Separability and the Twisted Frobenius Bimodule

Lars Kadison

1 Introduction

A classical notion in the theory of induced representations is that of a Frobenius extension. Let a *ring extension* A/S be a homomorphism of unital associative rings $S \to A$. We define A/S to be a *Frobenius extension* if A is isomorphic to its dual as S-A-bimodules,

$$ {}_S A_A \cong {}_S \mathrm{Hom}_S(A_S, S_S)_A, \tag{1}$$

and A_S is a finitely generated projective right S-module [17]. Now for any ring extension A/S where A_S is finite projective and S-module ${}_S M$, we have a natural isomorphism of A-modules, ${}_A A \otimes_S M \cong {}_A \mathrm{Hom}_S({}_S A^*, {}_S M)$ where $A^* := \mathrm{Hom}_S(A_S, S_S)$. It follows that A/S is a Frobenius extension if and only if the functors of induction and co-induction from S-modules into A-modules are naturally equivalent: for every ${}_S M$, there is a natural isomorphism of A-modules,

$$ {}_A A \otimes_S M \cong {}_A \mathrm{Hom}_S({}_S A, {}_S M). \tag{2}$$

Now the bimodules ${}_S A_A$, ${}_A A_S$, and ${}_S S_S$ in the isomorphism 1 are the natural ones; however, ${}_S S_S$ may be replaced by a left twisted bimodule ${}_\beta S_S$, where β is a ring automorphism of S and the left S-module structure on S is indicated by $s_1 \cdot s_2 := \beta(s_1)s_2$. This replacement by ${}_\beta S_S$ in the definition above defines more generally a β-Frobenius extension [22]. β-Frobenius extensions are characterized as those ring extensions A/S having a bimodule homomorphism $E : {}_S A_S \to {}_\beta S_S$, called a *Frobenius homomorphism*, as well as $2n$ elements $x_i, y_i \in A$, called a *dual base*, such that

$$ \sum_{i=1}^n \beta^{-1}(E(ax_i))y_i = a = \sum_{i=1}^n x_i E(y_i a) \tag{3}$$

for every $a \in A$ [25]: call $\{E, x_i, y_i\}$ a *β-Frobenius system.* This is equivalent to assuming ${}_S A$ finite projective and ${}_A A_S \cong {}_A\mathrm{Hom_S({}_SA, {}_SS)}_{\beta^{-1}}$, for we map $a \mapsto a(\beta^{-1}E)$ for every $a \in A$.

It may be proven that E and the Casimir element $\sum_i x_i \otimes y_i$ are unique up to multiplication by an invertible in the centralizer $C_A(S)$ of S in A. The *Nakayama automorphism* η of $C_A(S)$ is defined by

$$E(ad) = E(\eta(d)a)$$

for every $a \in A, d \in C_A(S)$. Then from Equations 3, $\eta(d) = \sum_i \beta^{-1}(E(x_i d))y_i$, and

$$\eta^{-1}(d) = \sum_i x_i E(dy_i). \tag{4}$$

The Nakayama automorphism is unique up to an inner automorphism by an invertible in $C_A(S)$. In case A/S is an algebra, β is automatically the identity, since for every $s \in S$,

$$\beta^{-1}(s) = \sum_i x_i E(y_i \beta^{-1}(s)) = s \sum_i x_i E(y_i) = s.$$

For example, a Hopf subalgebra K in a finite dimensional Hopf algebra H over a field is a free β-Frobenius extension by a theorem of Schneider [29]. By a theorem of Larson and Sweedler [18], the antipode is bijective, and H and K are Frobenius algebras with Frobenius homomorphisms which are left or right integrals in the dual algebra. The automorphism β of K is the following composition of the Nakayama automorphisms of H and K:

$$\beta = \eta_K \circ \eta_H^{-1}. \tag{5}$$

It follows from Equation 6 that η_H and η_H^{-1} restrict to inverse mappings of $K \to K$.

We describe a β-Frobenius system for H/K due to Fischman, Montgomery and Schneider [10]. From the formula [26, Theorem 2] for the antipode S it follows that, given a left integral $f \in H^*$ and right integral t_H in H such that $f(t_H) = 1$, $\{f, S^{-1}(t_{H(2)}), t_{H(1)}\}$ is a Frobenius system for H. Given right and left modular functions m_H and m_H^{-1}, a computation using Equation 4 determines η_H^{-1}, from which

$$\eta_H(a) = S^2(a \leftharpoonup m_H^{-1}), \tag{6}$$

for every $a \in H$. Let t_K be a right integral for K. Now by a theorem of Nichols and Zoeller in [23], H_K and ${}_KH$ are free. Then there exists $\hat{\Lambda} \in H$ such that $t_H = \hat{\Lambda} t_K$. Let $\Lambda := \eta_H(S^{-1}(\hat{\Lambda}))$. Then a β-Frobenius system for H/K is given by $\{E, S^{-1}\Lambda_{(2)}, \Lambda_{(1)}\}$ where

$$E(a) = \sum_{(a)} f(a_{(1)} S^{-1}(t_K)) a_{(2)}, \tag{7}$$

for every $a \in H$ [10].

In this paper we study a generalization of β-Frobenius extensions to Frobenius bimodules with two-sided twisting. These may be useful in the further study of subobjects of Hopf algebras and Hopf algebras over categories. In the framework of bimodules, Hom, and tensor, we prove an endomorphism ring theorem and its converse for certain twisted Frobenius bimodules in Section 2. In Section 3, we discuss a duality of the notions of separable extension and split extension A/S (a ring extension A/S is split when ${}_SA_S = {}_SS_S \oplus *$). The duality is cast in terms of the separable bimodules and endomorphism rings studied by Sugano in [32]. We observe an endomorphism ring theorem for projective split separable extensions. In Section 4, we characterize the twisted Frobenius bimodules that are separable in terms of data corresponding to a Frobenius system. The duality mentioned before results in two corollaries for separable β-Frobenius and split β-Frobenius extensions. We show that if A/S is a β-Frobenius extension, then A/S is split (separable) if and only if $\mathrm{End}(A_S)/A$ is separable (respectively, split). In Section 5, we discuss the question of when projective separability implies Frobenius. We give a Hopf algebra example and a matrix example of free separable β-Frobenius extensions with finite rank, which are not Frobenius extensions in the ordinary sense.

Preliminaries A ring R will mean a unital associative ring. A ring homomorphism sends 1 into 1. A right module M_R or left module ${}_RM$ is always unitary. Bimodules are associative with respect to the left and right actions.

