Spectral multipliers
for the Ornstein-Uhlenbeck semigroup

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Abstract

The setting of this paper is Euclidean space with the Gaussian measure. We let L be the associated Laplacian, by means of which the Ornstein-Uhlenbeck semigroup is defined. The main result is a multiplier theorem, saying that a function of L which is of Laplace transform type defines an operator of weak type (1,1) for the Gaussian measure. The (distribution) kernel of this operator is determined, in terms of an integral involving the kernel of the Ornstein-Uhlenbeck semigroup. This applies in particular to the imaginary powers of L. It is also verified that the weak type constant of these powers increases exponentially with the absolute value of the exponent.

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1 Introduction

We shall be working in a finite-dimensional Euclidean space $\mathbb{R}^d$, where we shall consider the Gaussian measure $\mathrm{d}\gamma(x) = e^{-|x|^2}\,\mathrm{d}x$ and also Lebesgue measure $\mathrm{d}x$. The operator $L = -\frac{1}{2}\Delta + x\cdot\nabla$, defined on the space of test functions (i.e., the space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$), has a self-adjoint extension to $L^2(\gamma)$, also denoted L. The spectral properties of L are well known: L is positive semi-definite with discrete spectrum $\{0,1,...\}$. For $d=1$, the eigenfunctions are the Hermite polynomials, given by

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2},\ n = 0,1,....$$

The eigenfunctions for arbitrary d are tensor products $H_\alpha = \otimes_{i=1}^d H_{\alpha_i}$, where α is a multiindex, and the corresponding eigenvalue is the length $|\alpha|$. For each nonnegative integer k, we shall denote by P_k the orthogonal projection of $L^2(\gamma)$ onto the subspace generated by the Hermite polynomials of degree k. The operator L is the infinitesimal generator of a "heat" semigroup, the so-called *Hermite* or *Ornstein-Uhlenbeck* semi-group $(e^{-tL})_{t\geq 0}$, defined in the spectral sense as

$$e^{-tL} = \sum_{k=0}^{+\infty} e^{-tk}P_k.$$

In other words, e^{-tL} is the bounded operator on $L^2(\gamma)$ which maps each H_α to $e^{-t|\alpha|}H_\alpha$. It can be shown that the semi-group $(e^{-tL})_{t\geq 0}$ is also given by the Mehler kernel

$$M_t(x,y) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\exp\left(-\frac{|e^{-t}x-y|^2}{1-e^{-2t}}\right),$$

in the sense that $e^{-tL}f(x) = \int M_t(x,y)f(y)\,\mathrm{d}y$ for $f\in L^2(\gamma)$. See [Me, pp. 172-3] and also the survey article [S3].

During the last two decades, a considerable effort has been devoted to the study of several operators naturally associated with the operator L and their boundedness properties between different spaces given by the Gaussian measure γ, mainly the spaces $L^p(\gamma)$ and weak$-L^1(\gamma)$. At this time, a whole area of research, which is sometimes called Gaussian Harmonic Analysis, is already well established. As a significant but incomplete sample of these investigations, we can cite the papers [S2], dealing with the maximal function associated to the semigroup, [F-G-S], [F-S], [G-S-T], [Gz], [My], [Mu2], [Pé], [Pi], [U] and our recent article [GMST], dealing with Riesz transforms of different orders.

In [GMST] we prove the weak type $(1,1)$ boundedness for the Riesz transforms of order 2. Our method consists in decomposing the operator in a "local" part, given by a kernel living close to the diagonal, and the remaining or "global" part. The local part is essentially a Calderón-Zygmund singular integral, and the global part is delicately analysed by estimating its size in different regions and using the "forbidden regions" method coming from [S2].

In the present paper we use the same philosophy to deal with multipliers of Laplace transform type (in the terminology of [St1]) associated to L. Their boundedness in $L^p(\gamma)$, for $1<p<\infty$, is a result of the general theory given in [St1]. The main result of this paper is the weak type $(1,1)$ of these multipliers, with respect to γ. Two main aspects distinguish the present work from [GMST]. First of all, we improve the treatment of the local part by making a smooth truncation and reducing the estimates to the general Calderón-Zygmund theory. Then the global part can be immediately bounded by the maximal Mehler kernel used by P. Sjögren in [S2], so that the delicate estimates of [GMST] to deal with the global part are now simply replaced by an appeal to the bounds in [S2].

We also investigate how to define the multiplier operator in terms of its kernel, as a limit of truncated integrals. In particular, we see under what conditions the multiplier is given by a principal value integral.

Boundedness is also proved for the maximal multiplier operator, via a vector-valued version of the estimates.

Our result applies in particular to the imaginary powers of L. Here the growth of the operator (quasi-)norm for large imaginary powers is of special interest, for reasons which we shall briefly describe.

The assumption that the function m is of Laplace transform type implies that m extends to a holomorphic function on the right half plane $\{z : \Re z > 0\}$, which is bounded on every sector $S_\theta = \{z : |\arg z| < \theta\}$, $0 < \theta < \pi/2$. Since the spectrum of the operator L on $L^1(\gamma)$ is the closed right half plane [D, pp. 115–116], it is natural to impose a holomorphy condition on the multiplier m if we want the operator $m(L)$ to be defined on $L^1(\gamma)$. On the other hand, since the spectrum of L on $L^p(\gamma)$, $1 < p < \infty$, is the set $\mathbb{N}$ of nonnegative integers, it seems too stringent to require holomorphy of the multiplier m to obtain boundedness of $m(L)$ on $L^p(\gamma)$.

In [M], S. Meda gave a sufficient condition for the existence of a nonholomorphic functional calculus for the generator A of a symmetric contraction semigroup on $L^p(M)$, $1 \le p \le \infty$, where M is a σ-finite measure space. He proved that, if the norms of the imaginary powers $A^{i\alpha}$ of the generator, as an operator on L^p, grow polynomially as $|\alpha|$ tends to infinity, then the multiplier operator $m(A)$ is bounded on L^p, provided that the function m satisfies a Hörmander condition of sufficiently high order on $(0, +\infty)$ [M, Theorem 4]. In particular there exist nonholomorphic functions m on $(0, +\infty)$ such that $m(A)$ is bounded on L^p. The polynomial growth condition for the imaginary powers is satisfied for instance by an invariant Laplacian or sublaplacian on groups of polynomial growth.

A standard method to obtain estimates of the norms of $A^{i\alpha}$ on L^p is via complex interpolation between a weak type (1,1) estimate and the L^2 estimate. The L^2 estimate is trivial, since by the spectral theorem the operators $A^{i\alpha}$ are unitary on L^2.

In view of these remarks it is important to obtain sharp estimates of the weak type (1,1) quasinorm of the imaginary powers of the operator L. Since L has a nontrivial kernel, to define imaginary powers we must first restrict L to the orthogonal complement of the kernel. This amounts to considering $L^{i\alpha}\Pi_0$, where $\Pi_0 = I - P_0$. This is a particular case of a multiplier operator $m(L)$ of Laplace transform type, to which we can apply our main result. From its proof, one can see that the weak type $(1,1)$ constant of $L^{i\alpha}\Pi_0$ increases at most exponentially as $|\alpha| \to \infty$. We prove that this estimate cannot be improved to polynomial growth. We also show that for negative powers of L there is not even a weak type $(1,1)$ estimate.

This paper is organized as follows. Section 2 contains the setup, the derivation of the kernel of the multiplier operator outside the diagonal and some estimates for it. In Section 3 the weak type $(1, 1)$ estimate is proved. Section 4 is devoted to the expression of the operator in terms of the kernel. The maximal multiplier operator is estimated in Section 5. Finally, Section 6 contains the discussion of the imaginary and negative powers of L.

