A Paley-Wiener Theorem for Convex Sets in $\mathbb{C}^n$

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Abstract

We study the Laplace transform on Hardy spaces on a class of convex domains in $\mathbb{C}^n$. We obtain a Paley-Wiener theorem with a norm that characterizes the entire functions of exponential type which occur as Laplace transforms. This is done by using the Fantappiè transform and the Borel transform to rewrite the Laplace transform and reduce the problem to known theorems in one complex variable.
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Chapter 1

Introduction

1.1 Background and formulation of the problem

Let X be some space of functions on a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, and let T be a continuous linear functional on this space. By continuity, the action of T is determined by its action on a dense subset of X. Often we will have that a linear system of exponentials $e^{\langle \cdot, z\rangle}$, where z belongs to some set S, is dense in X. If that is the case we will thus have that T is determined by its Laplace transform $\hat{T}(z) = T\left(e^{\langle \cdot, z\rangle}\right)$, $z \in S$. The Laplace transform will then be a function on S, and we get an isomorphism between the continuous linear functionals (the dual space) on X and a space of functions defined on S. The problem of describing the dual space in this way is very old.

For an example, consider the Hilbert space $L^2(-1,1)$ of square integrable functions on the interval $(-1,1)$. Every function $g \in L^2(-1,1)$ defines a continuous functional by

$$T(h) = \int_{-1}^{1} h(t)\overline{g(t)}dt, \quad h \in L^2(-1,1)$$

and the Laplace transform

$$\hat{T}(z) = T\left(e^{\langle \cdot, z\rangle}\right) = \int_{-1}^{1} e^{tz}\overline{g(t)}dt$$

is an entire function in $\mathbb{C}$. By the Cauchy-Schwarz inequality we immediately get that

$$|\hat{T}(z)| \leq Ce^{|\mathrm{Re}\, z|} \leq Ce^{|z|}, \tag{1.1}$$

and by the Plancherel theorem we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{T}(iy)|^2 dy = \|g\|^2_{L^2(-1,1)}.$$

The Paley-Wiener theorem is the converse of this.

Theorem 1.1 (Paley-Wiener). *Suppose that f(z) is an entire function in* $\mathbb{C}$ *such that*

$$|f(z)| \le Ce^{A|z|} \tag{1.2}$$

for some $A > 0$, *and*

$$\int_{-\infty}^{\infty} |f(iy)|^2\, dy < \infty. \tag{1.3}$$

Then there is a $g \in L^2(-A, A)$ *such that*

$$f(z) = \int_{-A}^{A} e^{tz}\overline{g(t)}\, dt.$$

There is another way of looking at this. Let us restrict the functional T to act on entire functions h. We then get a continuous functional on the space of entire functions, which is called an *analytic functional* (see also section 3.1 below). If we let γ be a contour around the interval $(-1, 1)$ in $\mathbb{C}$ and use Cauchy's formula we get

$$\begin{aligned} T(h) = \int_{-1}^{1} h(t)\overline{g(t)}dt &= \int_{-1}^{1}\left(\frac{1}{2\pi i}\int_\gamma \frac{h(z)}{z-t}dz\right)\overline{g(t)}dt \\ &= \frac{1}{2\pi i}\int_\gamma h(z)\left(\int_{-1}^{1}\frac{\overline{g(t)}}{z-t}dt\right)dz \end{aligned}$$

where the inner integral defines a function which is analytic in the complement of the interval $(-1, 1)$. If we for instance let T be the functional given by $g(t) \equiv 1$ we get

$$T(h) = \frac{1}{2\pi i}\int_\gamma h(z)\log\left(\frac{z+1}{z-1}\right)dz.$$

If ϕ is an analytic function in the complement of some other compact set K, then it defines an analytic functional μ in the same way by

$$\mu(h) = \int_\gamma h(z)\phi(z)dz \tag{1.4}$$

for some contour γ around K. Since this integral by Cauchy's theorem is independent of the contour γ we get that the functional μ is *carried* by the compact K in the sense that for every open set $\omega \supset K$

$$|\mu(h)| \le C_\omega \sup_{\zeta\in\omega}|h(\zeta)|.$$

In particular we get that $\hat{\mu}(z) = \mu(e^{z\cdot})$ is an entire function with

$$|\hat{\mu}(z)| \le C_\omega e^{H_\omega(z)}, \tag{1.5}$$

where

$$H_\omega(z) = \sup_{\zeta\in\omega}(\mathrm{Re}\, z\zeta)$$

is the *supporting function* of the set ω. The supporting function is 1-homogeneous, and it is easy to see that $H_\omega(e^{i\theta})$ is the length of the set ω projected on the ray from the origin with direction $e^{-i\theta}$. For example, the supporting function of the disk $D(0;R)$ is $H_{D(0;R)}(z) = R|z|$, and we saw in (1.1) that the supporting function of the interval $(-1,1)$ is $H_{(-1,1)}(z) = |\mathrm{Re}\, z|$.

The Polya theorem, a proof of which can be found in [1], is the converse of statement (1.5).

Theorem 1.2 (Polya). *If $K \subset \mathbb{C}$ is compact and convex, and $f(z)$ is an entire function which satisfies*

$$|f(z)| \le C_\omega e^{H_\omega(z)}$$

for every $\omega \supset K$, then it is the Laplace transform of a unique analytic functional μ which is carried by K.

If we return to the example above, the Polya theorem says that an entire function which satisfies (1.1) is the Laplace transform of an analytic functional μ carried by the interval $(-1,1)$. It can then be represented by a function ϕ as in (1.4). (In fact every analytic functional carried by K can be written as in (1.4) for some ϕ. For a discussion of analytic functionals in one dimension, see [1].)

The difference between the Polya theorem and the Paley-Wiener theorem is then that, when K is an interval, the latter gives an answer to when the integral in (1.4) can be replaced by an integral on the *boundary* of K, and not only on some curve in the complement.

In general, let K be a bounded convex domain, and let $E^2(K)$ be the Hardy space (also called the Smirnov space — see [6, chapter 10] for a discussion of Hardy spaces on general domains) of analytic functions on K that are limits of analytic polynomials with respect to the norm

$$\|h\|_{E^2(K)} = \left(\int_{\partial K} |h(t)|^2 d\sigma(t)\right)^{1/2},$$

where $d\sigma(t)$ denotes the arc-length element on ∂K. Let $P^2(K)$ be the class of entire functions which can be represented as

$$f(z) = \int_{\partial K} e^{zt}\overline{g(t)}d\sigma(t), \quad g \in E^2(K). \tag{1.6}$$

A function f is said to be of *exponential type* if it satisfies (1.2) for some A. The Paley-Wiener theorem then states that $f \in P^2(-A,A)$ if and only if f is of exponential type (where the A is the type of f), and f is square integrable on two rays orthogonal to the interval. In 1964 Levin [11, Appendix I] proved a generalization, which says that if K is a bounded, convex polygon then $f \in P^2(K)$ if and only if f is of exponential type and

$$\int_0^\infty |f(re^{i\theta_j})|^2 e^{-2H_K(re^{i\theta_j})} dr < \infty, \quad j = 1, \ldots, N, \tag{1.7}$$

where $-\theta_1, \ldots, -\theta_N$ are the directions of the normals to the sides of K.

If the set K is not a polygon, we do not have any directions which are distinguished in the above sense, and the condition on the function f must then include all rays. In [12] Likht gave the description of the class $P^2(K)$ when K is a circle, and in [9] Katsnel'son gave necessary conditions for f to belong to $P^2(K)$ for an arbitrary convex and compact K, and proved an analogue of Parseval's identity when K is a disc. In Theorem 4.1 below we give a generalization of this to $\mathbb{C}^n$.

In 1988 Lyubarskiĭ [14] solved the problem when K is a convex compact, with smooth boundary whose curvature is bounded away from 0 and ∞. Finally Lutsenko and Yulmukhametov [13] proved a theorem which can be formulated as follows.

Theorem 1.3. *Let K be a bounded convex domain in $\mathbb{C}$, and let $f(z)$ be an entire function. For f to admit the representation (1.6) it is necessary and sufficient that*

$$\int_{\mathbb{C}} |f(z)|^2 \frac{1}{D_K(z)} \Delta H_K(z) dm(z) < \infty \tag{1.8}$$

where

$$D_K(z) = \|e^{\lambda z}\|^2_{E^2(K)} = \int_{\partial K} e^{2\mathrm{Re}\,(\lambda z)} d\sigma(\lambda).$$

Moreover

$$\left(\int_{\mathbb{C}} |f(z)|^2 \frac{1}{D_K(z)} \Delta H_K(z) dm(z) \right)^{1/2} \sim \|g\|_{E^2(K)}. \tag{1.9}$$

In other words, the Laplace transform is an isomorphism between the Hilbert spaces $E^2(K)$ and $P^2(K)$, with the norms as in (1.9).

We can remark that if K satisfies the assumptions in Lyubarskiĭ [14] then the function $D_K(re^{i\theta}) \sim e^{2H_K(re^{i\theta})}/\sqrt{r}$ as $r \to \infty$, uniformly with respect to θ, and $\Delta H_K(z) \sim 1/|z|$. The condition (1.8) can then be written as

$$\int_0^{2\pi} \int_0^\infty |f(re^{i\theta})|^2 e^{-2H_K(re^{i\theta})}, r^{1/2} dr d\theta < \infty \tag{1.10}$$

which is exactly the (square of the) norm used by Lyubarskiĭ in [14]. Furthermore, if K is a polygon, and $\Delta H_K(z)$ is interpreted correctly, Theorem 1.3 reduces to Levin's theorem.

1.2 The results, and plan of the paper

In $\mathbb{C}^n$, the Laplace transform of the analytic functional μ will be defined in the same way as above, by letting μ act on the function $e^{z\cdot\zeta}$ with z as a parameter

$$\hat{\mu}(z) = \mu_\zeta(e^{z\cdot\zeta})$$

(where $z \cdot \zeta = \sum z_j\zeta_j$). The Polya theorem was generalized to $\mathbb{C}^n$ by Martineau [15]. Different proofs can be found e.g. in [4], [8, Section 4.7] and [3].

In this paper we will prove the following generalization of Theorem 1.3 to a class of compact domains in $\mathbb{C}^n$.

Theorem 1.4. *Let $K \subset \mathbb{C}^n$ be a bounded and strongly convex set with smooth boundary, and f an entire function. Then f is the Laplace transform of some $\psi \in E^2(K)$ if and only if*

$$\int |f(z)|^2 e^{-2H_K(z)}|z|^{n-1/2}(i\partial\bar{\partial}H_K)^n < \infty. \tag{1.11}$$

In that case we also have

$$\|\psi\|_{E^2(K)} \sim \left(\int |f(z)|^2 e^{-2H_K(z)}|z|^{n-1/2}(i\partial\bar{\partial}H_K)^n\right)^{1/2}. \tag{1.12}$$

When K is a so called *circled* set (see section 4.2 below) we can get a bit more. Using the function $\tau_n(r)$, which is defined in (4.9) and satisfies $\tau_n(r) \sim r^{n-1/2}$ as $r \to \infty$, we get inequalities which are sharp in the sense that we have equalities for the unit ball.

Theorem 1.5. *Let K be as in the previous theorem, and in addition circled. Then an entire function f is the Laplace transform of some $\psi \in E^2(K)$ if and only if*

$$\int |f(z)|^2 \tau_n(H_K(z)) e^{-2H_K(z)}(i\partial\bar{\partial}H_K)^n < \infty, \tag{1.13}$$

where τ_n is as in (4.9). We then have

$$c_n\|\psi\|_{E^2(K)} \leq \left(\int |f(z)|^2 \tau_n(H_K(z)) e^{-2H_K(z)}(i\partial\bar{\partial}H_K)^n\right)^{1/2} \leq c_{n,K}\|\psi\|_{E^2(K)}, \tag{1.14}$$

where the left constant only depends on the dimension and we have equality on both sides for the unit ball.

The problem of representing and characterizing the continuous linear functionals by their Laplace transforms is also considered in [5]. The spaces considered in that paper are variants of the so called Fock space of entire functions. So, while that paper deals with functions defined on Euclidean space of (real) dimension $2n$, we consider functions defined on a hypersurface of (real) dimension $2n-1$.

Since K is bounded and convex, it can always be given as

$$K = \{z \in \mathbb{C}^n : \rho(z) \leq 1\}$$

where ρ is 1-homogeneous and smooth (because K is smooth). The Hardy spaces need not be defined with respect to area measure on the boundary of the domain. There is at least one other natural measure, as suggested e.g. by the Cauchy-Fantappiè representation formula (2.6) below, namely the measure dS_K represented by the form $(2\pi i)^{-n}\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1}$ on ∂K. The norm in $E^2(K)$ will thus be

$$\|\psi\|_{E^2(K)} = \left(\int_{\partial K} |\psi(w)|^2 dS_K(w)\right)^{1/2}.$$

In other words, Theorem 1.4 says that f can be represented in the form

$$f(z) = \int_{\partial K} e^{z\cdot w}\overline{g(w)}dS_K(w), \quad g \in E^2(K),$$

if and only if f satisfies (1.11), and the Laplace transform is an isomorphism between the Hilbert spaces $E^2(K)$ and $P^2(K)$.

The reader may well notice that under our assumptions, the measure dS_K is equivalent to the ordinary surface measure, and the measure represented by the form $|z|^n(i\partial\bar{\partial}H_K(z))^n$ is equivalent to ordinary Lebesgue measure. One reason why we still choose to write the theorem like this, is that for circled domains, this enables us to obtain an isometry for the Borel transform (see section 4.2 below), which gives us Theorem 1.5. Another reason is that the form $(i\partial\bar{\partial}H_K(z))^n$ must probably be used for less smooth domains, like in Theorem 1.3.