An R-S bimodule M is denoted by ${}_RM_S$. Its right dual is defined by $M^* := \mathrm{Hom}(M_S, S_S)$, an S-R module where $sfr(m) := sf(rm)$. The left dual of M is ${}^*M := \mathrm{Hom}({}_RM, {}_RR)$ is also S-R module where $(m)(sfr) := [(ms)f]r$, the argument being conveniently placed on the left. Both $M \mapsto M^*$ and $M \mapsto {}^*M$ are contravariant functors of bimodule categories, sending ${}_R\mathcal{M}_S \to {}_S\mathcal{M}_R$.

If $R = S$ in the last paragraph, denote $\hat{M} := \mathrm{Hom_{S-S}(M, S)}$. Define the group of S-central elements by $M^S := \{m \in M |\ ms = sm,\ \forall \mathrm{s} \in \mathrm{S}\}$. Note that $(M^*)^S = ({}^*M)^S = \hat{M}$.

If ${}_RM_S$, ${}_RN_T$, ${}_TQ_S$ and ${}_SP_T$ are bimodules, then $M \otimes_S P$ receives the natural R-T bimodule structure indicated by $r(m \otimes n)t := rm \otimes nt$, and the group of right module homomorphisms $\mathrm{Hom_S(M_S, Q_S)}$ receives the natural T-R bimodule structure on $\mathrm{Hom_S(M_S, Q_S)}$ indicated by $(tfr)(m) := t(f(rm))$. The group of left module homomorphisms $\mathrm{Hom_R(M, N)}$ receives the natural S-T bimodule structure indicated by $(m)(sft) = ((ms)f)t$, where the argument is written to the left of the function (and if $M = N$, composition is "arrow-theoretic, " i.e., reverse from the usual). All bimodules arising from Hom and tensor in this paper are the natural ones unless otherwise indicated.

A *ring extension* A/S is a ring homomorphism $S \xrightarrow{\iota} A$. A ring extension is an *algebra* if S is commutative and ι factors into $S \to Z(A) \hookrightarrow A$ where $Z(A) := A^A$ is the center of A. A ring extension is *proper* if ι is 1-1, in which case identification is made.

The natural bimodule ${}_SA_S$ is given by $s \cdot a \cdot s' := \iota(s)a\iota(s')$. In particular, we consider the natural modules A_S and ${}_SA$. An adjective, such as right projective or projective, for the ring extension A/S refers to the same adjective for one or both of these natural modules. The structure map ι is usually suppressed.

2 Frobenius bimodules

Suppose S is a ring and $\beta : S \to S$ is a ring automorphism. A left S-module ${}_SM$ receives a new module action defined by

$$s \cdot_\beta m = \beta(s)m, \tag{8}$$

for every $s \in S, m \in M$, the new module being denoted by ${}_\beta M$, called the *β-twisted module*. Given another S-module ${}_SN$, note that $\mathrm{Hom_S({}_\beta M, {}_SN)} = \mathrm{Hom_S({}_SM, {}_{\beta^{-1}}N)}$, since $f \in \mathrm{Hom_S({}_\beta M, {}_SN)}$ satisfies $f(\beta(s)m) = sf(m)$ for every $s \in S$, equivalently $f(sm) = \beta^{-1}(s)f(m)\ \forall \mathrm{s} \in \mathrm{S}$. In particular, if ${}_SN \cong {}_\beta M$, then ${}_{\beta^{-1}}N \cong {}_SM$.

Similarly, we twist a right S-module P_S by β. Note that $P \otimes_S {}_\beta M = P_{\beta^{-1}} \otimes_S M$, whence $P_\beta \otimes_S {}_\beta M = P \otimes_S M$. If ${}_SM_S$ is a bimodule, note that $({}_\beta M_\beta)^S = M^S$.

Let A be another ring and suppose $\alpha : A \to A$ and β are inner automorphisms. A bimodule ${}_AM_S$ is then isomorphic to its twisted bimodule: ${}_AM_S \cong {}_\alpha M_\beta$. For if $u \in A^\circ$ and $v \in S^\circ$ are invertibles such that $ua = \alpha(a)u$ for every $a \in A$, and $sv = v\beta(s)$, then $m \mapsto umv$ defines an isomorphism ${}_AM_S \xrightarrow{\cong} {}_\alpha M_\beta$.

It is well-known that P_S is finite projective if and only if $P \otimes P^* \cong \mathrm{Hom_S(P_S, P_S)}$ given by $p \otimes f \mapsto pf$, where $pf : p' \mapsto pf(p')$ for every $p' \in P$. We need the following generalization of the forward implication.

Lemma 2.1 *If P_S is finite projective, then $\phi : P \otimes {}_\beta P^* \to \mathrm{Hom_S(P_\beta, P_S)}$ given by $p \otimes f \mapsto p(\beta^{-1} \circ f)$ is an isomorphism.*

Proof. Suppose $\{p_k, g_k\}$ is a finite projective base for P_S. Then an inverse mapping to ϕ is given by $\psi : d \mapsto \sum_k d(p_k) \otimes g_k$ for each $d \in \mathrm{Hom_S(P_\beta, P_S)}$. We note that $\phi \circ \psi = \mathrm{Id}$, since for each $p' \in P$ we have

$$\sum_k d(p_k)\beta^{-1}(g_k(p')) = d(\sum_k p_k g_k(p')) = d(p').$$

That $\psi \circ \phi = \mathrm{Id}$ follows from

$$\sum_k p(\beta^{-1}(f(p_k)) \otimes g_k = p \otimes \sum_k f(p_k)g_k = p \otimes f$$

for each $p \in P, f \in P^*$. $\square$

Suppose B and T are rings, and that $\alpha : B \to B$ and $\beta : T \to T$ are ring automorphisms.

Definition 2.1 *A bimodule ${}_BP_T$ is an α-β-Frobenius bimodule if*

1. *P_T and ${}_BP$ are finite projective modules, and*
2. *${}_T({}^*P)_B \cong {}_\beta(P^*)_\alpha$.*

If $\alpha = \mathrm{Id_B}$ and $\beta = \mathrm{Id_T}$, this definition recovers the untwisted Frobenius bimodules defined by Anderson and Fuller [2]. A notable class of examples of Frobenius bimodules come about as follows. By Theorem [21, 1.1] of Morita a faithfully balanced bimodule [2] satisfies the untwisted condition 2 of a Frobenius bimodule. Suppose P_R is a progenerator and $\mathcal{E}$ is defined to be the endomorphism ring $\mathrm{End_R(P_R)}$. Then it follows from Morita's Lemma [2, 17.8] that the natural bimodule ${}_{\mathcal{E}}P_R$ is faithfully balanced. It follows from [2, Lemma 17.7] that ${}_{\mathcal{E}}P_R$ also satisfies condition 1 of a Frobenius bimodule.

A ring extension A/S is a *β-Frobenius extension* if and only if the natural bimodule ${}_AA_S$ is a Id-β-Frobenius bimodule. This is clear since ${}_SA_A \cong {}_\beta(A_S)^*_A$ and A_S finite projective is a characterization of β-Frobenius extension [10, 22, 25].

