Cotangent cohomology of rational surface singularities

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Abstract

In this paper we show that the number of generators of the cotangent cohomology groups T^n_Y, $n \geq 2$, is the same for all rational surface singularities Y. For a large class of rational surface singularities, including quotient singularities, this number is also the dimension. For them we obtain an explicit formula for the Poincaré series $P_Y(t) = \sum \dim T^n_Y \cdot t^n$. In the special case of the cone over the rational normal curve we give the multigraded Poincaré series.

1. Introduction

The cotangent cohomology groups T^n with small n play an important role in the deformation theory of singularities: T^1 classifies infinitesimal deformations and the obstructions land in T^2. Originally constructed ad hoc, the correct way to obtain these groups is as the cohomology of the cotangent complex. This yields also higher T^n, which no longer have a direct meaning in terms of deformations.

In this paper we study these higher cohomology groups T^n_Y for rational surface singularities Y. For a large class of rational surface singularities, including quotient singularities, we obtain their dimension. For an explicit formula for the Poincaré series $P_Y(t)$, see (5.3).

Our methods are a combination of the following three items:

(1) We use the hyperplane section machinery of [Behnke–Christophersen] to move freely between surface singularities, partition curves, and fat points. It suffices to compute the cohomology groups T^n_Y for special singularities, to obtain the number of generators for all rational surface singularities.

(2) In many cases, cotangent cohomology may be obtained via Harrison cohomology, which is much easier to handle. Using a Noether normalisation the Harrison complex gets linear over a bigger ring than just $\mathbb{C}$ (which is our ground field).

(3) Taking for Y a cone over a rational normal curve, we may use the explicit description of T^n_Y obtained in [Altmann–Sletsjøe] by toric methods.

The descriptions in (2) and (3) complement each other and show that T^n_Y of the cone over the rational normal curve is concentrated in degree $-n$. This allows us to compute the dimension as Euler characteristic.

The paper is organised as follows. After recalling the definitions of cotangent and Harrison cohomology we review its computation for the case of the fat point of minimal multiplicity and give the

explicit formula for its Poincaré series (we are indebted to Duco van Straten and Ragnar Buchweitz for help on this point).
Section 3 describes the applications of Noether normalisation to the computation of Harrison cohomology. The main result is the degree bound for the cotangent cohomology of Cohen-Macaulay singularities of minimal multiplicity from below, cf. Corollary (3.4)(2).
In the next section toric methods are used to deal with the cone over the rational normal curve. In this special case we can bound the degree of the cohomology groups from above, too. As a consequence, we obtain complete information about the Poincaré series.
Finally, using these results as input for the hyperplane machinery we find in the last section the Poincaré series for the partition curves and obtain that their T^n is annihilated by the maximal ideal. This then implies that the number of generators of T^n_Y is the same for all rational surface singularities.

Notation: We would like to give the following guide line concerning the notation for Poincaré series. The symbol Q denotes those series involving Harrison cohomology of the actual space or ring with values in $\mathbb{C}$, while P always points to the usual cotangent cohomology of the space itself. Moreover, if these letters come with a tilde, then a finer grading than the usual $\mathbb{Z}$-grading is involved.

2. Cotangent cohomology and Harrison cohomology

(2.1) Let A be a commutative algebra of essentially finite type over a base-ring S. For any A-module M, one gets the *André–Quillen* or *cotangent cohomology groups* as

$$T^n(A/S, M) := H^n\big(\mathrm{Hom}_A(\mathbb{L}_*^{A/S}, M)\big)$$

with $\mathbb{L}_*^{A/S}$ being the so-called cotangent complex. We are going to recall the major properties of this cohomology theory. For the details, including the definition of $\mathbb{L}_*^{A/S}$, see [Loday].

If A is a smooth S-algebra, then $T^n(A/S, M) = 0$ for $n \geq 1$ and all A-modules M. For general A, a short exact sequence of A-modules gives a long exact sequence in cotangent cohomology. Moreover, the Zariski-Jacobi sequence takes care of ring homomorphisms $S \to A \to B$; for a B-module M it looks like

$$\cdots \longrightarrow T^n(B/A, M) \longrightarrow T^n(B/S, M) \longrightarrow T^n(A/S, M) \longrightarrow T^{n+1}(B/A, M) \longrightarrow \cdots$$

The cotangent cohomology behaves well under base change. Given a co-cartesian diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle\phi}\big\uparrow & & \big\uparrow \\ S & \longrightarrow & S' \end{array}$$

with ϕ flat, and an A'-module M', there is a natural isomorphism

$$T^n(A'/S', M') \cong T^n(A/S, M')\,.$$

If, moreover, S' is a flat S-module, then for any A-module M

$$T^n(A'/S', M \otimes_S S') \cong T^n(A/S, M) \otimes_S S'\,.$$

(2.2) To describe Harrison cohomology, we first recall Hochschild cohomology. While this concept works also for non-commutative unital algebras, we assume here the same setting as before. For an A-module M, we consider the complex

$$C^n(A/S, M) := \operatorname{Hom}_S(A^{\otimes n}, M)$$

with differential

$$(\delta f)(a_0, \ldots, a_n) := \\ a_0 f(a_1, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \ldots, a_{i-1}a_i, \ldots, a_n) + (-1)^{n+1} a_n f(a_0, \ldots, a_{n-1}) .$$

Hochschild cohomology $HH^n(A/S, M)$ is the cohomology of this complex. It can also be computed from the so-called *reduced* subcomplex $\overline{C}^\bullet(A/S, M)$ consisting only of those maps $f: A^{\otimes n} \to M$ that vanish whenever at least one of the arguments equals 1.

Definition: A permutation $\sigma \in S_n$ is called a $(p, n-p)$-shuffle if $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(n)$. Moreover, in the group algebra $\mathbb{Z}[S_n]$ we define the elements

$$\operatorname{sh}_{p,n-p} := \sum_{(p,\, n-p)\text{-shuffles}} \operatorname{sgn}(\sigma)\, \sigma \qquad \text{and} \qquad \operatorname{sh} := \sum_{p=1}^{n-1} \operatorname{sh}_{p,n-p} .$$

The latter element $\operatorname{sh} \in \mathbb{Z}[S_n]$ gives rise to the so-called shuffle invariant subcomplexes

$$C^n_{\text{sh}}(A/S, M) := \left\{ f \in \operatorname{Hom}_S(A^{\otimes n}, M) \;\middle|\; f(\operatorname{sh}(\underline{a})) = 0 \ \text{ for every } \ \underline{a} \in A^{\otimes n} \right\} \subset C^n(A/S, M)$$

and $\overline{C}^n_{\text{sh}}(A/S, M) \subset \overline{C}^n(A/S, M)$ defined in the same manner. Both complexes yield the same cohomology, which is called *Harrison cohomology*:

$$\operatorname{Harr}^n(A/S, M) := H^n\big(C^\bullet_{\text{sh}}(A/S, M)\big) = H^n\big(\overline{C}^\bullet_{\text{sh}}(A/S, M)\big) .$$

(2.3) The following well known result compares the cohomology theories defined so far. Good references are [Loday] or [Palamodov].

