STABILITY AND ANALYTICITY IN MAXIMUM-NORM FOR SIMPLICIAL LAGRANGE FINITE ELEMENT SEMIDISCRETIZATIONS OF PARABOLIC EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

V. Thomée and L.B. Wahlbin

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Abstract. Stability and analyticity estimates in maximum-norm are shown for spatially discrete finite element approximations based on simplicial Lagrange elements for the model heat equation with Dirichlet boundary conditions. The bounds are logarithm free and valid in arbitrary dimension and for arbitrary polynomial degree. The work continues an earlier study by Schatz et al. [5] in which Neumann boundary conditions were considered.

1. Introduction. In the earlier work Schatz et al. [5], stability and analyticity bounds in maximum-norm were derived for spatially semidiscrete finite element methods for parabolic equations with natural Neumann boundary conditions. The finite element spaces were given on quasiuniform partitions but were otherwise quite general. In the present paper we consider homogeneous Dirichlet boundary conditions in the specific context of simplicial Lagrange elements on a smooth convex domain; for notational convenience we shall treat only the model heat equation. The presentation will be based on that in [5] but, in our concrete situation, certain technical details will be less cumbersome, which will allow for a shorter but still selfcontained account.

Let Ω be a bounded convex domain in R^N, $N \geq 2$, with a sufficiently smooth boundary and consider the homogeneous heat equation with homogeneous Dirichlet boundary conditions,

$$u_t = \Delta u, \quad \text{in } \Omega \times (0,\infty),$$
$$u = 0, \quad \text{on } \partial\Omega \times (0,\infty), \qquad u(\cdot,0) = v, \quad \text{in } \Omega. \tag{1.1}$$

Let $h > 0$ be a parameter and let $\mathcal{T}_h$ be a partition of a subset of Ω into open disjoint face-to-face N-simplices $\tau = \tau_j^h$ with $max_j diam(\tau_j^h) = h$ such that their union determines a domain $\Omega_h \subset \Omega$ with all its boundary vertices on $\partial\Omega$ and, consequently, $\operatorname{dist}(x, \partial\Omega) \leq Ch^2$ for all $x \in \partial\Omega_h$. Further, let $S_h \subset H_0^1(\Omega)$ be the

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continuous piecewise polynomial functions of total degree $r-1, r \geq 2$, on Ω_h which vanish on the boundary,

$$S_h = \{\chi \in C(\Omega_h);\ \chi|_\tau \in \Pi_{r-1},\ \chi = 0 \quad \text{on } \Omega \setminus \Omega_h\}.$$

We then define the semigroup $E_h(t)$ on S_h as the solution operator of the semi-discrete analogue of (1.1),

$$(1.2) \qquad (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = 0, \quad \forall \chi \in S_h, t > 0, \quad \text{with } u_h(0) = v_h,$$

where $(\cdot,\cdot)$ is the L_2 inner product over Ω. Our object is to show that, under the appropriate assumptions, $E_h(t)$ is an analytic semigroup, uniformly in h. We recall that this is equivalent to a resolvent estimate for the generator of $E_h(t)$, the discrete Laplacian defined in S_h, and that this makes it possible to derive stability estimates for certain fully discrete methods for (1.1), cf. [4] and [7, Chapter 8].

We refer to [5] for a discussion of the earlier literature on this subject and to such applications. We emphasize that, as in [5], the bound in our main result stated below does not contain any factors $\log(1/h)$, which simplifies the application to the analysis of fully discrete schemes, and is essential when Gronwall's lemma is used in some arguments for integro-differential equations, cf., e.g., [1].

We shall make the assumption that the family of partitions is globally quasiuniform so that, with c_0 and C_0 fixed constants, $c_0 h \leq \rho(B_1(\tau)) \leq \rho(B_2(\tau)) \leq C_0 h$ for all simplices $\tau = \tau_j^h$, where $\rho(B_l(\tau)), l = 1, 2$, are the radii of the largest inscribed and smallest circumscribed balls, respectively. Our main result is then the following.

Theorem. *Let λ_1 be the smallest eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions, and let $\lambda_0 < \lambda_1$. Then under the above assumptions on S_h there is a constant C such that*

$$\|E_h(t)\|_{L_\infty} + t\|E_h'(t)\|_{L_\infty} \leq Ce^{-\lambda_0 t}, \quad \textit{for } t \geq 0.$$

2. Preliminaries. We first note that, because of the quasiuniformity of the partitions, we have the well-known inverse property, valid for all piecewise polynomials χ of fixed total degree and all simplices $\tau = \tau_j^h$,

$$(2.1) \qquad \|D^k \chi\|_{p,\tau} \leq Ch^{-(k-l)-N(\frac{1}{q}-\frac{1}{p})}\|D^l \chi\|_{q,\tau},\ 1 \leq p, q \leq \infty,\ k \geq l \geq 0.$$

Here and below we denote $\|D^k v\|_{p,D} = \sum_{|\alpha|=k} \|D^\alpha v\|_{p,D}$, where $\|v\|_{p,D} = \|v\|_{L_p(D)}$ for spatial domains D, with D omitted when $D = \Omega$.

Let I_h denote the standard piecewise linear interpolant defined by $(I_h w)(\xi) = w(\xi)$ for all vertices ξ and functions w which vanish on $\partial\Omega$. Then, as is well known, for each simplex τ,

$$(2.2)\ \|I_h w - w\|_{\infty,\tau} + h\|\nabla(I_h w - w)\|_{\infty,\tau} \leq Ch^s \|D^s w\|_{\infty,\tau},\ w \in C^2(\bar{\tau}),\ s = 1, 2.$$

For $s = 2$ and $N = 2$ or 3 we also have the corresponding property in L_2 based norms, but for $N \geq 4, I_h w$ is not well defined in $H^2(\tau)$; for $s = 1$ the L_2 analogue of (2.2) does not even hold for $N = 2$. In our technical work, however, we shall need an "interpolation" operator which has such approximation properties in L_2, and we therefore define $\tilde{I}_h w$ for $w \in \overset{o}{W}{}_1^1$ as a piecewise linear function in S_h such that, for interior vertices ξ,

$$(\tilde{I}_h w)(\xi) = \frac{1}{meas(B_\rho(\xi))} \int_{B_\rho(\xi)} w\, dx,$$

where $B_\rho(\xi)$ the ball with center ξ and radius $\rho = c_1 h$ small enough that $B_\rho(\xi)$ does not meet $\partial\Omega$. To describe the result, we also introduce balls $B_\rho(\xi)$ for the boundary vertices, and, for any simplex τ, set $N_h(\tau) = conv\ hull\ (\cup_{\xi \in \bar{\tau}} B_\rho(\xi)) \cap \Omega$.