2 The multiplier operator and its kernel

For every complex-valued function m defined on the set $\mathbb{N}$ of the nonnegative integers, the multiplier operator

$$m(L) = \sum_{k=0}^{+\infty} m(k) P_k$$

is a densely defined operator on $L^2(\gamma)$ with domain

$$\mathcal{D}_m = \left\{ f \in L^2(\gamma) : \sum_{k=0}^{+\infty} |m(k)|^2 \, ||P_k f||^2 < \infty \right\}.$$

The operator $m(L)$ is bounded on $L^2(\gamma)$ if and only if the function m is bounded and the norm of $m(L)$ qua operator on $L^2(\gamma)$ is the supremum of $|m|$. The projection P_0 onto the kernel of L is a bounded operator on each $L^p(\gamma)$, $1 \leq p < \infty$. By subtracting $m(0)P_0$ if necessary, we can thus assume that $m(0) = 0$. Following [St1], we say that the function m is of Laplace transform type if

$$m(k) = k \int_0^{+\infty} \phi(t) e^{-kt} \, \mathrm{d}t, \quad k > 0,$$

where ϕ is a bounded measurable function on $(0, +\infty)$. Performing the change of variables $e^{-t} = r$, we see that a function m is of Laplace transform type if and only if

$$m(k) = k \int_0^1 \psi(r) r^k \, \mathrm{d}r/r, \quad k > 0, \tag{2.1}$$

where $\psi(r) = \phi(-\log r)$. It follows from the general Littlewood-Paley theory for semigroups that, if m is of Laplace transform type, then $m(L)$ is bounded on all the spaces $L^p(\gamma)$, $1 < p < \infty$, [St1, Chapter IV, §6]. Our aim in this paper is to show that $m(L)$ is also of weak type $(1,1)$ with respect to the Gaussian measure. If m is of Laplace transform type, then $m(L)$ is a continuous operator from the space of test functions to the space of distributions on $\mathbb{R}^d$, and so it has a distributional kernel. In this section, we shall prove that, off the diagonal, this kernel has a density K_ψ with respect to the measure $\mathrm{d}\gamma(x) \otimes \mathrm{d}y$, which satisfies the standard Calderón-Zygmund estimates in a suitable neighbourhood of the diagonal. See Lemma 2.1 and Theorem 2.2 below.

Our starting point is the family of operators r^L, $0 \leq r < 1$, whose integral kernels

$$\mathcal{M}_r(x,y) = \pi^{-d/2} (1 - r^2)^{-d/2} \exp\left(- \frac{|rx - y|^2}{1 - r^2} \right)$$

may be obtained from the Mehler kernel by the change of variables $t = -\log r$. Thus

$$r^L f(x) = \int \mathcal{M}_r(x,y) \, f(y) \, \mathrm{d}y$$

for all test functions f. Since the Mehler kernel satisfies the heat equation $\partial_t M_t = -L_x M_t$, the kernel $\mathcal{M}_r$ satisfies the transformed equation $r\partial_r \mathcal{M}_r = L_x \mathcal{M}_r$. If $\psi \in L^\infty(\mathbb{R}^+)$, define

$$K_\psi(x,y) = \int_0^1 \psi(r)\, \partial_r \mathcal{M}_r(x,y)\, \mathrm{d}r.$$

For $t > 0$ the local region N_t is the neighbourhood of the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$N_t = \left\{ (x,y) : |x-y| < \frac{t}{1+|x|+|y|} \right\}.$$

Lemma 2.1 *If $x \neq y$ the integral defining K_ψ is absolutely convergent. Moreover, for $t > 0$ and each pair of multiindices α and β in $\mathbb{N}^d$ of lengths a and b, respectively, there exists a constant C such that*

$$|\partial_x^\alpha \partial_y^\beta K_\psi(x,y)| \le C \frac{||\psi||_\infty}{|x-y|^{d+a+b}}$$

for all $(x,y) \in N_t$, $x \neq y$.

Proof The formula

$$\partial_x^\alpha \partial_y^\beta \mathcal{M}_r(x,y) = \pi^{-d/2}(1-r^2)^{-(d+a+b)/2}(-r)^a H_{\alpha+\beta}\left(\frac{rx-y}{\sqrt{1-r^2}}\right) \exp\left(\frac{-|rx-y|^2}{1-r^2}\right) \tag{2.2}$$

is a consequence of the definition of the Hermite polynomials. Thus an elementary computation shows that the function $r \mapsto \partial_r \partial_x^\alpha \partial_y^\beta \mathcal{M}_r(x,y)$ is the product of the positive function $\pi^{-d/2}(1-r^2)^{-(d/2)-a-b-2} \exp\left(\frac{-|rx-y|^2}{1-r^2}\right)$ and of a polynomial in r of degree at most $2a+b+3$, whose coefficients depend on x and y. Hence, as a function of r, it changes sign a finite number of times and there exists a constant C such that

$$\int_0^1 |\psi(r)|\, |\partial_r \partial_x^\alpha \partial_y^\beta \mathcal{M}_r(x,y)|\, \mathrm{d}r \le C\, ||\psi||_\infty \max_{0\le r<1} \left|\partial_x^\alpha \partial_y^\beta \mathcal{M}_r(x,y)\right|, \tag{2.3}$$

for all x and y in $\mathbb{R}^d$. By (2.2) we have that

$$\left|\partial_x^\alpha \partial_y^\beta \mathcal{M}_r(x,y)\right| \le C(1-r^2)^{-(d+a+b)/2} \exp\left(-c_0 \frac{|rx-y|^2}{1-r^2}\right) \tag{2.4}$$

for some positive constant c_0. Since, in the local region N_t, one has

$$|rx-y|^2 \ge |x-y|^2 - 2(1-r)|x||x-y| \ge |x-y|^2 - 2(1-r)t,$$

the right-hand side in (2.4) can be estimated by

$$C(t)(1-r^2)^{-(d+a+b)/2} \exp\left(-c_0 \frac{|x-y|^2}{1-r^2}\right) \le C|x-y|^{-(d+a+b)}$$

for all (x,y) in N_t. This proves the lemma.

□

Theorem 2.2 *If the function m is of Laplace transform type, given by formula (2.1), then*

$$m(L) = \int_0^1 \psi(r)\, L\, r^L \,\mathrm{d}r/r, \tag{2.5}$$

where the integral converges in the weak operator topology of $L^2(\gamma)$. Moreover if f is a test function,

$$m(L)f(x) = \int K_\psi(x,y) f(y)\,\mathrm{d}y$$

for all x outside the support of f.

Proof For every pair of functions f, g in $L^2(\gamma)$, we shall denote by $\langle f, g\rangle_\gamma$ their inner product in $L^2(\gamma)$. Thus

$$\begin{aligned} \langle m(L)f, g\rangle_\gamma &= \sum_{k=1}^{\infty} m(k) \langle P_k f, g\rangle_\gamma \\ &= \sum_{k=1}^{\infty} k \int_0^1 \psi(r)\, r^k \,\mathrm{d}r/r\ \langle P_k f, g\rangle_\gamma \end{aligned}$$

Since $\sum_{k=0}^{\infty} |\langle P_k f, g\rangle_\gamma| \leq ||f||\,||g||$, we may interchange the order of summation and integration to get

$$\begin{aligned} \langle m(L)f, g\rangle_\gamma &= \int_0^1 \psi(r) \sum_{k=1}^{\infty} k r^k \langle P_k f, g\rangle_\gamma \,\mathrm{d}r/r \\ &= \int_0^1 \psi(r) \langle L\, r^L f, g\rangle_\gamma \,\mathrm{d}r/r. \end{aligned} \tag{2.6}$$

This proves (2.5). To compute the kernel of the operator $m(L)$, assume that f and g are test functions on $\mathbb{R}^d$. Then

$$\begin{aligned} \langle L\, r^L f, g\rangle_\gamma &= \langle r^L f, Lg\rangle_\gamma \\ &= \iint \mathcal{M}_r(x,y) f(y)\,\mathrm{d}y\ \overline{Lg(x)}\,\mathrm{d}\gamma(x) \\ &= \langle \mathcal{M}_r \,\mathrm{d}\gamma(x) \otimes \mathrm{d}y, L_x(\overline{g} \otimes f)\rangle \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the pairing between distributions and test functions on $\mathbb{R}^d \times \mathbb{R}^d$ and $\mathcal{M}_r \,\mathrm{d}\gamma(x) \otimes \mathrm{d}y$ is the distribution whose density with respect to the measure $\mathrm{d}\gamma(x) \otimes \mathrm{d}y$ is $\mathcal{M}_r$. Since the operator L is symmetric with respect to the Gaussian measure,

$$\begin{aligned} \langle L\, r^L f, g\rangle_\gamma &= \langle (L_x \mathcal{M}_r)\,\mathrm{d}\gamma(x) \otimes \mathrm{d}y, \overline{g} \otimes f\rangle \\ &= \iint r\partial_r \mathcal{M}_r(x,y)\, \overline{g}(x)\, f(y)\,\mathrm{d}y\,\mathrm{d}\gamma(x). \end{aligned}$$

Thus, by (2.6),

$$\langle m(L)f, g\rangle_\gamma = \int_0^1 \psi(r) \iint \partial_r \mathcal{M}_r(x,y)\, \overline{g}(x)\, f(y)\,\mathrm{d}\gamma(x)\,\mathrm{d}y\,\mathrm{d}r. \tag{2.7}$$

If f and g have disjoint supports, the triple integral in (2.7) is absolutely convergent in view of Lemma 2.1. Thus, by Fubini's theorem

$$\langle m(L)f, g\rangle_\gamma = \iint K_\psi(x,y)\, f(y)\, \mathrm{d}y\, \overline{g}(x)\, \mathrm{d}\gamma(x).$$

This proves that K_ψ is the restriction to the complement of the diagonal of the kernel of $m(L)$.

