ASYMPTOTIC BEHAVIOUR OF CONDITIONAL LAWS AND MOMENTS OF α-STABLE RANDOM VECTORS, WITH APPLICATION TO UPCROSSING INTENSITIES

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We prove weak convergence and convergence of certain moments for the conditional law $(X/u \mid Y=u)$ of an α-stable random vector (X,Y) when $\alpha\in(1,2)$. As an example of application we derive a new result in crossing theory for α-stable processes.

1. Introduction. Given an $\alpha\in(1,2)$ we write $Z\in S_\alpha(\sigma,\beta)$ when Z is a strictly α-stable random variable with Fourier transform (characteristic function)

$$(1.1)\quad \mathbf{E}\{\exp[i\theta Z]\} = \exp\{-|\theta|^\alpha\sigma^\alpha[1+i\,\beta\,\tau_\alpha\,\mathrm{sign}(\theta)]\} \quad \text{where} \quad \tau_\alpha\equiv\tan(\tfrac{\pi(2-\alpha)}{2}).$$

Here the scale $\sigma=\sigma_Z\geq 0$ and the skewness $\beta=\beta_Z\in[-1,1]$ are "free" parameters.

Given measurable functions $f,g:\mathbb{R}\to\mathbb{R}$ we put $f^{\langle\alpha\rangle}\equiv|f|^\alpha\mathrm{sign}(f)$ and define

$$\langle f\rangle\equiv\textstyle\int_{\mathbb{R}} f(x)\,dx,\quad \langle f\rangle_\alpha\equiv\langle f^{\langle\alpha\rangle}\rangle,\quad \|f\|_\alpha\equiv\langle|f|^\alpha\rangle^{1/\alpha}\quad\text{and}\quad\langle f,g\rangle_{n,\alpha}\equiv\langle f^n g^{\langle\alpha-n\rangle}\rangle$$

(where $n\in\mathbb{N}$ is required if $f\not\geq 0$). Moreover $\mathbb{L}^\alpha(\mathbb{R})\equiv\{(h:\mathbb{R}\to\mathbb{R}):\|h\|_\alpha<\infty\}$.

Let $\{\xi(t)\}_{t\in\mathbb{R}}$ denote an α-stable Lévy motion with skewness $\beta=-1$, so that $\xi(t)$ has independent stationary increments and $\xi(t)\in S_\alpha(|t|^{1/\alpha},-1)$. Assuming that $f,g\in\mathbb{L}^\alpha(\mathbb{R})$ it is then well known that the bivariate α-stable random variable

$$(1.2)\quad (X,Y)\equiv\left(\textstyle\int_{\mathbb{R}} f\,d\xi,\,\int_{\mathbb{R}} g\,d\xi\right)\quad\text{satisfies}\quad \theta X+\varphi Y\in S_\alpha\left(\|\theta f+\varphi g\|_\alpha,-\frac{\langle\theta f+\varphi g\rangle_\alpha}{\|\theta f+\varphi g\|_\alpha^\alpha}\right).$$

Further each bivariate α-stable vector (X,Y) has this representation in law for some choice of f and g: See, for example, Samorodnitsky and Taqqu (1994, Chapters 1-3) on these and other basic properties of α-stable random variables.

Of course, (when needed) random variables and processes that appear in the sequel are assumed to be defined on a common complete probability space $(\Omega,\mathfrak{F},\mathbf{P})$.

Research supported by NFR Grant M-AA/MA 09207-310.

AMS 1991 subject classifications. Primary 60E07, 60F99, 62E20; secondary 60G70, 60G99.

Key words and phrases. Moment, conditional moment, conditional probability, α-stable distribution, skewed α-stable distribution, crossing, upcrossing, upcrossing intensity.

Cioczek-Georges and Taqqu (1994, 1995a) showed that [thereby sharpening earlier findings of Samorodnitsky and Taqqu (1991) and Wu and Cambanis (1991)]

$$(1.3) \qquad \mathbf{E}\{|X|^{\varrho}\,|\,Y=u\}<\infty \quad \Leftrightarrow \quad \langle|f|,|g|\rangle_{\nu,\alpha}<\infty \begin{cases} \text{for some } \ \nu>\varrho \ \text{ when } \ \varrho<2 \\ \text{for} \qquad \nu=\varrho \ \text{ when } \ \varrho=2 \end{cases},$$

for a power $\varrho\in(0,2]$. (The condition on the right-hand side is void when $\varrho<\alpha$.)

In Section 2 we characterize the unique continuous (wrt. y) regular conditional law $(X\,|\,Y=y)$ for an α-stable vector (X,Y) by means of specifying its Fourier transform. When [as in (1.3)] making statements about conditional probabilities and expectations, we always assume that they are computed according to this law, and so there is no ambiguity concerning what versions these statements refer to.

In Section 3 we use a result of Albin (1997) to investigate the asymptotic behaviour of $\mathbf{E}\{X^2|\,Y=u\}$ as $u\to\infty$.

In Section 4 we use Fourier techniques to derive an upper bound for the moment $\mathbf{E}\{|X|^{\varrho}I_{\{|X|>\lambda\}}|\,Y=y\}$ when $\varrho\in[0,\alpha)$. Besides being of importance on its own, this bound is a crucial ingredient in the proofs of Sections 5 and 6.

In Section 5 we prove weak convergence for $(X/u\,|\,Y=u)$ as $u\to\infty$, together with convergence of the moments $\mathbf{E}\big\{|X/u|^{\varrho}I_{\{X/u>\lambda\}}\,\big|\,Y=u\big\}$ when $\varrho\in(0,\alpha)$ and, under the additional condition $\langle|f|,|g|\rangle_{2,\alpha}<\infty$, when $\varrho\in[\alpha,2)$. We also discuss convergence of probabilities and moments conditioned on $Y>u$.

The expected number of upcrossings of a level u by a stationary and differentiable symmetric α-stable ($S\alpha S$) process $\{\eta(t)\}_{t\in I}$ such that $(\eta'(0),\eta(0))$ possesses a continuous density function $f_{\eta'(0),\eta(0)}$ is given by Rice's formula

$$(1.4) \qquad \mu(I;u)=\mathrm{length}(I)\int_0^\infty x\,f_{\eta'(0),\eta(0)}(x,u)\,dx.$$

Michna and Rychlik (1995) proved this result under quite restrictive additional conditions, and Adler and Samorodnitsky (1997) extended it to a virtually optimal setting. See also Marcus (1989) and Adler, Samorodnitsky and Gadrich (1993).

In Section 6 we prove a verison of (1.4) without any requirements about stationarity, symmetry or existence of joint densities. Our proof is based on the bound for moments in Section 4 and on the counting-device for upcrossings described in Leadbetter, Lindgren and Rootzén (1983, Section 7.2). Despite the fact that our proof produces a more general result, it is considerably shorter and easier than proofs by previous authors.

2. Conditional distributions for α-stable random vectors. Choose functions $f_1,\ldots,f_n,g\in\mathbb{L}^\alpha(\mathbb{R})$ where $\|g\|_\alpha>0$, and consider the α-stable random vector

$$(2.1)\qquad (X,Y) = \big(X_1,\ldots,X_n,Y\big) = \big(\textstyle\int_{\mathbb{R}} f_1\,d\xi,\ldots,\int_{\mathbb{R}} f_n\,d\xi,\int_{\mathbb{R}} g\,d\xi\big).$$

In Proposition 1 below we prove existence of and characterize the unique regular conditional distributions $F_{X|Y}(\cdot\,|\,y)$ that depend continuously on $y\in\mathbb{R}$.

Let Z be an α-stable random vector in $\mathbb{R}^m$ with spectral measure Γ_Z [as defined in e.g., Samorodnitsky and Taqqu (1994, Section 2.3)]. By Kuelbs and Mandrekar (1974, Lemma 2.1), the linear dimension $L(dF_Z)$ of the support of the distribution of Z equals that of the support of Γ_Z $L(\Gamma_Z)$. Further the Fourier transform of Z is integrable (so that Z has a bounded and continuous density function) if $L(\Gamma_Z)=m$ [cf. Samorodnitsky and Taqqu (1994, Lemma 5.1.1)]. It follows that if Z has a density [so that $L(dF_Z)=m$], then Z has a bounded and continuous density. Henceforth we shall therefore without loss assume that α-stable density functions are bounded and continuous when they exist. In particular, the component Y of the vector (X,Y) defined in (2.1) has a bounded and continuous density f_Y since $L(\Gamma_Y)=1$ when $\|g\|_\alpha>0$ [cf. Samorodnitsky and Taqqu (1994, Example 2.3.3)].

Proposition 1. *Consider the α-stable random vector (X,Y) in $\mathbb{R}^{n+1}$ given by (2.1) where $\|g\|_\alpha>0$. Then there exists a unique family of distribution functions $\{F_{X|Y}(\cdot\,|\,y)\}_{y\in\mathbb{R}}$ on $\mathbb{R}^n$ with the properties that*

$$(2.2)\qquad \textstyle\int_{x\in\mathbb{R}^n} h(x)\; dF_{X|Y}(x\,|\,y) \quad \textit{is a version of} \quad \mathbf{E}\{h(X)\,|\,Y=y\}$$

for each measurable map $h:\mathbb{R}^n\to\mathbb{R}$ satisfying $\mathbf{E}\{|h(X)|\}<\infty$, and that

$$(2.3)\qquad F_{X|Y}(\cdot\,|\,y) \to_d F_{X|Y}(\cdot\,|\,y_0) \quad \textit{as} \quad y\to y_0 \quad \textit{(continuity).}$$

Further, writing $\langle\!\langle\theta,x\rangle\!\rangle=\theta_1x_1+\ldots+\theta_nx_n$ the law $F_{X|Y}(\cdot\,|\,y)$ has Fourier transform

$$(2.4)\qquad \phi_{X|Y}(\theta\,|\,y) \equiv \frac{1}{2\pi f_Y(y)}\int_{\mathbb{R}} e^{-i\varphi y}\,\mathbf{E}\{\exp[i(\langle\!\langle\theta,X\rangle\!\rangle+\varphi Y)]\}\,d\varphi \quad \textit{for } \theta\in\mathbb{R}^n,$$

and if (X,Y) possesses a density function $f_{X,Y}(x,y)$, then

$$(2.5)\qquad F_{X|Y}(\cdot\,|\,y) \quad \textit{has density function} \quad f_{X|Y}(\cdot\,|\,y)\equiv f_{X,Y}(\cdot,y)/f_Y(y).$$

Proposition 1 allows us to refer to conditional probabilities and expectations for α-stable random vectors in the same easy-going manner as for Gaussian vectors. The result does not seem to have been observed previously, but a related discussion is given in Samorodnitsky and Taqqu (1994, Section 5.1).

Convention. In the sequel conditional probabilities and expectations are assumed to be computed according to the law $F_{X|Y}(\cdot\,|\,y)$ specified through (2.4) [or (2.5)].

Proof of Proposition 1. As is well known [e.g., Breiman (1968, Section 4.3)], there exists a so-called regular family of distributions $\{F^{\mathrm{reg}}_{X|Y}(\cdot\,|\,y)\}_{y\in\mathbb{R}}$ such that

$$(2.6)\qquad \int_{x\in\mathbb{R}^n} h(x)\, dF^{\mathrm{reg}}_{X|Y}(x\,|\,y) \quad \text{is a version of} \quad \mathbf{E}\{h(X)\,|\,Y=y\}$$

whenever $\mathbf{E}\{|h(X)|\}<\infty$. Further observe the fact that [cf. (1.1) and (1.2)]

$$\big|\mathbf{E}\{\exp[i(\langle\!\langle\theta,X\rangle\!\rangle+\varphi Y)]\}\big| = \exp\{-\|\langle\!\langle\theta,f\rangle\!\rangle+\varphi g\|_\alpha^\alpha\} \le \exp\big\{-\big|\|\langle\!\langle\theta,f\rangle\!\rangle\|_\alpha-|\varphi|\|g\|_\alpha\big|^\alpha\big\}.$$

Since the density f_Y is continuous and locally bounded away from zero, it follows that $\phi_{X|Y}(\theta\,|\,y)$ is a well-defined continuous function of $(\theta,y)\in\mathbb{R}^{n+1}$, and that

$$\mathbf{E}\big\{e^{i\langle\!\langle\theta,X\rangle\!\rangle} I_{\{Y\in[a,b]\}}\big\} = \int_{(x,y)\,\in\,\mathbb{R}^n\times[a,b]} e^{i\langle\!\langle\theta,x\rangle\!\rangle}\, dF_{X,Y}(x,y) = \int_a^b \phi_{X|Y}(\theta\,|\,y)\, f_Y(y)\; dy$$

for $-\infty<a<b<\infty$. Hence $\phi_{X|Y}(\theta\,|\,y)$ is a version of $\mathbf{E}\{\exp[i\langle\!\langle\theta,X\rangle\!\rangle]\,|\,Y=y\}$, and in view of (2.6) we therefore conclude that

$$(2.7)\quad \phi_{X|Y}(\theta\,|\,y) = \int_{x\in\mathbb{R}^n} e^{i\langle\!\langle\theta,x\rangle\!\rangle}\, dF^{\mathrm{reg}}_{X|Y}(x\,|\,y) \quad \text{for } \theta\in\mathbb{Q}^n, \text{ for almost all } y\in\mathbb{R}.$$

By continuity in θ, (2.7) extends to all θ, and so $\phi_{X|Y}(\theta\,|\,y)$ is the Fourier transform of some distribution $F_{X|Y}(\cdot\,|\,y)$ for almost all y. By continuity in y, this statement in turn extends to all y, and (2.3) must hold. Further, since by (2.7) $F_{X|Y}(\cdot\,|\,y) =_d F^{\mathrm{reg}}_{X|Y}(\cdot\,|\,y)$ a.e., (2.6) implies (2.2).