We will refer to a Frobenius extension A/S where $\beta = \mathrm{Id_S}$ as simply a *Frobenius extension.* By a theorem of Nakayama and Tsuzuku in [22], given two automorphisms α, β of S, a proper β-Frobenius extension A/S is an α-Frobenius extension if and only if $\alpha\beta^{-1}(s) = usu^{-1}$ for some $u \in A^\circ$ and every $s \in S$, which we call an *A-inner automorphism of S.* In particular, a β-Frobenius extension A/S is a Frobenius extension if and only if β is A-inner.

If $\beta : T \to T$ is a ring automorphism, we will refer to an Id-β-Frobenius bimodule ${}_BP_T$ as a *β-Frobenius bimodule.* For any bimodule ${}_BQ_T$, a ring automorphism β of T has a *Q-extension* to a ring automorphism $\alpha : B \to B$ in case there is a bimodule isomorphism $\gamma : {}_BQ_T \xrightarrow{\cong} {}_\alpha Q_\beta$. For example, an *automorphism of a ring extension* is a Q-extension as follows: if B/T is a ring extension with structure homomorphism ι, ${}_BQ_T = {}_BB_T$ is the natural bimodule, and $\alpha \circ \iota = \iota \circ \beta$, then α induces a B-extension of β.

The following sets up some more notation for the next theorem, which generalizes the endomorphism ring theorem for β-Frobenius extensions [22]. Given a bimodule ${}_BP_T$, denote the endomorphism ring of P_T by $\mathcal{E} := \mathrm{End_T(P_T)}$. Note that there is a proper ring extension $B \to \mathcal{E}$, where $b \mapsto \lambda_b$, left multiplication by $b \in B$. Note too the natural bimodule ${}_{\mathcal{E}}P_T$ given by $fpt = f(p)t$ for every $f \in \mathcal{E}, p \in P, t \in T$.

Theorem 2.1 *Suppose ${}_BP_T$ is a β-Frobenius bimodule and $\alpha : B \to B$ is a P-extension of β. Then $\mathcal{E}$ is an α^{-1}-Frobenius extension of A.*

Proof. First, $\mathcal{E}_B$ is finite projective, since $\mathcal{E}_B \cong P \otimes_T P^*_B$ and both P_T and P^*_B are finite projective as a consequence of Definition 2.1.

Next we apply $\mathcal{E}_B \cong P \otimes_T P^*_B$, the Hom-Tensor relation, Definition 2.1 for the $\mathrm{Id_A}$-β-Frobenius bimodule P, reflexivity of ${}_BP$ and ${}_BP_T \cong {}_\alpha P_\beta$ (in that order) in a computation with the natural bimodules discussed in the preliminaries in Section 1:

$$
\begin{aligned}
{}_{\alpha^{-1}}\mathrm{Hom_B}(\mathcal{E}_\mathrm{B},\mathrm{B_B})_\mathcal{E} &= {}_{\alpha^{-1}}\mathrm{Hom_B}(\mathrm{Hom_T}(\mathrm{P_T},{}_\mathcal{E}\mathrm{P_T})_\mathrm{B},\ \mathrm{B_B})_\mathcal{E} \\
&\cong {}_{\alpha^{-1}}\mathrm{Hom_B}({}_\mathcal{E}\mathrm{P}\otimes_\mathrm{T}\mathrm{P}^*_\mathrm{B},\ \mathrm{B_B})_\mathcal{E} \\
&\cong {}_{\alpha^{-1}}\mathrm{Hom_T}({}_\mathcal{E}\mathrm{P_T},\ \mathrm{Hom_B}(({}_\mathrm{B}\mathrm{P_T})^*,\mathrm{B_B})_\mathrm{T})_\mathcal{E} \\
&\cong {}_B\mathrm{Hom_T}({}_\mathcal{E}\mathrm{P_T},\ \mathrm{Hom_B}({}_{\beta^{-1}}({}^*\mathrm{P})_\mathrm{B},{}_{\alpha^{-1}}\mathrm{B_B})_\mathrm{T})_\mathcal{E} \\
&\cong {}_B\mathrm{Hom_T}(\mathrm{P_T},\ {}_{\alpha^{-1}}\mathrm{P}_{\beta^{-1}})_\mathcal{E} \\
&\cong {}_B\mathrm{Hom_T}(\mathrm{P_T},\mathrm{P_T})_\mathcal{E} = {}_\mathrm{B}\mathcal{E}_\mathcal{E}.
\end{aligned}
$$

Whence ${}_B\mathcal{E}_\mathcal{E} \cong {}_{\alpha^{-1}}(\mathcal{E}_B)^*_\mathcal{E}$ and $\mathcal{E}/B$ is a α^{-1}-Frobenius extension. □

Corollary 2.1 *[22] Suppose A/S is a β-Frobenius extension and α extends β to A. Then* $\mathrm{End_S(A_S)}$ *is an α^{-1}-Frobenius extension of A.*

With the suppositions of the corollary, an α^{-1}-Frobenius system is given explicitly as follows. Since A_S is finite projective and ${}_SA_A \cong {}_\beta A^*_A$, we note that $\mathcal{E} := \mathrm{End(A_S)} \cong \mathrm{A}_\beta \otimes_\mathrm{S} \mathrm{A}$, since $\mathcal{E} \cong A \otimes_S A^*$. This extends to an isomorphism of rings if multiplication on $A_\beta \otimes_S A$ is given by

$$(a \otimes b)(c \otimes d) = aE(bc) \otimes d. \tag{9}$$

If ${}_SA_A \stackrel{\cong}{\to} {}_\beta A^*_A$ is given by $a \mapsto Ea$, where $E \in (A_S)^*$ is the image of 1, (x_i, f_i) is a projective base, or dual base, for A_S, and $f_i = Ey_i$, then $\{E, x_i, y_i\}$ is a β-Frobenius system for A/S [25]. Then $\sum_i x_i \otimes y_i$ is easily checked to be the unity element in $A_\beta \otimes_S A$. An α^{-1}-Frobenius homomorphism for $A_\beta \otimes_S A$ over A may easily be checked to be given by $E_1 : A_\beta \otimes_S A \to A$, $a \otimes b \mapsto \alpha^{-1}(a)b$ with dual base $\{x_i \otimes 1\}$ and $\{1 \otimes y_i\}$ (cf. [24, $\alpha = \mathrm{Id_A}$]). This leads to a lemma needed later:

Lemma 2.2 *Suppose A/S is a β-Frobenius extension with β extending to an automorphism α of A and A_S free of rank n. Then ${}_SA$, $\mathcal{E}_A$ and ${}_A\mathcal{E}$ are free of rank n.*