Theorem: *If $\mathbb{Q} \subset S$, then Harrison cohomology is a direct summand of Hochschild cohomology. Moreover, if A is a flat S-module, then*

$$T^n(A/S, M) \cong \operatorname{Harr}^{n+1}(A/S, M) .$$

(2.4) As an example, we consider the fat point Z_m ($m \geq 2$) with minimal multiplicity $d = m+1$. Let V be an m-dimensional $\mathbb{C}$-vector space and let $A = \mathcal{O}_{Z_m}$ be the ring $\mathbb{C} \oplus V$ with trivial multiplication $V^2 = 0$. First we compute the Hochschild cohomology $HH^\bullet(A/\mathbb{C}, A)$. The reduced complex is

$$\overline{C}^n(A/\mathbb{C}, A) = \operatorname{Hom}_{\mathbb{C}}(V^{\otimes n}, A) .$$

Because $ab = 0 \in A$ for all $a, b \in V$, the differential reduces to

$$(\delta f)(a_0, \ldots, a_n) = a_0 f(a_1, \ldots, a_n) + (-1)^{n+1} a_n f(a_0, \ldots, a_{n-1}) .$$

We conclude that $\delta f = 0$ if and only if $\operatorname{im} f \subset V$; hence

$$HH^n(A/\mathbb{C}, A) = \operatorname{Hom}\left(V^{\otimes n}, V\right) \Big/ \delta \operatorname{Hom}\left(V^{\otimes (n-1)}, \mathbb{C}\right) .$$

On the complex $\overline{C}^n(A/\mathbb{C},\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V^{\otimes n},\mathbb{C})$ the differential is trivial, so $\mathrm{Hom}\,(V^{\otimes n},\mathbb{C}) = HH^n(A/\mathbb{C},\mathbb{C})$. We finally obtain

$$HH^n(A/\mathbb{C},A) = HH^n(A/\mathbb{C},\mathbb{C})\otimes V \Big/ \delta_*\, HH^{n-1}(A/\mathbb{C},\mathbb{C})\;,$$

where the map δ_* is injective. For the Harrison cohomology, one has to add again the condition of shuffle invariance:

$$\mathrm{Harr}^n(A/\mathbb{C},A) = \mathrm{Harr}^n(A/\mathbb{C},\mathbb{C})\otimes V \Big/ \delta_*\ \mathrm{Harr}^{n-1}(A/\mathbb{C},\mathbb{C})\,.$$

Proposition: ([Schlessinger–Stasheff]) *Identifying the Hochschild cohomology $HH^\bullet(A/\mathbb{C},\mathbb{C})$ with the tensor algebra TV^* on the dual vector space V^*, the Harrison cohomology $\mathrm{Harr}^\bullet(A/\mathbb{C},\mathbb{C})\subset TV^*$ consists of the primitive elements in TV^*. They form a free graded Lie algebra L on V^* with V^* sitting in degree -1.*

Proof: The tensor algebra TV^* is a Hopf algebra with comultiplication

$$\Delta(x_1\otimes\cdots\otimes x_n) := \sum_p \sum_{(p,n-p)-\text{shuffles }\sigma} \mathrm{sgn}(\sigma)\,\big(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(p)}\big)\otimes\big(x_{\sigma(p+1)}\otimes\cdots\otimes x_{\sigma(n)}\big)\,.$$

It is the dual of the Hopf algebra T^cV with shuffle multiplication

$$(v_1\otimes\cdots\otimes v_p)*(v_{p+1}\otimes\cdots\otimes v_n) = \sum_{(p,n-p)-\text{shuffles }\sigma} \mathrm{sgn}(\sigma)\cdot v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}\,.$$

In particular, for any $f\in TV^*$ and $a,b\in T^cV$ one has $(\Delta f)(a,b) = f(a*b)$. Hence, the condition that f vanishes on shuffles is equivalent to $\Delta f = f\otimes 1 + 1\otimes f$, i.e. to f being primitive in TV^*. □

The dimension of $\mathrm{Harr}^n(A/\mathbb{C},\mathbb{C})$ follows now from the dimension of the space of homogeneous elements in the free Lie algebra, which was first computed in the graded case in [Ree].

Lemma: $\dim_{\mathbb{C}} \mathrm{Harr}^n(A/\mathbb{C},\mathbb{C}) = \frac{1}{n}\sum_{d|n}(-1)^{n+\frac{n}{d}}\mu(d)\,m^{\frac{n}{d}}$ *with μ denoting the Möbius function.*

Proof: In the free Lie algebra L on V^*, we choose an ordered basis p_i of the even degree homogeneous parts $L_{2\bullet}$ as well as an ordered basis q_i of the odd degree ones. Since TV^* is the universal enveloping algebra of L, a basis for TV^* is given by the elements of the form $p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}q_1^{s_1}\cdots q_l^{s_l}$ with $r_i\geq 0$ and $s_i = 0,1$. In particular, if $c_n := \dim L_{-n}$, then the Poincaré series of the tensor algebra

$$\sum_n \dim T^nV^*\cdot t^n = \sum_n m^n\,t^n = \frac{1}{1-mt}$$

may be alternatively described as

$$\prod_{n\text{ even}}(1+t^n+t^{2n}+\cdots)^{c_n}\prod_{n\text{ odd}}(1+t^n)^{c_n}\,.$$

Replacing t by $-t$ and taking logarithms, the comparison of both expressions yields

$$-\log(1+mt) \;=\; -\sum_{n\text{ even}} c_n\log(1-t^n) + \sum_{n\text{ odd}} c_n\log(1-t^n) \;=\; -\sum_n(-1)^n c_n\log(1-t^n)\,.$$

Hence

$$\sum_n \frac{1}{n}\,(-m)^n\,t^n = \sum_{d,\nu}(-1)^d\frac{1}{\nu}\,c_d\,t^{d\,\nu}\,,$$

and by comparing the coefficients we find

$$(-m)^n = \sum_{d|n}(-1)^d\,d\,c_d\,.$$

Now the result follows via Möbius inversion. □

We collect the dimensions in the Poincaré series

$$Q_{Z_m}(t) := \sum_{n\geq 1} \dim \operatorname{Harr}^n(\mathbb{C}\oplus V/\mathbb{C}, \mathbb{C}) \cdot t^n = \sum_{n\geq 1} c_n t^n \;.$$

3. Harrison cohomology via Noether normalisation

(3.1) Let Y be a Cohen-Macaulay singularity of dimension N and multiplicity d; denote by A its local ring. Choosing a *Noether normalisation*, i.e. a flat map $Y \to \mathbb{C}^N$ of degree d, provides a regular local ring P of dimension N and a homomorphism $P \to A$ turning A into a free P-module of rank d. Strictly speaking, this might only be possible after passing to an étale covering. Alternatively one can work in the analytic category, see [Palamodov] for the definition of analytic Harrison cohomology.