$\square$

3 The weak type estimate

Let μ denote either Lebesgue or Gauss measure on $\mathbb{R}^d$. In this section we shall consider a linear operator T mapping the space of test functions into the space of measurable functions on $\mathbb{R}^d$, satisfying the following assumptions:

(a) T either extends to a bounded operator on $L^q(\mu)$ for some q, $1 < q < \infty$, or is of weak type $(1,1)$ with respect to μ;

(b) there exists a measurable function K, defined in the complement of the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$, such that for every test function f

$$Tf(x) = \int K(x,y)\, f(y)\, \mathrm{d}y,$$

for all x outside the support of f;

(c) the function K satisfies the estimates

$$|K(x,y)| \leq \frac{C}{|x-y|^d}, \qquad |\partial_x K(x,y)| + |\partial_y K(x,y)| \leq \frac{C}{|x-y|^{d+1}},$$

for all (x,y) in the local region N_2, $x \neq y$.

Let φ be a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\varphi(x,y) = 1$ if $(x,y) \in N_1$, $\varphi(x,y) = 0$ if $(x,y) \notin N_2$ and

(3.1) $$|\partial_x \varphi(x,y)| + |\partial_y \varphi(x,y)| \leq C\, |x-y|^{-1} \quad \text{if} \quad x \neq y.$$

We define the global and local parts of the operator T by

$$\begin{aligned} T_{\mathrm{glob}} f(x) &= \int K(x,y)(1-\varphi(x,y))f(y)\, \mathrm{d}y, \\ T_{\mathrm{loc}} f(x) &= Tf(x) - T_{\mathrm{glob}} f(x). \end{aligned}$$

We shall prove that if the operator T satisfies assumptions (a), (b) and (c), then its local part is bounded on $L^p(\gamma)$ and on $L^p(\mathrm{d}x)$, for $1 < p < \infty$. Moreover T_{loc} is of weak type (1,1), both with respect to Lebesgue and Gauss measure (see Theorem 3.7 below). The boundedness of the local part of the multiplier operator $m(L)$ defined in the previous section will follow at once. We shall need the following covering lemma whose proof can be found in [GMST]; item 3. below is modified in a simple way. If B is a ball in $\mathbb{R}^d$, we shall denote by δB the ball that has the same centre as B and radius δ times that of B. We shall denote by $|E|$ the Lebesgue measure of a set E in $\mathbb{R}^d$.

Lemma 3.1 *There exists a collection of balls*

$$B_j = B(x_j, \frac{\kappa}{1+|x_j|}),$$

where $\kappa = 1/20$, such that

1. *the collection $\{B_j : j \in \mathbb{N}\}$ covers $\mathbb{R}^d$;*
2. *the balls $\{\frac{1}{4}B_j : j \in \mathbb{N}\}$ are pairwise disjoint;*
3. *for any $A > 0$, the collection $\{AB_j : j \in \mathbb{N}\}$ has bounded overlap, i.e., $\sup \sum_j \chi_{AB_j} < \infty$;*
4. *$B_j \times 4B_j \subset N_1$ for all $j \in \mathbb{N}$;*
5. *$x \in B_j \Rightarrow B(x, \frac{\kappa}{1+|x|}) \subset 4B_j$.*

Lemma 3.1 will be basic in passing from estimates with respect to Lebesgue measure to estimates with respect to the Gaussian measure and viceversa. Notice that the two measures are equivalent on each ball of the collection $\{4B_j : j \in \mathbb{N}\}$. More precisely, there exist two positive constants c and C such that for all $j \in \mathbb{N}$

$$c\, e^{-|x_j|^2} |E| \leq \gamma(E) \leq C\, e^{-|x_j|^2} |E| \tag{3.2}$$

for each measurable subset E of $4B_j$.

We shall also need the remark that

$$N_{1/7} \subset \bigcup_j (B_j \times 4B_j) \subset N_1. \tag{3.3}$$

This follows from 4. of Lemma 3.1 and the simple observation that, for x in B_j,

$$1 + |x_j| \leq \frac{21}{20}(1 + |x|).$$

Similarly, one verifies that

$$(x, y) \in N_2 \text{ and } x \in B_j \;\Rightarrow\; y \in AB_j \tag{3.4}$$

for some constant $A > 0$.

Lemma 3.2 *Let μ be a nonnegative Borel measure on $\mathbb{R}^d$. Given a sequence of nonnegative measurable functions (f_j), let $f = \sum_j \chi_{B_j} f_j$, where $\{B_j : j \in \mathbb{N}\}$ is the collection of balls in Lemma 3.1. Then*

$$\mu\{x : f(x) > \lambda\} \le \sum_j \mu\{x \in B_j : f_j(x) > \lambda/M\} \tag{3.5}$$

for all $\lambda > 0$, and

$$||f||_q \le M \left(\sum_j \int_{B_j} |f_j|^q \, \mathrm{d}\mu \right)^{1/q}, \tag{3.6}$$

for $1 \le q < \infty$.

Proof Since each point belongs to at most M balls B_j, the level set $\{x : f(x) > \lambda\}$ is contained in $\bigcup_j \{x \in B_j : f_j(x) > \lambda/M\}$. This proves (3.5). To prove (3.6), we simply observe that

$$\int_{\mathbb{R}^d} |f|^q \, \mathrm{d}\mu \le \int_{\mathbb{R}^d} \left(M \max_j |f_j| \chi_{B_j} \right)^q \mathrm{d}\mu \le M^q \int_{\mathbb{R}^d} \sum_j |f_j|^q \chi_{B_j} \, \mathrm{d}\mu.$$

□

Lemma 3.3 *Let μ denote either Lebesgue or Gauss measure on $\mathbb{R}^d$. Let T be a linear operator mapping the space of L^∞ functions with compact support into the space of measurable functions on $\mathbb{R}^d$. Given the covering $\{B_j\}$ in Lemma 3.1, we define the operator*

$$T^1 f(x) = \sum_j \chi_{B_j}(x) T(\chi_{4B_j} f)(x),$$

for measurable and locally bounded functions f. Then
(i) if T is of weak type $(1,1)$ with respect to the measure μ, it follows that T^1 is of weak type $(1,1)$ with respect to Lebesgue and Gauss measure; and
(ii) if T extends to a bounded operator on $L^q(\mu)$ for some $q, 1 < q < \infty$, it follows that T^1 is bounded on $L^q(dx)$ and on $L^q(\gamma)$.

Proof To prove (i), we observe that

$$\mu(\{x \in B_j : |T(\chi_{4B_j} f)(x)| > \lambda/M\}) \le \frac{C}{\lambda} \int_{4B_j} |f| \, \mathrm{d}\mu,$$

uniformly in j. Because of (3.2), this holds for both Lebesgue and Gauss measure. It is now enough to apply (3.5) and use the bounded overlap of the collection $\{4B_j\}$.

The proof of (ii) is analogous.

□

Remark Lemma 3.3 is also true in the vector-valued setting, in the following sense. Given two Banach spaces E and F and a linear operator T bounded from $L^1_E(\mathrm{d}\mu)$ to weak-$L^1_F(\mathrm{d}\mu)$, then defining the operator T^1 as in Lemma 3.3, we have that T^1 is bounded from $L^1_E(\mathrm{d}x)$ into weak-$L^1_F(\mathrm{d}x)$ and from $L^1_E(\mathrm{d}\gamma)$ into weak-$L^1_F(\mathrm{d}\gamma)$. Analogously, if T is bounded from $L^q_E(\mathrm{d}\mu)$ into $L^q_F(\mathrm{d}\mu)$, for some q, $1 < q < \infty$, then T^1 is bounded from $L^q_E(\mathrm{d}x)$ into $L^q_F(\mathrm{d}x)$ and from $L^q_E(\mathrm{d}\gamma)$ into $L^q_F(\mathrm{d}\gamma)$.

Proposition 3.4 *Under the assumptions* (a), (b) *and* (c) *made on* T *at the beginning of this section, the operator* T_{loc} *inherits from* T *either the* L^q- *boundedness or the weak type* $(1,1)$ *as the case might be. Besides, the corresponding boundedness holds for both Lebesgue and Gauss measure.*

Proof Assume that x is in the ball B_j from the covering in Lemma 3.1. Then

$$\begin{aligned} T_{\text{loc}}f(x) &= Tf(x) - T_{\text{glob}}f(x) \\ &= T(f\chi_{4B_j})(x) + T(f(1-\chi_{4B_j}))(x) - \int (1-\varphi)K(x,y)\,f(y)\,\mathrm{d}y \\ &= T(f\chi_{4B_j})(x) + \int (\varphi(x,y) - \chi_{4B_j}(y))K(x,y)\,f(y)\,\mathrm{d}y. \end{aligned}$$

By multiplying by χ_{B_j} and adding over j, we get

$$\begin{aligned} |T_{\text{loc}}f(x)| &\leq \left|\sum_j \chi_{B_j}(x)T(f\chi_{4B_j})(x)\right| + \int \sum_j \chi_{B_j}(x)|\varphi(x,y) - \chi_{4B_j}(y)||K(x,y)|\,|f(y)|\,\mathrm{d}y \\ &= |T^1(f)(x)| + T^2(|f|)(x), \end{aligned}$$

with T^1 as before and T^2 defined here. By assumption (a) on the operator and Lemma 3.3, we know that T^1 is either bounded on L^q for both measures or of weak type $(1,1)$ for both measures.