If a density $f_{X,Y}(x,y)$ is continuous in y, then the fact that f_Y is continuous and locally bounded away from zero and the theorem by Scheffé (1947) show that

$$\int_{x\in A} f_{X|Y}(x\,|\,y)\,dx \quad \text{is a continuous function of } y \text{ for every measurable } A\subseteq\mathbb{R}^n.$$

Hence the laws with densities $\{f_{X|Y}(\cdot\,|\,y)\}_{y\in\mathbb{R}}$ satisfy (2.2) and (2.3), and thus coincide with the laws $\{F_{X|Y}(\cdot\,|\,y)\}_{y\in\mathbb{R}}$ specified through (2.4). $\square$

3. Conditional second moments. Take functions f and g in (1.2) such that $\langle|f|,|g|\rangle_{2,\alpha}<\infty$ and either $g\ge 0$ or $g\le 0$ a.s. Extending results for the symmetric case by Wu and Cambanis (1991), and complementing results by Cioczek-Georges and Taqqu (1995b), it was then shown in Albin (1997, Theorem 1) that

$$(3.1)\quad \mathbf{E}\{X^2\,|\,Y=y\} = (\alpha-1)\bigg(\frac{\langle f,|g|\rangle_{2,\alpha}}{\|g\|_\alpha^\alpha} - \frac{\langle f,g\rangle_{1,\alpha}^2}{\|g\|_\alpha^{2\alpha}}\bigg)\int_{|y|}^\infty \frac{z\,f_Y(z)}{f_Y(y)}\,dz + \frac{\langle f,g\rangle_{1,\alpha}^2}{\|g\|_\alpha^{2\alpha}}\,y^2.$$

Note a minor error in Albin (1997): In equations 2.3 and 2.4 (as well as in later occurrences) $\int_y^\infty z\, f_Y(z)\, dz$ should be changed to $\int_{|y|}^\infty z\, f_Y(z)\, dz$.

In Theorem 1 and Corollary 1 below we use (3.1) to determine the asymptotic behaviour of $\mathbf{E}\{X^2|\,Y=u\}$ as $u\to\infty$.

Theorem 1. *Consider the α-stable random variable (X,Y) given by (1.2), where $\|g\|_\alpha>0$ and $\langle f,|g|\rangle_{2,\alpha}<\infty$. Further define the sets*

$$G^+\equiv\{x\in\mathbb{R}: g(x)>0\}\qquad \textit{and}\qquad G^-\equiv\{x\in\mathbb{R}: g(x)<0\}.$$

(i) *Suppose that $g\geq 0$ a.e., so that $\beta_Y=-1$. Then we have*

$$\lim_{u\to\infty} u^{-2}\,\mathbf{E}\{X^2|\,Y=u\} \;=\; \langle f,g\rangle_{1,\alpha}^2 \,/\, \langle g\rangle_\alpha^2.$$

When $\langle f,g\rangle_{1,\alpha}=0$ we further have

$$\lim_{u\to\infty} u^{(2-\alpha)/(\alpha-1)}\,\mathbf{E}\{X^2|\,Y=u\} \;=\; (\alpha-1)\left(\frac{\alpha\,\langle g\rangle_\alpha^{2-\alpha}}{\cos(\frac{\pi}{2}(2-\alpha))}\right)^{1/(\alpha-1)}\langle f,g\rangle_{2,\alpha}.$$

If $\langle f,g\rangle_{1,\alpha}=\langle f,g\rangle_{2,\alpha}=0$, then we have $f=0$ a.e., so that $X=0$ a.s.

(ii) *Suppose that $\langle g^-\rangle_\alpha>0$, so that $\beta_Y>-1$. Then we have*

$$\lim_{u\to\infty} u^{-2}\,\mathbf{E}\{X^2|\,Y=u\} \;=\; \langle fI_{G^-},g^-\rangle_{2,\alpha}\,/\,\langle g^-\rangle_\alpha.$$

When $\langle fI_{G^-},g^-\rangle_{2,\alpha}=0$ but $\langle fI_{G^+},g^+\rangle_{1,\alpha}>0$ we further have

$$\lim_{u\to\infty} u^{\alpha-2}\,\mathbf{E}\{X^2|\,Y=u\} \;=\; \frac{\Gamma(2\alpha-1)\,\langle fI_{G^+},g^+\rangle_{1,\alpha}^2}{\Gamma(\alpha-1)\,\cos(\frac{\pi}{2}(2-\alpha))\,\langle g^+\rangle_\alpha},$$

while $\langle fI_{G^-},g^-\rangle_{2,\alpha}=\langle fI_{G^+},g^+\rangle_{1,\alpha}=0$ but $\langle fI_{G^+},g^+\rangle_{2,\alpha}>0$ implies that

$$\lim_{u\to\infty}\mathbf{E}\{X^2|\,Y=u\} \;=\; 2\,(\alpha-1)\,\big[\langle f_+,g^+\rangle_{2,\alpha}\,/\,\langle g^+\rangle_\alpha\big]\,\mathbf{E}\big\{[S_\alpha(\|g^+\|_\alpha,-1)^+]^2\big\}.$$

If $\langle fI_{G^-},g^-\rangle_{2,\alpha}=\langle fI_{G^+},g^+\rangle_{2,\alpha}=0$, then we have $f=0$ a.e., so that $X=0$ a.s.

Proof of (i). By, for example, Samorodnitsky and Taqqu (1994, Chapter 1) we have

$$(3.2)\quad f_{S_\alpha(\sigma,-1)}(u)\sim A_\alpha\sigma^{-1}(u/\sigma)^{(2-\alpha)/(2(\alpha-1))}\exp\{-B_\alpha(u/\sigma)^{\alpha/(\alpha-1)}\}\ \text{ as } u\to\infty,$$

where $A_\alpha>0$ and $B_\alpha=(\alpha-1)\,[\cos(\frac{\pi}{2}(2-\alpha))/\alpha^\alpha]^{1/(\alpha-1)}$ are constants. Defining $w=w(u)\equiv[\alpha\,\langle g\rangle_\alpha/\cos(\frac{\pi}{2}(2-\alpha))]^{1/(\alpha-1)}\,u^{-1/(\alpha-1)}$, (3.2) and easy calculations show that $(u{+}xw)\,f_Y(u{+}xw)\,/\,(u\,f_Y(u))\to \mathrm{e}^{-x}$ and

$$(3.3)\qquad \textstyle\int_u^\infty y\,f_Y(y)\,dy = u\,w\,f_Y(u)\displaystyle\int_0^\infty\frac{(u{+}xw)\,f_Y(u{+}xw)}{u\,f_Y(u)}\,dx\sim u\,w(u)\,f_Y(u).$$

Using that $uw(u) = o(u^2)$ and $g = |g|$ a.e., (3.1) now yields the statement (i). $\square$

Proof of (ii). Writing $C_\alpha \equiv \alpha(\alpha-1)/[\Gamma(2-\alpha)\cos(\frac{\pi}{2}(2-\alpha))]$ we have

$$\lim_{u\to\infty} u^{\alpha+1} f_{S_\alpha(\sigma,\beta)}(u) = \tfrac{1}{2} C_\alpha (1+\beta)\,\sigma^\alpha \tag{3.4}$$

[e.g., Samorodnitsky and Taqqu (1994, Chapter 1)]. First assume that $g \le 0$ a.e., so that $\beta_Y = 1$ and $|g| = g^-$ a.s. Using (3.4) in an easy calculation, we then obtain

$$\int_u^\infty y\, f_Y(y)\, dy \sim (\alpha-1)^{-1} u^2 f_Y(u). \tag{3.5}$$

Inserting (3.5) in (3.1), and observing that since $\langle f, |g|\rangle_{2,\alpha} < \infty$ we must have $f = 0$ a.e. when $\langle f_-, g^-\rangle_{2,\alpha} = 0$, the statement (ii) of the theorem follows.

Now suppose that $\beta_Y < 1$ and let $(X_-, X_+) \equiv (\int_{\mathbb{R}} f_-\, d\xi, \int_{\mathbb{R}} f_+\, d\xi)$ and $(Y_-, Y_+) \equiv (\int_{\mathbb{R}} (-g^-)\, d\xi, \int_{\mathbb{R}} g^+\, d\xi)$, where $f_+ \equiv I_{G^+} f$ and $f_- \equiv I_{G^-} f$. Then we have

$$\begin{aligned} &\mathbf{E}\{X^2 \,|\, Y = u\} \\ &= \int_{\mathbb{R}} \Big(\mathbf{E}\{X_+^2 \,|\, Y_+ = x\} + \mathbf{E}\{X_-^2 \,|\, Y_- = u-x\} + 2\,\mathbf{E}\{X_+ \,|\, Y_+ = x\}\,\mathbf{E}\{X_- \,|\, Y_- = u-x\} \Big) \\ &\qquad\qquad \times \frac{f_{Y_+}(x)\, f_{Y_-}(u-x)}{f_Y(u)}\, dx \end{aligned}$$

[since $\langle f, |g|\rangle_{2,\alpha} < \infty$ implies that $I_{\mathbb{R}\setminus(G^+\cup G^-)} f = 0$ a.e.]. Using the formulae for linear regression [cf. Samorodnitsky and Taqqu (1994, equation 5.2.27)]

$$\mathbf{E}\{X_+ \,|\, Y_+ = y\} = \langle f_+, g^+\rangle_{1,\alpha}\, y / \langle g^+\rangle_\alpha \quad \text{and} \quad \mathbf{E}\{X_- \,|\, Y_- = y\} = \langle f_-, g^-\rangle_{1,\alpha}\, y / \langle g^-\rangle_\alpha$$

together with (3.1), we therefore obtain

(3.6)

$$\begin{aligned} \mathbf{E}\{X^2 \,|\, Y = u\} &= (\alpha-1)\left(\frac{\langle f_+, g^+\rangle_{2,\alpha}}{\langle g^+\rangle_\alpha} - \frac{\langle f_+, g^+\rangle_{1,\alpha}^2}{\langle g^+\rangle_\alpha^2} \right) \int_{\mathbb{R}} \int_{|x|}^\infty \frac{z\, f_{Y_+}(z)\, f_{Y_-}(u-x)}{f_Y(u)}\, dz dx \\ &\quad + (\alpha-1)\left(\frac{\langle f_-, g^-\rangle_{2,\alpha}}{\langle g^-\rangle_\alpha} - \frac{\langle f_-, g^-\rangle_{1,\alpha}^2}{\langle g^-\rangle_\alpha^2} \right) \int_{\mathbb{R}} \int_{|u-x|}^\infty \frac{z\, f_{Y_-}(z)\, f_{Y_+}(x)}{f_Y(u)}\, dz dx \\ &\quad + \frac{\langle f_+, g^+\rangle_{1,\alpha}^2}{\langle g^+\rangle_\alpha^2} \int_{\mathbb{R}} \frac{x^2 f_{Y_+}(x)\, f_{Y_-}(u-x)}{f_Y(u)}\, dx \\ &\quad + \frac{\langle f_-, g^-\rangle_{1,\alpha}^2}{\langle g^-\rangle_\alpha^2} \int_{\mathbb{R}} \frac{(u-x)^2 f_{Y_+}(x)\, f_{Y_-}(u-x)}{f_Y(u)}\, dx \\ &\quad + \frac{2\,\langle f_+, g^+\rangle_{1,\alpha}\, \langle f_-, g^-\rangle_{1,\alpha}}{\langle g^+\rangle_\alpha\, \langle g^-\rangle_\alpha} \int_{\mathbb{R}} \frac{x\, f_{Y_+}(x)\, (u-x)\, f_{Y_-}(u-x)}{f_Y(u)}\, dx. \end{aligned}$$

Here applications of (3.2)-(3.5) in straightforward calculations reveal that

$$
\begin{aligned}
&\frac{1}{f_Y(u)}\int_{\mathbb{R}}\int_{|x|}^{\infty} z\, f_{Y_+}(z)\, f_{Y_-}(u-x)\, dz dx \to \frac{2\,\langle g^-\rangle_\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha}\int_{\mathbb{R}}\int_{|x|}^{\infty} z\, f_{Y_+}(z)\, dz dx,\\
&\frac{u^{-2}}{f_Y(u)}\int_{\mathbb{R}}\int_{|u-x|}^{\infty} z\, f_{Y_-}(z)\, f_{Y_+}(x)\, dz dx \to \frac{1}{\alpha-1}\,\frac{2\,\langle g^-\rangle_\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha},\\
(3.7)\quad &\frac{u^{\alpha-2}}{f_Y(u)}\int_{\mathbb{R}} x^2 f_{Y_+}(x)\, f_{Y_-}(u-x)\, dx \to \frac{2\,\langle g^+\rangle_\alpha\,\langle g^-\rangle_\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha}\, C_\alpha \int_0^\infty \frac{x^{1-\alpha}\, dx}{(1+x)^{\alpha+1}},\\
&\frac{u^{-2}}{f_Y(u)}\int_{\mathbb{R}} (u-x)^2 f_{Y_+}(x)\, f_{Y_-}(u-x)\, dx \to \frac{2\,\langle g^-\rangle_\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha},\\
&\frac{u^{-1}}{f_Y(u)}\int_{\mathbb{R}} x\, f_{Y_+}(x)\,(u-x)\, f_{Y_-}(u-x) dx \to \frac{2\,\langle g^-\rangle_\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha}\,\mathbf{E}\{Y_+\}.
\end{aligned}
$$

Note that $\int_{\mathbb{R}}\int_{|x|}^{\infty} z\, f_{Y_+}(z)\, dz dx = 2\mathbf{E}\big\{[S_\alpha(\|g^+\|_\alpha,-1)^+]^2\big\}$, $(1+\beta_Y)\|g\|_\alpha^\alpha = 2\langle g^-\rangle_\alpha$ and $\int_0^\infty x^{1-\alpha}/(1+x)^{\alpha+1}\, dx = \Gamma(2-\alpha)\Gamma(2\alpha-1)\,/\,\Gamma(\alpha+1)$. Inserting (3.7) in (3.6) and using that $\langle f I_{G^-}, g^-\rangle_{2,\alpha}=0$ and $\langle f I_{G^+}, g^+\rangle_{2,\alpha}=0$ implies $\langle f_-, g^-\rangle_{1,\alpha}=0$ and $\langle f_+, g^+\rangle_{1,\alpha}=0$, respectively, the statement (ii) now follows. $\square$

An inspection of the proof of Theorem 1 shows that a version of the theorem applies in the case often encountered when X depends on u, that is, when

$$(3.8)\qquad (X,Y) = (X_u, Y) \equiv \big(\textstyle\int_{\mathbb{R}} f_u\, d\xi,\ \int_{\mathbb{R}} g\, d\xi\big) \quad \text{where} \quad f_u(\cdot), g(\cdot)\in \mathbb{L}^\alpha(\mathbb{R}).$$

Corollary 1. *Consider the α-stable random variable (X_u, Y) given by (3.8), where $\|g\|_\alpha>0$ and $\limsup_{u\to\infty}\langle f_u, |g|\rangle_{2,\alpha}<\infty$.*

(i) *Suppose that $g\geq 0$ a.e. and that $\liminf_{u\to\infty}\big|\langle f_u, g\rangle_{1,\alpha}\big|>0$. Then we have*

$$\mathbf{E}\{X_u^2 |\, Y=u\} \sim u^2\,\langle f_u, g\,\rangle_{1,\alpha}^2\,/\,\langle g\rangle_\alpha^2 \qquad \text{as } u\to\infty.$$

(ii) *Suppose that $\langle g^-\rangle_\alpha>0$ and that $\liminf_{u\to\infty}\langle f_u I_{G^-}, g^-\rangle_{2,\alpha}>0$. Then we have*

$$\mathbf{E}\{X_u^2 |\, Y=u\} \sim u^2\,\langle f_u I_{G^-}, g^-\rangle_{2,\alpha}\,/\,\langle g^-\rangle_\alpha \qquad \text{as } u\to\infty.$$

4. Bounds on conditional moments of order less than α. In Theorem 2 and Corollaries 2 and 3 below we derive bounds for $\mathbf{E}\{|X|^\varrho I_{\{|X|>\lambda\}}\,|\, Y=y\}$ via Fourier transforms. The usefulness of Fourier techniques when dealing with conditional α-stable moment was discovered by Samorodnitsky and Taqqu (1991) and Wu and Cambanis (1991). See also Cioczek-Georges and Taqqu (1994, 1995a, 1995b).