Proposition: *Let A be a free P-module as above. If M is any A-module, then $T^n(A/\mathbb{C}, M) \cong T^n(A/P, M)$ for $n \geq 2$. Moreover, the latter equals* $\operatorname{Harr}^{n+1}(A/P, M)$.

Proof: The Zariski-Jacobi sequence from (2.1) for $\mathbb{C} \to P \to A$ reads

$$\cdots \longrightarrow T^n(A/P, M) \longrightarrow T^n(A/\mathbb{C}, M) \longrightarrow T^n(P/\mathbb{C}, M) \longrightarrow T^{n+1}(A/P, M) \longrightarrow \cdots$$

As P is regular, we have $T^n(P/\mathbb{C}, M) = 0$ for $n \geq 1$ for all P-modules. On the other hand, since A is flat over P, we may use (2.3). □

(3.2) A rational surface singularity has minimal multiplicity, in the sense that $\operatorname{embdim} Y = \operatorname{mult} Y + \dim Y - 1$. In this situation we may choose coordinates $(z_1, \ldots, z_{d+1})$ such that the projection on the (z_d, z_{d+1})-plane is a Noether normalisation. Using the above language, this means that $P = \mathbb{C}[z_d, z_{d+1}]_{(z_d, z_{d+1})}$, and $\{1, z_1, \ldots, z_{d-1}\}$ provides a basis of A as a P-module.

More generally, for a Cohen-Macaulay singularity of minimal multiplicity we may take coordinates $(z_1, \ldots, z_{d+N-1})$ such that projection on the last N coordinates $(z_d, \ldots, z_{d+N-1})$ is a Noether normalisation.

Lemma: $\underline{m}_A^2 \subset \underline{m}_P \cdot \underline{m}_A$ *and* $(\underline{m}_P \cdot \underline{m}_A) \cap P \subset \underline{m}_P^2$.

Proof: Every product $z_i z_j \in \underline{m}_A^2$ may be decomposed as $z_i z_j = p_0 + \sum_{v=1}^{d-1} p_v z_v$ with some $p_v \in P$. Since $\{z_1, \ldots, z_{d+N-1}\}$ is a basis of $\underline{m}_A / \underline{m}_A^2$, we obtain $p_0 \in \underline{m}_P^2$ and $p_v \in \underline{m}_P$ for $v \geq 1$. The second inclusion follows from the fact that $z_1, \ldots, z_{d-1} \in A$ are linearly independent over P. □

Proposition: *For a Cohen-Macaulay singularity of minimal multiplicity d one has for $n \geq 1$ $T^n(A/\mathbb{C}, \mathbb{C}) = T^n(A/P, \mathbb{C}) = T^n(\mathbb{C}\oplus V/\mathbb{C}, \mathbb{C})$ with $V := \underline{m}_A / \underline{m}_P A$ being the $(d-1)$-dimensional vector space spanned by $z_1, \ldots, z_{d-1}$.*

Proof: The equality $T^n(A/\mathbb{C}, \mathbb{C}) = T^n(A/P, \mathbb{C})$ was already the subject of Proposition (3.1) with $M := \mathbb{C}$; it remains to treat the missing case of $n = 1$. Using again the Zariski-Jacobi sequence, we have to show that $T^0(A/\mathbb{C}, \mathbb{C}) \to T^0(P/\mathbb{C}, \mathbb{C})$ is surjective. However, since this map is dual to the homomorphism $\underline{m}_P / \underline{m}_P^2 \to \underline{m}_A / \underline{m}_A^2$, which is injective by the lemma above, we are done.
The second equality $T^n(A/P, \mathbb{C}) = T^n(\mathbb{C}\oplus V/\mathbb{C}, \mathbb{C})$ follows by base change, cf. (2.1):

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{C}\oplus V \\ {\scriptstyle \text{flat}}\uparrow & & \uparrow \\ P & \longrightarrow & \mathbb{C} \end{array}$$

□

(3.3) The previous proposition reduces the cotangent cohomology with $\mathbb{C}$-coefficients of rational surface singularities of multiplicity d to that of the fat point Z_m with $m = d-1$, discussed in (2.4).

Example: Denote by Y_d the cone over the rational normal curve of degree d. It may be described by the equations encoded in the condition

$$\operatorname{rank} \begin{pmatrix} z_0 & z_1 & \dots & z_{d-2} & z_{d-1} \\ z_1 & z_2 & \dots & z_{d-1} & z_d \end{pmatrix} \le 1 \, .$$

As Noether normalisation we take the projection on the (z_0, z_d)-plane. With $\deg z_i := [i, 1] \in \mathbb{Z}^2$, the local ring A_d of Y_d admits a $\mathbb{Z}^2$-grading. We would like to show how this grading affects the modules $T^\bullet(A_d/\mathbb{C}, \mathbb{C}) = T^\bullet(\mathbb{C} \oplus V/\mathbb{C}, \mathbb{C})$ (excluding T^0), i.e. we are going to determine the dimensions $\dim T^\bullet(\mathbb{C} \oplus V/\mathbb{C}, \mathbb{C})(-R)$ for $R \in \mathbb{Z}^2$.
We know that for every n

$$T^{n-1}(\mathbb{C} \oplus V/\mathbb{C}, \mathbb{C})(-R) = \operatorname{Harr}^n(\mathbb{C} \oplus V/\mathbb{C}, \mathbb{C})(-R) \subset T^n V^*(-R) = 0$$

unless $n = \operatorname{ht}(R)$, where $\operatorname{ht}(R) := R_2$ denotes the part carrying the standard $\mathbb{Z}$-grading. Hence, we just need to calculate the numbers

$$c_R := \dim \operatorname{Harr}^{\operatorname{ht}(R)}(\mathbb{C} \oplus V/\mathbb{C}, \mathbb{C})(-R)$$

and can proceed as in the proof of Proposition (2.4). Via the formal power series

$$\sum_{R \in \mathbb{Z}^2} \dim T^{\operatorname{ht}(R)} V^*(-R) \cdot x^R \in \mathbb{C}[|\mathbb{Z}^2|]$$

we obtain the equation

$$-\log\left(1 + x^{[1,1]} + \dots + x^{[d-1,1]}\right) = - \sum_{R \in \mathbb{Z}^2} (-1)^{\operatorname{ht}(R)} \, c_R \cdot \log(1 - x^R) \, .$$

In particular, if $\operatorname{ht}(R) = n$, then the coefficient of x^R in

$$(-1)^n \left(x^{[1,1]} + \dots + x^{[d-1,1]}\right)^n = (-1)^n \left(\frac{x^{[d,1]} - x^{[1,1]}}{x^{[1,0]} - 1}\right)^n$$

equals $\sum_{R'|R} (-1)^{\operatorname{ht}(R')} \operatorname{ht}(R') \cdot c_{R'}$. Again, we have to use Möbius inversion to obtain an explicit formula for the dimensions c_R.