The proof of Theorem 1.4 is carried out in two steps (as is the proof of Theorem 1.3 by Lutsenko and Yulmukhametov). For this, we have to give some background on different notions of convexity, like *strongly convex* sets as used in the theorem, and more specifically $\mathbb{C}$-*convexity*. This, and other preliminaries, will be presented in Chapter 2. In Chapter 3 we will define and discuss the so called Fantappiè transform and the Borel transform, and we will outline how they are used in the proof. In Chapter 4 we will prove the L^2-inequalities that are needed.

Chapter 2

Preliminaries

In this chapter we will make a brief discussion on various kinds of convexity and other preliminaries needed in what follows.

2.1 Convex sets

Remember that our set K from Theorem 1.4 and Theorem 1.5 is bounded and convex with smooth boundary. We will take K to be compact, and we can without loss of generality assume that $0 \in K$. Then it can always be given as

$$K = \{z : \rho(z) \leq 1\},$$

where the function ρ is 1-homogeneous. The smoothness and convexity of K implies that ρ is smooth. This representation for K will be used throughout the paper.

The *real tangent space* at a point $p \in \partial K$ is $T_p(\partial K) = \{v : \operatorname{Re} \partial\rho(p).v = 0\}$, and the *complex tangent space* is $T_p^{(1,0)}(\partial K) = \{v : \partial\rho(p).v = 0\}$. It is well known that for a domain with C^2 boundary, the usual geometric definition of convexity is equivalent to the analytic requirement that the Hessian of ρ is positive semidefinite restricted to the real tangent space at every $p \in \partial K$, i.e. that

$$2\operatorname{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) v_j v_k\right) + 2\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_k}(p) \bar{v_j} v_k \geq 0, \quad \forall v \in T_p(\partial K).$$

We will say that K is *strongly convex* if, for every $p \in \partial K$, this inequality is strict whenever $v \neq 0$.

2.2 The polar and the dual complement

For a $(1,0)$-form $\eta = \sum \eta_j dz_j$ and $z \in \mathbb{C}^n$ we shall write

$$\langle \eta, z \rangle = \sum \eta_j z_j.$$

Since K is convex, through every point in the complement $\mathbb{C}^n \setminus K$ there is a real hyperplane which does not meet the domain. Indeed, for $w \in \mathbb{C}^n \setminus K$ we can take the hyperplane

$$\{z : \operatorname{Re} \langle \partial\rho(w), w - z \rangle = 0\}.$$

The function ρ is 1-homogeneous, and Euler's Theorem implies that

$$2\operatorname{Re} \langle \partial\rho(w), w \rangle = \rho(w).$$

Hence the hyperplane above can be written as

$$\left\{ z : \operatorname{Re} \left\langle \frac{2\partial\rho(w)}{\rho(w)}, z \right\rangle = 1 \right\}.$$

We will sometimes, for convenience, identify $(1,0)$-forms with vectors, e.g. we identify $\partial\rho$ with $\left(\frac{\partial\rho}{\partial z_1}, \ldots, \frac{\partial\rho}{\partial z_n}\right)$. With $\xi = 2\partial\rho(w)/\rho(w)$ the hyperplane above can therefore be written as $\{z : \operatorname{Re} z \cdot \xi = 1\}$. Every real hyperplane which does not contain the origin can be written in this way, and if ξ defines a hyperplane contained in $\mathbb{C}^n \setminus K$ we thus have $\operatorname{Re} z \cdot \xi \neq 1$ for every $z \in K$. Actually, since $0 \in K$, we must have $\operatorname{Re} z \cdot \xi < 1$ for every $z \in K$. Indeed, if $\operatorname{Re} z \cdot \xi = c > 1$ for some $z \in K$, then $z/c \in K$ and $\operatorname{Re} (z/c) \cdot \xi = 1$.

Let

$$H_K(\xi) = \sup_{z \in K} \operatorname{Re} (z \cdot \xi) \tag{2.1}$$

be the *supporting function* of K. It is convex and 1-homogeneous, and if $|\xi| = 1$ then $H_K(\xi)$ is the length of the projection of K on the ray through the origin with direction $\overline{\xi}$.

Definition 2.1. The *polar* of a set $E \subset \mathbb{C}^n$ is the set

$$E^\circ = \{\xi : H_E(\xi) < 1\}.$$

(The polar is usually defined by a non strict inequality, but we want K° to be open.) By the above we see that K° can be interpreted as the set of real hyperplanes which do not meet K (plus the point 0). Since H_K is a convex function K° will be convex. It is bounded since $0 \in K$ implies that $H_K(\xi) \geq 0$ for every ξ, and by the Hahn-Banach theorem it follows that $K^{\circ\circ} = K$.

For a point in the complement of a general set $E \subset \mathbb{C}^n$, we cannot hope to find a real hyperplane through that point, which does not intersect E. We will say that E is *lineally convex* if through every point in the complement we can find a *complex* hyperplane which does not intersect E.

Every complex hyperplane which does not contain the origin can be written as $L_\xi = \{z : z \cdot \xi = 1\}$ for some $\xi \neq 0$. If we consider our set K we see, as above, that for every $w \in \mathbb{C}^n \setminus K$ we get that $\xi = 2\partial\rho(w)/\langle 2\partial\rho(w), w\rangle$ defines a complex hyperplane through w, which is contained in $\mathbb{C}^n \setminus K$.

Definition 2.2. The *dual complement* of $E \subset \mathbb{C}^n$ is the set

$$E^* = \{\xi : z \cdot \xi \neq 1, \forall z \in E\}.$$

Analogously to the polar, this can be interpreted as the set of complex hyperplanes (plus the point 0) which do not intersect E. In one variable this will simply become $E^* = \{1/z : z \in \mathbb{C} \setminus E\} \cup \{0\}$. We see that we always have $K^\circ \subset K^*$, but the inclusion is strict in general.

The set K^* will not be convex in general, but it does satisfy another convexity condition.

Definition 2.3. A set $E \subset \mathbb{C}^n$ is called $\mathbb{C}$-*convex* if $E \cap l$ is a connected and simply connected subset of E for every complex line l.

It is a highly nontrivial fact that any $\mathbb{C}$-convex set is in fact lineally convex (see [2] or [8]). For a set with C^1-boundary (so that it has a unique tangential hyperplane at every boundary point) lineal convexity also implies $\mathbb{C}$-convexity. Any convex set is $\mathbb{C}$-convex, which can be seen from the above definition, but the converse is false in general.

A much more thorough treatment of $\mathbb{C}$-convexity and its applications can be found in [2] or [8]. In [2] convexity and $\mathbb{C}$-convexity is discussed in projective space, where the point 0 does not have the peculiar and annoying role which it has in our statements.

It is not very difficult to see that the dual complement of a convex set (containing the origin) is $\mathbb{C}$-convex. In particular K^* is $\mathbb{C}$-convex. It is a nontrivial fact that actually if a $\mathbb{C}$-convex set contains the origin, then its dual complement is also $\mathbb{C}$-convex.

If E is given by a C^2 defining function r, i.e. $E = \{z : r(z) < 0\}$, there is an analytic condition for $\mathbb{C}$-convexity similar to the one for convexity. If E is $\mathbb{C}$-convex then

$$2\,\mathrm{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(p) v_j v_k\right) + 2\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial \bar{z}_j \partial z_k}(p) \bar{v}_j v_k \geq 0, \quad \forall v \in T_p^{(1,0)}(\partial K),$$

for every boundary point p (see [2]). Conversely, if this inequality is strict for every boundary point, then E is $\mathbb{C}$-convex. We will say that the set E is *strongly* $\mathbb{C}$*-convex* if the inequality is strict. In particular K, being strongly convex, is strongly $\mathbb{C}$-convex.

If $z \in K$ then $tz \in K$ for every $0 \leq t \leq 1$ since K is convex. This implies that if $\xi \in K^*$, then $t\xi \in K^*$ for every $0 \leq t \leq 1$. Hence K^* is star shaped with respect to the origin and is given as

$$K^* = \{\zeta : \rho^*(\zeta) < 1\},$$

for a 1-homogeneous function ρ^*.

If $w \in \mathbb{C}^n \setminus K$ we have seen that if we define

$$s(w) = 2\partial\rho(w)/\langle 2\partial\rho(w), w\rangle, \tag{2.2}$$

then $s(w)$ defines a complex hyperplane through w, which does not intersect K, and so $s(w) \in K^*$. Here, as well as on numerous other occasions, in a struggle not to clutter the notations we agree to identify e.g. the (1,0)-form $s_1 dz_1 + \cdots + s_n dz_n$ with the vector $(s_1, \ldots, s_n)$.

Fix a complex hyperplane L, which does not contain the origin. Consider domains of the type $\{z : \rho(z) \leq R\}$. By starting with R small and letting it grow, we can make L tangential to $\{z : \rho(z) \leq R\}$, at a point w, for some R_0. But since this set is smoothly bounded, L must then be the tangent complex hyperplane at w, which means that it is represented by $s(w)$. Thus we see that s is surjective from $\mathbb{C}^n \setminus \{0\}$ to $\mathbb{C}^n \setminus \{0\}$.

If we let $R < R_0$, then L will not meet the domain $\{z : \rho(z) \leq R\}$. If $R > R_0$ then L will intersect $\{z : \rho(z) \leq R\}$. Since this set is convex, this means that L cannot be tangential at any point (a convex set is always on one side of its tangent plane). Hence we see that L is tangent to $\{z : \rho(z) \leq R\}$ only for $R = R_0$. But since this set is strongly convex (and therefore strongly $\mathbb{C}$-convex), the function ρ restricted to the complex tangential directions, has a strict minimum at the tangent point w. This means that L is tangent to $\{z : \rho(z) \leq R_0\}$ *only* at w. Hence we see that the mapping s is also injective.

It may seem obvious from the construction, that given a smooth, strongly convex domain, its polar and dual complement are also smooth. On the other hand, there are smooth domains E, for which E^{**} do not even have C^1 boundary (see [2]). For the sake of completeness, we include the following lemma, which will be proved in Appendix A.

Lemma 2.4. *If K is a smoothly bounded, strongly convex domain in $\mathbb{C}^n$, then the polar K° and the dual complement K^* are also smooth.*

We can mention that the smoothness of K^* actually follows from the strong $\mathbb{C}$-convexity of K. See Remark A.2 in Appendix A.

There is obviously some symmetry in Lemma 2.4, in that $(K^*)^* = K$ and $(K^\circ)^\circ = K$ are also smooth. It is no surprise that we have the following, which will also be proved in Appendix A.

Proposition 2.5. *If K is a smoothly bounded, strongly convex domain in $\mathbb{C}^n$, then K° is also smoothly bounded and strongly convex.*
If K is a smoothly bounded, strongly $\mathbb{C}$-convex domain in $\mathbb{C}^n$, then K^* is also smoothly bounded and strongly $\mathbb{C}$-convex.

In the proof of Lemma 2.4 we show that s actually is a diffeomorphism. Since it is homogeneous of degree -1, its inverse s^* will also be homogeneous of degree -1. Since ρ and ρ^* are 1-homogeneous, this then implies that

$$\rho^*(\zeta) = \frac{1}{\rho(s^*(\zeta))},$$

since this is valid for $\zeta \in \partial K^*$. This shows that ρ^* is smooth.

If we restrict the mappings, then

$$s(w) : \partial K \to \partial K^*, \quad s^*(\zeta) : \partial K^* \to \partial K.$$

Analogously to $K^{\circ\circ} = K$ we have $K^{**} = K$ (this is valid for any lineally convex set). Hence any $w \in \partial K$ defines a complex hyperplane L_w, which is tangent to ∂K^*. Since $w \cdot s(w) = 1$ we have that $s(w) \in L_w$ so this is a plane tangential to ∂K^* at $s(w)$. But since ∂K^* is smooth by Lemma 2.4, it has a unique complex tangent plane. Therefore $w = s^*(s(w))$ represents the complex tangent plane to ∂K^* at $s(w)$. Since $s^*(\xi)$ apparently represents the complex tangent plane at ξ for any $\xi \in \partial K^*$ we therefore get

$$s^*(\xi) = 2\partial\rho^*(\xi)/\langle 2\partial\rho^*(\xi), \xi\rangle. \tag{2.3}$$

In the proof of our main theorems we will reduce to functions of one complex variable, and use the known theorems for the planar case. We know that $D^{**} = D$ for $\mathbb{C}$-convex domains D, but we also need to know what happens if we intersect with a complex line, and then take the dual complement of this planar domain. Let

$$D_a = \{\lambda \in \mathbb{C} : \lambda a \in D\}. \tag{2.4}$$

We will need the following lemma. The proof can be found in Appendix A.