Unlike the above mentioned authors, we study the case when $\varrho<\alpha$ and the existence of conditional moments (which was the problem they considered) is automatic. Our aim is instead to derive a bound for $\mathbf{E}\{|X|^\varrho\,|\,Y=y\}$ possessing the right rate as $y\to\infty$. Arguments are quite elementary, albeit a bit bulky.

Theorem 2. *Consider the α-stable random variable (X,Y) given by* (1.2) *where $\|g\|_\alpha>0$. If $\varrho\in[0,\alpha)$ we have*

$$\mathbf{E}\big\{|X|^\varrho I_{\{|X|>\lambda\}}\,\big|\,Y=y\big\}\;\leq\;\frac{K_\alpha}{\alpha-\varrho}\,\frac{\|f\|_\alpha^\alpha\,\lambda^{\varrho-\alpha}}{\max\{|y|,\|g\|_\alpha\}\,f_Y(y)}\,\mathrm{e}^{2\|f\|_\alpha^\alpha\lambda^{-\alpha}}$$

for $\lambda>0$ and $y\in\mathbb{R}$, where $K_\alpha>0$ is a constant (that depends on α only).

Proof. First observe that there exist constants $K_{1,\alpha},K_{2,\alpha}>0$ such that

$$\text{(4.1)}\qquad \left\{\begin{array}{l}\big|\langle tg\rangle_\alpha-\langle tg+sf\rangle_\alpha\big|\\ \big|\|tg\|_\alpha^\alpha-\|tg+sf\|_\alpha^\alpha\big|\end{array}\right.\;\leq\;K_{1,\alpha}\Big(\langle|f|,|g|\rangle_{1,\alpha}\,|s|\,|t|^{\alpha-1}+\|f\|_\alpha^\alpha\,|s|^\alpha\Big)$$

and

$$\text{(4.2)}\qquad \left\{\begin{array}{l}\big|2\,\langle tg\rangle_\alpha-\langle tg+sf\rangle_\alpha-\langle tg-sf\rangle_\alpha\big|\\ \big|2\,\|tg\|_\alpha^\alpha-\|tg+sf\|_\alpha^\alpha-\|tg-sf\|_\alpha^\alpha\big|\end{array}\right.\;\leq\;K_{2,\alpha}\,\|f\|_\alpha^\alpha\,|s|^\alpha$$

for $s,t\in\mathbb{R}$. The proofs of these inequalities rely on the elementary fact that

$$1+\alpha x\;\leq\;|1+x|^\alpha\;\leq\;1+\alpha x+K_{0,\alpha}\,|x|^\alpha\quad\text{for }x\in\mathbb{R},\quad\text{for some constant }K_{0,\alpha}>0:$$

To prove the (most difficult) inequality (4.2), for example, one notes that

$$0\;\leq\;\big|2\,(tg)^{\langle\alpha\rangle}-(tg+sf)^{\langle\alpha\rangle}-(tg-sf)^{\langle\alpha\rangle}\big|\;=\;|tg+sf|^\alpha+|tg-sf|^\alpha-2\,|tg|^\alpha\;\leq\;2K_{0,\alpha}|sf|^\alpha$$

when $|sf|<|tg|$, while

$$\left\{\begin{array}{l}\big|2\,(tg)^{\langle\alpha\rangle}-(tg+sf)^{\langle\alpha\rangle}-(tg-sf)^{\langle\alpha\rangle}\big|\\ \big|2\,|tg|^\alpha-|tg+sf|^\alpha-|tg-sf|^\alpha\big|\end{array}\right.\;\leq\;2\,(1+2^\alpha)\,|sf|^\alpha\quad\text{when}\quad|sf|\geq|tg|.$$

Adding things up it follows that (4.2) holds with $K_{2,\alpha}=\max\{2K_{0,\alpha},2\,(1+2^\alpha)\}$.

We will also need the elementary inequality

$$\begin{aligned}\text{(4.3)}\qquad&\big|2\,\cos(x)-\cos(y)-\cos(z)\big|\\ &=4\left|\cos^2(\tfrac{y-z}{4})\sin(\tfrac{2x-y-z}{4})\sin(\tfrac{2x+y+z}{4})-\sin^2(\tfrac{y-z}{4})\cos(\tfrac{2x-y-z}{4})\cos(\tfrac{2x+y+z}{4})\right|\\ &\leq\big|2x-y-z\big|+\tfrac{1}{4}|y-z|^2\end{aligned}$$

for $x, y, x \in \mathbb{R}$, and its corollary $|\cos(x) - \cos(y)| \leq |x-y|$, as well as the inequalities

$$(4.4) \qquad \begin{cases} |\mathrm{e}^{-x} - \mathrm{e}^{-y}| & \leq \mathrm{e}^{-(x\wedge y)}\, |x-y| \\ \left|2\mathrm{e}^{-x} - \mathrm{e}^{-y} - \mathrm{e}^{-z}\right| & \leq \mathrm{e}^{-(x\wedge y\wedge z)} \left(|2x-y-z| + 2|x-y|^2 + 2|x-z|^2\right) \end{cases}.$$

Combining these inequalities with (4.1)-(4.2) and using symmetry, we deduce that

$$(4.5) \quad \pm \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} |t|^\rho \left[\cos\Big(ty - \tau_\alpha \langle tg\rangle_\alpha\Big) \mathrm{e}^{-\|tg\|_\alpha^\alpha} - \cos\Big(ty - \tau_\alpha\langle sf+tg\rangle_\alpha\Big) \mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\right] \frac{dsdt}{s^{1+\varrho}}$$

$$= \pm \int_{s\in(0,1),\, t\in\mathbb{R}} |t|^\rho \left[\cos\Big(ty - \tau_\alpha\langle tg\rangle_\alpha\Big) - \cos\Big(ty - \tau_\alpha\langle sf+tg\rangle_\alpha\Big)\right]$$

$$\times \left(\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha} - \mathrm{e}^{-\|tg\|_\alpha^\alpha}\right) \frac{dsdt}{s^{1+\varrho}}$$

$$\pm \int_{s\in(0,1),\, t\in\mathbb{R}^+} |t|^\rho \cos\Big(ty - \tau_\alpha\langle tg\rangle_\alpha\Big) \left(2\,\mathrm{e}^{-\|tg\|_\alpha^\alpha} - \mathrm{e}^{-\|tg+sf\|_\alpha^\alpha} - \mathrm{e}^{-\|tg-sf\|_\alpha^\alpha}\right) \frac{dsdt}{s^{1+\varrho}}$$

$$\pm \int_{s\in(0,1),\, t\in\mathbb{R}^+} |t|^\rho \Big[2\,\cos\Big(ty - \tau_\alpha\langle tg\rangle_\alpha\Big) - \cos\Big(ty - \tau_\alpha\langle tg+sf\rangle_\alpha\Big)$$

$$- \cos\Big(ty - \tau_\alpha\langle tg-sf\rangle_\alpha\Big)\Big]\, \mathrm{e}^{-\|tg\|_\alpha^\alpha} \frac{dsdt}{s^{1+\varrho}}$$

$$\leq \tau_\alpha \int_{s\in(0,1),\, t\in\mathbb{R}} |t|^\rho \Big|\langle tg\rangle_\alpha - \langle sf+tg\rangle_\alpha\Big| \, \Big|\|tg\|_\alpha^\alpha - \|sf+tg\|_\alpha^\alpha\Big|\, \mathrm{e}^{-(\|sf+tg\|_\alpha^\alpha \wedge \|tg\|_\alpha^\alpha)} \frac{dsdt}{s^{1+\varrho}}$$

$$+ \int_{s\in(0,1),\, t\in\mathbb{R}^+} |t|^\rho \left(\Big|2\,\|tg\|_\alpha^\alpha - \|tg+sf\|_\alpha^\alpha - \|tg-sf\|_\alpha^\alpha\Big| + 2\Big|\|tg\|_\alpha^\alpha - \|tg+sf\|_\alpha^\alpha\Big|^2\right.$$

$$\left. + 2\Big|\|tg\|_\alpha^\alpha - \|tg-sf\|_\alpha^\alpha\Big|^2\right) \mathrm{e}^{-(\|tg+sf\|_\alpha^\alpha \wedge \|tg-sf\|_\alpha^\alpha \wedge \|tg\|_\alpha^\alpha)} \frac{dsdt}{s^{1+\varrho}}$$

$$+ \tau_\alpha \int_{s\in(0,1),\, t\in\mathbb{R}^+} |t|^\rho \left(\Big|2\,\langle tg\rangle_\alpha - \langle tg+sf\rangle_\alpha - \langle tg-sf\rangle_\alpha\Big|\right.$$

$$\left. + \tfrac{1}{4}\tau_\alpha\Big|\langle tg-sf\rangle_\alpha - \langle tg+sf\rangle_\alpha\Big|^2\right) \mathrm{e}^{-\|tg\|_\alpha^\alpha} \frac{dsdt}{s^{1+\varrho}}$$

$$\leq (\tau_\alpha^2 + 2\tau_\alpha + 4) \int_{s\in(0,1),\, t\in\mathbb{R}^+} t^\rho K_{1,\alpha}^2 \Big(\langle |f|, |g|\rangle_{1,\alpha}\, s\, t^{\alpha-1} + \|f\|_\alpha^\alpha s^\alpha\Big)^2 \mathrm{e}^{\|f\|_\alpha^\alpha - 2^{-\alpha}\|tg\|_\alpha^\alpha} \frac{dsdt}{s^{1+\varrho}}$$

$$+ (\tau_\alpha + 1) \int_{s\in(0,1),\, t\in\mathbb{R}^+} t^\rho K_{2,\alpha}\, \|f\|_\alpha^\alpha\, s^\alpha\, \mathrm{e}^{\|f\|_\alpha^\alpha - 2^{-\alpha}\|tg\|_\alpha^\alpha} \frac{dsdt}{s^{1+\varrho}}$$

$$= (\tau_\alpha^2 + 2\tau_\alpha + 4)\, K_{1,\alpha}^2\, \frac{2^{\rho+2\alpha-1}\, \Gamma(\frac{\rho+2\alpha-1}{\alpha})}{\alpha\,(2-\varrho)}\, \frac{\langle |f|, |g|\rangle_{1,\alpha}^2}{\|g\|_\alpha^{\rho+2\alpha-1}}\, \mathrm{e}^{\|f\|_\alpha^\alpha}$$

$$+ (\tau_\alpha^2+2\tau_\alpha+4)\,K_{1,\alpha}^2\,\frac{2^{\rho+\alpha+1}\,\Gamma(\frac{\rho+\alpha}{\alpha})}{\alpha\,(\alpha+1-\varrho)}\,\frac{\langle|f|,|g|\rangle_{1,\alpha}\|f\|_\alpha^\alpha}{\|g\|_\alpha^{\rho+\alpha}}\,\mathrm{e}^{\|f\|_\alpha^\alpha}$$

$$+ (\tau_\alpha^2+2\tau_\alpha+4)\,K_{1,\alpha}^2\,\frac{2^{\rho+1}\,\Gamma(\frac{\rho+1}{\alpha})}{\alpha\,(2\alpha-\varrho)}\,\frac{\|f\|_\alpha^{2\alpha}}{\|g\|_\alpha^{1+\rho}}\,\mathrm{e}^{\|f\|_\alpha^\alpha}$$

$$+ (\tau_\alpha+1)\,K_{2,\alpha}\,\frac{2^{\rho+1}\,\Gamma(\frac{\rho+1}{\alpha})}{\alpha\,(\alpha-\varrho)}\,\frac{\|f\|_\alpha^{\alpha}}{\|g\|_\alpha^{1+\rho}}\,\mathrm{e}^{\|f\|_\alpha^\alpha}$$

$$\leq \frac{K_{3,\alpha,\rho}\,\|f\|_\alpha^\alpha}{(\alpha-\varrho)\,\|g\|_\alpha^{1+\rho}}\,\mathrm{e}^{2\|f\|_\alpha^\alpha} \qquad \text{for} \quad \rho\geq 0, \qquad \text{for some constant} \quad K_{3,\alpha,\rho}>0$$

(that depends on α and ρ only). Here we used Hölder's inequality and the elementary fact that $\|f\|_\alpha^\kappa \leq \mathrm{e}^{\|f\|_\alpha^\alpha}$ when $\kappa\leq\alpha\mathrm{e}$ to obtain the last inequality. Replacing the inequality (4.3) with

$$\begin{aligned}
(4.6) \qquad & \big|2\,\sin(x)-\sin(y)-\sin(z)\big| \\
&= 4\left|\cos^2(\tfrac{y-z}{4})\,\sin(\tfrac{2x-y-z}{4})\,\cos(\tfrac{2x+y+z}{4}) + \sin^2(\tfrac{y-z}{4})\,\cos(\tfrac{2x-y-z}{4})\,\sin(\tfrac{2x+y+z}{4})\right| \\
&\leq \big|2x-y-z\big| + \tfrac{1}{4}|y-z|^2 \qquad \text{for} \quad x,y,x\in\mathbb{R},
\end{aligned}$$

the above computational scheme further carries over to prove the bound

$$\begin{aligned}
(4.7) \quad \pm \int_{\substack{s\in(0,1)\\ t\in\mathbb{R}}} t^{\langle\rho\rangle}\Big[\sin\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\mathrm{e}^{-\|tg\|_\alpha^\alpha} - \sin\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big)\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\Big]\frac{ds dt}{s^{1+\varrho}} \\
\leq \frac{K_{3,\alpha,\rho}\,\|f\|_\alpha^\alpha}{(\alpha-\varrho)\,\|g\|_\alpha^{1+\rho}}\,\mathrm{e}^{2\|f\|_\alpha^\alpha} \qquad \text{for} \quad \rho\geq 0.
\end{aligned}$$