Remarks:

(1) The multigraded Poincaré series

$$\widetilde{Q}_{Z_{d-1}}(x) := \sum_{R \in \mathbb{Z}^2} \dim \operatorname{Harr}^{\operatorname{ht}(R)}(\mathbb{C} \oplus V/\mathbb{C}, \mathbb{C})(-R) \cdot x^R = \sum_R c_R \, x^R$$

is contained in the completion of the semigroup ring $\mathbb{C}\left[\mathbb{Z}_{\ge 0}\,[1,1] + \mathbb{Z}_{\ge 0}\,[d-1,1]\right]$.

(2) The cohomology groups $\operatorname{Harr}^n(A_d/\mathbb{C}, \mathbb{C})(-R)$ vanish unless $n = \operatorname{ht}(R)$, even for $n = 1$. The corresponding Poincaré series $\widetilde{Q}_{Y_d}(x)$ equals $\widetilde{Q}_{Z_{d-1}}(x) + x^{[0,1]} + x^{[d,1]}$. The two additional terms arise from $\operatorname{Harr}^1(P/\mathbb{C}, \mathbb{C}) = T^0(P/\mathbb{C}, \mathbb{C})$ in the exact sequence of (3.1).

(3.4) Let Y be a Cohen-Macaulay singularity of minimal multiplicity $d \ge 3$.

Lemma: *The natural map $T^n(A/P, A) \to T^n(A/P, \mathbb{C})$ is the zero map.*

Proof: We compute $T^n(A/P, \bullet)$ with the reduced Harrison complex which sits in the reduced Hochschild complex. Using the notation of the beginning of (3.2), a reduced Hochschild $(n+1)$-cocycle f is, by P-linearity, determined by its values on the $(n+1)$-tuples of the coordinates z_1, $\dots$, z_{d-1}. Suppose $f(z_{i_0}, \dots, z_{i_n}) \notin \underline{m}_A$. Since $d \geq 3$, we may choose a z_k with $k \in \{1, \dots, d-1\}$ and $k \neq i_0$. Hence

$$0 = (\delta f)(z_{i_0}, \dots, z_{i_n}, z_k) = z_{i_0} f(z_{i_1}, \dots, z_k) \pm f(z_{i_0}, \dots, z_{i_n}) z_k$$
$$+ \text{ terms containing products } z_i z_j \text{ as arguments}.$$

Since $\underline{m}_A^2 \subset \underline{m}_P \cdot \underline{m}_A$ by Lemma (3.2), we may again apply P-linearity to see that the latter terms are contained in $\underline{m}_P \cdot A$. Hence, modulo $\underline{m}_P = \underline{m}_P + \underline{m}_A^2$, these terms vanish, but the resulting equation inside $V = \underline{m}_A / \underline{m}_P A$ contradicts the fact that z_{i_0} and z_k are linearly independent. $\square$

Corollary:

(1) *The map $T^n(A/P, \underline{m}_A) \to T^n(A/P, A)$ is surjective. In particular, every element of the group $T^n(A/P, A)$ may be represented by a cocycle $f: A^{\otimes(n+1)} \to \underline{m}_A$.*

(2) *If $P \to A$ is $\mathbb{Z}$-graded with $\deg z_i = 1$ for every i (such as for the cone over the rational normal curve presented in Example (3.3)), then $T^n(A/P, A)$ sits in degree $\geq -n$.*

4. The cone over the rational normal curve

(4.1) Let Y_d be the cone over the rational normal curve of degree $d \geq 3$. In Example (3.3) we have calculated the multigraded Poincaré series $\widetilde{Q}_{Y_d}(x) = \sum_R \dim \operatorname{Harr}^{\operatorname{ht}(R)}(A_d/\mathbb{C}, \mathbb{C})(-R) \cdot x^R$. The usual Poincaré series $Q_{Y_d}(t) = \sum_{n \geq 1} \dim \operatorname{Harr}^n(A_d/\mathbb{C}, \mathbb{C}) \cdot t^n$ is related to it via the substitution $x^R \mapsto t^{\operatorname{ht}(R)}$.

The goal of the present section is to obtain information about

$$P_{Y_d}(t) := \sum_{n \geq 1} \dim T^n(A_d/\mathbb{C}, A_d)(-R) \cdot t^n$$

or its multi graded version

$$\widetilde{P}_{Y_d}(x,t) := \sum_{n \geq 1} \sum_{R \in \mathbb{Z}^2} \dim T^n(A_d/\mathbb{C}, A_d)(-R) \cdot x^R t^n \in \mathbb{C}[|\mathbb{Z}^2|][|t|].$$

The first series may be obtained from the latter by substituting 1 for all monomials x^R, i.e. $P_{Y_d}(t) = \widetilde{P}_{Y_d}(1,t)$.

(4.2) In [Altmann–Sletsjøe], Proposition (5.2), combinatorial formulas have been obtained for the dimension of the vector spaces $T^n(-R) := T^n(A_d/\mathbb{C}, A_d)(-R)$. The point is that Y_d equals the affine toric variety Y_σ with σ the plane polyhedral cone

$$\sigma := \mathbb{R}_{\geq 0} \cdot (1,0) + \mathbb{R}_{\geq 0} \cdot (-1,d) = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0;\ dx + y \geq 0\} \subset \mathbb{R}^2.$$

The lattice containing the multidegrees R may be identified with the dual of the lattice $\mathbb{Z}^2$ inside $\mathbb{R}^2$, and the results of [Altmann–Sletsjøe] for this special cone may be described as follows:

(0) $T^0(-R)$ is two-dimensional if $R \leq 0$ on σ. It has dimension 1 if R is still non-positive on one of the σ-generators $(1,0)$ or $(-1,d)$, but yields exactly 1 at the other one. $T^0(-R)$ vanishes in every other case.