Lemma 2.6. *Let $\zeta \in \mathbb{C}^n \setminus \{0\}$. We have*

$$({K^*}_\zeta)^* = \left\{\lambda : \frac{\lambda}{|\zeta|^2}\bar{\zeta} \in proj_{\bar{\zeta}}K\right\}, \tag{2.5}$$

where $proj_{\bar{\zeta}}K$ is the orthogonal projection of K on the complex line through the origin and $\bar{\zeta}$. Furthermore

$$H_{({K^*}_\zeta)^*}(w) = H_K(w\zeta).$$

2.3 Notation, and a representation formula

Let $E \subset\subset \mathbb{C}^n$ be an open set with a C^2 defining function r. Assume that we can find a differential function ϑ with values in $\mathbb{C}^n$ such that

$$\vartheta(w) \cdot (w - z) \neq 0 \quad \text{for } w \in \partial E, z \in E.$$

For functions f holomorphic in the interior of E and continuous on the closure, we then have the Cauchy-Fantappiè representation formula

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial E} f(w) \frac{\vartheta(w) \wedge (\bar{\partial}\vartheta(w))^{n-1}}{\langle \vartheta(w), w - z \rangle^n}, \quad z \in E.$$

If E is lineally convex (in particular if E is convex or $\mathbb{C}$-convex), we know that for $w \in \partial E$, the condition $\langle \partial r(w), w - z \rangle = 0$ characterizes the complex tangent space at w. We therefore get that

$$\langle \partial r(w), w - z \rangle \neq 0 \quad \text{for } w \in \partial E, z \in E,$$

so that we can choose $\vartheta(w) = \partial r(w)$ and write

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial E} f(w) \frac{\partial r(w) \wedge (\bar{\partial}\partial r(w))^{n-1}}{\langle \partial r(w), w - z \rangle^n}, \quad z \in E. \tag{2.6}$$

A proof of the Cauchy-Fantappiè formula can be found in [16, section IV.3]. In particular it is valid for our convex set K, where we can take $\vartheta = s$.

The open unit ball in $\mathbb{C}^n$ will be denoted by B_n, $B_{n,R}$ will be the ball centred at 0 with radius R, and $B_n(z, R)$ will be the ball centred at z with radius R. The measure dm will always be the usual Lebesgue measure on $\mathbb{C}^n$.

The Hardy spaces $H^2(B_{n,R})$ will be the usual Hardy space in the ball $B_{n,R}$, with respect to surface measure on $\partial B_{n,R}$.

Formula (2.6) suggests another natural measure on ∂K, namely the measure dS_K represented by the form $(2\pi i)^{-n} \partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1}$. As we said in the introduction, we will use the measure dS_K on ∂K to define the Hardy space $E^2(K)$. The measure dS_{K^*} will be the measure represented by the form $(2\pi i)^{-n} \partial\rho^* \wedge (\bar{\partial}\partial\rho^*)^{n-1}$ on ∂K^* in the same way. For a strongly pseudoconvex domain, these measures are equivalent to the surface measure on the boundary, but they enable us to obtain an isometry for the Borel transform in section 4.2.

The measure dm will always be Lebesgue measure. The measure $d\sigma$ will denote surface measure on different hypersurfaces in $\mathbb{C}^n$ or curves in $\mathbb{C}$.

If the reader is worried that dS_K and dS_{K^*} are not even real, this can be seen by using the real operators d and $d^c = i(\bar{\partial} - \partial)$ to write

$$\frac{1}{(2\pi i)^n} \partial\rho(z) \wedge (\bar{\partial}\partial\rho(z))^{n-1} = \frac{1}{(4\pi)^n} d^c\rho(z) \wedge (dd^c\rho(z))^{n-1}.$$

This shows that dS_K and (in the same way dS_{K^*}) is real.

Recall that a $(2n-1)$-form ω in a neighbourhood of ∂K represents surface measure on ∂K if $\frac{d\rho(z)}{|d\rho(z)|} \wedge \omega(z) = dm(z)$ when $z \in \partial K$. It can be shown that actually $\frac{1}{(2i)^n} \frac{d\rho}{|d\rho|} \wedge \partial\rho(z) \wedge (\bar{\partial}\partial\rho(z))^{n-1}$ is $dm(z)$ multiplied with the determinant of the Levi form at z. Since K is strictly pseudoconvex (being strongly convex), this determinant is positive. Since K^* is also strictly pseudoconvex, the same argument shows that dS_{K^*} is positive.

Chapter 3

The Fantappiè transform and the Borel transform

The proof of our main theorem will be carried out in two parts, by means of the so called Fantappiè transform and the Borel transform. In this chapter we will define these transforms and explain how they will be used.

3.1 The Fantappiè transform and analytic functionals

For any open set $E \subset \mathbb{C}^n$, let $\mathcal{O}(E)$ be the vector space of holomorphic functions on E, endowed with the usual topology of uniform convergence on compact subsets. If E is instead a compact set, we will mean the space of all functions which are holomorphic in some neighbourhood of E. (The topology in that case is obtained as the inductive limit of the spaces $\mathcal{O}(U)$ for open neighbourhoods $U \supset E$.)

Definition 3.1. A continuous linear functional on $\mathcal{O}(E)$ is called an *analytic functional* on E. The space of all analytic functionals on E, that is, the dual space of $\mathcal{O}(E)$, will be denoted $\mathcal{O}'(E)$.

Let E be open. The continuity of an analytic functional μ implies that there is some compact subset $M \subset E$ such that

$$|\mu(h)| \leq C \sup_{z \in M} |h(z)|, \quad h \in \mathcal{O}(E). \tag{3.1}$$

Then, for every open $\omega \supset M$ in E, we obviously have

$$|\mu(h)| \leq C_\omega \sup_{z \in \omega} |h(z)|, \quad h \in \mathcal{O}(E). \tag{3.2}$$

Definition 3.2. An analytic functional μ on an open set $E \subset \mathbb{C}^n$ is said to be *carried* by the compact subset $M \subset E$ if for every neighbourhood $\omega \supset M$ in E, we have (3.2) for some constant C_ω.

If μ satisfies (3.1) then, by the Hahn-Banach theorem, μ can be extended to a continuous linear functional on $\mathcal{C}(E)$, which by the Riesz representation theorem is represented by a Radon measure $d\mu$ with support in M, so that

$$\mu(h) = \int_E h(z) d\mu(z), \quad h \in \mathcal{O}(E). \tag{3.3}$$

In particular

$$\mu\left(\frac{1}{(1 - z \cdot \zeta)^n}\right) = \int_E \frac{d\mu(z)}{(1 - z \cdot \zeta)^n}.$$

For every $\mu \in \mathcal{O}'(E)$ this is an analytic function for ζ such that $z \cdot \zeta \neq 1$ for all z in the support of $d\mu$. Since $d\mu$ is compactly supported in E, the function will be analytic in some neighbourhood of the compact set E^*, i.e. it is a function in $\mathcal{O}(E^*)$.

Definition 3.3. We define the Fantappiè transform $\mathcal{F} : \mathcal{O}'(E) \to \mathcal{O}(E^*)$ for an open or compact set $E \subset \mathbb{C}^n$ by

$$\mathcal{F}\mu(\zeta) = \mu\left(\frac{(n-1)!}{(1 - z \cdot \zeta)^n}\right).$$

(The Fantappiè transform is normally defined with respect to the function $1/(1 - z \cdot \zeta)$.) Analytic functionals and the Fantappiè transform are discussed in conjunction with the Polya-Martineau theorem in Section 4.7 of [8]. In the nice (but yet-to-be-published) treatment [3], different transforms, defined with respect to $1/(1 - z \cdot \zeta)^k, k \geq 1$, are used. It is a deep theorem that for an open or compact (and polynomially convex) set E, the Fantappiè transform is bijective precisely when E is $\mathbb{C}$-convex. Then it is also continuous.

Now let E be our compact and strongly convex set K. In this case the proof that the Fantappiè transform is surjective is not very difficult, and is obtained by a certain pairing between $\mathcal{O}(K)$ and $\mathcal{O}(K^*)$.

Remember the definition of s and s^* from Chapter 2. We will use the form

$$s(w) \wedge (\bar{\partial} s(w))^{n-1}.$$

Since $s(w)$ is a (1,0)-form, this will be an $(n, n-1)$-form in $\mathbb{C}^n \setminus K$. We can remark that if Q is any $(1,0)$-form and h any C^1 function, then

$$(hQ) \wedge (\bar{\partial}(hQ))^{n-1} = h^n Q \wedge (\bar{\partial} Q)^{n-1},$$

which follows immediately by observing that

$$hQ \wedge \bar{\partial}(hQ) = hQ \wedge (\bar{\partial} h \wedge Q + h\bar{\partial} Q) = hQ \wedge h\bar{\partial} Q.$$

In particular we have that

$$s(w) \wedge (\bar{\partial} s(w))^{n-1} = \frac{2^n}{\langle 2\partial\rho(w), w\rangle^n} \partial\rho(w) \wedge (\bar{\partial}\partial\rho(w))^{n-1}. \tag{3.4}$$

Now fix $g \in \mathcal{O}(K)$ and $\varphi \in \mathcal{O}(K^*)$. Then g is holomorphic in some neighbourhood of K, so take an open set $\Omega \supset K$ such that g is holomorphic in a neighbourhood of $\overline{\Omega}$. Then $s(\partial\Omega) \subset K^*$ and we can consider the integral

$$\int_{\partial\Omega} g(w)\varphi(s(w))s(w) \wedge (\bar{\partial}s(w))^{n-1}. \tag{3.5}$$

We want to show that this is independent of the particular choice of Ω. The function g is analytic, and the form $\varphi(s(w))s(w) \wedge (\bar{\partial}s(w))^{n-1}$ is of bidegree $(n, n-1)$, so we want to show that this form is $\bar{\partial}$-closed in $\mathbb{C}^n \setminus K$. This is simpler if we first consider the so called *incidence manifold* (this is used for inverting the Fantappiè transform in [3])

$$\Delta = \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n : z \cdot \zeta = 1\}. \tag{3.6}$$

Let

$$\nu = \sum \zeta_j dz_j$$

and consider the form

$$\nu \wedge (d\nu)^{n-1} \tag{3.7}$$

on Δ.

Lemma 3.4. *Let h be a smooth function on $\partial\Omega$. Then*

$$\int_{\partial\Omega} h(z)s(z) \wedge (\bar{\partial}s(z))^{n-1} = (-1)^n \int_{s(\partial\Omega)} h(s^*(\zeta))s^*(\zeta) \wedge (\bar{\partial}s^*(\zeta))^{n-1}.$$

Proof. Consider the submanifold Λ of Δ defined by

$$\Lambda = \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n : z \in \partial\Omega, \zeta = s(z)\},$$

and let $F : \partial\Omega \to \Lambda, z \mapsto (z, s(z))$. Then we get the pullback

$$F^*(\nu) = \sum s_j dz_j = s(z)$$

so that

$$F^*(\nu \wedge (d\nu)^{n-1}) = s \wedge (ds)^{n-1} = s \wedge (\bar{\partial}s)^{n-1},$$

where the last equality follows since $s \wedge (\bar{\partial}s)^{n-1}$ is a full form in z. Therefore

$$\int_{\partial\Omega} h(z)s(z) \wedge (\bar{\partial}s(z))^{n-1} = \int_{\Lambda} h(z)\nu \wedge (d\nu)^{n-1}.$$

But on Λ we have $z \cdot \zeta = 1$ so that

$$\sum \zeta_j dz_j = -\sum z_j d\zeta_j.$$

If we consider the mapping $\zeta \mapsto (s^*(\zeta), \zeta)$ from $s(\partial\Omega)$ to Λ, we therefore find in the same way that the form $\nu \wedge (d\nu)^{n-1}$ pulls back to $(-1)^n s^*(\zeta) \wedge (\bar{\partial}s^*(\zeta))^{n-1}$ on $s(\partial\Omega)$ and

$$\int_{\Lambda} h(z)\nu \wedge (d\nu)^{n-1} = (-1)^n \int_{s(\partial\Omega)} h(s^*(\zeta))s^*(\zeta) \wedge (\bar{\partial}s^*(\zeta))^{n-1}.$$

□

That the form $\varphi(s(w))s(w) \wedge (\bar{\partial}s(w))^{n-1}$ is closed ($\bar{\partial}$-closed by bidegree reasons) in $\mathbb{C}^n \setminus K$ now follows from the previous proof since we see that it is the pullback of the form $\varphi(\zeta)s^*(\zeta) \wedge (\bar{\partial}s^*(\zeta))^{n-1}$ in $K^* \setminus \{0\}$, where φ is analytic and the form $s^* \wedge (\bar{\partial}s^*)^{n-1}$ is $\bar{\partial}$-closed. The last fact is what the Cauchy-Fantappiè representation formula is built on, but we may repeat the proof here. In fact, if we let $s^*(\zeta) = \sum s_j^*(\zeta)d\zeta_j$ we have that

$$\begin{aligned} d(s^* \wedge (\bar{\partial}s^*)^{n-1}) &= \bar{\partial}(s^* \wedge (\bar{\partial}s^*)^{n-1}) = (\bar{\partial}s^*)^n \\ &= c_n(\bar{\partial}s_1^*(\zeta) \wedge d\zeta_1) \wedge \cdots \wedge (\bar{\partial}s_n^*(\zeta) \wedge d\zeta_n), \quad (3.8) \end{aligned}$$

and since

$$\langle s^*(\zeta), \zeta \rangle = \sum s_j^*(\zeta)\zeta_j = 1$$

we get that

$$\sum \zeta_j \, \bar{\partial}s_j^*(\zeta) = 0.$$

For $\zeta \neq 0$ this implies that the set $\{\bar{\partial}s_1^*(\zeta), \ldots, \bar{\partial}s_n^*(\zeta)\}$ is linearly dependent, which implies that (3.8) is 0.