Let X be a random variable with Fourier transform ϕ_X. Using the inequality

$$\textstyle\int_0^1 t^{-(1+\varrho)}\,[1-\cos(t)]\,dt \;\geq\; \int_0^1 t^{-(1+\varrho)}\,\frac{1}{4}t^2\,dt \;=\; \frac{1}{4}(2-\varrho)^{-1} \qquad \text{for} \quad \varrho\in[0,2),$$

together with a calculation inspired by Ramachandran and Rao (1968) and Ramachandran (1969) [cf. Samorodnitsky and Taqqu (1994, Theorem 5.1.2)], we obtain

$$\begin{aligned}
(4.8) \qquad \frac{1}{4(2-\varrho)}\,\mathbf{E}\big\{|X|^\varrho I_{\{|X|>1\}}\big\} &\leq \int_{|x|>1}\int_{t=0}^{t=|x|} |x|^\varrho\,\frac{1-\cos(t)}{t^{1+\varrho}}\,dt\,dF(x) \\
&\leq \int_{x\in\mathbb{R}}\int_{s=0}^{s=1}\frac{1-\cos(s|x|)}{s^{1+\varrho}}\,ds\,dF(x) \\
&= \int_0^1 \Re\big(1-\phi_X(s)\big)\,\frac{ds}{s^{1+\varrho}}.
\end{aligned}$$

Adapting this estimate to the context (1.2) and (2.4), (4.5) now shows that

$$
\begin{aligned}
(4.9)\qquad & \frac{2\pi f_Y(y)}{4(2-\varrho)}\,\mathbf{E}\big\{|X|^\varrho I_{\{|X|>1\}}\,\big|\,Y=y\big\} \\
&\le \int_0^1 \Re\bigg(\int_{\mathbb{R}} \mathrm{e}^{-ity}\Big[\phi_{X,Y}(0,t)-\phi_{X,Y}(s,t)\Big]\,dt\bigg)\frac{ds}{s^{1+\varrho}} \\
&= \int_{s\in(0,1),\,t\in\mathbb{R}} \bigg(\cos\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\,\mathrm{e}^{-\|tg\|_\alpha^\alpha}-\cos\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big)\,\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\bigg)\frac{dsdt}{s^{1+\varrho}} \\
&\le \frac{K_{3,\alpha,0}}{\alpha-\varrho}\,\frac{\|f\|_\alpha^\alpha}{\|g\|_\alpha}\,\mathrm{e}^{2\|f\|_\alpha^\alpha}.
\end{aligned}
$$

Invoking the inequality $\big||x|^{\alpha-1}-|y|^{\alpha-1}\big|\le|x-y|^{\alpha-1}$, it is easy to show that

$$\big||tg|^{\alpha-1}-|tg+sf|^{\alpha-1}\big|\le\big|(tg)^{\langle\alpha-1\rangle}-(tg+sf)^{\langle\alpha-1\rangle}\big|\le(1+2^{\alpha-1})|sf|^{\alpha-1}\le 2^\alpha|sf|^{\alpha-1}$$

by treating the cases $|tg|>|sf|$ and $|tg|\le|sf|$ separately. This in turn gives

$$
(4.10)\qquad \left.\begin{cases} \big|\langle g,tg\rangle_{1,\alpha}-\langle g,tg+sf\rangle_{1,\alpha}\big| \\ \big|\langle g,|tg|\rangle_{1,\alpha}-\langle g,|tg+sf|\rangle_{1,\alpha}\big|\end{cases}\right.\le 2^\alpha\,\langle|g|,|f|\rangle_{1,\alpha}\,|s|^{\alpha-1}\quad\text{for }\ s,t\in\mathbb{R}.
$$

Moreover integrations by part combine with an inspection of (4.5) to show that

$$
\begin{aligned}
(4.11)\qquad & \int_{s\in(0,1),\,t\in\mathbb{R}^+}\Big|2\,\langle g,|tg|\rangle_{1,\alpha}-\langle g,|tg+sf|\rangle_{1,\alpha}-\langle g,|tg-sf|\rangle_{1,\alpha}\Big|\,\mathrm{e}^{-\|tg\|_\alpha^\alpha}\frac{dsdt}{s^{1+\varrho}} \\
&= \|g\|_\alpha^\alpha\int_{s\in(0,1),\,t\in\mathbb{R}^+} t^{\alpha-1}\,\mathrm{sign}\Big(2\,\langle g,|tg|\rangle_{1,\alpha}-\langle g,|tg+sf|\rangle_{1,\alpha}-\langle g,|tg-sf|\rangle_{1,\alpha}\Big) \\
&\qquad\qquad\times\Big(2\,\langle tg\rangle_\alpha-\langle tg+sf\rangle_\alpha-\langle tg-sf\rangle_\alpha\Big)\mathrm{e}^{-\|tg\|_\alpha^\alpha}\frac{dsdt}{s^{1+\varrho}} \\
&\le \frac{K_{3,\alpha,\alpha-1}}{\alpha-\varrho}\,\|f\|_\alpha^\alpha\,\mathrm{e}^{2\|f\|_\alpha^\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.12)\qquad & \int_{s\in(0,1),\,t\in\mathbb{R}^+}\Big|2\,\langle g,tg\rangle_{1,\alpha}-\langle g,tg+sf\rangle_{1,\alpha}-\langle g,tg-sf\rangle_{1,\alpha}\Big|\,\mathrm{e}^{-\|tg\|_\alpha^\alpha}\frac{dsdt}{s^{1+\varrho}} \\
&= \|g\|_\alpha^\alpha\int_{s\in(0,1),\,t\in\mathbb{R}^+} t^{\alpha-1}\,\mathrm{sign}\Big(2\,\langle g,tg\rangle_{1,\alpha}-\langle g,tg+sf\rangle_{1,\alpha}-\langle g,tg-sf\rangle_{1,\alpha}\Big) \\
&\qquad\qquad\times\Big(2\,\|tg\|_\alpha^\alpha-\|tg+sf\|_\alpha^\alpha-\|tg-sf\|_\alpha^\alpha\Big)\mathrm{e}^{-\|tg\|_\alpha^\alpha}\frac{dsdt}{s^{1+\varrho}} \\
&\qquad+\frac{1}{\alpha}\int_{s\in(0,1)}\Big[\mathrm{sign}\Big(2\,\langle g,tg\rangle_{1,\alpha}-\langle g,tg+sf\rangle_{1,\alpha}-\langle g,tg-sf\rangle_{1,\alpha}\Big)
\end{aligned}
$$

$$\times \Big(2\,\|tg\|_\alpha^\alpha - \|tg+sf\|_\alpha^\alpha - \|tg-sf\|_\alpha^\alpha\Big)\mathrm{e}^{-\|tg\|_\alpha^\alpha}\Big]_{t=0}^{t=\infty} \frac{ds}{s^{1+\varrho}}$$

$$\le \frac{K_{3,\alpha,\alpha-1}}{\alpha-\varrho}\,\|f\|_\alpha^\alpha\,\mathrm{e}^{2\|f\|_\alpha^\alpha} + \frac{2}{\alpha(\alpha-\varrho)}\,\|f\|_\alpha^\alpha.$$

Integrating (4.9) by parts the estimates (4.3)-(4.7) and (4.10)-(4.12) show that

$$\left|\frac{2\pi\, y f_Y(y)}{4\alpha(2-\varrho)}\,\mathbf{E}\big\{|X|^\varrho I_{\{|X|>1\}}\,\big|\,Y=y\big\}\right| \tag{4.13}$$

$$\le \left|\frac{y}{\alpha}\int_0^1 \Re\Big(\int_{\mathbb{R}} \mathrm{e}^{-ity}\Big[\phi_{X,Y}(0,t)-\phi_{X,Y}(s,t)\Big]\,dt\Big)\,\frac{ds}{s^{1+\varrho}}\right|$$

$$= \left|\int_{s\in(0,1),\,t\in\mathbb{R}} \Im\Big(\frac{1}{\alpha}\,\mathrm{e}^{-ity}\,\frac{\partial}{\partial t}\Big[\phi_{X,Y}(0,t)-\phi_{X,Y}(s,t)\Big]\Big)\,dt\,\frac{ds}{s^{1+\varrho}}\right|$$

$$= \Bigg|\int_{s\in(0,1),\,t\in\mathbb{R}} \Big(\tau_\alpha\,\cos\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\,\langle g,|tg|\rangle_{1,\alpha}\,\mathrm{e}^{-\|tg\|_\alpha^\alpha}$$
$$-\tau_\alpha\,\cos\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big)\,\langle g,|sf+tg|\rangle_{1,\alpha}\,\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}$$
$$+\sin\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\,\langle g,tg\rangle_{1,\alpha}\,\mathrm{e}^{-\|tg\|_\alpha^\alpha}$$
$$-\sin\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big)\,\langle g,sf+tg\rangle_{1,\alpha}\,\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\Big)\,\frac{dsdt}{s^{1+\varrho}}\Bigg|$$

$$= \Bigg|\tau_\alpha \int_{s\in(0,1),\,t\in\mathbb{R}} \langle g,|tg|\rangle_{1,\alpha}\Big(\cos\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\,\mathrm{e}^{-\|tg\|_\alpha^\alpha}$$
$$-\cos\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big)\,\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\Big)\,\frac{dsdt}{s^{1+\varrho}}$$
$$-\tau_\alpha \int_{s\in(0,1),\,t\in\mathbb{R}} \Big(\cos\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)-\cos\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big)\Big)$$
$$\times\Big(\langle g,|tg|\rangle_{1,\alpha}-\langle g,|sf+tg|\rangle_{1,\alpha}\Big)\,\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\,\frac{dsdt}{s^{1+\varrho}}$$
$$+\tau_\alpha \int_{s\in(0,1),\,t\in\mathbb{R}} \cos\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\,\Big(\langle g,|tg|\rangle_{1,\alpha}-\langle g,|sf+tg|\rangle_{1,\alpha}\Big)$$
$$\times\Big(\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}-\mathrm{e}^{-\|tg\|_\alpha^\alpha}\Big)\,\frac{dsdt}{s^{1+\varrho}}$$
$$+\tau_\alpha \int_{s\in(0,1),\,t\in\mathbb{R}^+} \Big(2\,\langle g,|tg|\rangle_{1,\alpha}-\langle g,|tg+sf|\rangle_{1,\alpha}-\langle g,|tg-sf|\rangle_{1,\alpha}\Big)$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \times \cos\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\, \mathrm{e}^{-\|tg\|_\alpha^\alpha}\, \frac{dsdt}{s^{1+\varrho}} \\
&+ \int\limits_{s\in(0,1),\, t\in\mathbb{R}} \langle g, tg\rangle_{1,\alpha} \Bigg(\sin\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\, \mathrm{e}^{-\|tg\|_\alpha^\alpha} \\
&\qquad\qquad\qquad\qquad - \sin\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big)\, \mathrm{e}^{-\|sf+tg\|_\alpha^\alpha} \Bigg) \frac{dsdt}{s^{1+\varrho}} \\
&- \int\limits_{s\in(0,1),\, t\in\mathbb{R}} \Bigg(\sin\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big) - \sin\Big(ty-\tau_\alpha\langle sf+tg\rangle_\alpha\Big) \Bigg) \\
&\qquad\qquad\qquad\qquad \times \Big(\langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha} \Big)\, \mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\, \frac{dsdt}{s^{1+\varrho}} \\
&+ \int\limits_{s\in(0,1),\, t\in\mathbb{R}} \sin\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big) \Big(\langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha} \Big) \\
&\qquad\qquad\qquad\qquad \times \Big(\mathrm{e}^{-\|sf+tg\|_\alpha^\alpha} - \mathrm{e}^{-\|tg\|_\alpha^\alpha} \Big) \frac{dsdt}{s^{1+\varrho}} \\
&+ \int\limits_{s\in(0,1),\, t\in\mathbb{R}^+} \Big(2\, \langle g, tg\rangle_{1,\alpha} - \langle g, tg+sf\rangle_{1,\alpha} - \langle g, tg-sf\rangle_{1,\alpha} \Big) \\
&\qquad\qquad\qquad\qquad \times \sin\Big(ty-\tau_\alpha\langle tg\rangle_\alpha\Big)\, \mathrm{e}^{-\|tg\|_\alpha^\alpha}\, \frac{dsdt}{s^{1+\varrho}} \Bigg| \\
&\leq \tau_\alpha \frac{K_{3,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha\, \mathrm{e}^{2\|f\|_\alpha^\alpha} \\
&\quad + \tau_\alpha^2 \int\limits_{s\in(0,1),\, t\in\mathbb{R}} \Big| \langle tg\rangle_\alpha - \langle sf+tg\rangle_\alpha \Big| \Big| \langle g, |tg|\rangle_{1,\alpha} - \langle g, |sf+tg|\rangle_{1,\alpha} \Big|\, \mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\, \frac{dsdt}{s^{1+\varrho}} \\
&\quad + \tau_\alpha \int\limits_{s\in(0,1),\, t\in\mathbb{R}} \Big| \langle g, |tg|\rangle_{1,\alpha} - \langle g, |sf+tg|\rangle_{1,\alpha} \Big| \Big| \|sf+tg\|_\alpha^\alpha - \|tg\|_\alpha^\alpha \Big| \\
&\qquad\qquad\qquad\qquad \times \mathrm{e}^{-(\|sf+tg\|_\alpha^\alpha \wedge \|tg\|_\alpha^\alpha)}\, \frac{dsdt}{s^{1+\varrho}} \\
&\quad + \tau_\alpha \frac{K_{3,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha\, \mathrm{e}^{2\|f\|_\alpha^\alpha} \\
&\quad + \frac{K_{3,\alpha,\alpha-1}}{\alpha-\varrho} \|f\|_\alpha^\alpha\, \mathrm{e}^{2\|f\|_\alpha^\alpha} \\
&\quad + \tau_\alpha \int\limits_{s\in(0,1),\, t\in\mathbb{R}} \Big| \langle tg\rangle_\alpha - \langle sf+tg\rangle_\alpha \Big| \Big| \langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha} \Big|\, \mathrm{e}^{-\|sf+tg\|_\alpha^\alpha}\, \frac{dsdt}{s^{1+\varrho}} \\
&\quad + \int\limits_{s\in(0,1),\, t\in\mathbb{R}} \Big| \langle g, tg\rangle_{1,\alpha} - \langle g, sf+tg\rangle_{1,\alpha} \Big| \Big| \|sf+tg\|_\alpha^\alpha - \|tg\|_\alpha^\alpha \Big| \\
&\qquad\qquad\qquad\qquad \times \mathrm{e}^{-(\|sf+tg\|_\alpha^\alpha \wedge \|tg\|_\alpha^\alpha)}\, \frac{dsdt}{s^{1+\varrho}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{K_{3,\alpha,\alpha-1}+2/\alpha}{\alpha-\varrho}\,\|f\|_\alpha^\alpha\,\mathrm{e}^{2\|f\|_\alpha^\alpha}\\
&\le(\tau_\alpha+1)^2\int_{s\in(0,1),\,t\in\mathbb{R}} K_{1,\alpha}\Big(\langle|f|,|g|\rangle_{1,\alpha}|s|\,|t|^{\alpha-1}+\|f\|_\alpha^\alpha|s|^\alpha\Big)\,2^\alpha\,\langle|g|,|f|\rangle_{1,\alpha}|s|^{\alpha-1}\\
&\qquad\qquad\times\mathrm{e}^{\|f\|_\alpha^\alpha-\|g\|_\alpha^\alpha|\frac12 t|^\alpha}\,\frac{dsdt}{s^{1+\varrho}}\\
&+\frac{2(\tau_\alpha+1)K_{3,\alpha,\alpha-1}+2/\alpha}{\alpha-\varrho}\,\|f\|_\alpha^\alpha\,\mathrm{e}^{2\|f\|_\alpha^\alpha}\\
&=(\tau_\alpha+1)^2\,K_{1,\alpha}\,2^\alpha\,\langle|g|,|f|\rangle_{1,\alpha}\left(\frac{2^{\alpha+1}\,\langle|f|,|g|\rangle_{1,\alpha}}{\alpha(\alpha-\varrho)\,\|g\|_\alpha^\alpha}+\frac{4\Gamma(\frac1\alpha)\,\|f\|_\alpha^\alpha}{\alpha(2\alpha-1-\varrho)\,\|g\|_\alpha}\right)\mathrm{e}^{\|f\|_\alpha^\alpha}\\
&+\frac{2(\tau_\alpha+1)K_{3,\alpha,\alpha-1}+2/\alpha}{\alpha-\varrho}\,\|f\|_\alpha^\alpha\,\mathrm{e}^{2\|f\|_\alpha^\alpha}\\
&\le(\tau_\alpha+1)^2\,K_{1,\alpha}\,2^\alpha\,\|g\|_\alpha\|f\|_\alpha^{\alpha-1}\left(\frac{2^{\alpha+1}\,\|f\|_\alpha\|g\|_\alpha^{\alpha-1}}{\alpha(\alpha-\varrho)\,\|g\|_\alpha^\alpha}+\frac{4\Gamma(\frac1\alpha)\,\|f\|_\alpha^\alpha}{\alpha(\alpha-\varrho)\,\|g\|_\alpha}\right)\mathrm{e}^{\|f\|_\alpha^\alpha}\\
&+\frac{2(\tau_\alpha+1)K_{3,\alpha,\alpha-1}+2/\alpha}{\alpha-\varrho}\,\|f\|_\alpha^\alpha\,\mathrm{e}^{2\|f\|_\alpha^\alpha}\\
&=2\left[(\tau_\alpha+1)^2K_{1,\alpha}2^\alpha\left(\frac{2^\alpha\mathrm{e}^{-\|f\|_\alpha^\alpha}}{\alpha}+\frac{2\Gamma(\frac1\alpha)\,\|f\|_\alpha^{\alpha-1}\mathrm{e}^{-\|f\|_\alpha^\alpha}}{\alpha}\right)+(\tau_\alpha+1)K_{3,\alpha,\alpha-1}+\frac1\alpha\right]\\
&\qquad\qquad\times\frac{\|f\|_\alpha^\alpha}{\alpha-\varrho}\,\mathrm{e}^{2\|f\|_\alpha^\alpha}\\
&\le K_{4,\alpha}\,\frac{\|f\|_\alpha^\alpha}{\alpha-\varrho}\,\mathrm{e}^{2\|f\|_\alpha^\alpha}\qquad\text{for some constant}\quad K_{4,\alpha}>0\quad\text{(that depends on }\alpha\text{ only).}
\end{aligned}
$$