(1) $T^1(-R)$ is one-dimensional for $R = [1,1]$ and $R = [d-1,1]$; it is two-dimensional for the degrees in between, i.e. for $R = [2,1], \ldots, [d-2,1]$. Altogether this means that $\dim T^1 = 2d-4$.

(2) The vector space T^2 lives exclusively in the degrees of height two. More detailed, we have

$$\dim T^2(-R) = \begin{cases} k-2 & \text{for } R = [k,2] \text{ with } 2 \le k \le d-1 \\ d-3 & \text{for } R = [d,2] \\ 2d-k-2 & \text{for } R = [k,2] \text{ with } d+1 \le k \le 2d-2\,. \end{cases}$$

To formulate the result for the higher cohomology groups, we need some additional notation. If $R \in \mathbb{Z}^2$, then let K_R be the finite set

$$K_R := \{r \in \mathbb{Z}^2 \setminus \{0\} \,|\, r \ge 0 \text{ on } \sigma, \text{ but } r < R \text{ on } \sigma \setminus \{0\}\}\,.$$

Every such set $K \subset \mathbb{Z}^2$ gives rise to a complex $C^\bullet(K)$ with

$$C^n(K) := \left\{\varphi\colon \{(\lambda_1, \ldots, \lambda_n) \in K^n \,|\, \textstyle\sum_v \lambda_v \in K\} \to \mathbb{C} \,\middle|\, \varphi \text{ is shuffle invariant}\right\},$$

equipped with the modified, inhomogeneous Hochschild differential $d\colon C^n(K) \to C^{n+1}(K)$ given by

$$(d\varphi)(\lambda_0, \ldots, \lambda_n) := \\ \varphi(\lambda_1, \ldots, \lambda_n) + \sum_{v=1}^{n} (-1)^v \varphi(\lambda_0, \ldots, \lambda_{v-1} + \lambda_v, \ldots, \lambda_n) + (-1)^{n+1} \varphi(\lambda_0, \ldots, \lambda_{n-1})\,.$$

Denoting the cohomology of $C^\bullet(K)$ by $HA^\bullet(K)$, we may complete our list with the last point

(3) $T^n(-R) = HA^{n-1}(K_R)$ for $n \ge 3$.

Remark: The explicit description of $T^2(-R)$ does almost fit into the general context of $n \ge 3$. The correct formula is $T^2(-R) = HA^1(K_R)/(\operatorname{span}_{\mathbb{C}} K_R)^*$.

(4.3) The previous results on $T^n(-R)$ have two important consequences. Let $\Lambda := \{R \in \mathbb{Z}^2 \,|\, R \ge 0 \text{ on } \sigma\}$, $\Lambda_+ := \Lambda \setminus \{0\}$ and $\operatorname{int} \Lambda := \{R \in \mathbb{Z}^2 \,|\, R > 0 \text{ on } \sigma \setminus \{0\}\}$.

Proposition: *Let $n \ge 1$.*

(1) $T^n(-R) = 0$ *unless R is strictly positive on $\sigma \setminus \{0\}$, i.e. unless $R \in \Lambda_+$.*

(2) $T^n(-R) = 0$ *unless* $\operatorname{ht}(R) = n$. *In particular, T^n is killed by the maximal ideal of A_d.*

Proof: (1) If R is not positive on $\sigma \setminus \{0\}$, then $K_R = \emptyset$.
(2) If $n-1 \ge \operatorname{ht}(R)$, then $C^{n-1}(K_R) = 0$ for trivial reasons. Hence T^n sits in degree $\le (-n)$. But this is exactly the opposite inequality from Corollary (3.4)(2). □

In particular, we may shorten our Poincaré series to

$$\widetilde{P}_{Y_d}(x) := \sum_{\operatorname{ht}(R) \ge 1} \dim T^{\operatorname{ht}(R)}(A_d/\mathbb{C}, A_d)(-R) \cdot x^R \in \hat{A}_d = \mathbb{C}[|\Lambda|] \subset \mathbb{C}[|\mathbb{Z}^2|]\,.$$

We obtain $P_{Y_d}(t)$ from $\widetilde{P}_{Y_d}(x)$ via the substitution $x^R \mapsto t^{\operatorname{ht}(R)}$.

(4.4) Lemma:

(1) *Let $R \in \mathbb{Z}^2$ with* $\mathrm{ht}(R) \geq 3$*. Then*

$$\dim T^{\mathrm{ht}(R)}(-R) = \sum_{r \in \mathrm{int}\Lambda,\, R-r \in \Lambda_+} (-1)^{\mathrm{ht}(r)-1} \dim \mathrm{Harr}^{\mathrm{ht}(R)-\mathrm{ht}(r)}(A_d/\mathbb{C}, \mathbb{C})(r-R) + (-1)^{\mathrm{ht}(R)-1} \dim HA^1(K_R)\,.$$

(2) *For $R \in \mathbb{Z}^2$ with* $\mathrm{ht}(R) = 1$ *or* 2*, the right hand side of the above formula always yields zero.*

Proof: (1) The vanishing of $T_Y^n(-R)$ for $n \neq \mathrm{ht}(R)$ together with the equality $T_Y^n(-R) = HA^{n-1}(K_R)$ for $n \geq 3$ implies that the complex $C^\bullet(K_R)$ is exact up to the first and the $(\mathrm{ht}(R)-1)$-th place. In particular, we obtain

$$\dim T^{\mathrm{ht}(R)}(-R) = \sum_{n \geq 1} (-1)^{\mathrm{ht}(R)-1+n} \dim C^n(K_R) + (-1)^{\mathrm{ht}(R)-1} \dim HA^1(K_R)$$

where the sum is a finite one because $C^{\geq \mathrm{ht}(R)}(K_R) = 0$. Now the trick is to replace the differential of the inhomogeneous complex $C^\bullet(K_R)$ by its homogeneous part $d': C^n(K_R) \to C^{n+1}(K_R)$ defined as

$$(d'\varphi)(\lambda_0, \ldots, \lambda_n) := \sum_{v=1}^{n} (-1)^v \varphi(\lambda_0, \ldots, \lambda_{v-1} + \lambda_v, \ldots, \lambda_n)\,.$$

Then $\big(C^\bullet(K_R),\, d'\big)$ splits into a direct sum $\oplus_{r \in K_R} V^\bullet(-r)$ with

$$V^n(-r) := \Big\{\varphi: \big\{(\lambda_1, \ldots, \lambda_n) \in \Lambda_+^n \,\big|\, \textstyle\sum_v \lambda_v = r\big\} \to \mathbb{C} \,\Big|\, \varphi \text{ is shuffle invariant}\Big\}\,.$$

On the other hand, since $A_d = \mathbb{C}[\Lambda]$, we recognise this exactly as the reduced complex computing $\mathrm{Harr}^\bullet(A_d/\mathbb{C}, \mathbb{C})(-r)$. Hence,

$$\dim T^{\mathrm{ht}(R)}(-R) = \sum_{n \geq 1,\, r \in K_R} (-1)^{\mathrm{ht}(R)-1+n} \dim \mathrm{Harr}^n(A_d/\mathbb{C}, \mathbb{C})(-r) \;+\; (-1)^{\mathrm{ht}(R)-1} \dim HA^1(K_R).$$

Finally, we replace r by $R-r$ and recall that $\mathrm{Harr}^n(A_d/\mathbb{C}, \mathbb{C})(-r) = 0$ unless $n = \mathrm{ht}(r)$.