Lemma 2.1. *Under our above assuptions there exists a constant C such that, for all simplices $\tau = \tau_j^h$ and $1 \le p \le \infty$,*

$$\|\tilde{I}_h w - w\|_{p,\tau} \le Ch \|\nabla w\|_{p,N_h(\tau)}, \quad \text{for } w \in \overset{o}{W}{}_p^1(\Omega) \tag{2.3}$$

and

$$\begin{aligned} &\|\tilde{I}_h w - w\|_{p,\tau} + h\|\nabla(\tilde{I}_h w - w)\|_{p,\tau} \\ &\le Ch^2(\|\nabla w\|_{p,N_h(\tau)} + \|D^2 w\|_{p,N_h(\tau)}), \quad \text{for } w \in \overset{o}{W}{}_p^2(\Omega). \end{aligned} \tag{2.4}$$

Proof. Consider first a simplex τ which has no vertex on $\partial\Omega$. Then $\tilde{I}_h$ maps affine functions on $N_h(\tau)$ into their restrictions to τ, and since $N_h(\tau)$ is convex, (2.3) and (2.4) follow from the Bramble-Hilbert lemma and quasiuniformity.

For a simplex τ with one or more vertices on the boundary we shall first consider an auxiliary interpolant $\hat{I}_h$ for functions in $W_1^1(N_h(\tau))$, which do not necessarily vanish on $\partial\Omega$, into the linear functions on τ. For the interior vertices η of τ we set $\hat{I}_h w(\eta) = \tilde{I}_h w(\eta)$. For ξ a boundary vertex, let $F \ni \xi$ be the closure of an $(N-1)-$face of a simplex $\tau \in \mathcal{T}_h$ which has all its vertices on $\partial\Omega$, and define

$$\hat{I}_h w(\xi) = \frac{1}{meas(F)} \int_F \beta_\xi(s) w(s)\, ds,$$

where $\beta_\xi(s)$ is an affine function, which is bounded on F, uniformly in h, and chosen so that $\hat{I}_h \chi(\xi) = \chi(\xi)$ for χ affine (see Scott and Zhang [6]). By the standard Bramble-Hilbert argument it is then clear that (2.3) and (2.4) hold with $\tilde{I}_h$ replaced by $\hat{I}_h$. To prove (2.3) and (2.4) as stated, it now suffices to show that $\hat{I}_h w - \tilde{I}_h w$ may be appropriately bounded. We carry this out for the first term in (2.4); the proof for the gradient term follows by an inverse estimate and the proof of (2.3) is implicitly contained. Since $\tilde{I}_h w(\xi) = 0$ for boundary nodes this amounts to showing that

$$h^{N/p} |\hat{I}_h w(\xi)| \le Ch^2(\|\nabla w\|_{p,N_h(\tau)} + \|D^2 w\|_{p,N_h(\tau)}), \quad \text{for } 1 \le p \le \infty. \tag{2.5}$$

Let $\sigma = (\Omega \setminus \Omega_h) \cap N_h(\tau)$ denote the local skin-layer. We assume that, for the boundary vertices ξ, the radius ρ in $B_\rho(\xi)$ is taken large enough so that $N(\tau)$ contains a cylinder in Ω with base F and height ch. We have, using $w = 0$ on $\partial\Omega$, because the width of σ is $O(h^2)$ in the normal direction,

$$|\hat{I}_h w(\xi)| \le Ch^{-N+1} \int_F |w|\, ds \le Ch^{-N+1} \|\nabla w\|_{1,\sigma} \le Ch^{-N+3} \max_\nu \|\nabla w\|_{L_1(\nu)},$$

where ν are the intersections with σ of the planes parallel with F. For $p = \infty$ this implies (2.5) at once because $meas(\sigma) \le Ch^{N+1}$. For $p < \infty$ we shall use the easy to prove trace inequality

$$\|f\|_{L_1(\nu)} \le C(h^{-1}\|f\|_{1,N_h(\tau)} + \|\nabla f\|_{1,N_h(\tau)}).$$

Applying Hölder's inequality this shows, with $p^{-1} + q^{-1} = 1$,

$$\begin{aligned}\|\nabla w\|_{L_1(\nu)} &\le C(h^{-1}\|\nabla w\|_{1,N_h(\tau)} + \|D^2 w\|_{1,N_h(\tau)}) \\ &\le C(h^{N/q-1}\|\nabla w\|_{p,N_h(\tau)} + h^{N/q}\|D^2 w\|_{p,N_h(\tau)}),\end{aligned}$$

which proves (2.5) (with an extra factor h in the last term). □

It will be convenient to use a "smooth discrete delta-function".

Lemma 2.2. *Let $y \in \bar{\tau}$. Then there exists a function $\tilde{\delta} = \tilde{\delta}_y \in \mathcal{C}_0^\infty(\tau)$ such that*

$$\chi(y) = \int_\tau \chi\, \tilde{\delta}\, dx, \quad \forall \chi \in S_h.$$

Furthermore, there are constants $c > 0$ and C_k independent of y and h such that

$$dist\,(supp\ \tilde{\delta}, \partial\tau) \ge ch \tag{2.6}$$

and

$$\|D^k \tilde{\delta}\|_{\infty,\tau} \le C_k h^{-N-k}, \quad \forall k \ge 0. \tag{2.7}$$

Proof. Let $\hat{\tau}$ be the open unit size reference simplex in R^N and let $\omega \in \mathcal{C}_0^\infty(\hat{\tau})$ be fixed and nonnegative. Set $(\hat{v}, \hat{w})_\omega = \int_{\hat{\tau}} \hat{v}\, \hat{w}\, \omega\, d\hat{x}$ and let $\hat{\varphi}_1, \ldots, \hat{\varphi}_M$ be an orthonormal basis for the finite dimensional space Π_{r-1} under this inner product. Then $\hat{\delta}(\hat{x}) = \sum_{i=1}^M \hat{\varphi}_i(\hat{y})\, \varphi_i(\hat{x})\, \omega(\hat{x})$ has the required properties on $\hat{\tau}$, with $h = 1$. A standard affine mapping argument using quasiuniformity establishes the lemma as stated. □

Another basic ingredient of our proofs is the detailed behavior of the L_2-projection $P_h : L_1(\Omega) \to S_h$, cf., e.g., [8, in particular Lemma 7.2].

Lemma 2.3. *(i) With $\tilde{\delta}_y$ given in Lemma 2.2 we have*

$$(P_h v)(y) = (v, P_h \tilde{\delta}_y) \quad \text{for } y \in \Omega_h.$$

(ii) There exist constants C and $c > 0$ such that

$$|P_h \tilde{\delta}_y(x)| \le Ch^{-N} e^{-c|x-y|/h}, \quad \forall x, y \in \Omega_h.$$

(iii) There exists a constant C such that $\|P_h\|_{L_p} \le C$ for $1 \le p \le \infty$.

We shall also need the following estimates for the Green's function in the continuous problem (1.1), see, e.g., Èĭdel'man and Ivasišen [3].