Here we used Hölder's inequality to obtain the second last inequality, and the elementary fact that $\|f\|_\alpha^{\alpha-1}\mathrm{e}^{-\|f\|_\alpha^\alpha}\le 1$ to obtain the last inequality.

Combining (4.9) and (4.13) the theorem follows in the particular case when $\lambda=1$. However, the general case with an arbitrary positive λ can be reduced to the case $\lambda=1$ by elementary algebraic manipulations. □

Corollary 2. *Consider the α-stable random variable (X,Y) given by* (1.2) *where $\|g\|_\alpha>0$. If $\varrho\in(0,\alpha)$ we have*

$$
\mathbf{E}\{|X|^\varrho\,|\,Y=y\}\;\le\;\frac{K_\alpha\,\|f\|_\alpha^\varrho}{[(\alpha-\varrho)\,\max\{|y|,\|g\|_\alpha\}\,f_Y(y)]^{\varrho/\alpha}}\qquad\textit{for}\quad y\in\mathbb{R}.
$$

Proof. By Theorem 2 we have

$$
\mathbf{E}\{|X|^\varrho\,|\,Y=y\}\;\le\;\lambda^\varrho+\frac{K_\alpha}{\alpha-\varrho}\,\frac{\|f\|_\alpha^\alpha\,\lambda^{\varrho-\alpha}}{\max\{|y|,\|g\|_\alpha\}\,f_Y(y)}\,\mathrm{e}^{2\|f\|_\alpha^\alpha\lambda^{-\alpha}}\qquad\text{for}\quad\lambda>0.
$$

Taking $\lambda = \|f\|_\alpha / [(\alpha - \varrho)\ \max\{|y|, \|g\|_\alpha\} f_Y(y)]^{1/\alpha}$ it follows that

$$(4.14)\qquad \mathbf{E}\{|X|^\varrho \mid Y = y\} \le \frac{\|f\|_\alpha^\varrho \left[1 + K_\alpha\, \mathrm{e}^{2(\alpha-\varrho)\ \max\{|y|, \|g\|_\alpha\} f_Y(y)}\right]}{[(\alpha - \varrho)\ \max\{|y|, \|g\|_\alpha\} f_Y(y)]^{\varrho/\alpha}}.$$

Now note the fact that (3.4) implies (albeit not immediately so)

$$(4.15)\quad f_{S_\alpha(\sigma,\beta)}(x) \le D_\alpha\, \sigma^\alpha\, (|x| + \sigma)^{-(\alpha+1)} \quad \text{for } x \in \mathbb{R}, \ \text{ for some constant } D_\alpha > 0.$$

This yields $\max\{|y|, \|g\|_\alpha\} f_Y(y) \le D_\alpha$, which together with (4.14) proves the corollary. □

Corollary 3. *Consider the α-stable random variable (X_u, Y) given by* (3.8) *where $\langle g^-\rangle_\alpha > 0$. If $\varrho \in (0, \alpha)$ we have*

$$\limsup_{u\to\infty} \mathbf{E}\{|X_u/u|^\varrho \mid Y = u\} \le \frac{K_\alpha\ \limsup_{u\to\infty} \|f_u\|_\alpha^\varrho}{[(\alpha - \varrho)\ \langle g^-\rangle_\alpha]^{\varrho/\alpha}}.$$

Proof. Take $y = u$ in Corollary 2 (where f may depend on u), and use (3.4). □

5. Asymptotic behaviour of conditional probabilities and moments. In Theorem 3 we prove weak convergence of $(X/u \mid Y = u)$ as $u \to \infty$ by approximating X with a random variable $X^{(\ell,\varepsilon)}$ such that $\sigma_{X - X^{(\ell,\varepsilon)}} \to 0$ as $\ell \to \infty$ and $\varepsilon \downarrow 0$, and such that the limit $\lim_{u\to\infty} \mathbf{P}\{X^{(\ell,\varepsilon)}/u > \lambda \mid Y = u\}$ can be calculated. The remainder $X - X^{(\ell,\varepsilon)}$ is controlled via Corollary 3 and the Markov inequality.

Clearly, one expects the proof of convergence of $(X/u \mid Y = u)$ to be easier in the case $\alpha < 1$, than in the (usually more interesting) case when $\alpha > 1$. However, our bound on conditional moments in Corollary 3 is only valid when $\alpha > 1$. Thus our proof of Theorem 3 (which builds on Corollary 3) can only be adapted to the case $\alpha < 1$ if (a suitable version of) Corollary 3 is proved for that case.

Besides Corollary 3, the important mechanism in the proof of Theorem 3 is sub-exponentiality: The essential contribution to a large value for a sum of sub-exponential random variables comes from a single variable; cf. (5.4) below. [See Samorodnitsky (1988) and Rosiński and Samorodnitsky (1993) for earlier examples on the use of sub-exponentiality in asymptotic analysis of α-stable phenomena.]

Theorem 3. *Consider the α-stable random vector (X, Y) in $\mathbb{R}^{n+1}$ given by* (2.1) *where $\langle g^-\rangle_\alpha > 0$. Then we have (with obvious notation)*

$$(X/u \mid Y = u) \to_d Z \qquad \textit{where} \qquad \mathbf{P}\{Z \le z\} = \left\langle |g|^\alpha I_{\{x \in G^- :\, f(x)/g(x) \le z\}} \right\rangle \big/ \langle g^-\rangle_\alpha.$$

To explain how Theorem 3 follows from sub-exponentiality, we approximate the functions g^- and $f_i I_{G^-}$ [in (2.1)] by simple functions $\hat{g}^- = \sum_{j=1}^k g_j I_{E_j}$ and $\hat{f}_i^- = \sum_{j=1}^k f_i^{(j)} I_{E_j}$, where $\{E_j\}_{j=1}^k$ are disjoint Borelsets in $G^- = \{x\in\mathbb{R} : g(x) < 0\}$ and the distances $\|\hat{g}^- - g^-\|_\alpha$ and $\|\hat{f}_i^- - f_i I_{G^-}\|_\alpha$ are "small". Then we have $Y_- = \int_{\mathbb{R}} g^-\,d\xi \approx \hat{Y}_- = \int_{\mathbb{R}} \hat{g}^-\,d\xi$ (in the sence of convergence in probability), and

$$\mathbf{P}\{X/u\leq z\,|\,Y=u\} \approx \mathbf{P}\{X/u\leq z\,|\,Y_-=u\} \approx \mathbf{P}\{X/u\leq z\,|\,\hat{Y}_-=u\} \qquad \text{for } u \text{ large}$$

since the tail of $Y-Y_-$ [cf. (3.2)] is much lighter than that of Y_- [cf. (3.4)]. By sub-exponentiality and (3.4), the right-hand is asymptotically equivalent with

$$\frac{\sum_{j=1}^k \mathbf{P}\{X/u\leq z \mid g_j\int_{E_j} d\xi=u\}\; f_{g_j\int_{E_j} d\xi}(u)}{\sum_{j=1}^k f_{g_j\int_{E_j} d\xi}(u)}$$

$$\approx \frac{\sum_{j=1}^k \mathbf{P}\{f_i^{(j)}\int_{E_j} d\xi\,/\,u \leq z \mid g_j\int_{E_j} d\xi=u\}\; f_{g_j\int_{E_j} d\xi}(u)}{\sum_{j=1}^k f_{g_j\int_{E_j} d\xi}(u)}$$

$$\approx \sum_{j=1}^k I_{\{f_i^{(j)}/g_j\leq z\}}\, g_j^\alpha \int_{E_j} dx \Big/ \sum_{j=1}^k g_j^\alpha \int_{E_j} dx,$$

which, of course, is an approximation of $\big\langle |g|^\alpha I_{\{x\in G^-\,:\,f(x)/g(x)\leq z\}}\big\rangle / \langle g^-\rangle_\alpha$.