By the above we see that any $\varphi \in \mathcal{O}(K^*)$, by varying Ω in (3.5), defines an element in $\mathcal{O}'(K)$. We claim that this functional has φ as its Fantappiè transform (if we divide by a suitable constant). To show this, let μ be the analytic functional defined by φ and let $\xi \in K^*$. By Lemma 3.4, for a suitable $\Omega \supset K$, we have

$$\begin{aligned} \mathcal{F}\mu(\xi) &= \int_{\partial\Omega} \frac{(n-1)!}{(1 - w \cdot \xi)^n} \, \varphi(s(w)) \, s(w) \wedge (\bar{\partial}s(w))^{n-1} \\ &= \int_{s(\partial\Omega)} \frac{(n-1)!}{(s^*(\zeta) \cdot \zeta - s^*(\zeta) \cdot \xi)^n} \, \varphi(\zeta) \, s^*(\zeta) \wedge (\bar{\partial}s^*(\zeta))^{n-1} \\ &= \int_{s(\partial\Omega)} \frac{(n-1)!}{\langle s^*(\zeta), \zeta - \xi \rangle^n} \, \varphi(\zeta) \, s^*(\zeta) \wedge (\bar{\partial}s^*(\zeta))^{n-1} \\ &= \int_{s(\partial\Omega)} \frac{(n-1)!}{\langle s^*(\zeta), \zeta - \xi \rangle^n} \, \varphi(\zeta) \, s^*(\zeta) \wedge (\bar{\partial}s^*(\zeta))^{n-1} \\ &= c_n \varphi(\xi) \quad (3.9) \end{aligned}$$

where the last equality follows from the Cauchy-Fantappiè representation formula (2.6) above.

We will use this pairing and equation (3.9) later on.

3.2 The Borel transform

A function f in $\mathbb{C}^n$ is said to be of *exponential type* if

$$|f(z)| \leq Ce^{A|z|}, \quad z \in \mathbb{C}^n.$$

Definition 3.5. The *Borel transform* of an entire function of exponential type is the function

$$\mathcal{B}f(\zeta) = \int_0^\infty f(t\zeta)t^{n-1}e^{-t}dt. \tag{3.10}$$

If $E \subset \mathbb{C}^n$ is open and $\mu \in \mathcal{O}'(E)$ we have seen that μ is carried by some compact set $M \subset E$. Then $\hat{\mu}(z)$ is an entire function (this can be seen from the representation (3.3)) such that

$$|\hat{\mu}(z)| = |\mu(e^{z\cdot\zeta})| \leq C_\omega \sup_{\zeta\in\omega} |e^{z\cdot\zeta}| = C_\omega e^{H_\omega(z)}, \quad \forall\omega \supset M. \tag{3.11}$$

If f is an entire function satisfying (3.11), then the Borel transform of f converges and is holomorphic in the set $\{\zeta : H_M(\zeta) < 1\} = M^\circ$. We will now show that it has an analytic continuation to the set M^*. Let

$$\mathcal{B}_\theta f(\zeta) = \int_0^\infty f(te^{i\theta}\zeta)(te^{i\theta})^{n-1}e^{-te^{i\theta}}e^{i\theta}dt,$$

and notice that this is a holomorphic function in $\{\zeta : H_M(e^{i\theta}\zeta) < \mathrm{Re}\,(e^{i\theta})\}$. If $\zeta \in M^*$, then the image of

$$M \ni z \mapsto 1 - z\cdot\zeta$$

is a convex set (since the mapping is affine) in the plane, which avoids the origin (since $\zeta \in M^*$). If we rotate it by a suitable angle θ, this set will thus be contained in the right half plane, i.e.

$$\mathrm{Re}\,(e^{i\theta}(1 - z\cdot\zeta)) > 0, \quad z \in M,$$

which implies that

$$H_M(e^{i\theta}\zeta) = \sup_{z\in M} \mathrm{Re}\,(i\theta\zeta\cdot z) < \mathrm{Re}\,(e^{i\theta}),$$

and the set of all possible $e^{i\theta}$ is an interval on the unit circle. Hence, for every $\zeta \in M^*$, $\mathcal{B}_\theta f(\zeta)$ is defined for some θ, and is independent of θ by Cauchy's theorem. If $\mathcal{B}_\theta f(\zeta)$ is defined, then it is analytic in a neighbourhood of ζ, and in this way the Borel transform $\mathcal{B}f$ has an analytically continuation along any curve from the origin to $\zeta \in M^*$ (which by the Monodromy Theorem is unique).

3.3 Relations between the transforms

The reason why we are interested in the Fantappiè transform and the Borel transform in this context, is certain relations which exist between the two transforms and the Laplace transform.

Let E be our compact set K. If $\mu \in \mathcal{O}'(K)$ then we have seen in section 3.2 that $\hat{\mu}(z)$ is an entire function of exponential type, whose Borel transform $\mathcal{B}\hat{\mu}(\zeta)$ has an analytic continuation to K^*. If we again use that μ can be represented by a measure, we see that we in fact have

$$\begin{aligned}
\mathcal{B}\hat{\mu}(\zeta) &= \int_0^\infty \hat{\mu}(t\zeta)t^{n-1}e^{-t}dt \\
&= \int_0^\infty \left(\int e^{t\zeta\cdot z}d\mu(z)\right) t^{n-1}e^{-t}dt \\
&= \int d\mu(z) \int_0^\infty t^{n-1}e^{-t(1-\zeta\cdot z)}dt \\
&= (n-1)! \int \frac{d\mu(z)}{(1-\zeta\cdot z)^n} = \mathcal{F}\mu(\zeta).
\end{aligned}$$

Apparently we have the relation

$$\mathcal{B} \circ \mathcal{L} = \mathcal{F} \tag{3.12}$$

between the Borel-, Laplace- and Fantappiè transform respectively.

What we want to do now is to, as in the introduction, restrict ourselves to analytic functionals in $\mathcal{O}'(K)$ given by functions in $E^2(K)$. Every $g \in E^2(K)$ defines an analytic functional $\mu \in \mathcal{O}'(K)$ by

$$\mu(h) = \int_{\partial K} h(w)\overline{g(w)}\, dS_K(w), \quad h \in \mathcal{O}(K).$$

Its Fantappiè transform is an element of $\mathcal{O}(K^*)$, but we want to prove that actually $\mathcal{F}\mu \in E^2(K^*)$, and that the Fantappiè transform is an isomorphism between the normed spaces $E^2(K)$ and $E^2(K^*)$.

We will also show that if f is an entire function which satisfies the assumptions (1.11) in the main theorem, then we can define its Borel transform. We will prove that the Borel transform of such a function will belong to $E^2(K^*)$, and that the Borel transform is an isomorphism between the normed spaces $P^2(K)$ and $E^2(K^*)$, with the norms as in (1.12).

Chapter 4

L^2-inequalities

In this chapter we will demonstrate that the Borel transform gives an isomorphism between the space of certain entire functions of exponential type, and the space $E^2(K^*)$. We will do this by reducing to functions of one variable on complex lines through the origin. We will also prove that the Fantappiè transform is an isomorphism between the Hilbert spaces $E^2(K)$ and $E^2(K^*)$. As an introduction we will prove an analogue of Parseval's identity for the Laplace transform when the domain is a ball in $\mathbb{C}^n$.

4.1 An explicit identity for the ball

In [9] Katsnel'son proved, among other things, an analogue of Parseval's identity when the domain is a disc in $\mathbb{C}$. By generalizing his proof, we get the following result in $\mathbb{C}^n$.

For the proof we need two well known identities, which are calculated for instance in [17, chapter 1]. Here $\omega_n = 2\pi^n/(n-1)!$ is the area of the unit sphere ∂B_n and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and β are multi-indices, with $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

$$\int_{\partial B_n} z^\alpha \bar{z}^\beta d\sigma(z) = 0, \text{when } \alpha \neq \beta \tag{4.1}$$

$$\int_{\partial B_n} |z^\alpha|^2 d\sigma(z) = \omega_n \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} \tag{4.2}$$

Theorem 4.1. *Let f be an entire function and $R > 0$. Then f is the Laplace transform of a (unique) $\psi \in H^2(B_{n,R})$ if and only if*

$$\int |f(z)|^2 \tau(R|z|) e^{-2R|z|} dm(z) < \infty \tag{4.3}$$

where

$$\tau(r) = 2\int_0^\infty \frac{e^{-2tr}}{\sqrt{2t+t^2}} dt. \tag{4.4}$$

Furthermore, in that case

$$\|\psi\|^2_{H^2(B_{n,R})} = \frac{R}{\omega_n^2 (n-1)!^2} \int |f(z)|^2 \tau(R|z|) e^{-2R|z|} dm(z).$$

Proof. Let $R = 1$ and ψ be a function in $H^2(B_n)$ with the Taylor expansion

$$\psi(\zeta) = \sum_{\alpha \geq 0} c_\alpha \zeta^\alpha$$

If we let $m_{\alpha,n}$ be the right hand side in (4.2) we have

$$\|\psi\|^2_{H^2(B_n)} = \sum_{\alpha \geq 0} m_{\alpha,n} |c_\alpha|^2. \tag{4.5}$$

Let $f(z)$ be the Laplace transform of ψ. Since we have

$$(z \cdot \zeta)^k = (z_1 \zeta_1 + \cdots + z_n \zeta_n)^k = \sum_{|\beta|=k} \frac{k!}{\beta!} z^\beta \zeta^\beta$$

we get

$$\begin{aligned} f(z) &= (e^{z\cdot\zeta}, \psi(\zeta))_{H^2(B_n)} \\ &= \int_{\partial B_n} e^{z\cdot\zeta}\, \overline{\psi(\zeta)} d\sigma(\zeta) \\ &= \int_{\partial B_n} \sum_{k \geq 0} \frac{(z\cdot\zeta)^k}{k!} \sum_{\alpha \geq 0} \overline{c_\alpha} \overline{\zeta^\alpha}\, d\sigma(\zeta) \\ &= \sum_{\alpha \geq 0} \overline{c_\alpha} \frac{1}{\alpha!} m_{\alpha,n} z^\alpha \end{aligned} \tag{4.6}$$

and using the expression for $m_{\alpha,n}$

$$\begin{aligned} \int_{\partial B_n} |f(rz)|^2 d\sigma(z) &= \sum_{\alpha \geq 0} |c_\alpha|^2 \frac{1}{(\alpha!)^2} m^2_{\alpha,n} r^{2|\alpha|} m_{\alpha,n} \\ &= \sum_{k \geq 0} r^{2k} \left(\omega_n \frac{(n-1)!}{(n-1+k)!} \right)^2 \sum_{|\alpha|=k} m_{\alpha,n} |c_\alpha|^2. \end{aligned} \tag{4.7}$$

It is a nice exercise in calculus to prove that

$$\int_0^\infty \tau(r) r^{2k+1} e^{-2r} dr = (k!)^2 \tag{4.8}$$

which together with (4.5) and (4.7) gives us

$$\begin{aligned} \int |f(z)|^2 \tau(|z|) e^{-2|z|} dm(z) &= \int_0^\infty \tau(r) r^{2n-1} e^{-2r} \int_{\partial B_n} |f(rz)|^2 d\sigma(z) \\ &= \omega_n^2 (n-1)!^2 \sum_{\alpha \geq 0} m_{\alpha,n} |c_\alpha|^2 \\ &= \omega_n^2 (n-1)!^2 \|\psi\|^2_{H^2(B_n)}. \end{aligned}$$

For the sufficiency of condition (4.3) assume that f satisfies (4.3) with $R = 1$. Let

$$f(z) = \sum_{\alpha \geq 0} d_\alpha z^\alpha$$

and define, as suggested by (4.6),

$$\psi(\zeta) = \sum_{\alpha \geq 0} \overline{d_\alpha} \alpha! \frac{1}{m_{\alpha,n}} \zeta^\alpha.$$

Then $\psi \in H^2(B_n)$, and its Laplace transform is f.

This concludes the proof of the theorem when $R = 1$. The general case follows by a change of variables. □

4.2 The Borel transform for circled domains

As we said in section 3.3, we want to prove L^2-inequalities for the Borel transform. It turns out that the discussion is simpler when our domain K is a so called *circled* domain, in which case we can also obtain an isometry and not only norm equivalences, so will discuss this case first. There are two different ways of defining circled domains in the literature. Some authors do not distinguish between circled domains and Reinhardt domains, but we will mean the following:

Definition 4.2. A set $S \subseteq \mathbb{C}^n$ is called a *circled* set if $z \in S$ implies that $e^{i\theta} z \in S$ for all $\theta \in [0, 2\pi]$.

Let us modify the function τ from (4.4) and define

$$\tau_n(r) = 2r^n \int_0^\infty \frac{e^{-2tr}}{\sqrt{2t + t^2}} dt = r^n \tau(r). \tag{4.9}$$

We can now state the main theorem of this section.

Theorem 4.3. *Let K be a bounded, strongly convex domain which is smooth and circled. Then the Borel transform is an isomorphism between the space of entire functions f which satisfies*

$$\int |f(z)|^2 \tau_n(H_K(z)) e^{-2H_K(z)} (i\partial\bar{\partial} H_K)^n < \infty, \tag{4.10}$$

with τ_n as in (4.9), and $E^2(K^*)$. Furthermore,

$$\|\mathcal{B}f\|^2_{E^2(K^*)} = c_n \int |f(z)|^2 \tau_n(H_K(z)) e^{-2H_K(z)} (i\partial\bar{\partial} H_K)^n.$$

Notice the similarities with Theorem 4.1. The reason why we had to modify τ is that since H_K is homogeneous of degree 1, the form $(i\partial\bar{\partial}H_K)^n$ is homogeneous of degree $-n$. The other differences with Theorem 4.1 depends on the fact that in this theorem $\psi \in E^2(K^*)$, and the norm in this space is not defined with respect to surface measure.