Let us now consider T^2. This operator is defined by the kernel

$$H(x,y) = \sum_j \chi_{B_j}(x)|\varphi(x,y) - \chi_{4B_j}(y)||K(x,y)|,$$

which is supported in the region $N_2 \setminus N_{\frac{1}{7}}$ because of (3.3). Thus, by assumption (c), $|H(x,y)| \leq C(1+|y|)^d\chi_{N_2}(x,y)$. Let μ be either Lebesgue or Gauss measure. The strong type $(1,1)$ property of T^2 with respect to μ follows by Fubini's theorem and the fact that

$$\mu\left(B(y, \frac{2}{1+|y|})\right) \leq C(d)(1+|y|)^{-d}\frac{\mathrm{d}\mu}{\mathrm{d}y}(y).$$

The boundedness of T^2 on $L^\infty(\mu)$ follows from the fact that $T^2(|f|)(x)$ is dominated by the mean value of $|f|$ on the ball $B\left(x, \frac{2}{1+|x|}\right)$. By interpolation, it is bounded on L^p, $1 \leq p \leq \infty$, for both measures.

□

Let us record as a separate lemma the fact that, for an operator whose kernel lives near the diagonal, boundedness with respect to Lebesgue and Gauss measure are equivalent.

Definition 3.5 *We shall say that an operator* $S : \mathcal{C}_0^\infty \longrightarrow (\mathcal{C}_0^\infty)'$ *is* **local** *if its kernel is supported in* N_2.

Lemma 3.6 *If* S *is a local operator, then strong type* (p,p) *for Lebesgue and Gauss measures are equivalent. The same holds for weak type* (p,p), $1 \leq p \leq \infty$.

Proof From (3.4) we get $S(f) = \sum_j \chi_{B_j} S(\chi_{AB_j} f)$. The conclusion follows by Lemma 3.2 and the bounded overlap property, i.e. item 3. of Lemma 3.1. $\square$

Theorem 3.7 *Under the assumptions* (a), (b) *and* (c) *made on* T *at the beginning of this section, the operator* T_{loc} *is bounded on* $L^p(\gamma)$ *and on* $L^p(\mathrm{d}x)$ *for* $1 < p < \infty$. *Moreover,* T_{loc} *is of weak type* $(1,1)$ *both with respect to Lebesgue and Gauss measure.*

Proof By Proposition 3.4, T_{loc} is bounded in L^q for both measures or of weak type $(1,1)$ for both measures. Next we remark that T_{loc} is a Calderón-Zygmund operator. Indeed let $K_{\mathrm{loc}} = K\varphi$. Then for each test function f on $\mathbb{R}^d$

$$T_{\mathrm{loc}} f(x) = \int K_{\mathrm{loc}}(x,y)\, f(y)\, \mathrm{d}y$$

for all x outside the support of f. Moreover, by assumption (c) and (3.1), the kernel K_{loc} satisfies the estimates

$$|K_{\mathrm{loc}}(x,y)| \leq \frac{C}{|x-y|^d}, \qquad |\partial_x K_{\mathrm{loc}}(x,y)| + |\partial_y K_{\mathrm{loc}}(x,y)| \leq \frac{C}{|x-y|^{d+1}},$$

for all (x,y) in $\mathbb{R}^d$, $x \neq y$.

Notice that in the case when T_{loc} is bounded in L^q for some $q > 1$, the weak type $(1,1)$ follows by standard Calderón-Zygmund theory. Thus, we have the weak type $(1,1)$ property in all cases. From there, the boundedness of T_{loc} on L^p for all p, $1 < p < \infty$ can be deduced via a good-lambda inequality as in [C-F, proof of Theorem III]. Since T_{loc} is a local operator, the conclusion follows from Lemma 3.6.

$\square$

We can now state our main result.

Theorem 3.8 *If the function* m *is of Laplace transform type, then the multiplier operator* $m(L)$ *is of weak type* $(1,1)$ *with respect to the Gaussian measure.*

Proof By Lemma 2.1, the operator $m(L)$ satisfies assumptions (a), (b) and (c) from the beginning of the section with $q = 2$ and $\mu = \gamma$. Therefore, by Theorem 3.7, its local part is of weak type $(1,1)$ with respect to both Lebesgue and Gauss measure. We only have to prove that the global part of the operator is of weak type $(1,1)$ with respect to the Gaussian measure. But, by using (2.3), we have

$$\begin{aligned} |m(L)_{\mathrm{glob}} f(x)| &\leq C\|\psi\|_\infty \int \max_{0 \leq r < 1} \mathcal{M}_r(x,y)(1-\varphi(x,y))|f(y)|\,\mathrm{d}y \\ &\leq C\|\psi\|_\infty \int \max_{0 \leq r < 1} \mathcal{M}_r(x,y)(1-\chi_{N_1}(x,y))|f(y)|\,\mathrm{d}y \end{aligned}$$

and the latter operator is of weak type $(1,1)$ (see [S2]). $\square$

4 Pointwise expression for the operator in terms of its kernel

The operator $m(L)$ is in general not a principal value singular integral. In this section, we investigate how it can be expressed as a limit of the sum of a truncated integral and a variable multiple of the identity.

Theorem 4.1 *Let m be of Laplace transform type. There exists a bounded function α defined on $(0,+\infty)$ such that*

$$m(L)f(x) = \lim_{\varepsilon\to 0+}\left(\alpha(\varepsilon)f(x) + \int_{|x-y|>\varepsilon} K_\psi(x,y)f(y)\,\mathrm{d}y\right)$$

for almost all x, whenever f is a test function. If (ε_j) is a sequence tending to 0, such that $\lim_{j\to\infty}\alpha(\varepsilon_j)=\alpha_0$ exists, then

$$m(L)f(x) = \alpha_0 f(x) + \lim_{j\to\infty}\int_{|x-y|>\varepsilon_j} K_\psi(x,y)f(y)\,\mathrm{d}y.$$

If the function ψ is continuous at 1, then $\lim_{\varepsilon\to 0}\alpha(\varepsilon)=\psi(1)$ and $m(L)$ is the sum of $\psi(1)I$ and a principal value integral.

We begin by proving a technical lemma.

Lemma 4.2 *For any compact subset K of $\mathbb{R}^d$ and any multiindex β of length b, there exists a constant $C=C(K,b)$ such that for x and y in K*

$$\int_0^{1/2} |\partial_y^\beta\left(\mathcal{M}_r(x,y)-\mathcal{M}_0(x,y)\right)|\frac{\mathrm{d}r}{r} \le C \tag{4.1}$$

and

$$\int_{1/2}^{1} |\partial_y^\beta\left(\mathcal{M}_r(x,y)-\mathcal{M}_0(x,y)\right)|\frac{\mathrm{d}r}{r} \le C(1+h_b(|x-y|)) \tag{4.2}$$

where

$$h_b(\rho)=\begin{cases}\rho^{2-d-b} & \text{if } d+b>2;\\ \log(\rho) & \text{if } d+b=2;\\ 1 & \text{if } d+b=1.\end{cases}$$

Proof Since the function $(r,x,y)\mapsto\partial_y^\beta\mathcal{M}_r(x,y)$ is smooth in $[0,1/2]\times\mathbb{R}^d\times\mathbb{R}^d$, there exists a constant $C(K,b)$ such that for $0\le r\le 1/2$ and x, y in K

$$|\partial_y^\beta\left(\mathcal{M}_r(x,y)-\mathcal{M}_0(x,y)\right)| \le C(K,b)\,r.$$

This proves (4.1). Next we observe that by (2.4)

$$\int_{1/2}^{1}|\partial_y^\beta\mathcal{M}_r(x,y)|\frac{\mathrm{d}r}{r}\le C\int_{1/2}^{1}(1-r^2)^{-(d+b)/2}\exp\left(-c_0\frac{|rx-y|^2}{1-r^2}\right)\,\mathrm{d}r$$

for some positive constant c_0. Since $|rx-y|^2 \geq |x-y|^2 - C(K)(1-r)$ for x and y in K, the right-hand side is bounded by

$$C\int_{1/2}^{1}(1-r^2)^{-(d+b)/2}\exp\left(-c_0\frac{|x-y|^2}{1-r^2}\right)\,\mathrm{d}r \leq Ch_b(|x-y|).$$