(2) If $\mathrm{ht}(R) = 2$, then the right hand side equals $\#K_R - \#K_R = 0$. If $\mathrm{ht}(R) = 1$, then no summand at all survives. $\square$

(4.5) Let $F(x) := \sum_{v=1}^{d-1} x^{[v,1]} - x^{[d,2]} = \frac{x^{[d,1]} - x^{[1,1]}}{x^{[1,0]} - 1} - x^{[d,2]}$. Using the Poincaré series $\widetilde{Q}_{Y_d}(x)$ of (3.3), we obtain the following formula:

Theorem: *The multigraded Poincaré series of the cone over the rational normal curve of degree d equals*

$$\widetilde{P}_{Y_d}(x) \;=\; \frac{F(x) \cdot (\widetilde{Q}_{Y_d}(x) + 2)}{\big(x^{[0,1]} + 1\big)\big(x^{[d,1]} + 1\big)} - \frac{x^{[1,1]}}{x^{[0,1]} + 1} - \frac{x^{[d-1,1]}}{x^{[d,1]} + 1}\,.$$

Proof: The previous lemma implies that

$$\widetilde{P}_{Y_d}(x) = \sum_{\mathrm{ht}(R)=1,2} \dim T^{\mathrm{ht}(R)}(-R) \cdot x^R + \sum_{R \in \Lambda_+} (-1)^{\mathrm{ht}(R)-1} \dim HA^1(K_R) \cdot x^R + \sum_{\substack{R \in \Lambda_+ \\ r \in \mathrm{int}\Lambda,\, R-r \in \Lambda_+}} (-1)^{\mathrm{ht}(r)-1} \dim \mathrm{Harr}^{\mathrm{ht}(R-r)}(A_d/\mathbb{C}, \mathbb{C})(r-R) \cdot x^R\,.$$

Using the description of $T^1(-R)$ and $T^2(-R)$ from (4.2), including the remark at the very end, we obtain for the first two summands

$$\Big(2\sum_{v=1}^{d-1} x^{[v,1]} - x^{[1,1]} - x^{[d-1,1]}\Big) + \sum_{R\in\Lambda_+} (-1)^{\mathrm{ht}(R)-1} \dim \operatorname{span}(K_R) \cdot x^R\,,$$

which is equal to

$$2\sum_{R\in\mathrm{int}\Lambda} (-1)^{\mathrm{ht}(R)-1} x^R \;+\; \sum_{k\geq 1} (-1)^k x^{[1,k]} \;+\; \sum_{k\geq 1} (-1)^k x^{[kd-1,k]}\,.$$

The third summand in the above formula for $\widetilde{P}_{Y_d}(x)$ may be approached by summing over r first. Then, substituting $s := R - r \in \Lambda_+$ and splitting x^R into the product $x^r \cdot x^s$, we see that this summand is nothing else than

$$\Big(\sum_{r\in\mathrm{int}\Lambda} (-1)^{\mathrm{ht}(r)-1} x^r\Big) \cdot \widetilde{Q}_{Y_d}(x)\,.$$

In particular, we obtain

$$\widetilde{P}_{Y_d}(x) = \Big(\sum_{R\in\mathrm{int}\Lambda} (-1)^{\mathrm{ht}(R)-1} x^R\Big) \cdot \Big(\widetilde{Q}_{Y_d}(x) + 2\Big) \;+\; \sum_{k\geq 1} (-1)^k x^{[1,k]} \;+\; \sum_{k\geq 1} (-1)^k x^{[kd-1,k]}\,.$$

Finally, we should calculate the infinite sums. The latter two are geometric series; they yield $-x^{[1,1]}/\big(x^{[0,1]}+1\big)$ and $-x^{[d-1,1]}/\big(x^{[d,1]}+1\big)$, respectively. With the first sum we proceed as follows:

$$\begin{aligned}
\textstyle\sum_R (-1)^{\mathrm{ht}(R)-1} x^R &= -\textstyle\sum_{k\geq 1}\sum_{v=1}^{kd-1} (-1)^k x^{[v,k]}\\
&= -\textstyle\sum_{k\geq 1} (-1)^k x^{[1,k]}\, \big(x^{[kd-1,0]} - 1\big)/\big(x^{[1,0]}-1\big)\\
&= -\Big(\textstyle\sum_{k\geq 1} (-1)^k x^{[kd,k]} - \sum_{k\geq 1} (-1)^k x^{[1,k]}\Big)\Big/\Big(x^{[1,0]}-1\Big)\\
&= \Big(x^{[d,1]}/\big(1+x^{[d,1]}\big) - x^{[1,1]}/\big(1+x^{[0,1]}\big)\Big)\Big/\Big(x^{[1,0]}-1\Big)\\
&= \Big(x^{[d,1]} - x^{[d+1,2]} + x^{[d,2]} - x^{[1,1]}\Big)\Big/\Big(\big(x^{[d,1]}+1\big)\big(x^{[1,0]}-1\big)\big(x^{[0,1]}+1\big)\Big)\\
&= F(x)\Big/\Big(\big(x^{[d,1]}+1\big)\big(x^{[0,1]}+1\big)\Big)\,.
\end{aligned}$$

$\square$

(4.6) As example we determine the ht $=3$ part of $\widetilde{P}_{Y_d}(x)$. We need the first terms of $\widetilde{Q}_{Z_{d-1}}(x)$. By (3.3) the ht $=1$ part is just $x^{[1,1]}+\cdots+x^{[d-1,1]}$, whereas the ht $=2$ part is

$$\frac{1}{2}\big((x^{[1,1]}+\cdots+x^{[d-1,1]})^2 - (x^{[2,2]}+x^{[4,2]}+\cdots+x^{[2(d-1),2]})\big) = \frac{(x^{[d,1]}-x^{[1,1]})(x^{[d,1]}-x^{[2,1]})}{(x^{[1,0]}-1)^2(x^{[1,0]}+1)}\,.$$