Lemma 2.4. *The solution of (1.1) may be represented in terms of a Green's function $G(x,y;t), t>0, x,y\in\Omega,$ as*

$$u(x,t) = (E(t)v)(x) = \int_\Omega G(x,y;t)\, v(y)\, dy. \tag{2.8}$$

For any integer l_0, multi-integer l, and $T>0$ there exist constants C and $c>0$ such that

$$|D_t^{l_0} D_x^l G(x,y;t)| \le C(|x-y|+t^{1/2})^{-(N+2l_0+|l|)} e^{-c|x-y|^2/t}, \quad x,y\in\Omega,\ 0\le t\le T.$$

3. Proof of the Theorem. We begin by observing that the semigroup $E_h(t)$ is generated by the discrete Laplacian $\Delta_h : S_h \to S_h$ defined by

$$(\Delta_h\psi,\chi) = -(\nabla\psi,\nabla\chi),\ \forall\psi,\chi\in S_h.$$

To show the desired bound for $E_h(t)=e^{\Delta_h t}$ for t bounded away from zero, we apply the inequality $\|\chi\|_\infty \le C\|\Delta_h^q\chi\|_2,\ q>N/2$ (see [2, Lemma 4.1] and [5, (3.6)]), together with the obvious L_2 operator norm estimate (using $\lambda_{1,h}\ge\lambda_1>\lambda_0$)

$$\|\Delta_h^q E_h(t)\|_{L_2} \le \sup_{\lambda\ge\lambda_1} (\lambda^q e^{-\lambda t}) \le C t^{-q} e^{-\lambda_0 t},$$

and similarly for $E_h'(t)$. We may therefore assume below that $t\le 1$.

We shall first reduce the proof to certain estimates for approximate Green's functions and then prove these. Starting with the stability estimate, let x_0 be any point in Ω_h. This point will be fixed throughout this section and often suppressed in the notation. Let $x_0\in\bar\tau_0=\bar\tau_{j_0}^h$ and let $\tilde\delta=\tilde\delta_{x_0}$ be the regularized delta-function of Lemma 2.2 with $y=x_0$. We define the discrete Green's function $\Gamma_h=\Gamma_{x_0,h}(x,t)\in S_h$ by

$$\Gamma_{h,t}-\Delta_h\Gamma_h=0, \quad \text{for } t>0, \quad \text{with } \Gamma_h(0)=P_h\tilde\delta,$$

and we may then write

$$(E_h(t)v_h)(x_0) = (\Gamma_h(t), v_h).$$

Further, letting $\Gamma=\Gamma_{x_0}(x,t)$ be the solution of the continuous problem

$$\Gamma_t-\Delta\Gamma=0, \quad \text{for } t>0, \quad \text{with } \Gamma(0)=\tilde\delta,$$

we find, with $F=F_{x_0}(x,t)=\Gamma_h-\Gamma$,

$$(E_h(t)v_h)(x_0) = (F(t),v_h)+(\Gamma(t),v_h), \quad \text{for } v_h\in S_h.$$

Here $|(\Gamma(t),v_h)|\le\|v_h\|_\infty\|\Gamma(t)\|_1$ and, since $\Gamma(x,t)=\int_{\tau_0} G(x,y;t)\tilde\delta(y)\,dy$, we have by the Green's function estimates of Lemma 2.4 and by (2.7) that $\|\Gamma(t)\|_1\le C\|\tilde\delta\|_1\le C$. In order to secure the stability estimate, it hence remains to bound

$\|F(t)\|_{1,\Omega_h}$. But $F(t) = P_h\tilde{\delta} - \tilde{\delta} + \int_0^t F_s(s)ds$ and, by the stability of P_h in L_1 (Lemma 2.3), the desired bound would follow from

$$\text{(3.1)} \qquad \|F_t\|_{L_1(Q_h)} \le C, \quad \text{where } Q_h = \Omega_h \times [0,1].$$

Similarly, for the smoothing estimate of the Theorem we write

$$t(E_h'(t)v_h)(x_0) = (tF_t(t), v_h) + (t\Gamma_t(t), v_h),$$

and using again the Green's function estimates, and $tF_t(t) = \int_0^t (sF_s)_s\, ds$, the proof is reduced to showing that (since $(tF_t)_t = F_t + tF_{tt}$ and assuming (3.1)),

$$\text{(3.2)} \qquad \|tF_{tt}\|_{L_1(Q_h)} \le C.$$

We shall now prove (3.1) and (3.2). For brevity we shall suppress h in Q_h, etc. Thus Q will mean $\Omega_h \times [0,1]$ and all spatial domains occurring will be inside Ω_h, unless explicitly stated to the contrary. We shall decompose Q into "parabolic annuli". For this, let $d_j = 2^{-j}$, j integer, and let $Q_j = \{(x,t) \in Q; d_j \le \rho(x,t) \le 2d_j\}$, where $\rho(x,t) = \max(|x - x_0|, t^{1/2})$ denotes the parabolic distance to $(x_0, 0)$, and similarly $\Omega_j = \{x \in \Omega_h; d_j \le |x - x_0| \le 2d_j\}$. Then, with J_0 fixed small enough that $\rho \le 2d_{J_0} = 2^{1-J_0}$ in Q, and any $J_* > J_0$,

$$Q = Q_h = \Big(\bigcup_{j=J_0}^{J_*} Q_j \Big) \cup Q_*, \quad \text{where } Q_* = \{(x,t) \in Q : \rho(x,t) \le d_{J_*}\}.$$

We shall refer to Q_* as the "innermost" set, and ultimately we shall choose $J_* = J_*(h)$ such that $d_{J_*} \le \mu_* h < 2d_{J_*}$ for small h, where μ_* is a sufficiently large number to be fixed later. Note that then $J_* \approx C\log(1/h)$. Constants C will, as usual, change freely but will be independent of μ_* for μ_* large. We shall write $\sum_{*,j}$ when the innermost set is included and $\sum_j$ when it is not.

In this proof, almost all norms occuring will be L_2 based, and we shall write $\|v\|_D$ and $|||v|||_R$ for L_2-norms over space and space-time sets D and R.

Since $meas(Q_j) \le Cd_j^{2+N}$ we have by Cauchy-Schwarz's inequality

(3.3)
$$\|F_t\|_{L_1(Q_h)} \le C\sum_{*,j} d_j^{1+N/2} |||F_t|||_{Q_j} = C(\mu_* h)^{1+N/2} |||F_t|||_{Q_*} + C\sum_j d_j^{1+N/2}\mathcal{F}_j,$$

where we have introduced $\mathcal{F}_j = |||F_t|||_{Q_j} + d_j^{-1}|||\nabla F|||_{Q_j}$, with the latter term added for the purpose of a later kickback argument. To estimate the first term in (3.3) we note that, by extending the integration to the whole domain Q and applying standard energy arguments, we have, using also (2.7),

$$\text{(3.4)} \quad |||F_t|||_{Q_*} \le |||\Gamma_{h,t}|||_Q + |||\Gamma_t|||_Q \le \|\nabla P_h\tilde{\delta}\| + \|\nabla\tilde{\delta}\| \le Ch^{-1}\|\tilde{\delta}\| \le Ch^{-1-N/2},$$

so that $(\mu_* h)^{1+N/2}|||F_t|||_{Q_*} \le C\mu_*^{1+N/2}$.