There are several equivalent characterizations of circled domains. The statements in the next proposition are part of the reason why the problem at hand is simpler when the domain is circled.

Proposition 4.4. *For a convex set $K \subset \mathbb{C}^n$, defined by a 1-homogeneous function ρ as in (2.1), the following statements are equivalent:*

- *(i)* *K is circled*
- *(ii)* *K° is circled*
- *(iii)* *$\langle 2\partial\rho(z), z\rangle \in \mathbb{R}$ for every z*
- *(iv)* *$K \cap l$ is a disk, for every complex line l through the origin*
- *(v)* *$K^\circ = K^*$.*

Proof. If K is circled then

$$H_K(e^{i\theta}w) = \sup_{z\in K} \operatorname{Re}(z \cdot e^{i\theta}w) = \sup_{z\in K} \operatorname{Re}((e^{-i\theta}z)\cdot(e^{i\theta}w)) = H_K(w).$$

This proves that K° is circled, and since $K^{\circ\circ} = K$ we get in the same way that K is circled whenever K° is. Thus (i) and (ii) are equivalent. That (i) and (iv) are equivalent follows straight from the definitions.

We consider condition (iii) in $\mathbb{C}$ first. Let $z \in \partial K$. That $\langle 2\partial\rho(z), z\rangle \in \mathbb{R}$ is equivalent to $\operatorname{Re}\langle\partial\rho(z), iz\rangle = 0$. But this is the same as saying that the vector iz is parallel to the tangent space at z, which means that the ray through z and the origin is orthogonal to the tangent space at z. It is clear that ∂K is a circle if and only if this is valid for every boundary point. For K in $\mathbb{C}^n$, let l be the complex line $\{\lambda z : \lambda \in \mathbb{C}\}$ for some fixed $z \neq 0$. By considering the function $r(\lambda) = \rho(\lambda z)$ we find that $K \cap l$ is a disk if and only if $\langle\partial\rho(\lambda z), \lambda z\rangle \in \mathbb{R}$ for every λ. Therefore (iii) and (iv) are equivalent.

We know that

$$s(z) = 2\partial\rho(z)/\langle 2\partial\rho(z), z\rangle$$

maps $\mathbb{C}^n \setminus K$ surjectively onto $K^* \setminus \{0\}$ and similarly

$$z \mapsto 2\partial\rho(z)/\rho(z)$$

maps $\mathbb{C}^n \setminus K$ surjectively onto $K^\circ \setminus \{0\}$. Since we know that $\operatorname{Re}\langle 2\partial\rho(z), z\rangle = \rho(z)$, these two mappings coincide if $\langle 2\partial\rho(z), z\rangle \in \mathbb{R}$. Hence (iii) implies (v).

Assume that (iii) is not true for $K \subset \mathbb{C}$, and take $z \in \partial K$ so that $\langle 2\partial\rho(z), z\rangle$ is not real. As above, this means that the tangent line to K at z is not orthogonal to the ray through the origin and z. The support line $\{w : \operatorname{Re} w \cdot \bar{z} = H_K(\bar{z})\}$ is, on the other hand, orthogonal to this ray, and hence is not tangent to ∂K at z. Since all of K is on one side of the support line, this implies that

$$|z|^2 = \operatorname{Re} z \cdot \bar{z} < H_K(\bar{z})$$

or

$$1 < H_K\left(\frac{\bar{z}}{|z|^2}\right) = H_K\left(\frac{1}{z}\right).$$

Therefore $1/z$ is in the complement of K°, but since $z \in \partial K$ we have that $1/z \in \partial K^*$. We get that $K^* \neq K^\circ$. If (iii) is not true for $K \subset \mathbb{C}^n$ then there is a z such that $K_z = \{\lambda \in \mathbb{C} : \lambda z \in K\}$ is not a disk. This implies that $(K_z)^* \neq (K_z)^\circ$, and (in a similar way as in the proof of Lemma 2.6) this implies that $K^* \neq K^\circ$. Consequently (v) implies (iii). □

We will have need for two different formulas for change of variables. The first is an analogue of integration by polar coordinates, where the integral over the unit sphere is replaced by the integral over ∂K.

Lemma 4.5. *The map*

$$(r, w) \mapsto z = r 2\partial\rho(w), \quad r \in \mathbb{R}_+ \quad w \in \partial K$$

is one-to-one onto $\mathbb{C}^n \setminus \{0\}$*, with inverse*

$$w = 2\partial H_K, \quad r = H_K(z).$$

This gives us the formula

$$\int_{\mathbb{C}^n} \varphi(z)(i\partial\bar{\partial}H_K)^n = (i)^n(n-1)! \int_{\partial K} \int_0^\infty \varphi(r2\partial\rho(w)) r^{n-1} dr\ \partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1}.$$

Proof. We have already discussed the injectivity and surjectivity of the mapping s in chapter 2. That the mappings considered here are bijective follows in the same way. The calculations to validate the pullback of the integrals can be found in the proof of the Polya-Martineau theorem in [4]. □

We know that $\operatorname{Re}\langle 2\partial\rho(w), w\rangle = \rho(w)$, and if we have a look at Proposition 4.4 we see that $\langle 2\partial\rho(w), w\rangle$ in fact is real if the domain is circled. If we use (3.4) we can therefore write

$$s(w) \wedge (\bar{\partial}s(w))^{n-1} = 2^n \rho^{-n}(w)\partial\rho(w) \wedge (\bar{\partial}\partial\rho(w))^{n-1} \tag{4.11}$$

in this case. If our domain is circled, Proposition 4.4 says that K° is circled, and $K^* = K^\circ$, and we saw in the proof that then $s(z) = 2\partial\rho(z)$ for $z \in \partial K$. In view of formula (4.11), which then is valid also for ρ^*, Lemma 3.4 therefore takes a particularly simple form when the domain is circled.

Lemma 4.6. *If our set K is circled, then*

$$\int_{\partial K} h(2\partial\rho(z))\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1} = (-1)^n \int_{\partial K^*} h(\zeta)\partial\rho^* \wedge (\bar{\partial}\partial\rho^*)^{n-1}.$$

If we think back to Theorem 4.1 we may notice that for the unit ball $B_n = B_n^\circ = B_n^*$, which simplifies things. But, of course, it also helps to have an explicit orthogonal basis for the holomorphic functions! In the case of B_n we also have that the area measure is rotation invariant. We have an analogue for circled domains.

Lemma 4.7. *If r is the defining function for a circled domain D, then the form $\partial r \wedge (\bar\partial\partial r)^{n-1}$ is rotation invariant in the sense that if $F : \partial D \to \partial D$ is the mapping $z \mapsto e^{i\theta}z$, then*

$$F^*(\partial r \wedge (\bar\partial\partial r)^{n-1}) = \partial r \wedge (\bar\partial\partial r)^{n-1} \tag{4.12}$$

for every $\theta \in [0, 2\pi]$.

Proof. That D is circled means that $r(e^{i\theta}z) = r(z)$ for every θ. This implies that

$$\partial r = \partial(r(e^{i\theta}\cdot)) = e^{i\theta}\partial r(e^{i\theta}\cdot) = F^*\partial r$$

and

$$\bar\partial\partial r = \bar\partial\partial(r(e^{i\theta}\cdot)) = \bar\partial\partial r(e^{i\theta}\cdot) = F^*\bar\partial\partial r.$$

Formula (4.12) follows. □

As an immediate corollary we get the following lemma (which is also valid for K^* and ρ^* if K is circled).

Lemma 4.8. *If K is circled we have that*

$$\int_{\partial K} \varphi(w)\,\partial\rho \wedge (\bar\partial\partial\rho)^{n-1} = \int_{\partial K} \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}w)\,d\theta\,\partial\rho \wedge (\bar\partial\partial\rho)^{n-1}.$$

We now proceed to the proof of Theorem 4.3.

Proof of Theorem 4.3. Assume that f satisfies (4.10) and let us rewrite that condition. Using Lemma 4.5 we get

$$\int |f(z)|^2\tau_n(H_K(z))e^{-2H_K(z)}(i\partial\bar\partial H_K)^n =$$
$$c_n \int_{\partial K}\int_0^\infty |f(r2\partial\rho(w))|^2\tau(r)r^{2n-1}e^{-2r}dr\ \partial\rho \wedge (\bar\partial\partial\rho)^{n-1}.$$

We now use Lemma 4.6 to turn this into an integral over ∂K^*, and get that the last line is equal to

$$c_n \int_{\partial K^*}\int_0^\infty |f(r\zeta)|^2\tau(r)r^{2n-1}e^{-2r}dr\ \partial\rho^* \wedge (\bar\partial\partial\rho^*)^{n-1}.$$

If we let

$$f_\zeta(z) = f(z\zeta)z^{n-1}$$

for $z \in \mathbb{C}$, we can, again as K and $K^\circ = K^*$ are circled, use the rotation invariance in Lemma 4.8 to rewrite the last integral as

$$c_n \int_{\partial K^*} \int_0^{2\pi} \int_0^\infty |f_\zeta(re^{i\theta})|^2 \tau(r) r e^{-2r} dr\, d\theta\, \partial\rho^* \wedge (\bar\partial\partial\rho^*)^{n-1}. \qquad (4.13)$$

This implies that (for a.e. ζ at least)

$$\int_0^{2\pi} \int_0^\infty |f_\zeta(re^{i\theta})|^2 \tau(r) r e^{-2r} dr\, d\theta < \infty. \qquad (4.14)$$

We now want to show that f is of exponential type, so that the Borel transform of f is defined. We will prove that for every $\epsilon > 0$ there is a C_ϵ such that

$$|f(z)| \leq C_\epsilon\, e^{(1+\epsilon)H_K(z)}. \qquad (4.15)$$

It is obviously sufficient to prove this for z away from the origin. Fix $\epsilon > 0$ small. First of all, since f is holomorphic we have

$$\begin{aligned} |f(z)| &= \frac{1}{|B_n(z,\epsilon)|} \left| \int_{B_n(z,\epsilon)} f(w) dm(w) \right| \\ &\leq \frac{1}{|B_n(z,\epsilon)|^{1/2}} \left(\int_{B_n(z,\epsilon)} |f(w)|^2 dm(w) \right)^{1/2}. \end{aligned}$$

Since $H_K(z)$ is convex, it is Lipschitz continuous on the close unit ball in $\mathbb{C}^n$. This implies that when $w \in B_n(z,\epsilon)$ we have

$$\begin{aligned} |H_K(z) - H_K(w)| &= \left| |z| H_K\left(\frac{1}{|z|}z\right) - |w| H_K\left(\frac{1}{|w|}w\right) \right| \\ &\leq \left||z| - |w|\right| H_K\left(\frac{1}{|z|}z\right) + |w| \left| H_K\left(\frac{1}{|z|}z\right) - H_K\left(\frac{1}{|w|}w\right) \right| \\ &\leq C|z-w| + |w| C \left| \frac{1}{|z|}z - \frac{1}{|w|}w \right| \leq C\epsilon. \end{aligned}$$

Therefore we get that $e^{-2H_K(z)} \sim e^{-2H_K(w)}$ when $w \in B_n(z,\epsilon)$ so that

$$\begin{aligned} |f(z)| e^{-(1+\epsilon)H_K(z)} &\leq \frac{1}{|B_n(z,\epsilon)|^{1/2}} \left(\int_{B_n(z,\epsilon)} |f(w)|^2 e^{-2(1+\epsilon)H_K(z)} dm(w) \right)^{1/2} \\ &\leq C_\epsilon \left(\int_{B_n(z,\epsilon)} |f(w)|^2 e^{-2(1+\epsilon)H_K(w)} dm(w) \right)^{1/2}. \end{aligned}$$

It is easy to show that $\tau(r) \sim r^{-1/2}$ as $r \to \infty$, and this implies that

$$|f(z)| e^{-(1+\epsilon)H_K(z)} \leq C_\epsilon \left(\int_{B_n(z,\epsilon)} |f(w)|^2 \tau(H_K(w)) e^{-2H_K(w)} dm(w) \right)^{1/2}.$$

Since our domain is smooth and bounded we have that

$$\tau_n(H_K(z))(i\partial\bar{\partial}H_K)^n(z) \sim \tau(H_K(z))dm. \tag{4.16}$$

(In fact we have that $(i\partial\bar{\partial}\phi)^n = 2^n n!\, det(\phi_{j\bar{k}})dm$ for smooth functions ϕ.) If we use (4.16) and expand the area of integration to the whole of $\mathbb{C}^n$ we therefore get

$$|f(z)| \le C_\epsilon e^{(1+\epsilon)H_K(z)} \left(\int_{\mathbb{C}^n} |f(w)|^2 \tau_n(H_K(w)) e^{-2H_K(w)} (i\partial\bar{\partial}H_K)^n \right)^{1/2}.$$

This implies that the Borel transform $\mathcal{B}f(\zeta)$ converges for ζ such that $H_K(\zeta) < 1$, i.e. $\zeta \in K^\circ$, and as the domain K is circled, $K^\circ = K^*$. In short we have that

$$\psi(\zeta) := \mathcal{B}f(\zeta)$$

is holomorphic in K^*.