Proof. Let $\{x_i\}$ and $\{f_i\}$ be dual bases for A_S and $(A_S)^*$, $(i = 1, \ldots, n$ throughout) and $a \mapsto Ea$ ($\forall\, \mathrm{a} \in \mathrm{A}$) a Frobenius isomorphism ${}_SA_S \stackrel{\cong}{\to} \mathrm{Hom_S}({}_\mathrm{A}\mathrm{A_S}, {}_\beta\mathrm{S_S})$. Let $y_i \in A$ be determined by $Ey_i = f_i \in A^*$. Then

$\sum_j x_j E(y_j a) = a$ for every $a \in A$, whence $E(y_i x_j) = \delta_{ij}$, the Kronecker delta. Then $\{y_i\}$ is a basis for ${}_S A$, since $\sum_i \beta^{-1}(E(ax_i))y_i = a$ for all $a \in A$, and $\sum_i s_i y_i = 0$ for some $s_i \in S$ implies $E(\sum_i s_i y_i x_j) = \beta(s_j) = 0$ for each $j = 1, \ldots, n$.

By our remarks above, $\mathcal{E}/A$ has α^{-1}-Frobenius system $\{E_1, x_i \otimes 1, 1 \otimes y_i\}$. But $E_1((1 \otimes y_i)(x_j \otimes 1)) = E_1(E(y_i x_j) \otimes 1) = \alpha^{-1}(\delta_{ij}) = \delta_{ij}$, from which it follows that $\mathcal{E}_A$ and ${}_A\mathcal{E}$ are free of rank n. □

A converse of the endomorphism ring theorem is given next. Condition 10 is a weak version of Condition 2 for a Frobenius bimodule studied by Willard [35]. Condition 10 is satisfied if P_S is a generator module, as in our discussion following Definition 2.1.

Theorem 2.2 *Let A and S be rings and ${}_A P_S$ a bimodule such that P_S and ${}_A P$ are both finite projective. Let $\alpha : A \to A$ be a ring automorphism. Suppose $\mathcal{E} := \mathrm{End_S(P_S)}$ is an α-Frobenius extension of A, and*

$$ {}_S\mathrm{Hom}_{\mathcal{E}}({}_{\mathcal{E}}\mathrm{P},\, {}_{\mathcal{E}}\mathcal{E})_{\mathrm{A}} \cong {}_{\mathrm{S}}\mathrm{Hom_S(P_S,\, S_S)_A}. \tag{10} $$

Then ${}_A P_S$ is an α-Frobenius bimodule.

Proof. We apply the Hom-Tensor Relation, the necessary condition ${}_{\mathcal{E}}\mathcal{E}_A \cong {}_{\mathcal{E}}({}^*\mathcal{E})\alpha^{-1}$ for an α-Frobenius extension $\mathcal{E}/A$ and Condition 10 in that order:

$$
\begin{aligned}
{}_S({}^*P)_{\alpha^{-1}} = {}_S\mathrm{Hom_A({}_AP, {}_AA)}_{\alpha^{-1}} &\cong {}_S\mathrm{Hom_A({}_A}\mathcal{E} \otimes_{\mathcal{E}} \mathrm{P},\, \mathrm{{}_AA})_{\alpha^{-1}} \\
&\cong {}_S\mathrm{Hom}_{\mathcal{E}}({}_{\mathcal{E}}\mathrm{P_S},\, {}_{\mathcal{E}}\mathrm{Hom_A}({}_{\mathrm{A}}\mathcal{E}, \mathrm{{}_AA})_{\alpha^{-1}}) \\
&\cong {}_S\mathrm{Hom}_{\mathcal{E}}({}_{\mathcal{E}}\mathrm{P_S},\, {}_{\mathcal{E}}\mathcal{E}_{\mathrm{A}})_{\mathrm{A}} \\
&\cong {}_S\mathrm{Hom_S(P_S,\, S_S)_A} = (\mathrm{{}_AP_S})^*
\end{aligned}
$$

By assumption P is finite projective as an A-module and as an S-module, whence P is an α-Frobenius bimodule. □

Corollary 2.2 *Suppose A/S is a ring extension and α is an automorphism of the ring extension. If A_S is a finite projective generator and $\mathcal{E} := \mathrm{End_S(A_S)}$ is an α-Frobenius extension of A, then A/S is an α^{-1}-Frobenius extension.*

Proof. Since A_S is a generator, Willard's condition with ${}_A P_S = {}_A A_S$ is satisfied. The rest follows from the theorem and the lemma below. □

Lemma 2.3 *Suppose A/S is a ring extension with α an automorphism of the ring extension and β its restriction to S. If A_S is a finite projective and*

$$ {}_SA_{\alpha^{-1}} \cong {}_S\mathrm{Hom_S(A_S, S_S)_A}, \tag{11}$$

then A/S is an β^{-1}-Frobenius extension.

Proof. Let β be the automorphism of S such that $\alpha\iota = \iota\beta$ for the structure map $\iota : S \to A$ of the ring extension A/S. We wish to show that A/S has a β^{-1}-Frobenius system as defined in Equations 3.

Let $\psi : {}_SA_A \stackrel{\cong}{\to} {}_S\mathrm{Hom_S({}_\alpha A_S, S_S)_A}$ and $E := \psi(1)$. Then for every $s_1, s_2 \in S, a \in A$, we have $E(s_1as_2) = \beta^{-1}(s_1)E(a)s_2$. If $\{x_i, f_i\}$ is finite projective base for A_S, and $y_i \in A$ such that $\psi(\alpha^{-1}(y_i)) = E\alpha^{-1}(y_i) = f_i$, then $a = \sum_i x_iE(y_ia)$ for every $a \in A$. We compute for every $a, b \in A$:

$$\begin{aligned} \psi(\sum_i E(ax_i)\alpha^{-1}(y_i))(b) &= E(\sum_i \beta(E(ax_i)y_ib) \\ &= \sum_i E(ax_i)E(y_ib) \\ &= E(a\sum_i x_iE(y_ib)) \\ &= E(ab) = \psi(\alpha^{-1}(a))(b). \end{aligned}$$

Then $a = \sum_i \beta(E(ax_i))y_i$ for every $a \in A$, and $\{E, x_i, y_i\}$ is a β^{-1}-Frobenius system. $\square$

In closing this section, we remark that Frobenius properties, endomorphism ring theorems and converses for functors and categories have been studied by Morita in [21, 20].

3 Separable bimodules and a duality

Separable bimodules are defined in [32] as follows. Suppose ${}_TM_R$ is a bimodule. Define the evaluation map $\mathrm{ev} : M \otimes_R {}^*M \longrightarrow T$, by $m \otimes f \longmapsto (m)f$, and note that it is a T-bimodule homomorphism. T is *M-separable* over R, or M is separable, if ev is a split epi in ${}_T\mathcal{M}_T$.