Moreover, for any x and y in K

$$\int_{1/2}^{1}|\partial_y^\beta \mathcal{M}_0(x,y)|\frac{\mathrm{d}r}{r} \leq C.$$

This proves (4.2). $\square$

Proof of Theorem 4.1 By Theorem 2.2 , if f and g are test functions on $\mathbb{R}^d$,

$$\begin{aligned}
\langle m(L)f,g\rangle_\gamma &= \int_0^1 \psi(r)\langle r^L Lf,g\rangle_\gamma \frac{\mathrm{d}r}{r} \\
&= \int_0^1 \psi(r)\iint \mathcal{M}_r(x,y)Lf(y)\,\mathrm{d}y\;\overline{g}(x)\,\mathrm{d}\gamma(x)\frac{\mathrm{d}r}{r} \\
&= \int_0^1 \psi(r)\iint \left(\mathcal{M}_r(x,y)-\mathcal{M}_0(x,y)\right)Lf(y)\,\mathrm{d}y\;\overline{g}(x)\,\mathrm{d}\gamma(x)\frac{\mathrm{d}r}{r},
\end{aligned}$$

since $\int \mathcal{M}_0(x,y)Lf(y)\,\mathrm{d}y = \langle Lf,1\rangle_\gamma = 0$. By Lemma 4.2, the last triple integral is absolutely convergent. Thus, by Fubini's theorem

$$\langle m(L)f,g\rangle_\gamma = \int \overline{g}(x)\int_0^1 \psi(r)\int \left(\mathcal{M}_r(x,y)-\mathcal{M}_0(x,y)\right)Lf(y)\,\mathrm{d}y\frac{\mathrm{d}r}{r}\,\mathrm{d}\gamma(x).$$

This proves that for a.a. x in $\mathbb{R}^d$

$$m(L)f(x) = \int_0^1 \psi(r)\int_{\mathbb{R}^d}\left(\mathcal{M}_r(x,y)-\mathcal{M}_0(x,y)\right)\,Lf(y)\,\mathrm{d}y\frac{\mathrm{d}r}{r}. \tag{4.3}$$

For x fixed, let $\mathcal{N}_r(y) = \left(\mathcal{M}_r(x,y)-\mathcal{M}_0(x,y)\right)\gamma^{-1}(y)$. Thus

$$m(L)f(x) = \lim_{\varepsilon\to 0}\int_0^1 \psi(r)\int_{|x-y|>\varepsilon}\mathcal{N}_r(y)\,Lf(y)\,\mathrm{d}\gamma(y)\frac{\mathrm{d}r}{r}.$$

From Green's formula for L. one can deduce that

$$\begin{aligned}
\int_{|x-y|>\epsilon}\mathcal{N}_r(y)\,Lf(y)\,\mathrm{d}\gamma(y) &= \int_{|x-y|>\epsilon}L\mathcal{N}_r(y)\,f(y)\,\mathrm{d}\gamma(y) \\
&- \frac{1}{2}\int_{S_\epsilon}\partial_n f(y)\,\mathcal{N}_r(y)\gamma(y)\,\mathrm{d}\sigma(y) + \frac{1}{2}\int_{S_\epsilon}f(y)\,\partial_n\mathcal{N}_r(y)\gamma(y)\,\mathrm{d}\sigma(y) \\
&\qquad = I_1(\varepsilon,r)+I_2(\varepsilon,r)+I_3(\varepsilon,r),
\end{aligned}$$

where S_ϵ is the surface of the ball of radius ϵ centred at x, ∂_n the interior normal derivative and $\mathrm{d}\sigma$ the surface measure on S_ϵ. Since $L\mathcal{N}_r(y) = r\partial_r \mathcal{M}_r(x,y)\gamma^{-1}(y)$, we get

$$\begin{aligned}
\int_0^1 \psi(r)\ I_1(\varepsilon, r)\frac{\mathrm{d}r}{r} &= \int_0^1 \psi(r) \int_{|x-y|>\epsilon} r\partial_r \mathcal{M}_r(x,y)\, f(y)\, \mathrm{d}y \frac{\mathrm{d}r}{r} \\
&= \int_{|x-y|>\epsilon} f(y) \int_0^1 \psi(r)\partial_r \mathcal{M}_r(x,y)\, \mathrm{d}r\, \mathrm{d}y \\
&= \int_{|x-y|>\epsilon} K_\psi(x,y)\, f(y)\, \mathrm{d}y. \qquad (4.4)
\end{aligned}$$

To prove the first part of the theorem, we only need to show that, as ε tends to 0,

$$\int_0^1 \psi(r)\ I_2(\varepsilon, r)\frac{\mathrm{d}r}{r} = O(\rho_d(\varepsilon)) \qquad (4.5)$$

$$\int_0^{1/2} \psi(r)\ I_3(\varepsilon, r)\frac{\mathrm{d}r}{r} = O(\rho_d(\varepsilon)) \qquad (4.6)$$

$$\int_{1/2}^1 \psi(r)\ I_3(\varepsilon, r)\frac{\mathrm{d}r}{r} = \alpha(\varepsilon) f(x) + O(\rho_d(\varepsilon)), \qquad (4.7)$$

where $\rho_d(\varepsilon) = \varepsilon$ if $d \geq 3$, $\rho_2(\varepsilon) = \rho_1(\varepsilon) = \varepsilon(1 + |\log \varepsilon|)$ and α is a bounded function. In this proof, O symbols and constants C may depend on x. Aiming at (4.5), we conclude from Lemma 4.2 that

$$\left| \int_0^1 \psi(r)\ I_2(\varepsilon, r)\frac{\mathrm{d}r}{r} \right| \leq C\|\psi\|_\infty \max |\partial f|(1 + h_0(\varepsilon))\varepsilon^{d-1},$$

which is $O(\rho_d(\varepsilon))$ provided that $d \geq 2$. If $d = 1$, we get

$$\begin{aligned}
(4.8)\quad & \int_0^1 \psi(r)\ I_2(\varepsilon, r)\frac{\mathrm{d}r}{r} \\
= \; & -\frac{1}{2} f'(x-\epsilon)\gamma(x-\epsilon) \int_0^1 \psi(r)\, \mathcal{N}_r(x-\varepsilon)\frac{\mathrm{d}r}{r} + \frac{1}{2} f'(x+\epsilon)\gamma(x+\epsilon) \int_0^1 \psi(r)\, \mathcal{N}_r(x+\varepsilon)\frac{\mathrm{d}r}{r}.
\end{aligned}$$

Arguing as in the proof of Lemma 4.2, we see that for y in a compact set

$$\int_0^1 |\partial_y \mathcal{N}_r(y)| \frac{\mathrm{d}r}{r} \leq C(1 + |\log|x-y||).$$

Thus

$$\begin{aligned}
\left| \int_0^1 \psi(r)\, (\mathcal{N}_r(x-\varepsilon) - \mathcal{N}_r(x+\varepsilon))\, \frac{\mathrm{d}r}{r} \right| &\leq \|\psi\|_\infty \int_{x-\varepsilon}^{x+\varepsilon} \int_0^1 |\partial_y \mathcal{N}_r(y)| \frac{\mathrm{d}r}{r}\, \mathrm{d}y \\
&\leq C\|\psi\|_\infty \int_{-\varepsilon}^{\varepsilon} (1 + |\log|t||)\, \mathrm{d}t \\
&= O(\varepsilon(1 + |\log \varepsilon|)).
\end{aligned}$$

Since the function f and the density γ are smooth, this implies that both sides of (4.8) are $O(\varepsilon(1 + |\log \varepsilon|))$, and (4.5) is proved.