Inserting this in the formula for $\widetilde{P}_{Y_d}(x)$ we finally find the grading of $T^3_{Y_d}$:

$$\frac{(x^{[d,1]}-x^{[1,1]})(x^{[d-1,1]}-x^{[2,1]})(x^{[d-2,1]}-x^{[2,1]})}{(x^{[1,0]}-1)^2(x^{[2,0]}-1)}\,.$$

For even d we get the symmetric formula

$$(x^{[1,1]}+\cdots+x^{[d-1,1]})(x^{[2,1]}+x^{[3,1]}+\cdots+x^{[d-3,1]}+x^{[d-2,1]})(x^{[2,1]}+x^{[4,1]}+\cdots+x^{[d-4,1]}+x^{[d-2,1]})\,.$$

(4.7) Applying the ring homomorphism $x^R \mapsto t^{\mathrm{ht}(R)}$ to the formula of Theorem (4.5) yields:

Corollary: *The ordinary Poincaré series $P_{Y_d}(t)$ of the cone over the rational normal curve equals*

$$P_{Y_d}(t) \;=\; \Big(Q_{Y_d}(t)+2\Big)\cdot\frac{(d-1)\,t-t^2}{(t+1)^2} \;-\; \frac{2\,t}{t+1}\,.$$

5. Hyperplane sections

Choosing a Noether normalisation of an N-dimensional singularity Y means writing Y as total space of an N-parameter family. In this situation we have compared the cohomology of Y with that of the 0-dimensional special fibre. Cutting down the dimension step by step leads to the comparison of the cohomology of a singularity and its hyperplane section.

(5.1) First we recall the main points from [Behnke–Christophersen]. Let $f\colon Y\to\mathbb{C}$ be a flat map such that both Y and the special fibre H have isolated singularities. By T^n_Y and T^n_H we simply denote the cotangent cohomology $T^n(\mathcal{O}_Y/\mathbb{C},\mathcal{O}_Y)$ and $T^n(\mathcal{O}_H/\mathbb{C},\mathcal{O}_H)$, respectively.

Main Lemma: ([Behnke–Christophersen], (1.3)) *There is a long exact sequence*

$$T^1_H \longrightarrow T^2_Y \stackrel{\cdot f}{\longrightarrow} T^2_Y \longrightarrow T^2_H \longrightarrow T^3_Y \stackrel{\cdot f}{\longrightarrow} T^3_Y \longrightarrow \dots\,.$$

Moreover, $\dim T^2_Y\big/f\cdot T^2_Y = \tau_H - e_{H,Y}$ *with* $\tau_H := \dim T^1_H$ *and* $e_{H,Y}$ *denoting the dimension of the smoothing component containing f inside the versal base space of H.*

This lemma will be an important tool for the comparison of the Poincaré series $P_Y(t)$ and $P_H(t)$ of Y and H, respectively. However, since we are not only interested in the dimension, but also in the number of generators of the cohomology groups, we introduce the following notation. If M is a module over a local ring $(A,\underline{m}_A)$, then

$$\mathrm{cg}(M) := \dim_{\mathbb{C}} M\big/\underline{m}_A M$$

is the number of elements in a minimal generator set of M. By $P^{\mathrm{cg}}_Y(t)$ and $P^{\mathrm{cg}}_H(t)$ we denote the Poincaré series using "cg" instead of "dim". Similarly, $\tau^{\mathrm{cg}}_\bullet := \mathrm{cg}(T^1_\bullet)$.

Proposition:

(1) *Assume that $f\cdot T^n_Y=0$ for $n\geq 2$. Then* $P_H(t) = (1+1/t)\,P_Y(t) - \tau_Y\,(t+1) + e_{H,Y}\,t$.

(2) *If $\underline{m}_H\cdot T^n_H = 0$ for $n\geq 2$, then* $P^{\mathrm{cg}}_H(t) = (1+1/t)\,P^{\mathrm{cg}}_Y(t) - \tau^{\mathrm{cg}}_Y\,(t+1) + (\tau^{\mathrm{cg}}_H - \tau_H + e_{H,Y})\,t$.

Proof: In the first case the long exact sequence of the Main Lemma splits into short exact sequences

$$0 \longrightarrow T^n_Y \longrightarrow T^n_H \longrightarrow T^{n+1}_Y \longrightarrow 0$$

for $n\geq 2$. Moreover, the assumption that f annihilates T^2_Y implies that $e_{H,Y} = \tau_H - \dim T^2_Y$. For the second part we follow the arguments of [Behnke–Christophersen], (5.1). The short sequences have to be replaced by

$$0 \longrightarrow T^n_Y\big/f\cdot T^n_Y \longrightarrow T^n_H \longrightarrow \ker\big[f\colon T^{n+1}_Y \to T^{n+1}_Y\big] \longrightarrow 0\,.$$

Since T^{n+1}_Y is finite-dimensional, the dimensions of $\ker\big[f\colon T^{n+1}_Y\to T^{n+1}_Y\big]$ and $T^n_Y\big/f\cdot T^n_Y$ are equal. Now, the claim follows from the fact that $T^n_Y\big/f\cdot T^n_Y = T^n_Y\big/\underline{m}_Y\,T^n_Y$, which is a direct consequence of the assumption $\underline{m}_H\cdot T^n_H = 0$. □

(5.2) We would like to apply the previous formulas to partition curves $H(d_1,\ldots,d_r)$. They are defined as the wedge of the monomial curves $H(d_i)$ described by the equations

$$\operatorname{rank}\begin{pmatrix} z_1 & z_2 & \ldots & z_{d_i-1} & z_{d_i} \\ z_1 & z_3 & \ldots & z_{d_i} & z_1^2 \end{pmatrix} \le 1$$

([Behnke–Christophersen], 3.2). The point making partition curves so exciting is that they occur as the general hypersurface sections of rational surface singularities. Moreover, they sit right in between the cone over the rational normal curve Y_d and the fat point Z_{d-1} with $d := d_1 + \ldots + d_r$.