First some easy deductions. It follows from the definition that ${}_TM$ is a generator. Similarly, if M^* is a separable R-T bimodule with M_R reflexive, then M_R is a generator.

Proposition 3.1 *Suppose ${}_TM_R$ is a faithfully balanced bimodule. Then ${}_TM_R$ is separable $\Leftrightarrow$ ${}_TM$ is a generator $\Leftrightarrow$ M_R is finite projective.*

Proof. Since ${}_TM_R$ is faithfully balanced, the following square is commutative:

$$\begin{array}{ccc} {}_TM \otimes_R M_T^* & \xrightarrow{\cong} & {}_TM \otimes_R {}^*M_T \\ \eta \downarrow & & \downarrow \mathrm{ev} \\ {}_T\mathrm{End}(\mathrm{M_R})_\mathrm{T} & \xleftarrow{\cong} & {}_TT_T \end{array}$$

where the top horizontal arrow is given by $m \otimes f \mapsto m \otimes \Phi(f)$, $\Phi : {}_RM_T^* \xrightarrow{\cong} {}_R^*M_T$ being the isomorphism defined in the proof of [21, Theorem 1.1], the bottom arrow sends elements in T to their left multiplication operator, and η is the standard mapping $m \otimes f \mapsto mf \in \mathrm{End}(\mathrm{M_R})$.

Now one of the vertical arrows is epi if and only if the other is epi. But ev is epi iff ${}_TM$ is a generator, while η is epi iff M_R is finite projective. Furthermore, η is epi implies η is an isomorphism, whence M is separable iff M_R is finite projective. $\square$

Clearly, T is M-separable over R iff there is an element

$$e = \sum_{i=1}^{n} m_i \otimes f_i \in M \otimes_R {}^*M,$$

called a *M-separability element*, satisfying $te = et \quad \forall \mathrm{t} \in \mathrm{T}$ and $\sum_{i=1}^{n}(m_i)f_i = 1$.

Now let A/S be a ring extension. Two specializations of a separable bimodule are to be made in defining separable extension and split extension. First, let $M_1 = {}_AA_S$. Since $\mathrm{Hom}({}_\mathrm{A}\mathrm{A}, {}_\mathrm{A}\mathrm{A}) \cong \mathrm{A}$, A being M_1-separable over S is equivalent to the following definition. A ring extension A/S is a *separable extension* if

$$\mu_S : A \otimes_S A \to A, \quad a \otimes b \longmapsto ab,$$

is a split epimorphism of A-bimodules [12]. The M_1-separability element corresponds to the ordinary separability element, while separable algebras over commutative rings [6] correspond to the algebra case of separable extension.

Secondly, let $M_2 = {}_SA_A$ in the definition of separable bimodule. Note that $M_2 = {}^*M_1$. Since $A \otimes_A \mathrm{Hom}({}_\mathrm{S}\mathrm{A}, {}_\mathrm{S}\mathrm{S}) \xrightarrow{\mathrm{ev}} \mathrm{S}$ reduces to ${}^*A \to S$, $f \mapsto f(1)$, and ${}^*A^S = \hat{A}$, S being M_2-separable over A is equivalent to the following definition. A ring extension A/S is a *split extension* if there exists $E \in$

$\mathrm{Hom_{S-S}(A,S)}$ such that $E(1) = 1$. Call E a *conditional expectation* of the split extension A/S. Equivalently, A/S is a split extension if $A = S \oplus N$ as S-bimodules for some sub-bimodule N in A.

Split extensions are a noteworthy class of ring extensions. The left or right (respectively, weak) global dimension of rings of A and S in a split extension A/S satisfy

$$D(S) \leq D(A) + d_S(A), \tag{12}$$

where d_S is the projective (respectively, flat) dimension of the S-module A (cf. [16]). Secondly, the centralizer $C_A(S)$ is a split extension of the center of S (restrict E to $C_A(S)$). Thirdly, a characterization of split extension is that for every module N_S and arrow $N \to N'$, the natural monic $N_S \to N \otimes_S A_S$ is a split monomorphism and natural with respect to $N \to N'$ (cf. [27]). Similarly, for every module N_S the natural epi $\mathrm{Hom(A_S, N_S)_S \to N_S}$ is split.

Dual-like analogues of these propositions hold for separable extensions. If A/S is a projective separable extension, we have $D(A) \leq D(S)$ (cf. [12, Proposition 1.8]). Secondly, $C_A(S)$ is a split extension of the center of A [12, Lemma 2.2]. Thirdly, a characterization of separable extension is for every module M_A and arrow $M \to M'$, the natural epi $M \otimes_S A \to M$ is a split A-epimorphism and natural with respect to $M \to M'$ (cf. [27]). Similarly, the natural monic $M_A \to \mathrm{Hom_S(A_S, M_S)_A}$ is split for every A-module M. It follows from the last point and naturality that the natural inclusion $A \to \mathrm{End(A_S)}$ is a split extension if A/S is a separable extension.

The rest of this section will study a type of duality between separable extension and split extension A/S with respect to the endomorphism ring $\mathrm{End_S(A_S)}$. Given a bimodule ${}_RM_T$, let $\mathcal{E} := \mathrm{End_T(M_T)}$, a ring extension over R via left multiplication. Let $E := \mathrm{End({}_RM)}$ be the left endomorphism ring of M, a ring extension over T via right multiplication (with arrow-theoretic composition and arguments written to the left). The next theorem is equivalent to [32, Theorem 1, Proposition 2].

Theorem 3.1 *Suppose ${}_RM_T$ is a bimodule.*

1. *If R is M-separable over T, then $\mathcal{E}$ is a split extension over R.*

2. *If M_T is finite projective and $\mathcal{E}$ is a split extension over R, then R is M-separable over T.*

3. *If ${}_RM$ is finite projective and R is M-separable over T, then E is a separable extension of T.*

4. *If ${}_RM$ is a generator and E is a separable extension of T, then R is M-separable over T.*

The proof is contained in [32]. The explicit forms of the conditional expectations and separability elements may be useful and are given below in their respective order.

1. Let $\sum_i m_i \otimes f_i$ be an M-separability element. Then a conditional expectation ${}_R\mathcal{E}_R \to {}_RR_R$ is given by $f \mapsto \sum_i (f(m_i))f_i$ for every $f \in \mathcal{E}$.

2. Let $\{m_i, f_i\}$ be a finite projective base for M_T and let $F : {}_R\mathcal{E}_R \to {}_RR_R$ be a conditional expectation. For each $m \in M$, let mf_i denote the element in $\mathcal{E}$ given by $m' \mapsto mf_i(m')$. Define $g_i \in \mathrm{Hom_R({}_RM, {}_RR)}$ by $g_i(m) = F(mf_i)$. Then $\sum_i m_i \otimes g_i$ is an M-separability element.