A similar argument proves (4.6). To prove (4.7) we observe that

$$\gamma(y)\partial_n\mathcal{N}_r(y) = \partial_n\mathcal{M}_r(x,y) + 2\langle y,n\rangle\mathcal{M}_r(x,y).$$

Arguing as in the proof of (4.5), one can easily see that the second summand here will give a contribution to the integral in (4.7) which is $O(\rho_d(\varepsilon))$. To analyse the contribution of the first summand, we shall compare it with a convolution kernel . Let

$$\phi(s) = 2\pi^{-d/2}(1-r^2)^{-d/2-1}\langle sx-y,n\rangle \exp\left(-\frac{|sx-y|^2}{1-r^2}\right).$$

Then

$$\begin{aligned}
\partial_n\mathcal{M}_r(x,y) &= \phi(1) - \int_r^1 \phi'(s)\,\mathrm{d}s \\
&= 2\pi^{-d/2}(1-r^2)^{-d/2-1}|x-y| \exp\left(-\frac{|x-y|^2}{1-r^2}\right) - E_r(x,y).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{1/2}^1 \psi(r)\ I_3(\varepsilon,r)\frac{\mathrm{d}r}{r} \\
&= \pi^{-d/2}\varepsilon\int_{1/2}^1 \psi(r)\ (1-r^2)^{-d/2-1}\exp\left(-\frac{\varepsilon^2}{1-r^2}\right)\frac{\mathrm{d}r}{r}\int_{S_\varepsilon} f(y)\,\mathrm{d}\sigma(y) \\
&\quad + \frac{1}{2}\int_{1/2}^1 \psi(r)\int_{S_\varepsilon} E_r(x,y)f(y)\,\mathrm{d}\sigma(y)\frac{\mathrm{d}r}{r} + O(\rho_d(\varepsilon)), \quad \varepsilon\to 0.
\end{aligned} \tag{4.9}$$

Observe that $\int_{S_\varepsilon} f(y)\,\mathrm{d}\sigma(y) = \omega_d\varepsilon^{d-1}\left(f(x)+O(\varepsilon)\right)$, where ω_d is the measure of the unit sphere in $\mathbb{R}^d$. If we perform the change of variables $\varepsilon^2 = (1-r^2)t$, we conclude that the first summand in the right-hand side of (4.9) equals $\alpha(\varepsilon)\left(f(x)+O(\varepsilon)\right)$, where

$$\alpha(\varepsilon) = \frac{1}{2}\pi^{-d/2}\omega_d\int_{4\varepsilon^2/3}^\infty \psi(\sqrt{1-\varepsilon^2/t})(1-\varepsilon^2/t)^{-1}t^{d/2-1}e^{-t}\,\mathrm{d}t. \tag{4.10}$$

To estimate the integral of the error term $E_r(x,y)$ we put $A = \frac{sx-y}{\sqrt{1-r^2}}$ and we observe that

$$\begin{aligned}
|\phi'(s)| &= 2\pi^{-d/2}(1-r^2)^{-d/2-1}|\langle x,n\rangle - 2\langle A,n\rangle\langle A,x\rangle| \exp\left(-|A|^2\right) \\
&\leq C|x|(1-r^2)^{-d/2-1}\exp(-c_0|A|^2),
\end{aligned}$$

where c_0 is some small positive constant. Thus

$$\begin{aligned}
|E_r(x,y)| &\leq C|x|(1-r^2)^{-d/2}\sup_{r\leq s<1}\exp\left(-c_0\frac{|sx-y|^2}{1-r^2}\right) \\
&\leq C|x|e^{2c_0|x|\varepsilon}(1-r^2)^{-d/2}\exp\left(-c_0\frac{\varepsilon^2}{1-r^2}\right),
\end{aligned}$$

because $|sx-y|^2 \geq |x-y|^2 - 2(1-r)|x|\varepsilon$, for $y \in S_\varepsilon$ and $r \leq s < 1$.

Therefore, if $d \geq 2$,

$$\int_{1/2}^{1} \psi(r) \int_{S_\varepsilon} f(y) E_r(x,y) \, \mathrm{d}\sigma(y) \frac{\mathrm{d}r}{r} = O(\rho_d(\varepsilon)).$$

If $d=1$, the integral over S_ε is the sum of the values of the integrand at the points $y = x \pm \varepsilon$. Let $A(t) = \frac{sx-(x+t)}{\sqrt{1-r^2}}$. Then

$$\begin{aligned} & E_r(x, x+\varepsilon) + E_r(x, x-\varepsilon) \\ &= 2\pi^{-1/2}(1-r^2)^{-3/2} x \int_r^1 \left((2A^2(\varepsilon)-1)e^{-A^2(\varepsilon)} - (2A^2(-\varepsilon)-1)e^{-A^2(-\varepsilon)} \right) \mathrm{d}s. \end{aligned}$$

An argument similar to the one used in the estimate of $E_r(x,y)$ shows that

$$|\partial_t \left((2A^2(t)-1)e^{-A^2(t)} \right)| \leq C(x,\varepsilon)(1-r^2)^{-1/2} \exp\left(-c_0 \frac{t^2}{1-r^2} \right),$$

for $|t| \leq \varepsilon$, where the constant $C(x,\varepsilon)$ stays bounded for x and ε bounded. Thus

$$|E_r(x,x+\varepsilon) + E_r(x,x-\varepsilon)| \leq C(x,\varepsilon)(1-r^2)^{-1} \int_0^\varepsilon \exp\left(-c_0 \frac{t^2}{1-r^2} \right) \mathrm{d}t,$$

and

$$\begin{aligned} & \left| \int_{1/2}^1 \psi(r) \left(E_r(x,x+\varepsilon) + E_r(x,x-\varepsilon) \right) \frac{\mathrm{d}r}{r} \right| \\ & \leq C \|\psi\|_\infty \int_{1/2}^1 (1-r^2)^{-1} \int_0^\varepsilon e^{-c_0 \frac{t^2}{1-r^2}} \, \mathrm{d}t \, \frac{\mathrm{d}r}{r}. \end{aligned}$$

The right hand side here can easily be estimated by $C\|\psi\|_\infty \varepsilon(1 + |\log \varepsilon|)$. Since the function f is smooth, this suffices to prove (4.7) also for $d=1$. The first claim of the theorem follows, and the other two claims are easy consequences of the first and of the expression for α in (4.10). $\square$

5 Maximal Operators

Let T be a linear operator satisfying the assumptions (a), (b) and (c) at the beginning of Section 3. Let T^* be the operator defined by

$$T^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\epsilon} K(x,y) f(y) \, \mathrm{d}y \right|.$$

Notice that the function T^*f is measurable since the supremum over all $\varepsilon > 0$ coincides with the supremum over all rational $\varepsilon > 0$, because the operator T satisfies assumption (c). We consider the vector-valued (in fact $l^\infty-$valued) operator S given by

$$Sf(x) = \left\{ \int K(x,y) \chi_{\{|x-y|>\varepsilon\}} f(y) \, \mathrm{d}y \right\}_{\varepsilon \in \mathbb{Q}^+} = \left\{ \int K_\varepsilon(x,y) f(y) \, \mathrm{d}y \right\}_{\varepsilon \in \mathbb{Q}^+}.$$

It is clear that $T^* f(x) = \|Sf(x)\|_{l^\infty}$. In order to prove that T^* maps $L^1(\gamma)$ into weak$-L^1(\gamma)$, it is enough to prove that S maps $L^1(\gamma)$ into weak$-L^1_{l^\infty}(\gamma)$.

To handle S, we define, in analogy with Section 3, the operators S_{glob} and S_{loc}, given by

$$\begin{aligned} S_{\mathrm{glob}} f(x) &= \left\{ \int K_\varepsilon(x,y)(1-\varphi(x,y)) f(y)\,\mathrm{d}y \right\}_\varepsilon \\ S_{\mathrm{loc}} f(x) &= Sf(x) - S_{\mathrm{glob}} f(x). \end{aligned}$$

Theorem 5.1 S_{loc} *maps* $L^1(\gamma)$ *into* weak$-L^1_{l^\infty}(\gamma)$.

Proof Observe that

$$S_{\mathrm{loc}} f(x) = \left\{ \int K(x,y)\varphi(x,y)\chi_{\{|x-y|>\varepsilon\}} f(y)\,\mathrm{d}y \right\}_\varepsilon .$$

In other words, S_{loc} is the $l^\infty-$valued version of the maximal operator associated to the Calderón-Zygmund operator T_{loc}.

It now follows that S_{loc} maps $L^1(\,\mathrm{d}x)$ into weak$-L^1(\,\mathrm{d}x)_{l^\infty}$; see Theorem 5.20(iv) of Chapter II of [GR], whose proof can be found in Section 4 of Chapter V. (This is written only for convolution operators, but it can be extended to our case.) In a more general context, this is also proved in [St2, Corollary 2 of Section 7, Chapter I, page 36].