Theorem: *Let $H := H(d_1,\ldots,d_r)$ be a partition curve. For $n \ge 2$ the modules T^n_H are annihilated by the maximal ideal $\underline{m}_H$. The corresponding Poincaré series is*

$$P_H(t) \;=\; \frac{d-1-t}{t+1}\,Q_{Z_{d-1}}(t) + \tau_H\,t - (d-1)^2\,t\,.$$

Proof: We write $Y := Y_d$ and $Z := Z_{d-1}$. The idea is to compare $P_H(t)$ and $P^{\rm cg}_H(t)$ which can be calculated from $P_Y(t)$ and $P^{\rm cg}_Z(t)$, respectively. Firstly, since $\underline{m}_Y T^n_Y = 0$ for $n \ge 1$, we obtain from Proposition (5.1)(1) and Corollary (4.7) that

$$\begin{aligned} P_H(t) &= (1+1/t)\,P_Y(t) - \tau_Y\,(t+1) + e_{H,Y}\,t \\ &= \Big(Q_Y(t)+2\Big)\frac{(d-1)-t}{t+1} - 2 - (2d-4)(t+1) + e_{H,Y}\,t \\ &= \frac{d-1-t}{t+1}\,Q_Z(t) - 2\,t\,(d-1) + e_{H,Y}\,t\,, \end{aligned}$$

where we used that $Q_Y(t) = Q_Z(t) + 2\,t$. On the other hand, since $\underline{m}_Z T^n_Z = 0$ for all n, we can use the second part of Proposition (5.1) to get

$$P_Z(t) \;=\; P^{\rm cg}_Z(t) \;=\; (1+1/t)\,P^{\rm cg}_H(t) - \tau^{\rm cg}_H\,(t+1) + e_{Z,H}\,t\,.$$

The calculations of (2.4) give us $P_Z(t)$ explicitly: we have $\dim T^n_Z = (d-1)c_{n+1} - c_n$ and $\dim T^0_Z = (d-1)^2$. Therefore

$$P_Z(t) \;=\; \frac{d-1-t}{t}\,Q_Z(t) - (d-1)^2\,.$$

Hence,

$$P^{\rm cg}_H(t) = \frac{d-1-t}{t+1}\,Q_Z(t) + \tau^{\rm cg}_H\,t - \frac{t}{t+1}\Big((d-1)^2 + e_{Z,H}\,t\Big)\,.$$

Finally, we use that $\tau_H - e_{H,Y} = (d-1)(d-3)$ and $e_{Z,H} = (d-1)^2$ (see [Behnke–Christophersen], (4.5) and (6.3.2) respectively). This implies the $P_H(t)$-formula of the theorem as well as

$$P_H(t) - P^{\rm cg}_H(t) \;=\; \big(\tau_H - \tau^{\rm cg}_H\big)\,t\,.$$

In particular, if $n \ge 2$, then the modules T^n_H have as dimension the number of generators, i.e. they are killed by the maximal ideal. □

Corollary: *The number of generators of $T^{\ge 2}$ is the same for all rational surface singularities with fixed multiplicity d.*

Proof: Apply again Proposition (5.1)(2). □

(5.3) We have seen that $\dim T^n = \operatorname{cg} T^n$ $(n \ge 2)$ for the cone over the rational normal curve. This property holds for a larger class of singularities, including quotient singularities.

Theorem: *Let Y be a rational surface singularity such that the projectivised tangent cone has only hypersurface singularities. Then the dimension of T^n for $n \geq 3$ equals the number of generators.*

Proof: Under the assumptions of the theorem the tangent cone $\overline{Y}$ of Y has also finite-dimensional T^n, $n \geq 2$. With $d := \operatorname{mult}(Y)$ we shall show that $\dim T^n_{\overline{Y}} = \dim T^n_{Y_d}$ for $n \geq 3$. As Y is a deformation of its tangent cone $\overline{Y}$, semi-continuity implies that $\dim T^n_Y = \dim T^n_{Y_d}$, which equals the number of generators of T^n_Y. The advantage of working with $\overline{Y}$ is that it is a homogeneous singularity, so Corollary (3.4)(2) applies.

The general hyperplane section $\overline{H}$ of $\overline{Y}$ is in general a non-reduced curve. In fact, it is a wedge of curves described by the equations

$$\operatorname{rank} \begin{pmatrix} z_1 & z_2 & \dots & z_{d_i-1} & z_{d_i} \\ z_1 & z_3 & \dots & z_{d_i} & 0 \end{pmatrix} \leq 1 \,,$$

which is the tangent cone to the curve $H(d_i)$. The curve $\overline{H}$ is also a special section of the cone over the rational normal curve; to see this it suffices to take the cone over a suitable divisor of degree d on $\mathbb{P}^1$. Applying the Main Lemma to $\overline{H}$ and Y_d we obtain the short exact sequences

$$0 \longrightarrow T^n_{Y_d} \longrightarrow T^n_{\overline{H}} \longrightarrow T^{n+1}_{Y_d} \longrightarrow 0 \,,$$

which show that for $n \geq 2$ the dimension of $T^n_{\overline{H}}$ is the same as that of a partition curve of multiplicity d. Moreover, as the module $T^n_{Y_d}$ is concentrated in degree $-n$, $T^{n+1}_{Y_d}$ in degree $-(n+1)$ and the connecting homomorphism, being induced by a coboundary map, has degree -1, it follows that $T^n_{\overline{H}}$ is concentrated in degree $-n$.

We now look again at $\overline{H}$ as hyperplane section of $\overline{Y}$. The short exact sequence corresponding to the second one in the proof of Proposition (5.1), yields that $\ker \left[f : T^{n+1}_{\overline{Y}} \to T^{n+1}_{\overline{Y}} \right]$ is concentrated in degree $-(n+1)$ for $n \geq 2$. The part of highest degree in $T^{n+1}_{\overline{Y}}$ is contained in this kernel, as multiplication by f increases the degree. On the other hand, $T^{n+1}_{\overline{Y}}$ sits in degree $\geq -(n+1)$ by Corollary (3.4)(2). Therefore $T^{n+1}_{\overline{Y}}$ is concentrated in degree $\geq -(n+1)$ and its dimension equals the number of generators, which is the same as for all rational surface singularities of multiplicity d. $\square$

Note that we cannot conclude anything in the case $n = 2$ and in fact the result does not hold for the famous counterexample ([Behnke–Christophersen] 5.5).

Corollary: *For quotient singularities the dimension of T^n, $n \geq 2$, depends only on the multiplicity. In particular, the Poincaré series is*

$$P(t) \;=\; \Big(Q_{Y_d}(t) + 2 \Big) \cdot \frac{(d-1)\,t - t^2}{(t+1)^2} \;-\; \frac{2\,t}{t+1} - (\tau - 2d + 4)t \,.$$

Proof: For $n = 2$ this is [Behnke–Christophersen], (Theorem 5.1.1.(3)). If $n \geq 3$, then we use the previous theorem. In the formula of Corollary (4.7) we have then only to introduce a correction term for $\tau = \dim T^1$. $\square$

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