Proof of Theorem 3. By considering the random vectors $\big(\pm X_1,\ldots,\pm X_n,Y\big)$, convergence for $(X/u\,|\,Y=u)$ will follow provided that we can prove

$$\lim_{u\to\infty} \mathbf{P}\{X/u>\lambda\,|\,Y=u\} = \big\langle |g|^\alpha I_{\{x\in G^-\,:\,f(x)/g(x)>\lambda\}}\big\rangle / \langle g^-\rangle_\alpha \tag{5.1}$$

for continuity points $\lambda>0$ (with components $\lambda_1,\ldots,\lambda_n>0$) of the distribution of Z. To that end we define (again using obvious notation)

$$\begin{cases} Y_{k,\varepsilon}^{(+)} \equiv \int_{\mathbb{R}} I_{A_k} g\,d\xi \quad \text{where} \quad A_k \equiv \big\{x\in G^+ : (k-1)\varepsilon\, g(x) < f(x) \leq k\varepsilon\, g(x)\big\} \\ Y_{k,\varepsilon}^{(-)} \equiv \int_{\mathbb{R}} I_{B_k} g\,d\xi \quad \text{where} \quad B_k \equiv \big\{x\in G^- : k\varepsilon\, g(x) < f(x) \leq (k-1)\varepsilon\, g(x)\big\} \end{cases}$$

for $\varepsilon>0$ and $k\in\mathbb{Z}^n$. Further let $G^0\equiv\big\{x\in\mathbb{R} : g(x)=0\big\}$, $X^{(0)}\equiv\int_{\mathbb{R}} I_{G^0} f\,d\xi$,

$$f^{(\ell,\varepsilon)} \equiv \sum_{\|k\|\leq\ell} I_{A_k} k\varepsilon\, g + \sum_{\|k\|\leq\ell} I_{B_k} k\varepsilon\, g \qquad \text{and} \qquad f^{(\varepsilon)} \equiv \sum_{k\in\mathbb{Z}^n} I_{A_k} k\varepsilon\, g + \sum_{k\in\mathbb{Z}^n} I_{B_k} k\varepsilon\, g,$$

where $\|k\| = \max\{|k_1|,\ldots,|k_n|\}$ for $k\in\mathbb{Z}^n$. Setting $f^{(\pm)}\equiv I_{G^+\cup G^-} f$ we then have $|f_i^{(\varepsilon)} - f_i^{(\pm)}| \leq \varepsilon\,|g|$ for $i=1,\ldots,n$, and in particular each component $f_i^{(\varepsilon)}\in\mathbb{L}^\alpha(\mathbb{R})$. Writing $X_{k,\varepsilon}^{(+)}\equiv k\varepsilon\, Y_{k,\varepsilon}^{(+)}$ and $X_{k,\varepsilon}^{(-)}\equiv k\varepsilon\, Y_{k,\varepsilon}^{(-)}$ we may thus define

$$X^{(\ell,\varepsilon)} \equiv \int_{\mathbb{R}} f^{(\ell,\varepsilon)}\,d\xi = \sum_{\|k\|\leq\ell} (X_{k,\varepsilon}^{(+)}+X_{k,\varepsilon}^{(-)}) \quad \text{and} \quad X^{(\varepsilon)} \equiv \int_{\mathbb{R}} f^{(\varepsilon)}\,d\xi = \sum_{k\in\mathbb{Z}^n} (X_{k,\varepsilon}^{(+)}+X_{k,\varepsilon}^{(-)}).$$

To proceed, we note that by Corollary 3 and since $X^{(0)}$ is independent of Y,

$$\begin{aligned}
\limsup_{\ell\to\infty}\limsup_{u\to\infty}\mathbf{E}\big\{|(X-X^{(\ell,\varepsilon)})_i/u| \,\big|\, Y=u\big\}
&\le \limsup_{u\to\infty}\mathbf{E}\big\{|(X-X^{(\pm)})_i/u| \,\big|\, Y=u\big\}\\
&\qquad + \limsup_{u\to\infty}\mathbf{E}\big\{|(X^{(\pm)}-X^{(\varepsilon)})_i/u| \,\big|\, Y=u\big\}\\
&\qquad + \limsup_{\ell\to\infty}\limsup_{u\to\infty}\mathbf{E}\big\{|(X^{(\varepsilon)}-X^{(\ell,\varepsilon)})_i/u| \,\big|\, Y=u\big\}\\
&\le \limsup_{u\to\infty}\mathbf{E}\big\{|X_i^{(0)}/u| \,\big|\, Y=u\big\}\\
&\qquad + K_\alpha\,\|f_i^{(\pm)}-f_i^{(\varepsilon)}\|_\alpha\big/[(\alpha-1)\,\langle g^-\rangle_\alpha]^{1/\alpha}\\
&\qquad + \limsup_{\ell\to\infty}K_\alpha\,\|f_i^{(\varepsilon)}-f_i^{(\ell,\varepsilon)}\|_\alpha\big/[(\alpha-1)\,\langle g^-\rangle_\alpha]^{1/\alpha}\\
&\le 0\\
&\qquad + K_\alpha\,\varepsilon\,\|g\|_\alpha\big/[(\alpha-1)\,\langle g^-\rangle_\alpha]^{1/\alpha}\\
&\qquad + 0
\end{aligned}$$

[where $X^{(\pm)}\equiv\int_{\mathbb{R}}f^{(\pm)}\,d\xi$]. For each vector $\delta=(\delta_1,\ldots,\delta_n)>0$ we therefore have

$$\tag{5.2}\begin{aligned}
&\limsup_{u\to\infty}\mathbf{P}\{X/u>\lambda \mid Y=u\}\\
&\le \limsup_{\ell\to\infty}\limsup_{u\to\infty}\Big(\mathbf{P}\big\{X^{(\ell,\varepsilon)}/u>\lambda-\delta \,\big|\, Y=u\big\}+\sum_{i=1}^{n}\frac{1}{\delta_i}\mathbf{E}\big\{|(X-X^{(\ell,\varepsilon)})_i/u| \,\big|\, Y=u\big\}\Big)\\
&\le \limsup_{\ell\to\infty}\limsup_{u\to\infty}\mathbf{P}\big\{X^{(\ell,\varepsilon)}/u>\lambda-\delta \,\big|\, Y=u\big\}\;+\;O(\varepsilon)\qquad\text{as }\ \varepsilon\downarrow 0,
\end{aligned}$$

and

$$\tag{5.3}\begin{aligned}
&\liminf_{u\to\infty}\mathbf{P}\{X/u>\lambda \mid Y=u\}\\
&\ge \liminf_{\ell\to\infty}\liminf_{u\to\infty}\Big(\mathbf{P}\big\{X^{(\ell,\varepsilon)}/u>\lambda+\delta \,\big|\, Y=u\big\}-\sum_{i=1}^{n}\frac{1}{\delta_i}\mathbf{E}\big\{|(X-X^{(\ell,\varepsilon)})_i/u| \,\big|\, Y=u\big\}\Big)\\
&= \liminf_{\ell\to\infty}\liminf_{u\to\infty}\mathbf{P}\big\{X^{(\ell,\varepsilon)}/u>\lambda+\delta \,\big|\, Y=u\big\}\;-\;O(\varepsilon)\qquad\text{as }\ \varepsilon\downarrow 0.
\end{aligned}$$

Now observe that by calculations similar to those that are featured in (3.7),

$$\mathbf{P}\big\{|X_{k,\varepsilon}^{(+)}/u|>|\lambda| \,\big|\, Y=u\big\}=\int\limits_{\{x\in\mathbb{R}\,:\,|k\varepsilon x|>|\lambda|u\}}\frac{f_{Y_{k,\varepsilon}^{(+)}}(x)\,f_{Y-Y_{k,\varepsilon}^{(+)}}(u-x)}{f_Y(u)}\,dx\to 0$$

as $u\to\infty$ for $k\in\mathbb{Z}^n$ and $\lambda\in\mathbb{R}^n\setminus\{0\}$, while

$$\mathbf{P}\big\{X^{(-)}_{k,\varepsilon}/u>\lambda \;\big|\; Y=u\big\} = \int\limits_{\{x\in\mathbb{R}\,:\,k\varepsilon x>\lambda u\}} \frac{f_{Y^{(-)}_{k,\varepsilon}}(x)\, f_{Y-Y^{(-)}_{k,\varepsilon}}(u-x)}{f_Y(u)}\,dx$$

$$\to \begin{cases} 0 & \text{for} \quad k\varepsilon-\lambda \not> 0 \\ \dfrac{2\,\|I_{B_k} g\|_\alpha^\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha} & \text{for} \quad k\varepsilon-\lambda>0 \end{cases}$$

when $\lambda=(\lambda_1,\ldots,\lambda_n)>0$. For $\lambda,\gamma\neq 0$ and distinct $j,k\in\mathbb{Z}^n$ we further have

$$(5.4) \qquad \mathbf{P}\Big\{|X^{(\pm)}_{j,\varepsilon}/u|>|\lambda|,\ |X^{(\pm)}_{k,\varepsilon}/u|>|\gamma| \;\Big|\; Y=u\Big\}$$

$$= \int\limits_{\{(x,y)\in\mathbb{R}^2\,:\,|j\varepsilon x|>|\lambda|u,\ |k\varepsilon y|>|\gamma|u\}} \frac{f_{Y^{(\pm)}_{j,\varepsilon}}(x)\, f_{Y^{(\pm)}_{k,\varepsilon}}(y)\, f_{Y-Y^{(\pm)}_{j,\varepsilon}-Y^{(\pm)}_{k,\varepsilon}}(u-x-y)}{f_Y(u)}\,dxdy \to 0.$$

Given vectors $\lambda,\delta>0$ satisfying $\lambda-4\delta>0$, these asymptotic relations show that

$$(5.5) \qquad \limsup_{u\to\infty} \mathbf{P}\big\{X^{(\ell,\varepsilon)}/u>\lambda-\delta \;\big|\; Y=u\big\}$$

$$\leq \limsup_{u\to\infty} \mathbf{P}\Big\{\sum_{\|k\|\leq\ell} X^{(-)}_{k,\varepsilon}/u>\lambda-2\delta \;\Big|\; Y=u\Big\}$$

$$+\limsup_{u\to\infty} \sum_{i=1}^{n} \sum_{\|k\|\leq\ell} \mathbf{P}\Big\{(X^{(+)}_{k,\varepsilon})_i/u>\frac{\delta_i}{(2\ell+1)^n} \;\Big|\; Y=u\Big\}$$

$$= \limsup_{u\to\infty} \mathbf{P}\Big\{\sum_{\|k\|\leq\ell} X^{(-)}_{k,\varepsilon}/u>\lambda-2\delta,\ \bigcup_{\|j\|\leq\ell}\Big\{|X^{(-)}_{j,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n}\Big\} \;\Big|\; Y=u\Big\}$$

$$\leq \limsup_{u\to\infty} \mathbf{P}\Big\{\sum_{\|k\|\leq\ell} X^{(-)}_{k,\varepsilon}/u>\lambda-2\delta,$$

$$\bigcup_{\|i\|\leq\ell}\Big[\Big\{|X^{(-)}_{i,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n}\Big\}\cap\bigcap_{\|j\|\leq\ell,\,j\neq i}\Big\{|X^{(-)}_{j,\varepsilon}/u|\leq\frac{|\delta|}{(2\ell+1)^n}\Big\}\Big] \;\Big|\; Y=u\Big\}$$

$$+\limsup_{u\to\infty} \mathbf{P}\Big\{\sum_{\|k\|\leq\ell} X^{(-)}_{k,\varepsilon}/u>\lambda-2\delta,$$

$$\bigcup_{\|i\|\leq\ell}\Big[\Big\{|X^{(-)}_{i,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n}\Big\}\cap\bigcup_{\|j\|\leq\ell,\,j\neq i}\Big\{|X^{(-)}_{j,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n}\Big\}\Big] \;\Big|\; Y=u\Big\}$$

$$\leq \limsup_{u\to\infty} \sum_{\|i\|\leq\ell} \mathbf{P}\big\{X^{(-)}_{i,\varepsilon}/u>\lambda-3\delta \;\big|\; Y=u\big\}$$

$$+\limsup_{u\to\infty} \sum_{\|i\|\leq\ell} \sum_{\|j\|\leq\ell,\,j\neq i} \mathbf{P}\Big\{|X^{(-)}_{i,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n},\ |X^{(-)}_{j,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n} \;\Big|\; Y=u\Big\}$$

$$\leq \sum_{\{i\in\mathbb{Z}^n\,:\,\|i\|\leq\ell,\,i\varepsilon>\lambda-4\delta\}} \frac{2\,\|I_{B_i} g\|_\alpha^\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha}$$

$$\to \big\langle |g|^\alpha I_{\{x\in G^-:\, f(x)/g(x)\,>\,\lambda-4\delta\}}\big\rangle \big/ \langle g^-\rangle_\alpha \quad \text{as} \quad \ell\to\infty \quad \text{and} \quad \varepsilon\downarrow 0$$

(in that order). Similarly we obtain

$$\text{(5.6)} \qquad \liminf_{u\to\infty} \mathbf{P}\big\{X^{(\ell,\varepsilon)}/u>\lambda+\delta \;\big|\; Y=u\big\}$$

$$\geq \liminf_{u\to\infty} \mathbf{P}\Bigg\{\sum_{\|k\|\leq\ell} X^{(-)}_{k,\varepsilon}/u>\lambda+2\delta \;\Bigg|\; Y=u\Bigg\}$$

$$\quad - \limsup_{u\to\infty} \sum_{i=1}^{n} \sum_{\|k\|\leq\ell} \mathbf{P}\Bigg\{(X^{(+)}_{k,\varepsilon})_i/u<-\frac{\delta_i}{(2\ell+1)^n} \;\Bigg|\; Y=u\Bigg\}$$

$$\geq \liminf_{u\to\infty} \mathbf{P}\Bigg\{\bigcup_{\|k\|\leq\ell} \big\{X^{(-)}_{k,\varepsilon}/u>\lambda+3\delta\big\} \;\Bigg|\; Y=u\Bigg\}$$

$$\quad - \limsup_{u\to\infty} \mathbf{P}\Bigg\{\bigcup_{\|k\|\leq\ell}\Bigg[\big\{X^{(-)}_{k,\varepsilon}/u>\lambda+3\delta\big\}\cap \bigcup_{\|j\|\leq\ell,\, j\neq k}\Big\{|X^{(-)}_{j,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n}\Big\}\Bigg]\;\Bigg|\; Y=u\Bigg\}$$

$$\geq \liminf_{u\to\infty} \sum_{\|k\|\leq\ell} \mathbf{P}\big\{X^{(-)}_{k,\varepsilon}/u>\lambda+3\delta \;\big|\; Y=u\big\}$$

$$\quad - \limsup_{u\to\infty} \sum_{\|k\|\leq\ell} \sum_{\|j\|\leq\ell,\, j\neq k} \mathbf{P}\Big\{X^{(-)}_{k,\varepsilon}/u>\lambda+3\delta,\; X^{(-)}_{j,\varepsilon}/u>\lambda+3\delta \;\Big|\; Y=u\Big\}$$

$$\quad - \limsup_{u\to\infty} \sum_{\|k\|\leq\ell} \sum_{\|j\|\leq\ell,\, j\neq k} \mathbf{P}\Bigg\{X^{(-)}_{k,\varepsilon}/u>\lambda+3\delta,\; |X^{(-)}_{\ell,\varepsilon}/u|>\frac{|\delta|}{(2\ell+1)^n} \;\Bigg|\; Y=u\Bigg\}$$

$$\geq \sum_{\{k\in\mathbb{Z}^n:\,\|k\|\leq\ell\,,\,k\varepsilon\,>\,\lambda+4\delta\}} \frac{2\,\|I_{B_k}g\|_\alpha^\alpha}{(1+\beta_Y)\,\|g\|_\alpha^\alpha}$$

$$\to \big\langle |g|^\alpha I_{\{x\in G^-:\, f(x)/g(x)\,>\,\lambda+4\delta\}}\big\rangle \big/ \langle g^-\rangle_\alpha \quad \text{as} \quad \ell\to\infty \quad \text{and} \quad \varepsilon\downarrow 0.$$