3. Let $\{x_i, h_i\}$ be a finite projective base for ${}_RM$, and $\sum_i m_j \otimes_T f_j$ be an M-separability element. For each $f \in {}^*M$ and $m \in M$, let $fm \in E$ be given by $m' \mapsto [(m')f]m$. Then $\sum_{i,j} h_i m_j \otimes_T f_j x_i$ is an separability element for E/T.

4. Given generator ${}_RM$, let $\sum_k (y_k)h_k = 1_R$ where $h_k \in {}^*M$ and $y_k \in M$. Given E/T separable, let $\sum_i F_i \otimes G_i \in E \otimes_T E$ be a separability element. Then $\sum_{i,k} (y_k)F_i \otimes G_i h_k$ is an M-separability element, where $G_i h_k \in \mathrm{Hom_R({}_RM, {}_RR)}$ is arrow-theoretic composition.

If A/S is a ring extension and A_S is a progenerator, the theorem implies that $\mathrm{End_S(A_S)/A}$ is a separable (split) extension if and only if A/S is a split (respectively, separable) extension. As a corollary we obtain a type of endomorphism ring theorem-with-converse for projective split separable extensions. Let $\mathcal{E}' := \mathrm{End_S(A_S)}$ for a ring extension A/S.

Corollary 3.1 *Suppose that A/S is a right finite projective ring extension. If A/S is a split separable extension, then $\mathcal{E}'$ is a left projective split separable extension of A. Conversely, if A/S is a right progenerator extension and $\mathcal{E}'$ is a split separable extension of A, then A/S is a split separable extension.*

Proof. The corollary follows from letting $M = A$ in the theorem, first as the natural A-S bimodule, then as the induced S^{op}-A^{op} bimodule. Note that $\mathrm{End}({}_{\mathrm{S^{op}}}\mathrm{A}) \cong \mathcal{E}'^{\mathrm{op}}$, and that any ring extension is separable (respectively, split) iff its opposite ring extension is separable (respectively, split). Finally, ${}_A\mathcal{E}' \cong {}_AA \otimes_S A^*$ is finite projective since ${}_SA^*$ is. □

4 Separable Frobenius bimodules

In this section, we will obtain a characterization of the twisted Frobenius bimodules that are separable in terms of data corresponding to a Frobenius system. We will obtain two corollaries characterizing those β-Frobenius extensions that are split or separable. We will then use this to prove a duality result like Theorem 3.1 for β-Frobenius extensions.

We set up some notation for the next theorem. Let $\alpha : B \to B$ and $\beta : T \to T$ be ring automorphisms. Suppose ${}_BP_T$ is an α-β-Frobenius bimodule with bimodule isomorphism $\Psi : {}_\beta(P^*)_\alpha \xrightarrow{\cong} {}^*_TP_B$. Let $\{p_k, g_k\}$ be a finite projective base for P_T, and $e_k := \Psi(g_k)$. Note that $e_k \in {}^*P$ and we write the argument in P to the left of e_k.

Theorem 4.1 *Suppose P is a Frobenius bimodule with the notation above. Then B is P-separable over T if and only if there is $d \in \mathrm{Hom}({}_\alpha\mathrm{P}_\beta, {}_\mathrm{B}\mathrm{P}_\mathrm{T})$ such that*

$$\sum_k (d(p_k))e_k = 1_B. \tag{13}$$

Proof. ($\Leftarrow$) Let Φ be the composite isomorphism

$${}_BP \otimes_T {}^*_{\beta^{-1}}P_{\alpha^{-1}} \xrightarrow{1\otimes\Psi^{-1}} {}_BP \otimes_T P^*_B \xrightarrow{\cong} {}_B\mathrm{End_T(P_T)_B}$$

Note that $\sum_k p_k \otimes e_k \mapsto \sum_k p_k \otimes g_k \mapsto \mathrm{Id_P}$ under Φ. Since $\mathrm{Id_P} \in \mathrm{End(P_T)^B}$ and d induces the homomorphism,

$${}_BP_T \otimes {}^*_{\beta^{-1}}P_{\alpha^{-1}} \xrightarrow{d\otimes 1} {}_{\alpha^{-1}}P_{\beta^{-1}} \otimes_T {}^*_{\beta^{-1}}P_{\alpha^{-1}} = {}_BP \otimes_T {}^*P_B,$$

it follows that $e := \sum_k d(p_k) \otimes e_k \in (P \otimes_T {}^*P)^B$. By Equation 13 then, e is a P-separability element.

($\Rightarrow$) Let e is a P-separability element in $(P \otimes {}^*P)^B$. Let Φ' be the composite isomorphism,

$$ {}_BP \otimes_T {}^*P_B \stackrel{1\otimes\Psi^{-1}}{\longrightarrow} {}_BP \otimes_T {}_\beta P^*_\alpha \stackrel{\cong}{\longrightarrow} \mathrm{Hom_T}(\mathrm{P}_\beta, \mathrm{P_T})_\alpha, $$

where the last mapping is the natural isomorphism given by $p \otimes f \mapsto p(\beta^{-1} \circ f)$. Then $d := \Phi'(e) \in \mathrm{Hom_{B-T}}({}_\alpha\mathrm{P}_\beta, {}_\mathrm{B}\mathrm{P_T})$. Moreover, $d = \Phi'(\sum_k d(p_k) \otimes e_k)$ since

$$ \Phi'(\sum_k d(p_k) \otimes e_k) = \sum_k d(p_k)\beta^{-1} \circ g_k = d \circ \sum_k p_k g_k = d. $$

Hence, $e = \sum_k d(p_k) \otimes e_k$ and Equation 13 follows. □

Given a ring extension A/S and automorphism β' of S, we define a twisted version of the centralizer of S in A by

$$ C_A^{\beta'}(S) := \{d' \in A |\, sd' = d'\beta'(s), \forall\, \mathrm{s} \in \mathrm{S}\} \tag{14} $$

Corollary 4.1 *Suppose A/S is a β'-Frobenius extension with β'-Frobenius system $\{E, x_i, y_i\}$. Then A/S is a separable extension if and only if there is $d' \in C_A^{\beta'^{-1}}(S)$ such that $\sum_i x_i d' y_i = 1_A$.*

Proof. Let $B = A$, $T = S$, $\alpha = \mathrm{Id_A}$, $\beta = \beta'$ and ${}_BP_T = {}_AA_S$ in the theorem. Since (x_i, Ey_i) is a finite projective base for A_S, it follows that $e_i = y_i$. But $d \in \mathrm{Hom_{A-S}}({}_\mathrm{A}\mathrm{A}_\beta, {}_\mathrm{A}\mathrm{A_S}) \cong \mathrm{C_A^{\beta^{-1}}(S)}$ via $f \mapsto f(1)$: set $d' = d(1)$. It follows that Equation 13 becomes $\sum_i x_i d y_i = 1_A$. We have noted in an earlier section that A is ${}_AA_S$-separable over S if and only if A/S is a separable extension. □

In case $\beta' = \mathrm{Id_S}$, we obtain [12, Proposition 2.18].