To bound the $\mathcal{F}_j$ we shall use local energy estimates for functions $e = z_h - z$ satisfying

$$\text{(3.5)} \qquad (e_t, \chi) + (\nabla e, \nabla\chi) = 0, \quad \forall \chi \in S_h,\ t > 0;$$

note that this equation holds for $e = F$ and F_t. We set $Q_j' = Q_{j-1} \cup Q_j \cup Q_{j+1}$ and correspondingly for Ω_j'. The following proposition will be proven in Section 4. Recall that all domains occurring are inside Ω_h in the spatial directions.

Proposition 3.1. *For any $q \geq 0$ there exists C such that the following holds. Let $z_h \in S_h$ and let z be smooth, and assume that $e = z_h - z$ satisfies (3.5). Then, with $\zeta = I_h z - z$,*

(3.6) $$|||e_t|||_{Q_j} + d_j^{-1}|||\nabla e|||_{Q_j} \leq C\big(G_j(e(0)) + H_j(\zeta) + h^q d_j^{-q}|||e_t|||_{Q_j'} + d_j^{-2}|||e|||_{Q_j'}\big),$$

where

$$G_j(v) = \|\nabla v\|_{\Omega_j'} + d_j^{-1}\|v\|_{\Omega_j'},$$
$$H_j(\zeta) = d_j|||\nabla \zeta_t|||_{Q_j'} + |||\zeta_t|||_{Q_j'} + d_j^{-1}|||\nabla\zeta|||_{Q_j'} + d_j^{-2}|||\zeta|||_{Q_j'}.$$

Application of this proposition to $F = \Gamma_h - \Gamma$ shows, with $\gamma_h = I_h\Gamma - \Gamma$,

$$\mathcal{F}_j \leq C(G_j(F(0)) + H_j(\gamma_h) + h^q d_j^{-q}|||F_t|||_{Q_j'} + d_j^{-2}|||F|||_{Q_j'}).$$

We shall show that, with $q = 2 + N/2$,

(3.7) $$G_j(F(0)) + H_j(\gamma_h) + h^q d_j^{-q}|||F_t|||_{Q_j'} \leq Chd_j^{-N/2-2}$$

so that

(3.8) $$\sum_j d_j^{N/2+1}\mathcal{F}_j \leq Ch\sum_j d_j^{-1} + C\sum_j d_j^{N/2-1}|||F|||_{Q_j'}.$$

We shall then apply a duality argument to bound $|||F|||_{Q_j'}$ in such a way that the last term in (3.8) is bounded by $C\mu_*^{N/2} + C\mu_*^{-1}\sum_j d_j^{N/2+1}\mathcal{F}_j$, whence

(3.9) $$\sum_j d_j^{N/2+1}\mathcal{F}_j \leq C(\mu_*^{N/2} + \mu_*^{-1}) + C\mu_*^{-1}\sum_j d_j^{N/2+1}\mathcal{F}_j.$$

For μ_* sufficiently large the latter term may be kicked back so that, with μ_* fixed in this way, (3.3) and (3.4) show (3.1).

For the proof of (3.7) we begin with $G_j(F(0))$. Since $F(0) = P_h\tilde{\delta} - \tilde{\delta} = P_h\tilde{\delta}$ on Ω_j' (if μ_* is large enough), we obtain, using an inverse estimate and $d_j \geq h$ (where, with a slight abuse of notation, we may assume that Ω_j' is a meshdomain), and the decay estimates of Lemma 2.3,

(3.10) $$G_j(P_h\tilde{\delta}) \leq Ch^{-1}\|P_h\tilde{\delta}\|_{\Omega_j'} \leq Ch^{-1-N}d_j^{N/2}e^{-cd_j/h} \leq Chd_j^{-N/2-2}.$$

We next consider the approximation terms $H_j(\gamma_h)$ in (3.7). For the highest order term we obtain by Hölder's inequality, the maximum-norm approximation error estimates (2.2), the Green's function estimates of Lemma 2.4, and (2.7),

(3.11) $$d_j|||\nabla\gamma_{h,t}|||_{Q_j'} \leq Cd_j^{2+N/2}|||\nabla\gamma_{h,t}|||_{\infty,Q_j'} \leq Cd_j^{2+N/2}h|||D^2\Gamma_t|||_{\infty,Q_j''}$$
$$\leq Cd_j^{2+N/2}hd_j^{-N-4}|||\tilde{\delta}|||_1 \leq Chd_j^{-N/2-2}, \quad \text{where } Q_j'' = \cup_{i=-2}^{2}Q_{j+i};$$

and where we also use the notation $|||\cdot|||_{\infty,R} = \|\cdot\|_{L_\infty(R)}$. For the lowest order term we have similarly

(3.12) $$d_j^{-2}|||\gamma_h|||_{Q_j'} \leq Cd_j^{-1+N/2}h^2|||D^2\Gamma|||_{\infty,Q_j''} \leq Ch^2d_j^{-N/2-3} \leq Chd_j^{-N/2-2}.$$

The remaining terms in $H_j(\gamma_h)$ are bounded in a similar fashion.

To complete the proof of (3.7), we use standard energy arguments (cf. (3.4)) to derive, with $q = 2 + N/2$,

(3.13) $$h^q d_j^{-q}|||F_t|||_Q \leq h^q d_j^{-q}(|||\Gamma_{h,t}|||_Q + |||\Gamma_t|||_Q) \leq Chd_j^{-2-N/2}.$$

Having thus shown (3.8) we now estimate $|||F|||_{Q_j'}$ by a duality argument.

Lemma 3.1. *We have*

$$|||F|||_{Q'_j} \le Chd_j^{-N/2} + C\mu_*^{-1}\sum_{*,i} d_i^2 \min(d_{i-j}^{1+N/2},1)\mathcal{F}_i. \tag{3.14}$$

Proof. In this proof we write $[v,w]$ for the inner product in $L_2(Q)$ where $Q = \Omega\times[0,1]$, i.e., the whole space domain is now included. We have

$$|||F|||_{Q'_j} = \sup\{[F,v]; \mathrm{supp}\, v\subset Q'_j,\ |||v|||_{Q'_j}=1\}.$$

For each fixed such v, let w solve the backward (dual) problem

$$-w_t - \Delta w = v, \quad \text{with } w=0 \quad \text{on } \partial\Omega, \quad w(1)=0 \quad \text{in } \Omega.$$

Integrating by parts we then obtain

$$[F,v] = (F(0),w(0)) + [F_t,w] + [\nabla F,\nabla w]. \tag{3.15}$$

Here, for any $\chi\in S_h$ (and μ_* large enough),

$$\begin{aligned}(F(0),&w(0)) = (P_h\tilde\delta - \tilde\delta, w(0)-\chi)\\ &= (P_h\tilde\delta, w(0)-\chi)_{\Omega''_j} + (P_h\tilde\delta-\tilde\delta, w(0)-\chi)_{\Omega_h\setminus\Omega''_j} = J_1+J_2.\end{aligned}$$