Now (5.1) follows from combining (5.2) and (5.3) with (5.5) and (5.6). □

Corollary 4. *Let* $\{f_u:\mathbb{R}\to\mathbb{R}^n\}_{u>0}$ *be a family of maps with components* $f_{u,i}\in\mathbb{L}^\alpha(\mathbb{R})$ *for* $i=1,\ldots,n$, *and consider the* $\mathbb{R}^{n+1}$*-valued* α*-stable random vector*

$$(X_u,Y) \;=\; \big(\textstyle\int_{\mathbb{R}} f_u\,d\xi,\int_{\mathbb{R}} g\,d\xi\big) \;=\; \big(\int_{\mathbb{R}} f_{u,1}\,d\xi,\ldots,\int_{\mathbb{R}} f_{u,n}\,d\xi,\int_{\mathbb{R}} g\,d\xi\big)$$

where $g\in\mathbb{L}^\alpha(\mathbb{R})$ *with* $\langle g^-\rangle_\alpha>0$. *If* $f_{u,i}\to_{\mathbb{L}^\alpha(\mathbb{R})} f_i$ *as* $u\to\infty$ *for* $i=1,\ldots,n$, *for some map* $f:\mathbb{R}\to\mathbb{R}^n$, *we then have*

$$(X_u/u\,|\,Y=u)\to_d Z \quad \textit{where} \quad \mathbf{P}\{Z\leq z\} = \big\langle |g|^\alpha I_{\{x\in G^-:\,f(x)/g(x)\leq z\}}\big\rangle \big/ \langle g^-\rangle_\alpha.$$

Proof. Writing $X\equiv\int_{\mathbb{R}} f\,d\xi$, Corollary 3 shows that $\mathbf{E}\big\{|(X_u-X)_i/u|\;\big|\;Y=u\big\}\to 0$ for $i=1,\ldots,n$. Hence the corollary follows from Theorem 3. □

When the conditional law $(X/u \mid Y=u)$ converges weakly, convergence of moments of order $\varrho\in(0,\alpha)$ follows from Corollary 3, while convergence of moments of order $\varrho\in[\alpha,2)$ follows from Theorem 1 if $\langle|f|,|g|\rangle_{2,\alpha}<\infty$. Moreover, probabilities and moments conditoned on the event that $Y>u$ also converge:

Corollary 5. *Consider the α-stable random variable (X_u,Y) given by (3.8) where $f_u\to_{\mathbb{L}^\alpha(\mathbb{R})} f$ as $u\to\infty$ and $\langle g^-\rangle_\alpha>0$. Further suppose that $\varrho\in(0,\alpha)$, or that $\varrho\in[\alpha,2)$ and $\limsup_{u\to\infty}\langle|f_u|,|g|\rangle_{2,\alpha}<\infty$. Then we have*

$$(5.7)\quad \lim_{u\to\infty}\mathbf{E}\big\{|X_u/u|^\varrho I_{\{X_u/u>\lambda\}}\,\big|\,Y=u\big\} = \big\langle|f|^\varrho|g|^{\alpha-\varrho}I_{\{x\in G^-:\,f(x)/g(x)>\lambda\}}\big\rangle\big/\langle g^-\rangle_\alpha$$

for continuity points $\lambda\in\mathbb{R}$ of the function on the right-hand side. Moreover

$$\begin{aligned}
\lim_{u\to\infty}\mathbf{E}\big\{[(X_u/u)^+]^\varrho\,\big|\,Y=u\big\} &= \langle(f^{\langle\varrho\rangle}g^{\langle\alpha-\varrho\rangle})^+I_{G^-}\rangle/\langle g^-\rangle_\alpha,\\
\lim_{u\to\infty}\mathbf{E}\big\{[(X_u/u)^-]^\varrho\,\big|\,Y=u\big\} &= \langle(f^{\langle\varrho\rangle}g^{\langle\alpha-\varrho\rangle})^-I_{G^-}\rangle/\langle g^-\rangle_\alpha,\\
\lim_{u\to\infty}\mathbf{E}\big\{|X_u/u|^\varrho\,\big|\,Y=u\big\} &= \langle|f|^\varrho|g|^{\alpha-\varrho}I_{G^-}\rangle/\langle g^-\rangle_\alpha,\\
\lim_{u\to\infty}\mathbf{E}\big\{(X_u/u)^{\langle\varrho\rangle}\,\big|\,Y=u\big\} &= \langle f^{\langle\varrho\rangle}g^{\langle\alpha-\varrho\rangle}I_{G^-}\rangle/\langle g^-\rangle_\alpha.
\end{aligned}$$

Proof. Since $(X_u/u \mid Y=u)\to_d Z$ by Corollary 4, it follows that

$$\big(|X_u/u|^\varrho I_{\{X_u/u>\lambda\}}\,\big|\,Y=u\big)\to_d |Z|^\varrho I_{\{Z>\lambda\}}\quad\text{when}\quad \big\langle|g|^\alpha I_{\{x\in G^-:\,f(x)/g(x)=\lambda\}}\big\rangle=0.$$

Further we obviously have

$$\big\langle|g|^\alpha I_{\{x\in G^-:\,f(x)/g(x)=\lambda\}}\big\rangle = \big\langle|f|^\varrho|g|^{\alpha-\varrho}I_{\{x\in G^-:\,f(x)/g(x)=\lambda\}}\big\rangle/|\lambda|^\varrho\quad\text{for}\quad\lambda\neq0.$$

Hence continuity points $\lambda\neq0$ for the right-hand side of (5.7) also are continuity points for $\big\langle|g|^\alpha I_{\{x\in G^-:\,f(x)/g(x)>\lambda\}}\big\rangle$. The fact that (5.7) holds for continuity points $\lambda\neq0$ thus follows if the family $\big\{\big(|X_u/u|^\varrho I_{\{X_u/u>\lambda\}}\,\big|\,Y=u\big)\big\}_{u>0}$ is uniformly integrable. However, by Corollary 3 when $\varrho<\alpha$, and by Corollary 1 when $\varrho\in[\alpha,2)$ and $\limsup_{u\to\infty}\langle|f_u|,|g|\rangle_{2,\alpha}<\infty$, we have $\limsup_{u\to\infty}\mathbf{E}\big\{|X_u/u|^\rho\,\big|\,Y=u\big\}<\infty$ for some $\rho>\varrho$. By elementary considerations this establishes uniform integrability.

By application of (5.7), for continuity points $\lambda\neq0$, we readily obtain

$$\begin{aligned}
\langle(f^{\langle\varrho\rangle}g^{\langle\alpha-\varrho\rangle})^+I_{G^-}\rangle/\langle g^-\rangle_\alpha &= \big\langle|f|^\varrho|g|^{\alpha-\varrho}I_{\{x\in G^-:\,f(x)/g(x)>0\}}\big\rangle/\langle g^-\rangle_\alpha - \limsup_{\varepsilon\downarrow0}|\varepsilon|^\varrho\\
&\le \liminf_{\varepsilon\downarrow0}\big\langle|f|^\varrho|g|^{\alpha-\varrho}I_{\{x\in G^-:\,f(x)/g(x)>\varepsilon\}}\big\rangle/\langle g^-\rangle_\alpha
\end{aligned}$$

$$\leq \limsup_{\varepsilon\downarrow 0} \liminf_{u\to\infty} \mathbf{E}\big\{ |X_u/u|^\varrho I_{\{X_u/u>\varepsilon\}} \,\big|\, Y=u \big\}$$
$$\leq \liminf_{u\to\infty} \mathbf{E}\big\{ [(X_u/u)^+]^\varrho \,\big|\, Y=u \big\}$$
$$\leq \limsup_{u\to\infty} \mathbf{E}\big\{ [(X_u/u)^+]^\varrho \,\big|\, Y=u \big\}$$
$$\leq \liminf_{\varepsilon\uparrow 0} \limsup_{u\to\infty} \mathbf{E}\big\{ |X_u/u|^\varrho I_{\{X_u/u>\varepsilon\}} \,\big|\, Y=u \big\}$$
$$\leq \limsup_{\varepsilon\uparrow 0} \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\, f(x)/g(x)>\varepsilon\}} \big\rangle / \langle g^-\rangle_\alpha$$
$$\leq \big\langle |f|^\varrho |g|^{\alpha-\varrho} I_{\{x\in G^-:\, f(x)/g(x)>0\}} \big\rangle / \langle g^-\rangle_\alpha + \limsup_{\varepsilon\uparrow 0} |\varepsilon|^\varrho.$$

This completes the proof of (5.7), and by application of what has already been proved to the variable $-X_u$, the proof of the whole corollary. □

Example. For a moving average process $X(t) = \int_{x\in\mathbb{R}} g(t+x)\, d\xi(x)$ where $g \in \mathbb{L}^\alpha(\mathbb{R})$ with $\langle g^-\rangle_\alpha > 0$, we have $\big(X(t_1)/u, \ldots, X(t_n)/u \,\big|\, X(0)=u\big) \to_d Z$ where

$$\mathbf{P}\{Z \leq z\} = \big\langle I_{\{x\in G^-:\, g(t_1+x)/g(x)\leq z_1,\, \ldots,\, g(t_n+x)/g(x)\leq z_n\}} |g|^\alpha \big\rangle \,/\, \langle g^-\rangle_\alpha. \quad \#$$

Under the hypothesis of Corollary 4, (3.4) and Corollary 4 imply that

$$\mathbf{P}\{X_u/u > z \,|\, Y>u\} = \int_1^\infty \mathbf{P}\left\{ \frac{X_u}{y\,u} > \frac{z}{y} \,\middle|\, Y = y\,u \right\} \frac{u\, f_Y(y\,u)\; dy}{\mathbf{P}\{Y>u\}}$$
$$\to \int_1^\infty \big\langle |g|^\alpha I_{\{x\in G^-:\, f(x)/g(x) > z/y\}} \big\rangle \,\frac{\alpha\; dy}{y^{\alpha+1}} \Big/ \langle g^-\rangle_\alpha.$$

In the special case when $f_u = f$ and $X_u = X$ this was shown via a direct argument by Samorodnitsky (1988, Theorem 3.1). We now adress convergence of moments:

Corollary 6. *Consider the α-stable random variable (X_u, Y) given by* (3.8) *where $f_u \to_{\mathbb{L}^\alpha(\mathbb{R})} f$ as $u\to\infty$ and $\langle g^-\rangle_\alpha > 0$. For each choice of $\varrho\in(0,\alpha)$ we then have*

$$\lim_{u\to\infty} \mathbf{E}\big\{ |X_u/u|^\varrho I_{\{X_u/u>\lambda\}} \,\big|\, Y>u \big\}$$
$$= \int_{y=1}^{y=\infty} \int_{x\in G^-} |f(x)|^\varrho\, |g(x)|^{\alpha-\varrho}\, I_{\{f(x)/g(x)>\lambda/y\}}\, \frac{\alpha\, dx dy}{y^{\alpha+1-\varrho}}$$
$$= \int_{x\in G^-} \left([f(x)^+]^\varrho\, |g(x)|^{\alpha-\varrho} - \frac{f(x)^{\langle\varrho\rangle}\, |g(x)|^{\alpha-\varrho}}{[1 \vee (\lambda g(x)/f(x))]^{\alpha-\varrho}} \right) \frac{\alpha\, dx}{\alpha-\varrho} \qquad \textit{for } \lambda\in\mathbb{R}.$$

Proof. In view of the obvious fact that

$$\mathbf{E}\left\{ \left|\frac{X_u}{u}\right|^\varrho I_{\{X_u/u>\lambda\}} \middle| Y>u \right\} = \int_1^\infty \mathbf{E}\left\{ \left|\frac{X_u}{uy}\right|^\varrho I_{\{X_u/(uy)>\lambda/y\}} \middle| Y=uy \right\} \frac{y^\varrho u f_Y(yu)\, dy}{\mathbf{P}\{Y>u\}},$$

the corollary follows from (5.7) and a change of the order of integration in the resulting limit if we can establish dominated convergence. However, dominated convergence is a simple consequence of (3.4) and Corollary 3. □

6. Upcrossings of α-stable processes. Choose a finite interval $I=[a,b]$ and a family of maps $\{f_t(\cdot)\in\mathbb{L}^\alpha(\mathbb{R}) : t\in I\}$, and consider the α-stable process

$$\eta(t) \equiv \text{ separable version of } \int_{-\infty}^{\infty} f_t(x)\,d\xi(x) \quad \text{for } t\in I: \tag{6.1}$$

It is well known that each non-pathological strictly α-stable process has this representation in law [e.g., Samorodnitsky and Taqqu (1994, Theorem 13.2.1)].

We shall assume that $\eta(t)$ is uniformly $\mathbf{P}$-differentiable, which means that

$$\lim_{\varepsilon\downarrow 0}\sup\Big\{\big\|(t-s)^{-1}[f_t(\cdot)-f_s(\cdot)-(t-s)f_s'(\cdot)]\big\|_\alpha : s,t\in I,\ 0<|t-s|\le\varepsilon\Big\}=0 \tag{6.2}$$

for some family of maps $\{f_t'(\cdot)\in\mathbb{L}^\alpha(\mathbb{R}) : t\in I\}$. Further we require that

$$\inf_{t\in I}\|f_t(\cdot)\|_\alpha>0 \qquad \text{and} \qquad \sup_{t\in I}\|f_t'(\cdot)\|_\alpha<\infty. \tag{6.3}$$

For a non-zero stationary process $\{\eta(t)\}_{t\in\mathbb{R}}$, (6.2) and (6.3) boil down to

$$\lim_{t\to 0}\big\|t^{-1}[f_t(\cdot)-f_0(\cdot)-tf_0'(\cdot)]\big\|_\alpha=0 \quad \text{for some} \quad f_0'(\cdot)\in\mathbb{L}^\alpha(\mathbb{R}). \tag{6.4}$$

In their study of stationary $S\alpha S$-processes, Adler and Samorodnitsky (1997) assume that $f_{(\cdot)}(x)$ is absolutely continuous for almost all x with $\int_0^1 \|\frac{\partial}{\partial t}f_t(\cdot)\|_\alpha\,dt<\infty$: The difference between this requirement and (6.4) appears to be minusculous.

Observe that, choosing a power $\varrho\in(1,\alpha)$, (6.2) readily gives

$$\begin{aligned}\mathbf{E}\{|\eta(t)-\eta(s)|^\varrho\} &\le \sup_{\beta\in[-1,1]}\mathbf{E}\big\{|S_\alpha(1,\beta)|^\varrho\big\}\,\big[\|f_t-f_s-(t-s)f_s'\|_\alpha+|t-s|\,\|f_s'\|_\alpha\big]^\varrho\\ &\le \text{constant}\times|t-s|^\varrho \qquad \text{for } s,t\in I.\end{aligned}$$

A well-known and classic argument [e.g., Cramér and Leadbetter (1967, Section 4.2)] therefore shows that $\eta(t)$ has continuous sample paths a.s.