Corollary 4.2 *Suppose A/S is a β'-Frobenius extension with β'-Frobenius system $\{E, x_i, y_i\}$. Then A/S is a split extension if and only if there is $d' \in C_A^{\beta'}(S)$ such that $E(d') = 1$.*

Proof. Let $B = S$, $T = A$, $\alpha = \beta'$, $\beta = \mathrm{Id_A}$ and ${}_BP_T = {}_SA_A$ in the theorem. We have seen that S is ${}_SA_A$-separable over A if and only if A/S is a split extension. With 1 the trivial projective base for A_A, E is the image of 1 under $\Psi : x \mapsto Ex$. Note that $d \in \mathrm{Hom}({}_{\beta'}\mathrm{A_A}, {}_\mathrm{S}\mathrm{A_A}) \cong \mathrm{C_A^{\beta'}(S)}$ via $f \mapsto f(1)$. Set $d' = d(1)$. Placing the argument of $E \in \hat{A}$ to the right this time, Equation 13 becomes $E(d') = 1_S$. □

The next proposition again illustrates the duality between split and separable extension with respect to the endomorphism ring. Three of the four implications below follow from Theorem 3.1. We must prove the fourth implication as a β-Frobenius extension need not satisfy the generator hypothesis in Theorem 3.1. The proof uses the conditions given in Corollaries 4.1 and 4.2. (Similar proofs may be provided for the other three implications [14].) Let $\mathcal{E}' := \mathrm{End_S(A_S)}$.

Proposition 4.1 *Suppose A/S is a ring extension, $\alpha : A \to A$ is an automorphism of the ring extension A/S, and $\beta : S \to S$ is its restriction to S. If A/S is a β-Frobenius extension, then:*

1. *A/S is a separable extension if and only if $\mathcal{E}'/A$ is a split extension.*
2. *A/S is a split extension if and only if $\mathcal{E}'/A$ is a separable extension.*

Proof. That A/S is a separable extension iff $\mathcal{E}'/A$ is a split extension follows from Theorem 3.1, parts 1 and 2, where $R = A$, $T = S$, and $M = {}_A A_S$.

That A/S is a split extension implies $\mathcal{E}'/A$ is a separable extension follows from Theorem 3.1, part 3, where $R = S^{\mathrm{op}}$ (the opposite ring), $T = A^{\mathrm{op}}$, and $M = {}_{S^{\mathrm{op}}} A_{A^{\mathrm{op}}}$. Then $E = \mathcal{E}'^{\,\mathrm{op}}$ is a separable extension of A^{op}, which is equivalent to $\mathcal{E}'/A$ being a separable extension.

To prove that $\mathcal{E}'/A$ is a separable extension implies that A/S is a split extension, suppose $\beta : S \to S$ denotes the restriction of α to S, and that $\{E, x_i, y_i\}$ is a β-Frobenius system for A/S. Recall from an earlier section that $A_\beta \otimes_S A \cong \mathcal{E}$ *as rings*, if multiplication in $A_\beta \otimes_S A$ is given by

$$(a \otimes b)(c \otimes d) := a \otimes \beta^{-1}(E(bc))d = aE(bc) \otimes d.$$

Note that $1_{\mathcal{E}'} = \sum_i x_i \otimes y_i$ under this identification. Recall the α^{-1}-Frobenius system $(E_1,\, x_i \otimes 1,\, 1 \otimes y_i)$ for $\mathcal{E}'/A$ in the endomorphism ring theorem above.

Note that the mapping $\eta : A_\beta \otimes_S A \to A$ given by $a \otimes b \mapsto aE(b)$ is well-defined since $E(sa) = \beta(s)E(a)$ for every $s \in S, a \in A$. By Corollary 4.1, there is an element $e := \sum_j a_j \otimes b_j \in C^{\alpha}_{\mathcal{E}'}(A)$ such that $\sum_i (x_i \otimes 1)e(1 \otimes y_i) = \sum_i x_i \otimes y_i$. Then $d := \sum_j a_j E(b_j) \in C'^{\beta}_A(S)$, and arises as follows:

$$\begin{aligned}
\sum_{i,j} (x_i \otimes 1)(a_j \otimes b_j)(1 \otimes y_i) &= \sum_i (x_i \otimes 1)(d \otimes y_i) \\
&= \sum_i x_i E(d) \otimes y_i \\
&= 1_{\mathcal{E}}.
\end{aligned}$$

By applying η to the last equation above, we obtain

$$\sum_i x_i E(d) E(y_i) = \sum_i x_i E(y_i) E(d) = E(d) = 1,$$

since $sd = d\beta(s)$ and $\sum_i x_i E(y_i a) = a$ for every $s \in S, a \in A$. It follows from Corollary 4.2 that A/S is a split extension. □

5 A Hopf algebra example

The question of when a separable extension is a Frobenius extension is one of the open questions concerning ring extensions [31]. Eilenberg and Nakayama showed in [7] that a separable algebra over a field is a Frobenius algebra. Endo and Watanabe extended the reduced trace of central simple algebras to central separable algebras (over a commutative ring k) and showed that projective, separable k-algebra is a symmetric (Frobenius) algebra in [8]. Sugano proves in [31] that a one-sided projective split H-separable [11] extension A/S is a symmetric Frobenius extension, and that a centrally projective separable extension is a Frobenius extension. In both cases, the problem is reduced to showing that the centralizer $C_S(A)$ is a projective separable algebra. Sugano shows in [33] that a projective H-separable extension of a simple ring is a Frobenius extension. It is for example not known if a projective split separable extension is a Frobenius extension [14]. By Theorems 3.1 and 2.2, it is almost equivalent to ask if a separable two-sided projective bimodule M such that M^* is separable is an untwisted Frobenius bimodule.

In this section, we will provide Hopf algebra and matrix examples of finitely free, separable β-Frobenius extension A/S, where β is not A-inner.

We first consider an example coming from Hopf algebras. Let ψ be a primitive N'th root of unity in a field k and $\frac{1}{N} \in k$. The *Taft k-algebra H* is generated by two elements g and x such that

$$g^N = 1, \quad x^N = 0,$$

and the following simple anti-commutation rule holds:

$$xg = \psi g x \tag{15}$$

Thus, H is an N^2 dimensional algebra with elements of the form $\sum_{i,j=0}^{N-1} a_{ij} x^i g^j$. Note the Wedderburn decomposition $H = K \oplus J$ where K is the separable subalgebra generated by g, a cyclic group algebra of dimension N, and J is the nilpotent radical ideal (x) satisfying $J^N = 0$.