We get that

$$\lambda^{-n}\psi(\frac{1}{\lambda}\zeta) = \int_0^\infty f(\frac{t}{\lambda}\zeta)\left(\frac{t}{\lambda}\right)^{n-1} e^{-t}\frac{dt}{\lambda} = \int_0^\infty f_\zeta(\frac{t}{\lambda})e^{-t}\frac{dt}{\lambda} \tag{4.17}$$

which converges for λ such that $(1/\lambda)\zeta \in K^\circ = K^*$. But since K° is circled, this means by Proposition 4.4 that $(1/\lambda)$ belongs to some disk, and the unit disk when $\zeta \in \partial K^*$.

Let $\varphi_\zeta(\lambda)$ be the left hand side in (4.17). For a fixed $\zeta \in \partial K^*$ this is, as we have said, holomorphic when $\lambda \in \mathbb{C} \setminus \overline{B_1}$, and so (by (1.4)) defines an analytic functional carried by the unit disk. The right hand side of (4.17) is the Borel transform in $\mathbb{C}$ of the function f_ζ (the way it is usually defined in $\mathbb{C}$, which differs slightly from our definition (3.10) above). In $\mathbb{C}$ is it well known that this Borel transform is the inverse of the Laplace transform (see for instance the proof of the Polya-Martineau theorem in [1, Theorem 5.1]), so that $\widehat{\varphi_\zeta} = f_\zeta$ (where $\widehat{\varphi_\zeta}$ is the Laplace transform of the analytic functional defined by φ_ζ).

We want to use Theorem 4.1 to relate the norms of $\varphi_\zeta(\lambda)$ and f_ζ, but $\varphi_\zeta(\lambda)$ is now holomorphic in the complement of the unit disk. Katsnel'son's original theorem in [9] is formulated in this way and so gives us what we want. (If we want to use Theorem 4.1 we can take the Cauchy transform of the function $\overline{\varphi_\zeta(\lambda)}$ to get a function in the unit disk, which represents the same analytic functional by integration on the boundary of the unit disk.)

Anyway, neglecting these technical details which really only involves the definitions we choose, we get that the integral in (4.14) equals the norm of φ_ζ, i.e.

$$\begin{aligned} c\int_0^{2\pi}\int_0^\infty |f_\zeta(re^{i\theta})|^2\tau(r)re^{-2r}dr\,d\theta &= \|\varphi_\zeta(\lambda)\|^2_{L^2(\partial B_1)} = \|\varphi_\zeta(\frac{1}{\lambda})\|^2_{L^2(\partial B_1)} \\ &= \int_0^{2\pi} |\psi(e^{i\theta}\zeta)|^2 d\theta \end{aligned}$$

where the second equality simply follows since the radius of the circle is 1.

Thus the integral in (4.13) equals

$$c_n \int_{\partial K^*} \int_0^{2\pi} |\psi(e^{i\theta}\zeta)|^2 d\theta\, \partial\rho^* \wedge (\bar{\partial}\partial\rho^*)^{n-1}$$

and using Lemma 4.8 again to remove the integral of θ, we have proved what we wanted.

We will now show that the Borel transform is surjective from the functions satisfying (4.10) to $E^2(K^*)$, so let $\psi \in E^2(K^*)$. Put

$$K_\epsilon = K + \epsilon B_n.$$

As in (3.5) we define an analytic functional in $\mathcal{O}'(K)$ by

$$\mu(h) = \int_{\partial K_\epsilon} h(w)\psi(s(w))s(w) \wedge (\bar{\partial}s(w))^{n-1}, \quad h \in \mathcal{O}(K),$$

where the integral is independent of the choice of shell to integrate over, since the form is $\bar{\partial}$-closed. We get that $\hat{\mu}$ is an entire function of exponential type, and in section 3.2 we saw that $\mathcal{B}\hat{\mu}$ has an analytic continuation to K^*. In section 3.3 we showed that

$$\mathcal{B}\hat{\mu} = \mathcal{F}\mu.$$

But we showed in (3.9) that actually $\mathcal{F}\mu = c_n\psi$. Hence

$$\mathcal{B}\hat{\mu} = \psi$$

and that the entire function $\hat{\mu}$ really satisfies condition (4.10) follows from $\psi \in E^2(K^*)$ by the same calculations as above. Hence the Borel transform is surjective. □

4.3 The Borel transform for non circled domains

If K is not circled, we of course lose a lot of the symmetry we used in the previous section. For instance, the form $s(w) \wedge (\bar{\partial}s(w))^{n-1}$ will be as in (3.4) where $\langle 2\partial\rho(w), w\rangle$ now is not real. The imaginary part of this will, in some sense, measure how far K is from being circled. The rotation invariance in Lemma 4.8 fails to hold, but if we use the notation ${K^*}_\zeta = \{\lambda \in \mathbb{C} : \lambda\zeta \in K^*\}$ from Lemma 2.6, we can at least get the following for K^* (and similarly for K):

Lemma 4.9. *If K^* is strongly $\mathbb{C}$-convex and smoothly bounded then*

$$\int_{\partial K^*} \varphi(\zeta)\, dS_{K^*}(\zeta) \sim \int_{\partial K^*} \int_{\partial({K^*}_\zeta)} \varphi(\lambda\zeta)\, d\sigma(\lambda)\, dS_{K^*}(\zeta)$$

for $\varphi \in C(\partial K^*)$.

Proof. The condition that K^* should be strongly $\mathbb{C}$-convex, is simply to guarantee that the measure dS_{K^*} is equivalent to ordinary surface measure, which it is in any strictly pseudoconvex domain. Let $F : \partial B_n \to \partial K^*$ be the map $z \mapsto z/\rho^*(z)$ with inverse $\zeta \mapsto \zeta/|\zeta|$. We get

$$\begin{aligned}
\int_{\partial K^*} \varphi(\zeta)\, dS_{K^*}(\zeta) &= \int_{\partial B_n} \varphi\left(\frac{1}{\rho^*(z)} z\right) F^*(dS_{K^*}) \\
&\sim \int_{\partial B_n} \varphi\left(\frac{1}{\rho^*(z)} z\right) d\sigma(z) \\
&= \int_{\partial B_n} \frac{1}{2\pi} \int_0^{2\pi} \varphi\left(\frac{e^{i\theta}}{\rho^*(e^{i\theta} z)} z\right) d\theta\, d\sigma(z).
\end{aligned}$$

If we let $\psi(z)$ be the inner integral in the last line, we get in the same way that the last line equals

$$\begin{aligned}
\int_{\partial K^*} \psi\left(\frac{1}{|\zeta|}\zeta\right) (F^{-1})^*(d\sigma) &\sim \int_{\partial K^*} \psi\left(\frac{1}{|\zeta|}\zeta\right) dS_{K^*}(\zeta) \\
&= \int_{\partial K^*} \frac{1}{2\pi} \int_0^{2\pi} \varphi\left(\frac{e^{i\theta}/|\zeta|}{\rho^*(e^{i\theta}\zeta/|\zeta|)}\zeta\right) d\theta\, dS_{K^*}(\zeta) \\
&= \int_{\partial K^*} \frac{1}{2\pi} \int_0^{2\pi} \varphi\left(\frac{e^{i\theta}}{\rho^*(e^{i\theta}\zeta)}\zeta\right) d\theta\, dS_{K^*}(\zeta) \\
&\sim \int_{\partial K^*} \int_{\partial(K^*{}_\zeta)} \varphi(\lambda\zeta)\, d\sigma(\lambda)\, dS_{K^*}(\zeta).
\end{aligned}$$

The measure dS_{K^*} is, as we said, equivalent to the surface measure, but the constants (as well as the constants in the pullback then) will depend on the eigenvalues of the Hessian of ρ^*, i.e. "how convex" K is. ☐

We will now state the main theorem of this section.

Theorem 4.10. *Let K be a bounded, smooth and strongly convex domain. Then the Borel transform is an isomorphism between the space of entire functions f which satisfies*

$$\int |f(z)|^2 e^{-2H_K(z)} |z|^{n-1/2} (i\partial\bar{\partial} H_K)^n < \infty, \tag{4.18}$$

and $E^2(K^*)$. Furthermore,

$$\|\mathcal{B}f\|^2_{E^2(K^*)} \sim \int |f(z)|^2 e^{-2H_K(z)} |z|^{n-1/2} (i\partial\bar{\partial} H_K)^n. \tag{4.19}$$

We will again need to pull back integration from ∂K to ∂K^*, as in Lemma 4.6. When the domain was circled, we saw from (4.11) that

$$\frac{1}{(2\pi i)^n} s(w) \wedge (\bar{\partial} s(w))^{n-1} = \frac{2^n}{(2\pi i)^n} \rho^{-n} \partial\rho(w) \wedge (\bar{\partial}\partial\rho(w))^{n-1}$$

is a real form. This will not be the case when K is not circled, and we will instead be interested in the modulus of these measures.

Lemma 4.11. *Let h be a smooth function on ∂K. Then*

$$\int_{\partial K} h(z)|s(z) \wedge (\bar{\partial} s(z))^{n-1}| = (-1)^n \int_{\partial K^*} h(s^*(\zeta))|s^*(\zeta) \wedge (\bar{\partial} s^*(\zeta))^{n-1}|.$$

Proof. This follows *mutatis mutandis* as in the proof of Lemma 3.4. □

We can now prove Theorem 4.10. The proof follows the proof of Theorem 4.3 very closely, except that we will have to settle for norm equivalences instead of identities. These can be derived in a similar fashion to what we have done above, but we will not indulge ourselves in technical details.

Proof of Theorem 4.10. We will rewrite condition (4.18). Instead of the change of variables in Lemma 4.5 we will use

$$(r, w) \mapsto z = r\, s(w), \quad r \in \mathbb{R}_+ \quad w \in \partial K$$

with inverse

$$w = \rho^*(z) s^*(z), \quad r = \rho^*(z).$$

That this mapping is one-to-one onto $\mathbb{C}^n \setminus \{0\}$ follows exactly as the bijectiveness of the mapping s in section 2.2.

We get that

$$\int |f(z)|^2 e^{-2H_K(z)} |z|^{n-1/2} (i\partial\bar{\partial} H_K)^n \sim \\ \int_{\partial K} \int_0^\infty |f(rs(w))|^2 r^{2n-3/2} e^{-2H_K(rs(w))}\, dr\, |s(w) \wedge (\bar{\partial} s(w))^{n-1}|$$

and using Lemma 4.11 this equals

$$\int_{\partial K^*} \int_0^\infty |f(r\zeta)|^2 r^{2n-3/2} e^{-2H_K(r\zeta)}\, dr\, |s^*(\zeta) \wedge (\bar{\partial} s^*(\zeta))^{n-1}| \\ \sim \int_{\partial K^*} \int_0^\infty |f(r\zeta)|^2 r^{2n-3/2} e^{-2H_K(r\zeta)}\, dr\, dS_{K^*}(\zeta).$$

Let, as in the proof of Theorem 4.3,

$$f_\zeta(z) = f(z\zeta) z^{n-1}.$$

If we use Lemma 4.9, we see that the last integral is similar to

$$\int_{\partial K^*} \int_0^\infty \int_{\partial (K^*{}_\zeta)} |f_\zeta(r\lambda)|^2 r^{1/2} e^{-2H_K(r\lambda\zeta)} d\sigma(\lambda)\, dr\, dS_{K^*}(\zeta). \tag{4.20}$$

From Lemma 2.6 we remember that $M = (K^*{}_\zeta)^*$ is some planar domain with supporting function $H_M(w) = H_K(w\zeta), w \in \mathbb{C}$. Hence we get that

$$\begin{aligned}\int_{\partial(K^*{}_\zeta)} |f_\zeta(r\lambda)|^2 e^{-2H_K(r\lambda\zeta)} d\sigma(\lambda) &\sim \int_{\partial(K^*{}_\zeta)^*} |f_\zeta(\frac{r}{w})|^2 e^{-2H_K(\frac{r}{w}\zeta)} d\sigma(w) \\ &= \int_{\partial M} |f_\zeta(\frac{r}{w})|^2 e^{-2H_M(\frac{r}{w})} d\sigma(w)\end{aligned}$$

(where the constants will depend on the diameter of $K^*{}_\zeta$) so that

$$\begin{aligned}&\int_0^\infty \int_{\partial(K^*{}_\zeta)} |f_\zeta(r\lambda)|^2 r^{1/2} e^{-2H_K(r\lambda\zeta)} d\sigma(\lambda)\, dr \\ &\quad\sim \int_0^\infty \int_{\partial M} |f_\zeta(\frac{r}{w})|^2 r^{1/2} e^{-2H_M(\frac{r}{w})} d\sigma(w)\, dr.\end{aligned}$$

In $\mathbb{C}$, the change of variables we used in the beginning of the proof is simply $(r, w) \mapsto r/w$. If we use this in the last line, we get that this is similar to

$$\int_{\mathbb{C}} |f_\zeta(w)|^2 |w|^{1/2} e^{-2H_M(w)} \Delta H_M(w) dm(w).$$

We want to show that the Borel transform of f is defined. Exactly as in the proof of Theorem 4.3 we get that

$$|f(z)| \leq C_\epsilon\, e^{(1+\epsilon)H_K(z)} \tag{4.21}$$

for every $\epsilon > 0$. This implies that the Borel transform of f converges for ζ such that $H_K(\zeta) < 1$, i.e. $\zeta \in K^\circ$, and in section 3.2 we proved that it has an analytic continuation to K^*. Hence we see that

$$\psi(\zeta) := \mathcal{B}f(\zeta)$$

is holomorphic in K^*.