Choosing $\chi = \tilde I_h w(0)$ with $\tilde I_h$ as in Lemma 2.1, using $\|P_h\tilde\delta\|_{\Omega''_j} \le Cd_j^{N/2}\|P_h\tilde\delta\|_{\infty,\Omega''_j}$ the exponential decay properties of P_h with $dist(x_0,\Omega''_j)\ge cd_j$, and the standard a priori estimate $\|\nabla w(0)\|\le |||v|||_Q = 1$,

$$|J_1| \le Cd_j^{N/2}h^{-N}e^{-cd_j/h}h\|\nabla w(0)\| \le Cd_j^{N/2}h^{-N+1}e^{-cd_j/h} \le Chd_j^{-N/2}.$$

Since P_h is stable in L_1 and $\|\tilde\delta\|_{1,\tau}\le C$,

$$|J_2| \le C\|\tilde I_h w(0) - w(0)\|_{\infty,\Omega_h\setminus\Omega''_j} \le Ch\|\nabla w(0)\|_{\infty,D_j},$$

where we used (2.3), and where D_j is a set containing $\Omega\setminus\Omega''_j$ but whose distance to Ω'_j is greater than cd_j. By Duhamel's principle we have $w(0) = \int_0^1 E(s)v(s)\,ds$, where $E(s)$ is defined in (2.8), and since $supp\, v(s)\subset\Omega'_j$, we have by Lemma 2.4 that $\|\nabla E(s)v(s)\|_{\infty,D_j} \le Cd_j^{-N-1}\|v(s)\|_{1,\Omega'_j} \le Cd_j^{-N/2-1}\|v(s)\|$. Since $v(s)=0$ for $s\ge(4d_j)^2$, integration and the Cauchy-Schwarz inequality show $\|\nabla w(0)\|_{\infty,D_j}\le Cd_j^{-N/2}|||v|||_Q = Cd_j^{-N/2}$. Thus $|(F(0),w(0))|\le Chd_j^{-N/2}$, which bounds the first term on the right in (3.15) by the first term on the right in (3.14).

We now consider the remaining two terms on the right of (3.15). Setting

$$\mathcal{B} = [\Gamma_t, w]_{(\Omega\setminus\Omega_h)\times[0,1]} + [\nabla\Gamma,\nabla w]_{(\Omega\setminus\Omega_h)\times[0,1]}$$

we have, recalling that $Q_h = (\cup_j Q_j)\cup Q_*$ and using Lemma 2.1,

$$\begin{aligned}[F_t,w]+[\nabla F,\nabla w] &= [F_t, w-\tilde I_h w] + [\nabla F, \nabla(w-\tilde I_h w)]\\ &\le C\sum_{*,i}(|||F_t|||_{Q_i}|||\tilde I_h w - w|||_{Q_i} + |||\nabla F|||_{Q_i}|||\nabla(\tilde I_h w - w)|||_{Q_i}) + \mathcal{B}\\ &\le C\sum_{*,i}(h^2|||F_t|||_{Q_i} + h|||\nabla F|||_{Q_i})|||D^2 w|||_{Q'_i} + \mathcal{B}.\end{aligned}$$

Here, since $h/d_i \le \mu_*^{-1} \le 1$, $h^2 |||F_t|||_{Q_i} + h|||\nabla F|||_{Q_i} \le C\mu_*^{-1} d_i^2 \mathcal{F}_i$. To show $|||D^2 w|||_{Q_i'} \le C \min(d_{i-j}^{1+N/2}, 1)$ we begin with $i \ge j+3$ and we write as above $w(t) = \int_t^1 E(s-t)v(s)\,ds$. From Lemma 2.4 we have in this case $|||D^2 E(s-t)v(s)|||_{\infty,Q_i'} \le Cd_j^{-N-2} d_j^{N/2} \|v(s)\|$ whence $|||D^2 w|||_{Q_i'} \le C d_i^{N/2+1} d_j^{-N/2-1} = C d_{i-j}^{N/2+1}$. For $i < j+3$ we use the standard a priori estimate $|||D^2 w|||_{Q_i'} \le |||D^2 w||| \le C|||v||| = C$.

It remains to estimate $\mathcal{B}$. Since $\Gamma_t = \Delta\Gamma$ we have integrating by parts

$$|\mathcal{B}| = \Big| \int_0^1 \int_{\partial\Omega_h} \frac{\partial\Gamma}{\partial n} w\,ds\,dt \Big| \le \sum_{*,i} \int_{(\partial\Omega_h \times [0,1]) \cap Q_i} \Big|\frac{\partial\Gamma}{\partial n}\Big|\, |w|\,ds\,dt.$$

Here, we may estimate $|\partial\Gamma/\partial n| \le C d_i^{-N-1}$ on $(\partial\Omega_h \times [0,1]) \cap Q_i$; this is also valid when $Q_i = Q_*$ due to (2.6). Integrating $\partial w/\partial n$ in the direction normal to $\partial\Omega$, we also have, using that the width of $\Omega \setminus \Omega_h$ in this direction is $O(h^2)$ and the Cauchy-Schwarz inequality, since $meas(((\Omega\setminus\Omega_h)\times[0,1]) \cap Q_i') \le C h^2 d_i^{N+1}$,

$$\int_{(\partial\Omega_h \times [0,1]) \cap Q_i} |w|\,ds\,dt \le C \int_{((\Omega\setminus\Omega_h)\times[0,1]) \cap Q_i'} |\nabla w|\,dx\,dt \le C h d_i^{N/2+1/2} |||\nabla w|||_{Q_i'}.$$

In the same way as above $|||\nabla w|||_{Q_i'} \le C d_i^{N/2+1} d_j^{-N/2}$ for $i \ge j+3$, and since $w(t) = 0$ for $t > (4d_j)^2$, a standard energy argument shows that $|||\nabla w|||_{Q_i'} \le |||\nabla w||| \le C d_j$, which we use for $i < j+3$. Thus

$$|\mathcal{B}| \le Ch\Big(\sum_{*,i \ge j+3} d_i^{1/2} d_j^{-N/2} + \sum_{i<j+3} d_i^{-N/2-1/2} d_j \Big) \le Chd_j^{-N/2+1/2} \le Chd_j^{-N/2}.$$

This completes the proof of the lemma. □

Continuing the proof of Proposition 3.1, Lemma 3.1 shows for the last term in (3.8)

$$\sum_j d_j^{N/2-1} |||F|||_{Q_j'} \le Ch \sum_j d_j^{-1} + C\mu_*^{-1} \sum_j d_j^{N/2-1} \sum_{*,i} d_i^2 \min(d_{i-j}^{1+N/2}, 1)\mathcal{F}_i.$$

Clearly the first term is bounded by $Chd_{J_*}^{-1} \le \mu_*^{-1}$ and, after changing the order of summation and estimating elementary geometric sums, the second is bounded by

$$C\mu_*^{-1} \sum_{*,i} d_i^2 \mathcal{F}_i \Big(\sum_{j \le i} d_i^{1+N/2} d_j^{-2} + \sum_{j>i} d_j^{N/2-1} \Big) \le C\mu_*^{-1} \sum_{*,i} d_i^{N/2+1} \mathcal{F}_i.$$

Here by (3.4) $|||F_t|||_{Q_*} \le Ch^{-1-N/2}$ and similarly $|||\nabla F|||_{Q_*} \le Ch^{-N/2}$ so that the term corresponding to Q_* is bounded by $C\mu_*^{-1} d_*^{N/2+1} h^{-1-N/2} = C\mu_*^{N/2}$. We have thus estimated the last term in (3.8) as desired so that (3.9) is shown. As indicated above the proof of (3.1) is thus complete.