Proposition 5.1 *The algebra extension H/K is a split β-Frobenius extension, where β is not an H-inner automorphism of K.*

Proof. We have noted that H/K is a split extension: indeed, one may choose a conditional expectation that is a ring homomorphism.

Furthermore, it is well-known that H is a non-cocommutative Hopf algebra with comultiplication given by $\Delta(g) = g \otimes g$ and $\Delta(x) = 1 \otimes x + x \otimes g$, counit by $\epsilon(g) = 1$ and $\epsilon(x) = 0$, and antipode by $S(g) = g^{-1}$ and $S(x) = -xg^{-1}$ [34].[1] Then

$$S^2(x) = gxg^{-1}, \tag{16}$$

while $S^2(g) = g$.[2]

The group algebra K is clearly a Hopf subalgebra of H. By the theorem of Fischman, Montgomery and Schneider [10] discussed in Section 1, H is a β-Frobenius extension of K. By Equation 5, $\beta := \eta_K \circ \eta_H^{-1}$ and we must determine the Nakayama automorphisms η_K, η_H. Since K is commutative, it is clear that $\eta_K = \mathrm{Id_K}$. By Equation 6, η_H is given in terms of the modular function m_H and S^2. m_H is nontrivial since H is not unimodular, having right integral

$$t_H = x^{N-1} \sum_{j=0}^{N-1} g^j \tag{17}$$

and left integral $\sum_j g^j x^{N-1}$. Since the right modular function $m_H : H \to k$ is given by $wt_H = m_H(w)t_H$ for every $w \in H$, we have that $m_H(x) = 0$ and $m_H(g) = \psi$. It follows that

$$\eta_H^{-1}(a) = S^{-2}(a \leftharpoonup m_H) = \sum_{(a)} m_H(a_{(1)}) g^{-1} a_{(2)} g,$$

for every $a \in H$. Whence

$$\eta_H^{-1}(x) = m_H(1)g^{-1}xg + m_H(x)g = \psi x, \quad \eta_H^{-1}(g) = \psi g. \tag{18}$$

[1] The proof that $\Delta(x)^N = 0$ uses the quantum binomial formula (cf. [30]).

[2] Note that S^2 is an inner automorphism of order N, so that H is an example of Hopf algebras with antipodes of arbitrary even order as N varies [34].

Since $\beta = \eta_H^{-1}$ restricted to K, we have

$$\beta(g^i) = \psi^i g^i, \tag{19}$$

for every i.

Suppose that there is $u \in H$ such that $\beta(g) = ugu^{-1} = \psi g$. Then $\epsilon(u)\epsilon(g)\epsilon(u)^{-1} = \psi\epsilon(g)$, whence $\psi = 1$, a contradiction. Thus, β is not H-inner. □

It follows from the theorem of Nakayama and Tsuzuku described in Section 2 that H/K is not a Frobenius extension. Since β is extended by the automorphism η_H^{-1} of H it follows from Corollary 2.1 that $\mathcal{E} := \mathrm{End}(\mathrm{H_K})$ is an η_H-Frobenius extension of H. Since H/K is split, so that H_K is a generator, it follows from Corollary 2.2 that η_H is not $\mathcal{E}$-inner. By Proposition 4.1, $\mathcal{E}/H$ is a separable extension, since H/K is a split, β-Frobenius extension. Since the natural modules H_K and ${}_KH$ are free by the theorem of Nichols-Zoeller in [23], it follows from Lemma 2.2 (or other means) that $\mathcal{E}_H$ and ${}_H\mathcal{E}$ are free too. We have proven:

Corollary 5.1 *$\mathcal{E}/H$ is a finitely free, separable η_H-Frobenius extension where η_H is not $\mathcal{E}$-inner.*

With regard to the open problem mentioned above, it should be noted that $\mathcal{E}/H$ is not a split extension. For if it were a split extension, H/K would be a separable extension by another application of Proposition 4.1. Then by transitivity of separability [12] K being a separable algebra implies that H is separable, a contradiction.

The Taft algebra H is part of a more general class of Hopf algebras constructed from a biproduct defined by Radford in [28]. That H is the Radford biproduct of K with a subalgebra isomorphic to $K' := \{\sum_i a_{i0}x^i\}$ follows from [28, Theorem 3]. In particular, it is easy to see from Equation 15 that H is a crossed product algebra of the cyclic group $G := \{1, g, g^2, \ldots, g^{N-1}\}$ acting on K' in an obvious way.

A matrix example. A second example of free separable β-Frobenius extension results from applying the same theory to [22, Example 2, p. 95]. Let

K be a field and $a, b, c, d \in K$. The following notation is convenient:

$$[a,b,c,d] := \begin{pmatrix} a & 0 & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & d & a \end{pmatrix}$$

Consider the algebra $A := \{[a,b,c,d] |\, a,b,c,d \in K\}$ with ordinary matrix multiplication and addition. Consider the subalgebra $S = \{[a,b,0,0] |\, a,b \in K\}$. Let $\beta : S \to S$ be the automorphism defined by $\beta([a,b,0,0]) = [b,a,0,0]$ for every $a, b \in K$. One checks that β is not A-inner. Then A/S is easily checked to be a β-Frobenius extension, with β-Frobenius homomorphism $E([a,b,c,d]) := [c,d,0,0]$ and dual base $\{1_A, u\}$, $\{u, 1_A\}$, where $u := [0,0,1,1]$. Moreover, it is easy to check that A_S and ${}_SA$ are both free with basis $\{1, u\}$.

We note that A/S is a split, though not separable, extension. We compute by hand that $C_A^\beta(S) = \{[0,0,c,d] |\, c,d \in K\}$. Then $1_A \in E(C_A^\beta(S))$, so that Corollary 4.2 implies that A/S is a split extension. Noting that $C_A^\beta(S)u + uC_A^\beta(S) = \{0\}$, Corollary 4.1 implies A/S is not separable, since $\beta^2 = 1$.

Now the ring automorphism $\alpha : A \to A$ given by $\alpha([a,b,c,d]) := [b,a,d,c]$ extends the automorphism β of S. Also $\alpha^2 = 1$. Then by Theorem 2.1 and Lemma 2.2, $\mathcal{E} := \mathrm{End_S(A_S)}$ is a free rank 2 α-Frobenius extension over A. That α is not $\mathcal{E}$-inner follows from Corollary 2.2. $\mathcal{E}/A$ is a separable, though not a split, extension by Proposition 4.1.

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