Writing $\eta'(t)=\int_{\mathbb{R}} f_t'\,d\xi$, Theorem 4 below states that the expected number of upcrossings of a level u by $\{\eta(t)\}_{t\in I}$ is given by

$$\mu(I;u)=\int_I \mathbf{E}\{\eta'(t)^+ \,|\, \eta(t)=u\}\,f_{\eta(t)}(u)\,dt. \tag{6.5}$$

Rice (1944, 1945) proposed this formula for differentiable processes. Under additional technical conditions, proofs were given by Leadbetter (1966) and Marcus

(1977) for stationary and non-stationary processes, respectively, but although natural and reasonable, even in the stationary case these conditions are so forbidding that they have been verified for very few processes except Gaussian ones. Indeed, when Adler and Samorodnitsky (1997) verify Marcus' conditions for $S\alpha S$-processes (via a 10-page argument), the key-ingredient in their proof is that symmetric α-stable processes allow representations as mixtures of centered Gaussian processes.

If $(\eta'(t),\eta(t))$ has a density $f_{\eta'(t),\eta(t)}(x,y)$ that is a continuous function of y for almost all $t\in I$, then Proposition 1 and (2.5) combine with (6.5) to show that

$$\mu(I;u) \;=\; \int_I \big[\int_0^\infty x\, f_{\eta'(t),\eta(t)}(x,u)\,dx\big]\,dt.$$

Our proof of (6.5) builds on Lemmas 7.2.1 and 7.2.2 in Leadbetter, Lindgren and Rootzén (1983). Albeit these lemmas are stated for stationary processes, only the last paragraph of the proof of Lemma 7.2.2 (iii) uses stationarity. All other arguments are valid for processes $\{\eta(t)\}_{t\in I}$ possessing continuous paths a.s. and a continuous univariate marginal distribution function $F_{\eta(t)}$ at each $t\in I$. Further the mesh used when approximating $\eta(t)$ with a step process need not be uniform.

For each family of sequences $\big\{a=s_0^{(n)}\le s_1^{(n)}\le\ldots\le s_n^{(n)}\le s_{n+1}^{(n)}=b\,:\,n\in\mathbb{N}\big\}$ such that $q_k^{(n)}\equiv s_k^{(n)}-s_{k-1}^{(n)}$ satisfy $\lim_{n\to\infty}\sup_{1\le k\le n+1} q_k^{(n)}=0$ we thus have

$$\mu(I;u) \;=\; \lim_{n\to\infty}\sum_{k=1}^{n+1}\mathbf{P}\big\{\eta(s_{k-1}^{(n)})<u<\eta(s_k^{(n)})\big\}. \tag{6.6}$$

Theorem 4. *Consider the process* $\{\eta(t)\}_{t\in I}$ *given by* (6.1) *where the maps* $\big\{f_t(\cdot)\in\mathbb{L}^\alpha(\mathbb{R})\,:\,t\in I\big\}$ *satisfy* (6.2) *and* (6.3). *Then the expected number of upcrossings* $\mu(I;u)$ *of the level* u *by* $\{\eta(t)\}_{t\in I}$ *satisfies Rice's formula* (6.5).

Proof. Take $\delta\in(0,1)$ and $\Delta\in(1,\infty)$, and let $t_k^{(n)}=a+k(b-a)/n$ for $k=0,\ldots,n$. Define a mesh $\{s_k^{(n)}\}_{k=0}^{n+1}$ by setting $s_0^{(n)}=a$, choosing $s_k^{(n)}\in(t_{k-1}^{(n)},t_k^{(n)}]$ so that

$$\int_\delta^\Delta \mathbf{P}\Big\{\eta'(s_k^{(n)})>(1-\delta)x\;\Big|\;\eta(s_k^{(n)})=u+q_k^{(n)}x\Big\}\,f_{\eta(s_k^{(n)})}\big(u+q_k^{(n)}x\big)\,dx\;-\;\delta$$

$$\le \inf_{s\in(t_{k-1}^{(n)},t_k^{(n)}]}\int_\delta^\Delta \mathbf{P}\Big\{\eta'(s)>(1-\delta)x\,\Big|\,\eta(s)=u+(s-s_{k-1}^{(n)})x\Big\}\,f_{\eta(s)}\big(u+(s-s_{k-1}^{(n)})x\big)\,dx$$

for $k=1,\ldots,n$ [where $q_k^{(n)}=s_k^{(n)}-s_{k-1}^{(n)}$ as above], and setting $s_{n+1}^{(n)}=b$. Also note that the Markov inequality combines with Corollary 2 and (4.15) to show that

$$\mathbf{P}\big\{\eta'(s)>(1-\delta)x\;\big|\;\eta(s)=u+(s-s_{k-1}^{(n)})x\big\}\,f_{\eta(s)}\big(u+(s-s_{k-1}^{(n)})x\big) \tag{6.7}$$

$$\leq \frac{K_\alpha\, D_\alpha^{1-1/\alpha}\, \|f_s'\|_\alpha}{(\alpha-1)^{1/\alpha}\, \|f_s\|_\alpha\, (1-\delta)x} \qquad \text{for } x>0 \text{ and } s\in I.$$

Using (2.3) and dominated convergence [guaranteed by (6.7)], we therefore obtain

$$(6.8) \quad \sum_{k=1}^{n+1} q_k^{(n)} \int_\delta^\Delta \mathbf{P}\Big\{\eta'(s_k^{(n)}) > (1-\delta)x \;\Big|\; \eta(s_k^{(n)}) = u + q_k^{(n)} x\Big\}\, f_{\eta(s_k^{(n)})}\big(u+q_k^{(n)}x\big)\, dx$$

$$\leq \sum_{k=1}^{n} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \int_\delta^\Delta \mathbf{P}\Big\{\eta'(s) > (1-\delta)x \,\Big|\, \eta(s) = u+(s-s_{k-1}^{(n)})x\Big\}\, f_{\eta(s)}\big(u+(s-s_{k-1}^{(n)})x\big)\, dx$$

$$+ [2(b-a)/n]\,(\Delta-\delta)\, \sup\nolimits_{x\in[\delta,\Delta]} f_{\eta(b)}(u+q_{n+1}^{(n)}x)$$

$$+ \delta(b-a)$$

$$\to \int_a^b \int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1-\delta)x \mid \eta(s)=u\}\, dx\, f_{\eta(s)}(u)\, ds \;+\; 0 \;+\; \delta(b-a) \qquad \text{as } n\to\infty$$

$$\to \int_a^b \mathbf{E}\{\eta'(s)^+ \mid \eta(s)=u\}\, f_{\eta(s)}(u)\, ds \qquad \text{as } \delta\downarrow 0 \text{ and } \Delta\uparrow\infty.$$

Defining a second mesh by setting $s_0^{(n)} = a$, choosing $s_k^{(n)} \in (t_{k-1}^{(n)}, t_k^{(n)}]$ so that

$$\int_\delta^\Delta \mathbf{P}\Big\{\eta'(s_k^{(n)}) > (1+\delta)x \;\Big|\; \eta(s_k^{(n)}) = u+q_k^{(n)}x\Big\}\, f_{\eta(s_k^{(n)})}\big(u+q_k^{(n)}x\big)\, dx \;+\; \delta$$

$$\geq \sup_{s\in(t_{k-1}^{(n)}, t_k^{(n)}]} \int_\delta^\Delta \mathbf{P}\Big\{\eta'(s) > (1+\delta)x \,\Big|\, \eta(s) = u+(s-s_{k-1}^{(n)})x\Big\}\, f_{\eta(s)}\big(u+(s-s_{k-1}^{(n)})x\big)\, dx,$$

for $k=1,\ldots,n$, and setting $s_{n+1}^{(n)} = b$, (2.3) and Fatou's Lemma further show that

$$(6.9) \quad \sum_{k=1}^{n+1} q_k^{(n)} \int_\delta^\Delta \mathbf{P}\Big\{\eta'(s_k^{(n)}) > (1+\delta)x \;\Big|\; \eta(s_k^{(n)}) = u + q_k^{(n)} x\Big\}\, f_{\eta(s_k^{(n)})}\big(u+q_k^{(n)}x\big)\, dx$$

$$\geq \sum_{k=1}^{n} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \int_\delta^\Delta \mathbf{P}\Big\{\eta'(s) > (1+\delta)x \,\Big|\, \eta(s) = u+(s-s_{k-1}^{(n)})x\Big\}\, f_{\eta(s)}\big(u+(s-s_{k-1}^{(n)})x\big)\, dx$$

$$- \delta(b-a)$$

$$\to \int_a^b \int_\delta^\Delta \mathbf{P}\{\eta'(s) > (1+\delta)x \mid \eta(s)=u\}\, dx\, f_{\eta(s)}(u)\, ds \;-\; \delta(b-a) \qquad \text{as } n\to\infty$$

$$\to \int_a^b \mathbf{E}\{\eta'(s)^+ \mid \eta(s)=u\}\, f_{\eta(s)}(u)\, ds \qquad \text{as } \delta\downarrow 0 \text{ and } \Delta\uparrow\infty.$$

Now choose a power $\varrho \in (1,\alpha)$. Using the Markov inequality together with Corollary 2 and (4.15) as when establishing (6.7), (6.6) then yields

$$\mu([a,b];u)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n+1} \int_0^\infty q_k^{(n)}\, \mathbf{P}\Big\{ \frac{\eta(s_k^{(n)})-\eta(s_{k-1}^{(n)})}{q_k^{(n)}} > x \,\Big|\, \eta(s_k^{(n)}) = u + q_k^{(n)} x \Big\} f_{\eta(s_k^{(n)})}(u+q_k^{(n)}x)\, dx$$

$$\le \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_0^\delta q_k^{(n)}\, D_\alpha \, \|f_{s_k^{(n)}}\|_\alpha^{-1}\, dx$$

$$+ \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_\delta^\Delta q_k^{(n)}\, \mathbf{P}\Big\{ \eta'(s_k^{(n)}) > (1-\delta)x \,\Big|\, \eta(s_k^{(n)}) = u + q_k^{(n)} x \Big\}\, f_{\eta(s_k^{(n)})}(u+q_k^{(n)}x)\, dx$$

$$+ \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_\delta^\Delta q_k^{(n)}\, \frac{K_\alpha\, D_\alpha^{1-1/\alpha}\, \big\|(-q_k^{(n)})^{-1}[f_{s_{k-1}^{(n)}} - f_{s_k^{(n)}} - (-q_k^{(n)}) f'_{s_k^{(n)}}]\big\|_\alpha}{(\alpha-1)^{1/\alpha}\, \|f_{s_k^{(n)}}\|_\alpha\, (\delta\, x)}\, dx$$

$$+ \limsup_{n\to\infty} \sum_{k=1}^{n+1} \int_\Delta^\infty q_k^{(n)}\, \frac{K_\alpha\, D_\alpha^{1-\varrho/\alpha}\, \big\|(q_k^{(n)})^{-1}[f_{s_k^{(n)}} - f_{s_{k-1}^{(n)}}]\big\|_\alpha^\varrho}{(\alpha-\varrho)^{\varrho/\alpha}\, \|f_{s_k^{(n)}}\|_\alpha\, x^\varrho}\, dx.$$

Sending $\delta\downarrow 0$ and $\Delta\uparrow\infty$, and invoking (6.2)-(6.3) and (6.8), it follows that

$$\mu([a,b];u) \le \int_a^b \mathbf{E}\{\eta'(s)^+ \,|\, \eta(s)=u\}\, f_{\eta(s)}(u)\, ds.$$

In a similar but somewhat simpler way we obtain

$$\mu([a,b];u)$$
$$\ge \liminf_{n\to\infty} \sum_{k=1}^{n} \int_\delta^\Delta q_k^{(n)}\, \mathbf{P}\Big\{ \eta'(s_k^{(n)}) > (1+\delta)x \,\Big|\, \eta(s_k^{(n)}) = u + q_k^{(n)} x \Big\}\, f_{\eta(s_k^{(n)})}\big(u+q_k^{(n)}x\big)\, dx$$

$$- \limsup_{n\to\infty} \sum_{k=1}^{n} \int_\delta^\Delta q_k^{(n)}\, \frac{K_\alpha\, D_\alpha^{1-1/\alpha}\, \big\|(-q_k^{(n)})^{-1}[f_{s_{k-1}^{(n)}} - f_{s_k^{(n)}} - (-q_k^{(n)}) f'_{s_k^{(n)}}]\big\|_\alpha}{(\alpha-1)^{1/\alpha}\, \|f_{s_k^{(n)}}\|_\alpha\, (\delta\, x)}\, dx.$$

Sending $\delta\downarrow 0$ and $\Delta\uparrow\infty$, and using (6.2)-(6.3) and (6.9), we therefore conclude

$$\mu([a,b];u) \ge \int_a^b \mathbf{E}\{\eta'(s)^+ \,|\, \eta(s)=u\}\, f_{\eta(s)}(u)\, ds. \quad \square$$

In the particular case when $\eta(t)$ is stationary, (6.6) readily yields

$$\mu(I;u) \;=\; \mathrm{length}(I)\, \lim_{s\downarrow 0} s^{-1}\, \mathbf{P}\{\eta(-s) < u < \eta(0)\}. \tag{6.10}$$

Taking off from (6.10) rather than (6.6), and using the Markov inequality and Corollary 2, the proof of (6.5) then reduces to just a few lines of elementary calculations.

Remark. In the proof of Theorem 4 it is not crucial that bounds invoked on conditional moments possess the right rate as $u\to\infty$, and the bound (4.9), which for symmetric variables is implicit in Samorodnitsky and Taqqu (1991) [see also Samorodnitsky and Taqqu (1994, Section 5.1)], is sufficient. Since Samorodnitsky and Taqqu proved a version of (4.9) [with $\|f\|_\alpha^\alpha$ replaced by $\langle |f|, |g| \rangle_{\varrho,\alpha}$; cf. (1.3)] valid for moments of order $\varrho > \alpha$ of symmetric variables; it seems clear that an approach via (6.10) offers a much simplified proof of the result by Adler and Samorodnitsky (1997) concerning Rice's formula for stationary $S\alpha S$-processes also when $\alpha \leq 1$.

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