We now turn to (3.2). Here, since $t \le 4d_j^2$ on Q_j, we have

$$\|tF_{tt}\|_{L_1(Q)} \le C(\mu_* h)^{3+N/2} |||F_{tt}|||_{Q_*} + C \sum_j d_j^{3+N/2} |||F_{tt}|||_{Q_j}. \tag{3.16}$$

Similarly to (3.4) we have

$$|||F_{tt}|||_{Q_*} \le |||\Gamma_{h,tt}|||_Q + |||\Gamma_{tt}|||_Q \le Ch^{-3-N/2}, \tag{3.17}$$

so that the first term on the right of (3.17) is bounded by $C\mu_*^{3+N/2}$. To estimate the second we apply Proposition 3.1 to F_{tt} to obtain

$$|||F_{tt}|||_{Q_j} \le C(G_j(F_t(0)) + H_j(\gamma_{h,t}) + h^q d_j^{-q}|||F_{tt}|||_{Q_j'} + d_j^{-2}|||F_t|||_{Q_j'}).$$

We now show that

$$G_j(F_t(0)) + H_j(\gamma_{h,t}) + h^q d_j^{-q}|||F_{tt}|||_{Q_j'} \le Chd_j^{-N/2-4} \tag{3.18}$$

and hence, with the sum in $\mathcal{F}_j$ bounded from the above,

$$\sum_j d_j^{3+N/2}|||F_{tt}|||_{Q_j} \le Ch\sum_j d_j^{-1} + C\sum_j d_j^{1+N/2}\mathcal{F}_j \le C. \tag{3.19}$$

Together (3.16), (3.17), and (3.19) complete the proof of (3.2).

For (3.18) we have $F_t(0) = \Delta_h P_h\tilde\delta - \Delta\tilde\delta = -\Delta_h P_h\tilde\delta$ on Ω_j' and hence, cf. (3.10),

$$G_j(F_t(0)) \le Ch^{-1}\|\Delta_h P_h\tilde\delta\|_{\Omega_j'} = Ch^{-1}\sup(\Delta_h P_h\tilde\delta, v),$$

where the supremum is taken over v with supp $v \subset \Omega_j'$ and $\|v\|_{\Omega_j'} = 1$. For each such v, we obtain by considering separately the contributions from Ω_j'' and $\Omega_h \setminus \Omega_j''$, by inverse estimates and decay properties and L_1-stability of the L_2-projection, see (2.1) and Lemma 2.3,

$$\begin{aligned}&|(\Delta_h P_h\tilde\delta, v)| = |(\nabla P_h\tilde\delta, \nabla P_h v)|\\ &\le Ch^{-2}\big(\|P_h\tilde\delta\|_{\Omega_j''}\,\|P_h v\| + \|P_h\tilde\delta\|_{1,\Omega_h}\,\|P_h v\|_{\infty,\Omega_h\setminus\Omega_j''}\big) \le Ch^{-2-N}d_j^{N/2}e^{-cd_j/h}.\end{aligned}$$

Thus

$$G_j(F_t(0)) \le Ch^{-3-N}d_j^{N/2}e^{-cd_j/h} \le Chd_j^{-N/2-4}.$$

Finally, the terms $H_j(\gamma_{h,t})$ and $h^q d_j^{-q}|||F_{tt}|||_{Q_j'}$ are bounded as in (3.11),(3.12), and (3.13) with $q = 4+N/2$, in the latter case using (3.17). The proof is thus complete. □

4. Local energy estimates. To show Proposition 3.1 we now prove the following.

Lemma 4.1. *Let $D \subset \Omega_h, I = [t_0, t_1] \subset [0,1]$, and consider the space-time cylinder $Q = D \times I$. Let $D_d = \{x \in \Omega_h; dist(x, D) \le d\}$, $I_{d^2} = [t_0 - d^2, t_1] \cap [0,1]$, and $Q_d = D_d \times I_{d^2}$. Let $e = z_h - z$ with $z_h \in S_h$ satisfy (3.5). Then for any $q \ge 1$ there exist C and $c > 0$ such that, for $d \ge ch$,*

$$\begin{aligned}|||e_t|||_Q + d^{-1}|||\nabla e|||_Q \le\ & C\kappa_d(\|\nabla e(0)\|_{D_d} + d^{-1}\|e(0)\|_{D_d})\\ &+ C(d|||\nabla z_t|||_{Q_d} + |||z_t|||_{Q_d} + d^{-1}|||\nabla z|||_{Q_d} + d^{-2}|||z|||_{Q_d})\\ &+ C(h/d)^q|||e_t|||_{Q_d} + Cd^{-2}|||e|||_{Q_d}.\end{aligned}$$

Here, $\kappa_d = 1$ *if* $t_0 < d^2$, $\kappa_d = 0$ *otherwise.*

Proof of Proposition 3.1. Since the sets Q_j in Proposition 3.1 are built up of the two cylinders $\Omega_j \times [0, d_j^2]$ and $\{x \in \Omega_h; |x - x_0| \le 2d_j\} \times [d_j^2, 4d_j^2]$, (3.6) with $H_j(z)$ instead of $H_j(\zeta)$ follows by application of Lemma 4.1 with $t_0 = 0$ for the first cylinder and $t_0 = d_j^2$ for the second, with $d = d_j/2 = d_{j+1}$ for both. Writing $e = z_h - z = (z_h - I_h z) - (z - I_h z)$ establishes it as stated. □

Proof of Lemma 4.1. For simplicity of presentation we shall first treat the case of piecewise linears, $r = 2$. The case $r \ge 3$ will be considered at the end.

Let $\omega_1(x) \in \mathcal{C}(\bar{\Omega}_h)$ be a nonnegative piecewise linear function on $\mathcal{T}_h$ (not necessarily vanishing on $\partial\Omega_h$) such that $\omega_1 = 1$ on D, $|\nabla\omega_1| \le Cd^{-1}$, and $\omega_1 = 0$ outside D_d; such a functions exists if $d \ge 2h$, say. Let $\omega_2(t) \in \mathcal{C}^1[0, t_1]$ be nonnegative with $\omega_2(t) = 1$ on I, $|\omega_2'| \le Cd^{-2}$, and $\omega_2 = 0$ outside I_{d^2} when $t_0 \ge d^2$, and set $\omega(x,t) = \omega_1(x)\omega_2(t)$. Note that Q_d is obtained by enlarging Q using the parabolic distance.

