epsilon}) - \sum_{k=1}^{\lfloor nx\rfloor} E(\xi_{k,n}^2(x))$ converges to 0 as $n\to\infty$. We have for $n\ge n_1$

$$
|E(\xi_{k,n}^2(x)\mathbb{1}_{|\xi_{k,n}(x)|\le\epsilon}) - E(\xi_{k,n}^2(x))| \le \frac{n\lambda_{1,\lfloor nx\rfloor}^2}{\sqrt{2\pi}}\int_{\frac{\epsilon}{\sqrt{n}\lambda_{1,\lfloor nx\rfloor}}}^{\infty}\frac{u^2}{\sqrt{u}}\exp(-u/2)du.
$$

Thus, we obtain for n large enough

$$
|E(\xi_{k,n}^2(x)\mathbb{1}_{|\xi_{k,n}(x)|\le\epsilon}) - E(\xi_{k,n}^2(x))| \le \frac{\epsilon^2}{\sqrt{2\pi}n}\exp(-\sqrt{n}/2).
$$

Therefore as $n\to\infty$

$$
\sum_{k=1}^{\lfloor nx\rfloor}|E(\xi_{k,n}^2(x)\mathbb{1}_{|\xi_{k,n}(x)|\le\epsilon}) - E(\xi_{k,n}^2(x))| \le \frac{\epsilon^2}{\sqrt{2\pi}}\exp(-\sqrt{n}/2).
$$

- Finally, we use the following decomposition

$$\sum_{k=1}^{\lfloor nx\rfloor} E((\xi^{\epsilon}_{k,n}(x))^2) = \sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x)\mathbb{1}_{|\xi_{k,n}(x)|\le\epsilon}) + \epsilon\sum_{k=1}^{\lfloor nx\rfloor}(P(\xi_{k,n}(x)>\epsilon)-P(\xi_{k,n}(x)<-\epsilon)) \quad (23)$$

As already shown, the second term in the right-hand side of (23) converges to 0 as $n\to\infty$ and we use the decomposition

$$\sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x)\mathbb{1}_{|\xi_{k,n}(x)|\le\epsilon}) = \sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x)) + \left(\sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x)\mathbb{1}_{|\xi_{k,n}(x)|\le\epsilon}) - \sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x))\right).$$

According to the first stage, $\sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x))$ converges to $2\int_0^x \alpha^2(x)dx$, and according to the second stage $\left(\sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x)\mathbb{1}_{|\xi_{k,n}|\le\epsilon}) - \sum_{k=1}^{\lfloor nx\rfloor} E(\xi^2_{k,n}(x))\right)$ converges to 0 as $n\to\infty$. Therefore, we can conclude that the left-hand side of (23) converges to $2\int_0^x \alpha^2(x)dx$ as $n\to\infty$.

□

Acknowledgement

This work was done while OLIVIER PERRIN was visiting the department of Mathematical Statistics at Chalmers University of Technology and Göteborg University in Sweden. This visit was partially supported by the European Union's research network "Statistical and Computational Methods for the Analysis of Spatial Data", grant ERB-FMRX-CT96-0096.

References

[1] R.L. Adler and R. Pyke, Uniform quadratic variation of Gaussian processes, Stochastic Processes and their Applications 48 (1993) 191-209.

[2] G. Baxter, A strong limit theorem for Gaussian processes, Proceedings of the American Mathematical Society 7 (1956) 522-527.

[3] P. Brugière, Estimation de la variance d'une diffusion, C. R. Acad. Sci. Tome 312 Série I (1991) 999-1004.

[4] D. Dacunha-Castelle and M. Duflo, Probability and Statistiques, Volume II Springer-Verlag (1986).

[5] V. Genon-Catalot, C. Laredo and D. Picard, Non-parametric estimation of the diffusion coefficient by wavelets methods, Scand. J. of Statist 19 (1992) 317-335.

[6] E.G. Gladyshev, A new limit theorem for processes with Gaussian increments, Theory Probab. Appl. 6 (1961) 52-61.

[7] X. Guyon, Variations de champs gaussiens stationnaires : application à l'identification, Probability Theory and Related Fields 75 (1987) 179-193.

[8] X. Guyon and J.R. León, Convergence en loi des H-variations d'un processus gaussien stationnaire sur $\mathbb{R}$, Annales de l'Institut Henri Poincaré 25 (1989) 265-282.

[9] R.A. Horn and C.R. Johnson, Topics in matrix analysis, Cambridge University Press (1991).

[10] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland Amsterdam (1989).

[11] J. Istas, Estimating the singularity fonction of a Gaussian process with applications, Scandinavian Journal of Statistics 23 (1996) 581-596.

[12] J. Istas and G. Lang, Quadratic variations and estimation of the local Hölder index of a Gaussian process, Annales de l'Institut Henri Poincaré 33 4 (1997) 407-436.

[13] N.L. Johnson and S. Kotz (1970), Continuous univariate distributions-2, Houghton Mifflin Company (1970).

[14] J.R. León and J. Ortega, Weak convergence of different types of variations for biparametric Gaussian processes, in: Colloquia Math. Soc. J. Bolyai 57, Limit theorem in Proba. and Stat. (Pecs, Hungary 1989) pp. 349-364.

[15] P. Lévy, Le mouvement brownien plan, Amer. J. Math 62 (1940) 487-550.

[16] J. Neveu, Bases mathématiques du calcul des probabilités, Masson 2nd éd (1980).

[17] O. Perrin, Modèle de covariance d'un Processus Non-Stationnaire par Déformation de l'Espace et Statistique, Ph.D. thesis, Université de Paris I Panthéon-Sorbonne, Paris (1997).

[18] O. Perrin and R. Senoussi (a), Reducing Non-stationary processes to stationarity by a time deformation, Preprint, No. 35, Dept. of Mathematics, Chalmers University of Technologie and Göteborg University (Göteborg, Sweden, 1998).

[19] O. Perrin and R. Senoussi (b), Reducing Non-stationary spatial random fields to stationarity and isotropy by a space deformation, Preprint, Dept. of Mathematics, Chalmers University of Technologie and Göteborg University (Göteborg, Sweden, 1998).

[20] P.D. Sampson and P. Guttorp, Nonparametric estimation of nonstationary spatial covariance structure, Journal of the American Statistical Association 87 (1992) 108-119.

[21] P. Soulier, Déviations quadratiques pour des estimateurs de la variance d'une diffusion, C. R. Acad. Sci. Tome 313 Série I (1991) 783-786.