We get

$$\lambda^{-n}\psi(\frac{1}{\lambda}\zeta) = \int_0^\infty f(\frac{t}{\lambda}\zeta)\left(\frac{t}{\lambda}\right)^{n-1} e^{-t}\,\frac{dt}{\lambda} = \int_0^\infty f_\zeta(\frac{t}{\lambda}) e^{-t}\,\frac{dt}{\lambda} \tag{4.22}$$

which converges at least for λ such that $(1/\lambda)\zeta \in K^\circ$, i.e. $H_K((1/\lambda)\zeta) = H_M(1/\lambda) < 1$. The expression in (4.22) is therefore analytic for $1/\lambda \in M^\circ$, and as above it can be seen to have an analytic continuation to $\{\lambda : 1/\lambda \in M^* = (K^*)_\zeta\}$, i.e. to M^c. Let $\varphi_\zeta(\lambda)$ be the left hand side in (4.22). It is thus analytic in M^c and defines an analytic functional carried by M. As in the proof of Theorem 4.3, the right hand side of (4.22) is the Borel transform of f_ζ (the way it is usually defined in $\mathbb{C}$ — see [1]), and as we have mentioned, it is well known that this Borel transform is the inverse of the

Laplace transform in $\mathbb{C}$. Therefore $\widehat{\varphi_\zeta} = f_\zeta$ (where $\widehat{\varphi_\zeta}$ is really the Laplace transform of the analytic functional defined by φ_ζ).

Lutsenko and Yulmukhametov prove Theorem 1.3 above in two steps, where the second step consists in showing that the Borel transform is an isomorphism between the entire functions which satisfies (1.8), and functions which are analytic in the complement of the domain, and square integrable on the boundary (the Hardy space in the complement).

Thus using Theorem 2 in [13] we get that

$$\begin{aligned}\int_{\mathbb{C}} |f_\zeta(w)|^2 |w|^{1/2} e^{-2H_M(w)} \Delta H_M(w) dm(w) &\sim \|\varphi_\zeta(\lambda)\|^2_{L^2(\partial M)} \\ &\sim \|\varphi_\zeta(1/\lambda)\|^2_{L^2(\partial M^*)} \\ &= \|\lambda^{-n}\psi(\lambda\zeta)\|^2_{L^2(\partial (K^*)_\zeta)} \\ &\sim \|\psi(\lambda\zeta)\|^2_{L^2(\partial (K^*)_\zeta)}\end{aligned}$$

where the change of variables $\lambda \mapsto 1/\lambda$ in the second line will give rise to constants which depends on the diameter of the set M, and the constants in the last line will depend on an upper and lower bound for the radius of K^* in different directions.

Using all this, the integral in (4.20) is similar to

$$\int_{\partial K^*} \int_{\partial (K^*)_\zeta} |\psi(\lambda\zeta)|^2 \, d\sigma(\lambda) \, dS_{K^*}(\zeta) \sim \int_{\partial K^*} |\psi(\zeta)|^2 \, dS_{K^*}(\zeta)$$

where we used Lemma 4.9.

Thus, if f satisfies (4.18) we have proved that its Borel transform $\mathcal{B}f$ satisfies (4.19). That the Borel transform is surjective follows exactly as in the proof of Theorem 4.3. □

4.4 Inequalities for the Fantappiè transform

We have seen in section 3.3 that every $g \in E^2(K)$ defines an element of $\mathcal{O}'(K)$, so that its Fantappiè transform is holomorphic in the interior of K^*. We shall prove that it is actually a function in $E^2(K^*)$, and that the Fantappiè transform is an isomorphism between the Hilbert spaces $E^2(K)$ and $E^2(K^*)$. The proof of this will be modelled on the proof of Theorem 2.5 in [5].

Theorem 4.12. *The Fantappiè transform is an isomorphism between $E^2(K)$ and $E^2(K^*)$. For every $g \in E^2(K)$ we have*

$$c_n \|g\|_{E^2(K)} \le \|\mathcal{F}g\|_{E^2(K^*)} \le c_{n,K} \|g\|_{E^2(K)} \tag{4.23}$$

(the constant on the left side depends only on the dimension), with equality on both sides if K is the unit ball.

Proof. Analogously to what we did in section 3.1 we will construct a pairing between $E^2(K)$ and $E^2(K^*)$. Consider the submanifold

$$\Lambda = \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n : z \in \partial K, \zeta = s(z)\}$$

of the incidence manifold (which was defined in (3.6)). We define

$$\langle h, \psi \rangle_\Lambda = \int_\Lambda h(z)\psi(\zeta)\nu \wedge (d\nu)^{n-1}, \quad h \in E^2(K), \quad \psi \in E^2(K^*),$$

where the form $\nu \wedge (d\nu)^{n-1}$ was defined in (3.7). In (the proof of) Lemma 3.4 we saw that we have the two parametrizations

$$z \mapsto (z, s(z)), \quad z \in \partial K,$$

and

$$\zeta \mapsto (s^*(\zeta), \zeta), \quad \zeta \in \partial K^*,$$

of Λ, and that the form $\nu \wedge (d\nu)^{n-1}$ pulls back to $s \wedge (\bar\partial s)^{n-1}$ and $\pm s^* \wedge (\bar\partial s^*)^{n-1}$, respectively. If we use Cauchy-Schwarz inequality we therefore get that

$$|\langle h, \psi \rangle_\Lambda|^2 \leq \int_{\partial K} |h(z)|^2 |s(z) \wedge (\bar\partial s(z))^{n-1}| \int_{\partial K^*} |\psi(\zeta)|^2 |s^*(\zeta) \wedge (\bar\partial s^*(\zeta))^{n-1}|.$$

Now look at the expression for $s \wedge (\bar\partial s)^{n-1}$ in (3.4). We have that

$$\operatorname{Re} \langle 2\partial\rho(z), z\rangle = \rho(z) = 1$$

on ∂K, so $|\langle 2\partial\rho(z), z\rangle| \geq 1$ on ∂K (and the same for ρ^* on ∂K^*). Therefore we have

$$|\langle h, \psi \rangle_\Lambda| \leq c_n \|h\|_{E^2(K)} \cdot \|\psi\|_{E^2(K^*)}.$$

Thus every fixed $\psi \in E^2(K^*)$ defines a bounded linear functional on the Hilbert space $E^2(K)$, with norm not exceeding $c_n\|\psi\|_{E^2(K^*)}$. By the Riesz representation theorem, this functional is represented by a unique $\beta(\psi) \in E^2(K)$ with

$$\|\beta(\psi)\|_{E^2(K)} \leq c_n \|\psi\|_{E^2(K^*)}. \tag{4.24}$$

Using the inner product in $E^2(K)$ we can write

$$(h, \beta(\psi))_{E^2(K)} = \langle h, \psi \rangle_\Lambda, \quad h \in E^2(K).$$

If we let

$$h_\xi(z) = \frac{(n-1)!}{(1 - z \cdot \xi)^n}$$

for ξ in (the interior of) K^* we have in particular that

$$\mathcal{F}(\beta(\psi))(\xi) = (h_\xi, \beta(\psi))_{E^2(K)} = \langle h_\xi, \psi \rangle_\Lambda = c_n \psi(\xi), \tag{4.25}$$

where we demonstrated the last equality in (3.9) (if we just write the pairing in the first line of (3.9) using our manifold Λ instead).

If we accept for the moment that the Fantappiè transform really maps $E^2(K)$ into $E^2(K^*)$ we have now in (4.25) proved that it is surjective onto $E^2(K^*)$. Then the left inequality in (4.23) follows from (4.24), since it is known (see [8] or [3]) that the Fantappiè transform is injective even as a mapping from $\mathcal{O}'(K)$.

What we have yet to prove is the right inequality in (4.23), which also demonstrates that the Fantappiè transform maps $E^2(K)$ into $E^2(K^*)$. Let us define

$$\alpha_1(z) = \langle \partial\rho(z), z\rangle^n, \quad \alpha_2(\zeta) = \langle \partial\rho^*(\zeta), \zeta\rangle^n.$$

If we use Lemma 3.4 and the expression for $s \wedge (\bar\partial s)^{n-1}$ in (3.4), we can write

$$\begin{aligned}
\mathcal{F}(g)(\xi) &= \int_{\partial K} \frac{(n-1)!}{(1-z\cdot\xi)^n}\overline{g(z)}\, dS_K(z) \\
&= c_n \int_{\partial K} \frac{\overline{g(z)}\alpha_1(z)}{(1-z\cdot\xi)^n}\, s(z)\wedge(\bar\partial s(z))^{n-1} \\
&= c_n \int_{\partial K^*} \frac{\overline{g(s^*(\zeta))}\alpha_1(s^*(\zeta))}{(1-s^*(\zeta)\cdot\xi)^n} s^*(\zeta)\wedge(\bar\partial s^*(\zeta))^{n-1} \\
&= c_n \int_{\partial K^*} \frac{\overline{g(s^*(\zeta))}\alpha_1(s^*(\zeta))}{\langle s^*(\zeta), \zeta-\xi\rangle^n}\, s^*(\zeta)\wedge(\bar\partial s^*(\zeta))^{n-1} \\
&= c_n \int_{\partial K^*} \frac{\overline{g(s^*(\zeta))}\alpha_1(s^*(\zeta))}{\langle \partial\rho^*(\zeta), \zeta-\xi\rangle^n}\, \partial\rho^*(\zeta)\wedge(\bar\partial\partial\rho^*(\zeta))^{n-1}. \quad (4.26)
\end{aligned}$$

If we let

$$H(\varphi)(\xi) = \int_{\partial K^*} \frac{\varphi(\zeta)}{\langle \partial\rho^*(\zeta), \zeta-\xi\rangle^n}\, \partial\rho^*(\zeta)\wedge(\bar\partial\partial\rho^*(\zeta))^{n-1}, \qquad (4.27)$$

we can rewrite the above as

$$\mathcal{F}(g) = c_n H((\overline{g}\cdot\alpha_1)\circ s^*).$$

The operator H is a Henkin-Ramirez type of projection operator on holomorphic functions. The boundedness of such type of operators, in the more general case of strictly pseudoconvex domains, is considered in Kerzman and Stein's paper [10]. Since the domain K^* is strongly $\mathbb{C}$-convex, it is a forteriori strictly pseudoconvex, but in a domain which is only pseudoconvex the denominator in (4.27) can be zero, and the somewhat naïve projection operator H (it correlates to the Cauchy-Fantappiè representation formula valid in convex or $\mathbb{C}$-convex domains) has no meaning. The operators considered in [10] are consequently (and also because the authors actually prove more than the boundedness property and need more symmetry) defined with a different kernel than our operator H. A different proof of the boundedness of those operators can, among other things, also be found in the first part of [7]. For strongly convex or strongly $\mathbb{C}$-convex domains the operator H does have meaning though, and in this case the kernel actually satisfies the assumptions in [10] or [7], and the boundedness of the operator H follows.

The operator H is bounded on L^2 with respect to surface measure $d\sigma$, which in our case is equivalent to the measure dS_{K^*}. This implies that

$$\begin{aligned}
\|\mathcal{F}(g)\|_{E^2(K^*)} &= c_n\|H((\overline{g}\cdot\alpha_1)\circ s^*)\|_{E^2(K^*)} \\
&\leq c_{n,K}\|H((\overline{g}\cdot\alpha_1)\circ s^*)\|_{L^2(\partial K^*,d\sigma)} \\
&\leq c_{n,K}\|(\overline{g}\cdot\alpha_1)\circ s^*\|_{L^2(\partial K^*,d\sigma)} \\
&\leq c_{n,K}\left(\int_{\partial K^*}|(\overline{g}\cdot\alpha_1)\circ s^*(\zeta)|^2\,dS_{K^*}(\zeta)\right)^{1/2} \\
&= c_{n,K}\left(\int_{\partial K^*}|(\overline{g}\cdot\alpha_1)\circ s^*(\zeta)|^2|\alpha_2(\zeta)|\,|s^*\wedge(\bar\partial s^*)^{n-1}|\right)^{1/2} \\
&= c_{n,K}\left(\int_{\partial K}|(\overline{g}\cdot\alpha_1)(z)|^2|\alpha_2(s(z))|\,|s\wedge(\bar\partial s)^{n-1}|\right)^{1/2} \\
&\leq c_{n,K}\left(\int_{\partial K}|g(z)|^2 dS_K(z)\right)^{1/2} \\
&= c_{n,K}\|g\|_{E^2(K)},
\end{aligned} \tag{4.28}$$

where we used Lemma 3.4 to change the integration to ∂K. We have now proved the right hand side of (4.23),

If K is the unit ball B_n, then $K = K^*$. We get that $\rho^*(z) = \rho(z) = |z|$, so

$$\partial\rho^*(z) = \partial\rho(z) = \frac{1}{2|z|}\sum \bar z_j dz_j,$$

and $\alpha_1(z)$ and $\alpha_2(z)$ are constant on ∂B_n. In this case will have $\beta(\psi)(z) = \overline{\psi(\bar z)}$ (if we divide by a suitable constant). That the Fantappiè transform of this function is ψ can actually be seen from (4.26), since then on ∂B_n we will have $s(z) = s^*(z) = \bar z$. The last line of (4.26) will in this case just be

$$c_n\int_{\partial K^*}\frac{\psi(\zeta)}{\langle\partial\rho^*(\zeta),\zeta-\xi\rangle^n}\,\partial\rho^*(\zeta)\wedge(\bar\partial\partial\rho^*(\zeta))^{n-1}$$

and by the Cauchy-Fantappiè formula, this equals $c_n\psi(\xi)$. Therefore we have equality in (4.24). In the same way we have that $\mathcal{F}(g)(z) = \overline{g(\bar z)}$, so that we have equalities all the way in (4.28). Hence we have equality on both sides of (4.23). □

4.5 Proof of the main theorems

We now have everything we need for Theorem 1.4 and Theorem 1.5. From section 3.3 we know that we have the relation

$$\mathcal{B} \circ \mathcal{L} = \mathcal{F}$$

if we consider the Fantappiè transform $\mathcal{F}$ as a mapping from $\mathcal{O}'(K)$ to $\mathcal{O}(K^*)$, the Laplace transform as a mapping from $\mathcal{O}'(K)$ to the space of entire functions of exponential type, and the Borel transform $\mathcal{B}$ as acting on entire functions of exponential type.