Consider the identity

$$\tag{4.1} \tfrac{1}{2}\frac{d}{dt}\|\omega^2 e\|^2 + \|\omega^2\nabla e\|^2 = (e_t, \omega^4 z_h) - (\omega^4 e_t, z) + 2(\omega^3\,\omega_t\, e, e)$$

$$+(\nabla e, \nabla(\omega^4 z_h)) - 4(\omega^3\nabla e, \nabla\omega\, z_h) - (\omega^4\nabla e, \nabla z) = \sum_{j=1}^{6} J_j.$$

We have by simple estimates (for brevity we omit subscripts D_d and Q_d)

$$|J_2| + |J_3| \le \tfrac{1}{2}\epsilon^2 d^2\|\omega^4 e_t\|^2 + C_\epsilon d^{-2}\|z\|^2 + Cd^{-2}\|e\|^2,$$
$$|J_5| \le Cd^{-1}\|\omega^2\nabla e\|\,\|z_h\| \le \tfrac{1}{4}\|\omega^2\nabla e\|^2 + Cd^{-2}\|e\|^2 + Cd^{-2}\|z\|^2,$$
$$|J_6| \le \tfrac{1}{4}\|\omega^2\nabla e\|^2 + C\|\nabla z\|^2.$$

Using the error equation (3.5) we obtain

$$J_1 + J_4 = (e_t, \omega^4 z_h - \chi) + (\nabla e, \nabla(\omega^4 z_h - \chi)) = K_1 + K_2.$$

Choosing $\chi = I_h(\omega^4 z_h)$ where I_h is now the standard interpolant, we have the superconvergent type estimates

$$\tag{4.2} \|\omega^4 z_h - \chi\| \le Chd^{-1}\|z_h\|, \quad \|\nabla(\omega^4 z_h - \chi)\| \le C(hd^{-2}\|z_h\| + hd^{-1}\|\nabla z_h\|);$$

In fact, it suffices to show these bounds for each τ, and for the first one we have by the maximum-norm error estimate (2.2), the fact that $D^2 z_h = 0$ on each τ, an inverse property for z_h, and $h/d \le C$,

$$\|\omega^4 z_h - I_h(\omega^4 z_h)\|_\tau \le Ch^{N/2}\|\omega^4 z_h - I_h(\omega^4 z_h)\|_{\infty,\tau} \le Ch^{N/2+2}\|D^2(\omega^4 z_h)\|_{\infty,\tau}$$
$$\le Ch^{N/2}(h^2 d^{-2}\|z_h\|_{\infty,\tau} + h^2 d^{-1}\|\nabla z_h\|_{\infty,\tau})$$
$$\le Ch^{N/2} hd^{-1}\|z_h\|_{\infty,\tau} \le Chd^{-1}\|z_h\|_\tau;$$

the second bound is shown in the same way. We hence have

$$|K_1| \le Chd^{-1}\|e_t\|\,\|z_h\| \le Ch^2\|e_t\|^2 + Cd^{-2}(\|e\|^2 + \|z\|^2),$$
$$|K_2| \le Chd^{-1}\|\nabla e\|(d^{-1}\|z_h\| + \|\nabla z_h\|)$$
$$\le Chd^{-1}\|\nabla e\|^2 + Cd^{-2}(\|e\|^2 + \|z\|^2) + C\|\nabla z\|^2.$$

Entering our bounds for the J_j into (4.1) and kicking back $\|\omega^2\nabla e\|^2$, we obtain

$$\frac{d}{dt}\|\omega^2 e\|^2 + \|\omega^2\nabla e\|^2 \le \epsilon^2 d^2\|\omega^4 e_t\|^2 + C_\epsilon d^{-2}\|z\|^2 + C\|\nabla z\|^2 + Cd^{-2}\|e\|^2 + Ch^2\|e_t\|^2 + Chd^{-1}\|\nabla e\|^2.$$

Hence integrating over I_{d^2}, taking square roots, and multiplying by d^{-1} we obtain

$$(4.3)\quad d^{-1}|||\omega^2\nabla e||| \le d^{-1}\kappa_d\|\omega^2 e(0)\| + \epsilon|||\omega^4 e_t||| + C_\epsilon(d^{-1}|||\nabla z||| + d^{-2}|||z|||) + C(h/d)^{1/2}(|||e_t||| + d^{-1}|||\nabla e|||) + Cd^{-2}|||e|||.$$

This bounds $|||\nabla e|||_Q$ as in the lemma, with $q = 1/2$, except for the additional term $(h/d)^{1/2}d^{-1}|||\nabla e|||$ and the term $|||\omega^4 e_t|||$ which we now turn to. We have

$$(4.4)\quad \|\omega^4 e_t\|^2 + \tfrac{1}{2}\frac{d}{dt}\|\omega^4\nabla e\|^2 = (e_t, \omega^8 z_{h,t}) - (\omega^8 e_t, z_t) + 4(\omega^7\omega_t\nabla e, \nabla e) + (\nabla e, \nabla(\omega^8 z_{h,t})) - (\nabla e, \nabla(\omega^8 z_t)) - 8(\nabla e, \omega^7\nabla\omega\, e_t) = \sum_{j=1}^{6} J_j'.$$

Here, by elementary estimates,

$$|J_2'| + |J_3'| \le \tfrac{1}{8}\|\omega^4 e_t\|^2 + C\|z_t\|^2 + Cd^{-2}\|\omega^2\nabla e\|^2,$$
$$|J_5'| = |8(\nabla e, \omega^7\nabla\omega\, z_t + \omega^8\nabla z_t)| \le Cd^{-1}\|\omega^2\nabla e\|\,\|z_t\| + C\|\omega^2\nabla e\|\,\|\nabla z_t\| \le Cd^{-2}\|\omega^2\nabla e\|^2 + C\|z_t\|^2 + Cd^2\|\nabla z_t\|^2,$$
$$|J_6'| \le Cd^{-1}\|\omega^2\nabla e\|\,\|\omega^4 e_t\| \le \tfrac{1}{8}\|\omega^4 e_t\|^2 + Cd^{-2}\|\omega^2\nabla e\|^2.$$

Using the error equation we have now, with $\chi = I_h(\omega^8 z_{h,t})$,

$$J_1' + J_4' = (e_t, \omega^8 z_{h,t} - \chi) + (\nabla e, \nabla(\omega^8 z_{h,t} - \chi)) = K_1' + K_2'.$$

Using $z_{h,t} = e_t + z_t$ we find, cf. (4.2),

$$|K_1'| \le Chd^{-1}\|e_t\|\,\|z_{h,t}\| \le Chd^{-1}\|e_t\|^2 + C\|z_t\|^2.$$

To study K_2' we have to be a little bit more delicate in the superapproximation argument. We note that by our choice of ω_1 the functions involved are polynomials on each simplex τ so that we may use the inverse inequalities of (2.1). Thus, using (2.2) and $D^\alpha z_{h,t} = 0$ for $|\alpha| = 2$,

$$\|\nabla(\omega^8 z_{h,t} - \chi)\|_\tau \le Ch^{N/2}\|\nabla(\omega^8 z_{h,t} - \chi)\|_{\infty,\tau} \le Ch^{N/2+1}\|D^2(\omega^8 z_{h,t})\|_{\infty,\tau}$$
$$\le Ch^{N/2}(hd^{-2}\|\omega^6 z_{h,t}\|_{\infty,\tau} + Cd^{-1}\|\omega^7 z_{h,t}\|_{\infty,\tau})$$
$$\le Ch^{N/2}d^{-1}\|\omega^6 z_{h,t}\|_{\infty,\tau} \le Cd^{-1}\|\omega^6 z_{h,t}\|_\tau.$$