In order to prove boundedness with respect to the Gaussian measure, we continue as in the proof of Proposition 3.4. In particular we have

$$\begin{aligned} \|S_{\mathrm{loc}} f(x)\|_{l^\infty} &\leq \sum_j \chi_{B_j}(x) \left\| S_{\mathrm{loc}}(f\chi_{4B_j})(x) \right\|_{l^\infty} \\ &+ \int \sum_j \|\{K_\varepsilon(x,y)\}_\varepsilon\|_{l^\infty} \chi_{B_j}(x) |\varphi(x,y) - \chi_{4B_j}(y)|\, |f(y)|\,\mathrm{d}y \\ &= \|S^1_{\mathrm{loc}} f(x)\|_{l^\infty} + S^2(|f|)(x). \end{aligned}$$

Now by the remark after Lemma 3.3, S^1_{loc} maps $L^1(\gamma)$ into weak-$L^1_{l^\infty}(\gamma)$. On the other hand as $\|\{K_\varepsilon(x,y)\}_\varepsilon\|_{l^\infty} = |K(x,y)|$, we have that S^2 coincides with the operator T^2 introduced in the proof of Proposition 3.4. In particular S^2 is bounded from $L^1(\gamma)$ into $L^1(\gamma)$. Hence S_{loc} maps $L^1(\gamma)$ into weak$-L^1_{l^\infty}(\gamma)$. □

We assume that m is of Laplace transform type. Let α be the bounded function of Theorem 4.1. The maximal operator associated with $m(L)$ is defined by

$$m(L)^* f(x) = \sup_{\varepsilon>0} \left| \alpha(\varepsilon) f(x) + \int_{|x-y|>\varepsilon} K_\psi(x,y) f(y)\,\mathrm{d}y \right| .$$

Theorem 5.2 *If the function* m *is of Laplace transform type, then the maximal operator* $m(L)^*$ *associated with the multiplier operator* $m(L)$ *is of weak type* $(1,1)$ *with respect to the Gaussian measure. Moreover, whenever* f *is a function in* $L^1(\gamma)$,

$$m(L)f(x) = \lim_{\varepsilon\to 0+} \left(\alpha(\epsilon) f(x) + \int_{|x-y|>\epsilon} K_\psi(x,y) f(y)\,\mathrm{d}y \right), \qquad \text{a.a. } x.$$

Proof Because of Theorem 4.1 and a standard approximation argument, we only have to prove that $m(L)^*$ is of weak type $(1,1)$ with respect to the Gaussian measure. Since α is a bounded function, it is enough to show that the operator

$$T^*f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K_\psi(x,y) f(y)\,\mathrm{d}y \right|$$

is of weak type $(1,1)$ with respect to γ. But this is equivalent to proving that the operator

$$S(f)(x) = \left\{ \int K_\psi(x,y) \chi_{\{|x-y|>\varepsilon\}} f(y)\,\mathrm{d}y \right\}_\varepsilon$$

maps $L^1(\gamma)$ into weak-$L^1_{l^\infty}(\gamma)$. As the kernel of $m(L)$ satisfies assumptions (a), (b) and (c), we can apply Theorem 5.1 to conclude that S_{loc} maps $L^1(\gamma)$ into weak-$L^1_{l^\infty}(\gamma)$. On the other hand, it is clear from (2.3) that

$$\begin{aligned} \|S_{\mathrm{glob}} f(x)\|_{\ell^\infty} &\le \int \left\| K_\psi(x,y)\chi_{\{|x-y|>\varepsilon\}}(1-\varphi(x,y)) \right\|_{\ell^\infty} |f(y)|\,\mathrm{d}y \\ &\le \int |K_\psi(x,y)|(1-\varphi(x,y))|f(y)|\,\mathrm{d}y \\ &\le C\|\psi\|_\infty \int \max_{0\le r<1} \mathcal{M}_r(x,y)(1-\chi_{N_1}(x,y))|f(y)|\,\mathrm{d}y. \end{aligned}$$

The latter operator is of weak type $(1,1)$ (see [S2]). $\square$

6 Negative and imaginary powers of L

Writing as in the introduction $\Pi_0 = I - P_0$, we shall prove negative results for the imaginary powers $L^{i\alpha}\Pi_0$, $\alpha \in \mathbb{R}$, and the negative powers $L^{-b}\Pi_0$, $b > 0$.

If in Theorem 2.2 we let $\psi(r) = \Gamma(1-i\alpha)^{-1}(-\log r)^{-i\alpha}$ for some $\alpha \in \mathbb{R}\setminus\{0\}$ and interpret $m(0)$ as 0, the resulting operator $m(L)$ will be $L^{i\alpha}\Pi_0$. It is thus in particular of weak type $(1,1)$. Considering the proof of Theorem 3.8, one can see that the constant involved increases at most exponentially as $|\alpha| \to \infty$. This cannot be improved to polynomial growth, as our next result shows.

Theorem 6.1 *For some $c > 0$, the weak type (1, 1) quasinorm of $L^{i\alpha}\Pi_0$ is bounded from below by $e^{c|\alpha|}$, when $\alpha \in \mathbb{R}$ with $|\alpha|$ large.*

Proof We denote by $c > 0$ and $C < \infty$ various constants depending only on the dimension. In order to be able to integrate by parts in the expression for the kernel, we consider the operator $L^{i\alpha}\Pi_0 + P_0$, whose kernel is

$$K_{-i\alpha}(x,y) = \frac{1}{\Gamma(1-i\alpha)} \int_0^1 \left(\log\frac{1}{r}\right)^{-i\alpha} \partial_r \left(\mathcal{M}_r(x,y) - (1-r)\mathcal{M}_0(x,y)\right)\,\mathrm{d}r.$$

After an integration by parts, we get

$$K_{-i\alpha}(x,y)=\frac{\pi^{-d/2}}{\Gamma(-i\alpha)}\int_0^1\left(\log\frac{1}{r}\right)^{-i\alpha-1}\left((1-r^2)^{-\frac{d}{2}}e^{-\frac{|rx-y|^2}{1-r^2}}-(1-r)e^{-|y|^2}\right)\frac{\mathrm{d}r}{r}.$$

We apply this operator to L^1 functions approximating a point mass at a point y with $\eta=|y|$ large. Since the above kernel is to be integrated against $L^1(\gamma)$ functions and Lebesgue measure, we need only integrate it against the measure $f=e^{\eta^2}\delta_y$ and verify that the weak$-L^1(\gamma)$ quasinorm of the resulting function $(L^{i\alpha}\Pi_0+P_0)f(x)$ has exponential growth as $|\alpha|\to\infty$.

Consider the value $(L^{i\alpha}\Pi_0+P_0)f(x)=K_{-i\alpha}(x,y)e^{\eta^2}$ at points $x=\xi y/\eta+v$ with $\xi\in\mathbb{R}$ and $v\perp y$, where $\eta/3<\xi<2\eta/3$ and $|v|<1$. As in [GMST], one has

$$(L^{i\alpha}\Pi_0+P_0)f(x)=\frac{\pi^{-d/2}}{\Gamma(-i\alpha)}\int_0^1\left(\log\frac{1}{r}\right)^{-i\alpha-1}\left(e^{\xi^2}(1-r^2)^{-\frac{d}{2}}e^{-\frac{|\xi-r\eta|^2}{1-r^2}-\frac{r^2|v|^2}{1-r^2}}-(1-r)\right)\frac{\mathrm{d}r}{r}.$$

We shall split the integral here into several parts. The main part will be

$$I_1=\frac{\pi^{-d/2}}{\Gamma(-i\alpha)}e^{\xi^2}\int_{|\xi-r\eta|<A}\left(\log\frac{1}{r}\right)^{-i\alpha-1}(1-r^2)^{-\frac{d}{2}}e^{-\frac{|\xi-r\eta|^2}{1-r^2}-\frac{r^2|v|^2}{1-r^2}}\frac{\mathrm{d}r}{r},$$

where $A=A(d)$ is a large constant to be determined. In I_1 the variable of integration r runs over an interval of length $2A/\eta$, and so $\arg(\log 1/r)^{-i\alpha}=-\alpha\log\log 1/r$ varies by at most $C|\alpha|A/\eta<<1$ if $|\alpha|/\eta$ is small enough. By the change of variables $s=\xi-r\eta$, we then easily get

$$|I_1|>c\frac{e^{\xi^2}}{|\Gamma(-i\alpha)|}\frac{1}{\eta}\int_{|s|<A}e^{-cs^2}\,\mathrm{d}s\;\geq\;c_0\frac{e^{\xi^2}}{|\Gamma(-i\alpha)|}\frac{1}{\eta},$$

for some constant c_0 depending only on d. Let

$$I_2=\frac{\pi^{-d/2}}{\Gamma(-i\alpha)}e^{\xi^2}\int_{|\xi-r\eta|>A,\,\frac{1}{6}<r<\frac{5}{6}}\left(\log\frac{1}{r}\right)^{-i\alpha-1}(1-r^2)^{-\frac{d}{2}}e^{-\frac{|\xi-r\eta|^2}{1-r^2}-\frac{r^2|v|^2}{1-r^2}}\frac{\mathrm{d}r}{r}.$$

Changing variables as above, we obtain

$$|I_2|\leq C\frac{e^{\xi^2}}{|\Gamma(-i\alpha)|}\frac{1}{\eta}\int_{|s|>A}e^{-cs^2}\,\mathrm{d}s.$$