Now, if K is circled we get, by Theorem 4.12 for the Fantappiè transform and Theorem 4.3 for the Borel transform, that the Laplace transform is an isomorphism between the Hilbert spaces $E^2(K)$ and $P^2(K)$ (with the norms as in (1.12)). If f satisfies the conditions in Theorem 1.5 then $\mathcal{B}f \in E^2(K^*)$ with $\|\mathcal{B}f\|_{E^2(K^*)} = c_n \|f\|_{P^2(K)}$. Theorem 4.12 implies that there is a unique $\psi \in E^2(K)$ such that $\mathcal{F}\psi = \mathcal{B}f$, and

$$c_n \|\psi\|_{E^2(K)} \leq \|\mathcal{B}f\|_{E^2(K^*)} \leq c_{n,K} \|\psi\|_{E^2(K)},$$

with equality on both sides for the unit ball. Hence

$$c_n \|\psi\|_{E^2(K)} \leq \|f\|_{P^2(K)} \leq c_{n,K} \|\psi\|_{E^2(K)},$$

with equalities for the unit ball, and Theorem 1.5 is proved.

If K is not circled we can instead use Theorem 4.10 for the Borel transform. For $\psi \in E^2(K)$ we get that

$$\|\mathcal{L}\psi\|_{P^2(K)} \sim \|\mathcal{B}\mathcal{L}\psi\|_{E^2(K^*)} = \|\mathcal{F}\psi\|_{E^2(K^*)} \sim \|\psi\|_{E^2(K)},$$

which proves (1.12) in Theorem 1.4.

Appendix A

Smoothness of K° and K^*

Proof of Lemma 2.4. If K is defined as in (2.1), then as we have seen above

$$\psi = 2\left(\frac{\partial\rho}{\partial z_1}, \ldots, \frac{\partial\rho}{\partial z_n}\right) =: 2\partial\rho$$

maps ∂K to ∂K°, and $\varphi(z) = \partial\rho(z)/\langle\partial\rho(z), z\rangle$ maps ∂K to ∂K^*. The strong convexity of K implies that these mappings are injective from ∂K to $\mathbb{C}^n$ (see the discussion in section 2.2). If we can prove that they are also immersions, i.e. that their ranks as mappings from ∂K are maximal, then a well known theorem says that their images are smooth manifolds, just like ∂K, and that the mappings are actually diffeomorphisms.

We start with the mapping ψ and let

$$D\psi = \frac{\partial\left(\operatorname{Re}\psi_1, \operatorname{Im}\psi_1, \ldots, \operatorname{Im}\psi_n\right)}{\partial\left(x_1, y_1, \ldots, y_n\right)}$$

be the Jacobian matrix of ψ at $p \in \partial K$. We have to show that if $v \in T_p(\partial K)$ is a nonzero tangent vector, $v = (a_1, b_1, \ldots, b_n)$, then $D\psi(v) \neq 0$. Extending the Jacobian to a mapping from the complexification $\mathbb{C}T_p(\partial K)$ of the tangent space and making the change of basis $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right\} \mapsto \left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z_1}}, \ldots, \frac{\partial}{\partial \bar{z_n}}\right\}$ (or by using the chain rule), we see that this is equivalent to (all derivatives at $p \in \partial K$)

$$\begin{bmatrix} \frac{\partial\psi_1}{\partial z_1} & \frac{\partial\psi_1}{\partial \bar{z_1}} & \frac{\partial\psi_1}{\partial z_2} & \cdots \\ \frac{\partial\bar{\psi_1}}{\partial z_1} & \frac{\partial\bar{\psi_1}}{\partial \bar{z_1}} & \frac{\partial\bar{\psi_1}}{\partial z_2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} v_1 \\ \bar{v_1} \\ \vdots \end{bmatrix} \neq 0$$

That is

$$\begin{bmatrix} \frac{\partial^2\rho}{\partial z_1 \partial z_1} & \frac{\partial^2\rho}{\partial \bar{z_1} \partial z_1} & \frac{\partial^2\rho}{\partial z_2 \partial z_1} & \cdots \\ \frac{\partial^2\rho}{\partial z_1 \partial \bar{z_1}} & \frac{\partial^2\rho}{\partial \bar{z_1} \partial \bar{z_1}} & \frac{\partial^2\rho}{\partial z_2 \partial \bar{z_1}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} v_1 \\ \bar{v_1} \\ \vdots \end{bmatrix} \neq 0 \tag{A.1}$$

where $v_j = a_j + ib_j$. We shall call this matrix $\tilde{D}\psi$.

On the other hand, the strong convexity of K means that for every nonzero $v \in T_p(\partial K)$

$$0 \quad < \quad 2\,\mathrm{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k} v_j v_k\right) + 2\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_k} \bar{v_j} v_k$$

or equivalently

$$\begin{bmatrix} v_1 & \bar{v_1} & \cdots \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \rho}{\partial z_1 \partial z_1} & \frac{\partial^2 \rho}{\partial \bar{z_1} \partial z_1} & \frac{\partial^2 \rho}{\partial z_2 \partial z_1} & \cdots \\ \frac{\partial^2 \rho}{\partial z_1 \partial \bar{z_1}} & \frac{\partial^2 \rho}{\partial \bar{z_1} \partial \bar{z_1}} & \frac{\partial^2 \rho}{\partial z_2 \partial \bar{z_1}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} v_1 \\ \bar{v_1} \\ \vdots \end{bmatrix} > 0. \qquad (A.2)$$

Comparing (A.1) and (A.2) we see that since the matrix $\tilde{D}\psi$ is positive definite when restricted to $T_p(\partial K)$, no nonzero tangent vector can be a nullvector, which is what we wanted to show.

We now turn our attention to the mapping φ. When $n = 1$ this mapping is simply $z \mapsto 1/z$, so we let $n > 1$. For simplicity we choose coordinates so that $\partial\rho = (1, 0, \ldots, 0)$ at p. (Observe that then Re (p_1) $= \mathrm{Re}\,\langle \partial\rho(p), p\rangle = \rho(p)/2 \neq 0$.) We then have

$$\begin{aligned} d\varphi_k(p) \quad = \quad & \frac{1}{p_1}\left(\sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k} dz_j + \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_k} d\bar{z}_j\right) - \\ & - \delta_{1k}\frac{1}{p_1^2}\left(dz_1 + \sum_{j,l=1}^{n} p_l \frac{\partial^2 \rho}{\partial z_j \partial z_l} dz_j\right) - \delta_{1k}\frac{1}{p_1^2}\sum_{j,l=1}^{n} p_l \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_l} d\bar{z}_j \end{aligned}$$

(δ_{1k} being the Kronecker delta) so that in the matrix $\tilde{D}\varphi$, except for the factor $1/p_1$, only the first two rows differ from $\tilde{D}\psi$. In our coordinates we have $T_p^{(1,0)}(\partial K) = \{v : \partial\rho.v = 0\} = \{v : v_1 = 0\}$, so if we look at equation (A.2) again we easily convince ourselves that no nonzero $v \in T_p^{(1,0)}(\partial K)$ can be a nullvector of $\tilde{D}\varphi$.

Now let $v \notin T_p^{(1,0)}(\partial K)$ so that $v_1 \neq 0$. We have

$$\tilde{D}\varphi \begin{bmatrix} v_1 \\ \bar{v_1} \\ \vdots \end{bmatrix} \quad = \quad \begin{bmatrix} (d\varphi_1).v \\ \overline{(d\varphi_1).v} \\ (d\varphi_2).v \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{v_1}{p_1^2} - \sum_{l=2}^{n} \frac{p_l}{p_1^2} \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_l} v_j - \sum_{l=2}^{n} \frac{p_l}{p_1^2} \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_l} \bar{v_j} \\ \cdots \\ \frac{1}{p_1} \left(d \frac{\partial \rho}{\partial z_2} \right) .v \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{v_1}{p_1^2} - \sum_{l=2}^{n} \frac{p_l}{p_1^2} \left(d \frac{\partial \rho}{\partial z_l} \right) .v \\ \cdots \\ \frac{1}{p_1} \left(d \frac{\partial \rho}{\partial z_2} \right) .v \\ \vdots \end{bmatrix}.$$

Thus if all the entries $3,\ldots,2n$ are zero, at least the first two are nonzero, and v can not be a nullvector. □

Remark A.1. You may notice that for ψ in the preceding proof, we only proved that the Jacobian is non singular when restricted to $T_p(\partial K)$, but that we found that the Jacobian for φ is non singular as a mapping from $\mathbb{C}^n$. This is natural. Since the function ρ is 1-homogeneous, the mapping ψ is 0-homogeneous, and the matrix in (A.1) has a nullspace of (real) dimension at least one. The mapping φ on the other hand is homogeneous of degree -1 and it is injective from $\mathbb{C}^n \setminus \{0\}$ to $\mathbb{C}^n \setminus \{0\}$. □

Remark A.2. When we showed in the above proof that the mapping φ was an immersion, we really only used that the matrix in (A.2) was positive definite restricted to the complex tangent space, i.e. that the Hessian of ρ was positive definite on the complex tangent space. This is exactly the condition that the domain is strongly $\mathbb{C}$-convex, so what we really have proved is that if the smooth domain K is strongly $\mathbb{C}$-convex, then K^* is smooth. □

Proof of Proposition 2.5. Let K be smooth and strongly convex. We know from the previous lemma that $K^\circ = \{z : H_K(z) < 1\}$ is smooth, and we want to show that the Hessian of H_K is positive definite restricted to the real tangent space at every point of ∂K°. Since the polar of a convex domain is always convex, we know that the Hessian is positive semidefinite, so we only need to prove that it is nonsingular. In the proof of the previous lemma we showed that the mapping

$$\psi(z) = 2\partial\rho(z) : \partial K \to \partial K^\circ$$

is a diffeomorphism, so that the Jacobian matrix $D(\psi^{-1})$ is nonsingular. But geometrically the mapping ψ^{-1} is given by

$$\psi^{-1}(w) = 2\partial H_K(w).$$

Thus the Jacobian matrix of ∂H_K is nonsingular, which exactly as in the previous proof is equivalent to that the Hessian of H_K is nonsingular.

In the $\mathbb{C}$-convex case we want to show that the Hessian of ρ^* is positive definite when restricted to the complex tangent space at any boundary point. This follows in the same way (in view of Remark A.2) if we just replace convex by $\mathbb{C}$-convex and the real tangent space by the complex tangent space. If φ is the mapping from ∂K to ∂K^* we get that the matrix $\tilde{D}(\varphi^{-1})$ is nonsingular. But the inverse is given by

$$\varphi^{-1}(w) = \frac{1}{\langle \partial\rho^*(w), w\rangle}.\partial\rho^*(w)$$

Just notice now that, as in the previous proof, the matrix $\tilde{D}(\varphi^{-1})$ is nonsingular (on the whole of $\mathbb{C}^n$) precisely when the Hessian of ρ^* is nonsingular restricted to the complex tangent space. □

Proof of Lemma 2.6. Let M be the set on the right hand side of (2.5). The orthogonal projection is by the Hermitian inner product $(z,w) = z\cdot\bar{w}$ on $\mathbb{C}^n$, and every $z \in \mathbb{C}^n$ can be written

$$z = \frac{\gamma}{|\zeta|^2}\bar{\zeta} + z_1$$

for some $\gamma \in \mathbb{C}$ and some $z_1 \in \mathbb{C}^n$ with $(z_1, \bar{\zeta}) = z_1\cdot\zeta = 0$. Using this we have that for $\lambda \neq 0$

$$\begin{aligned}
\lambda \in ({K^*}_\zeta)^* &\Leftrightarrow \frac{1}{\lambda} \notin {K^*}_\zeta \\
&\Leftrightarrow \exists z \in K : z\cdot\frac{1}{\lambda}\zeta = \frac{\gamma}{\lambda} = 1 \\
&\Leftrightarrow \exists z \in K : z = \frac{\lambda}{|\zeta|^2}\bar{\zeta} + z_1,\ (z_1,\bar{\zeta}) = 0 \\
&\Leftrightarrow \lambda \in M.
\end{aligned}$$

We also have that

$$\begin{aligned}
H_M(w) &= \sup_{\lambda\in M} \operatorname{Re}(\lambda\cdot w) = \sup_{\lambda\in M} \operatorname{Re}\left(\frac{\lambda}{|\zeta|^2}\bar{\zeta}\cdot w\zeta\right) \\
&= \sup_{\lambda\in M,\ (z_1,\bar{\zeta})=0} \operatorname{Re}\left(\left(\frac{\lambda}{|\zeta|^2}\bar{\zeta} + z_1\right)\cdot w\zeta\right) \\
&= \sup_{z\in K} \operatorname{Re}\,(z\cdot w\zeta) \\
&= H_K(w\zeta),
\end{aligned}$$

and the proof is complete. □

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