Since $\nabla e = \nabla z_h - \nabla z$, it follows that

$$|K_2'| \le Cd^{-1}\|\nabla z\|\,\|\omega^6 z_{h,t}\| + Cd^{-1}\sum_\tau \|\nabla z_h\|_\tau\|\omega^6 z_{h,t}\|_\tau = K_{21}' + K_{22}'.$$

Here, writing again $e_t = z_{h,t} - z_t$,

$$|K'_{21}| \le \tfrac{1}{8}\|\omega^4 e_t\|^2 + C\|z_t\|^2 + Cd^{-2}\|\nabla z\|^2.$$

For K'_{22} we note that since ∇z_h is constant on τ,

$$\|\nabla z_h\|_\tau \|\omega^6 z_{h,t}\|_\tau \le Ch^{N/2}\nabla z_h|_\tau \|\omega^2\|_{\infty,\tau}\|\omega^4 z_{h,t}\|_\tau \le C\|\omega^2\nabla z_h\|_\tau\|\omega^4 z_{h,t}\|_\tau,$$

and hence

$$|K'_{22}| \le Cd^{-1}\|\omega^2\nabla z_h\|\|\omega^4 z_{h,t}\| \le \tfrac{1}{8}\|\omega^4 e_t\|^2 + \|z_t\|^2 + Cd^{-2}\|\omega^2\nabla e\|^2 + Cd^{-2}\|\nabla z\|^2.$$

Using our estimates for the J'_j in (4.4), kicking back $\|\omega^4 e_t\|^2$, integrating over I_{d^2}, and taking square roots we arrive at

$$\begin{aligned} |||\omega^4 e_t||| \le \kappa_d\|\omega^4\nabla e(0)\| + Cd^{-1}|||\omega^2\nabla e||| + C(h/d)^{1/2}|||e_t||| \\ + C(d|||\nabla z_t||| + |||z_t||| + d^{-1}|||\nabla z|||). \end{aligned}$$

Adding a sufficiently small multiple of this inequality to (4.3), kicking back $|||\omega^2\nabla e|||$, and then taking ϵ small enough that $|||\omega^4 e_t|||$ may also be kicked back, we obtain

$$\begin{aligned} |||\omega^4 e_t||| + d^{-1}|||\omega^2\nabla e||| &\le C\kappa_d(\|\omega^4\nabla e(0)\| + d^{-1}\|\omega^2 e(0)\|) \\ &\quad + C(d|||\nabla z_t||| + |||z_t||| + d^{-1}|||\nabla z||| + d^{-2}|||z|||) \\ &\quad + C(h/d)^{1/2}(|||e_t||| + d^{-1}|||\nabla e|||) + Cd^{-2}|||e|||, \end{aligned}$$

which shows the lemma with $q = 1/2$, with the additional term $(h/d)^{1/2}d^{-1}|||\nabla e|||$. The case of a general q follows by iteration (and changing d), now with the additional term $(h/d)^q d^{-1}|||\nabla e|||$. For $q \ge 1$ this may be eliminated using the inequality $|||\nabla e||| \le |||\nabla z||| + |||\nabla z_h||| \le |||\nabla z||| + Ch^{-1}(|||e||| + |||z|||)$ and $h \le d$.

In the case of general simplicial Lagrange elements of order $r \ge 3$, we estimate instead the quantity $|||\omega^{2r}e_t||| + d^{-1}|||\omega^r\nabla e|||$. For the standard pointwise interpolant $I_{h,r}$ to the principal lattice we have, corresponding to (2.2),

$$\|I_{h,r}w - w\|_{\infty,\tau} + h\|\nabla(I_{h,r}w - w)\|_{\infty,\tau} \le Ch^r\|D^r w\|_{\infty,\tau},$$

for functions which vanish on $\partial\Omega_h \cap \bar\tau$. As a consequence, (4.2) still holds (with 4 replaced by $2r$). In the argument for K'_2, since $D^\alpha z_{h,t} = 0$ for $|\alpha| = r$, it is not difficult to see that now

$$h^{r-1}\|D^r(\omega^{4r}z_{h,t})\|_{\infty,\tau} \le Cd^{-1}\|\omega^{3r}z_{h,t}\|_{\infty,\tau}.$$

Finally, instead of using that ∇z_h is constant on τ as in the piecewise linear case, we now use that $\|\psi_1\|_{\infty,\tau}\|\psi_2\|_{\infty,\tau} \le C_{q,N}\|\psi_1\psi_2\|_{\infty,\tau}$ for $\psi_1, \psi_2 \in \Pi_q$ to show

$$\|\nabla z_h\|_\tau\|\omega^{3r}z_{h,t}\|_\tau \le Ch^{N/2}\|\nabla z_h\|_{\infty,\tau}\|\omega^r\|_{\infty,\tau}\|\omega^{2r}z_{h,t}\|_\tau \le C\|\omega^r\nabla z_h\|_\tau\|\omega^{2r}z_{h,t}\|_\tau.$$

The proof of Lemma 4.1 is then complete. ☐

References

1. N.Yu. Bakaev, S. Larsson, and V. Thomée, *Backward Euler type methods for parabolic integro-differential equations in Banach space*, to appear in RAIRO M2AN (1998).
2. J.H. Bramble, A.H. Schatz, V. Thomée, and L.B. Wahlbin, *Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations*, SIAM J. Numer. Anal. **14** (1977), 218-241.
3. S.D. Èĭdel'man, and S.D. Ivasišen, *Investigation of the Green's matrix for a homogeneous parabolic boundary value problem*, Trans. Moscow Math. Soc. **23** (1970), 179-242.
4. C. Palencia, *Maximum-norm analysis of completely discrete finite element methods for parabolic problems*, SIAM J. Numer. Anal. **33** (1996), 1654-1668.
5. A.H. Schatz, V. Thomée, and L.B. Wahlbin, *Stability, analyticity, and almost best approximation in maximum-norm for parabolic finite element equations*, Comm. Pure Appl. Math **51** (1998), 1349-1385.
6. L. R. Scott and S. Zhang, *Finite element interpolation of non-smooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), 483-493.
7. V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer Series in Computational Mathematics, Vol. 25, Springer-Verlag, Berlin Heidelberg New York, 1997.
8. L.B. Wahlbin, *Local behavior in finite element methods*, Handbook of Numerical Analysis, Vol II, Finite Element Methods (Part 1), P.G. Ciarlet and J.L. Lions, Eds, Elsevier, 1991, 355-522.

V. Thomée, Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

E-mail address: thomee@math.chalmers.se

L. B. Wahlbin, Department of mathematics, Cornell University, Ithaca N.Y. 14853, USA

E-mail address: wahlbin@math.cornell.edu