With A large enough, we thus have

$$|I_2|<|I_1|/6.$$

We further set

$$I_3=\frac{\pi^{-d/2}}{\Gamma(-i\alpha)}e^{\xi^2}\int_{\frac{5}{6}}^1\left(\log\frac{1}{r}\right)^{-i\alpha-1}(1-r^2)^{-\frac{d}{2}}e^{-\frac{|\xi-r\eta|^2}{1-r^2}-\frac{r^2|v|^2}{1-r^2}}\frac{\mathrm{d}r}{r}.$$

For these r, one has $|\xi - r\eta| > \eta/6$, and so

$$|I_3| \le C\frac{e^{\xi^2}}{|\Gamma(-i\alpha)|}\int_{\frac{5}{6}}^{1}(1-r)^{-1-d/2}e^{-\frac{c\eta^2}{1-r}}\,\mathrm{d}r \le C\frac{e^{\xi^2}}{|\Gamma(-i\alpha)|}\int_{\frac{5}{6}}^{1}e^{-c\eta^2}\,\mathrm{d}r \le |I_1|/6$$

for large η. The next part of $(L^{i\alpha}\Pi_0 + P_0)f(x)$ we consider is

$$I_4 = \frac{\pi^{-d/2}}{\Gamma(-i\alpha)}e^{\xi^2}\int_{1/\eta^2}^{\frac{1}{6}}\left(\log\frac{1}{r}\right)^{-i\alpha-1}(1-r^2)^{-\frac{d}{2}}e^{-\frac{|\xi-r\eta|^2}{1-r^2}-\frac{r^2|v|^2}{1-r^2}}\frac{\mathrm{d}r}{r}.$$

Here again $|\xi - r\eta| > \eta/6$, and for large η

$$|I_4| \le C\frac{e^{\xi^2}}{|\Gamma(-i\alpha)|}\int_{1/\eta^2}^{\frac{1}{6}}e^{-c\eta^2}\frac{\mathrm{d}r}{r} \le |I_1|/6.$$

Defining

$$I_5 = \frac{\pi^{-d/2}}{\Gamma(-i\alpha)}\int_{1/\eta^2}^{1}\left(\log\frac{1}{r}\right)^{-i\alpha-1}(1-r)\frac{\mathrm{d}r}{r},$$

we see that

$$|I_5| \le C\frac{\log\log\eta}{|\Gamma(-i\alpha)|} \le |I_1|/6$$

when η is large, since $\xi > \eta/3$. What remains to be estimated is

$$I_6 = \frac{\pi^{-d/2}}{\Gamma(-i\alpha)}\int_0^{1/\eta^2}\left(\log\frac{1}{r}\right)^{-i\alpha-1}\left((1-r^2)^{-\frac{d}{2}}e^{\frac{-r^2\xi^2-r^2\eta^2-r^2|v|^2+2r\xi\eta}{1-r^2}} - (1-r)\right)\frac{\mathrm{d}r}{r}.$$

In the numerator of the exponent of e here, the term $2r\xi\eta$ will dominate for large η, and so this exponent is positive and bounded by $Cr\eta^2$. Therefore

$$\begin{aligned}
|I_6| &\le \frac{C}{|\Gamma(-i\alpha)|}\int_0^{1/\eta^2}\left((1+Cr^2)e^{Cr\eta^2} - 1 + r\right)\frac{\mathrm{d}r}{r} \\
&\le \frac{C}{|\Gamma(-i\alpha)|}\int_0^{1/\eta^2} r\eta^2\frac{\mathrm{d}r}{r} \le \frac{C}{|\Gamma(-i\alpha)|} \le |I_1|/6.
\end{aligned}$$

Summing up, we conclude that

$$|(L^{i\alpha}\Pi_0 + P_0)f(x)| > \frac{I_1}{6} \ge c\frac{e^{\xi^2}}{|\Gamma(-i\alpha)|}\frac{1}{\eta} \ge c\frac{e^{(\eta/3)^2}}{|\Gamma(-i\alpha)|}\frac{1}{\eta}$$

in the set $\{x : \eta/3 < \xi < 2\eta/3, \quad |v| < 1\}$. The γ measure of this set is at least $c\eta^{-1}e^{-(\eta/3)^2}$. Thus we get for the weak$-L^1$ quasinorm

$$\| (L^{i\alpha}\Pi_0 + P_0)f \|_{L^{1,\infty}(\gamma)} \ge \frac{c}{|\Gamma(-i\alpha)|}\frac{1}{\eta^2}.$$

In the above argument, we must have η large and $|\alpha|/\eta$ smaller than some constant depending only on d. This allows us to have $\eta^{-2} \ge c|\alpha|^{-2}$, so that

$$\| (L^{i\alpha}\Pi_0 + P_0)f \|_{L^{1,\infty}(\gamma)} \ge \frac{c}{|\Gamma(-i\alpha)|}\frac{1}{\alpha^2} > e^{c|\alpha|}$$

for some c and large $|\alpha|$. The result follows, since P_0 is of strong type (1,1). $\square$

Proposition 6.2 *For any $b > 0$, the operator $L^{-b}\Pi_0$ is not of weak type $(1,1)$ with respect to γ.*

Proof Because of Theorem 2.2, the operator $L(L+\varepsilon I)^{-b-1}$, $\varepsilon > 0$, has kernel

$$\frac{1}{\Gamma(b+1)} \int_0^1 r^\varepsilon \left(\log \frac{1}{r}\right)^b \partial_r (\mathcal{M}_r(x,y) - \mathcal{M}_0(x,y)) \, \mathrm{d}r.$$

Integrating by parts and letting $\varepsilon \to 0$, we see that the kernel of $L^{-b}\Pi_0$ is

$$K_b(x,y) = \frac{\pi^{-d/2}}{\Gamma(b)} \int_0^1 \left(\log \frac{1}{r}\right)^{b-1} \left((1-r^2)^{-\frac{d}{2}} e^{-\frac{|rx-y|^2}{1-r^2}} - e^{-|y|^2}\right) \frac{\mathrm{d}r}{r}.$$

We remark that this expression can also be obtained from Lemma 2.2 of [GMST].

Choosing y with $\eta = |y|$ large as in the preceding proof, we see that it is enough to verify that the weak$-L^1(\gamma)$ quasinorm of $K_b(.,y)e^{\eta^2}$ tends to ∞ as $\eta \to \infty$. Notice that

$$\pi^{d/2}\Gamma(b)K_b(x,y)e^{\eta^2} = \int_0^1 \left(\log \frac{1}{r}\right)^{b-1} \left((1-r^2)^{-\frac{d}{2}} e^{\frac{-r^2|x|^2 - r^2\eta^2 + 2rx\cdot y}{1-r^2}} - 1\right) \frac{\mathrm{d}r}{r}. \tag{6.1}$$

We shall consider x with $|x| < 1$ and $x \cdot y < 0$. If $1/\eta < r < \varepsilon$ for some $\varepsilon = \varepsilon(d) > 0$, we have

$$(1-r^2)^{-\frac{d}{2}} e^{\frac{-r^2|x|^2 - r^2\eta^2 + 2rx\cdot y}{1-r^2}} \le e^{Cr^2} e^{-1} < 1/2.$$

Thus

$$\int_{1/\eta}^{\varepsilon} \left(\log \frac{1}{r}\right)^{b-1} \left((1-r^2)^{-\frac{d}{2}} e^{\frac{-r^2|x|^2 - r^2\eta^2 + 2rx\cdot y}{1-r^2}} - 1\right) \frac{\mathrm{d}r}{r}$$
$$\le -\frac{1}{2} \int_{1/\eta}^{\varepsilon} \left(\log \frac{1}{r}\right)^{b-1} \frac{\mathrm{d}r}{r} = -\frac{1}{2b} \left((\log \eta)^b - (\log 1/\varepsilon)^b\right) \le -\frac{1}{4b} (\log \eta)^b$$

for large η. Further,

$$\left| \int_0^{1/\eta} \left(\log \frac{1}{r}\right)^{b-1} \left((1-r^2)^{-\frac{d}{2}} e^{\frac{-r^2|x|^2 - r^2\eta^2 + 2rx\cdot y}{1-r^2}} - 1\right) \frac{\mathrm{d}r}{r} \right|$$
$$\le C \int_0^{1/\eta} \left(\log \frac{1}{r}\right)^{b-1} (r^2 + \eta^2 r^2 + \eta r) \frac{\mathrm{d}r}{r} \le C (\log \eta)^{b-1}.$$

The remaining part of the integral in (6.1) is for large η bounded by

$$C \int_\varepsilon^1 (1-r)^{b-1} \left((1-r^2)^{-\frac{d}{2}} e^{-\frac{1}{1-r^2}} + 1\right) \frac{\mathrm{d}r}{r} \le C.$$

Altogether, this means that $|K_b(x,y)e^{\eta^2}| \ge c(\log \eta)^b$ in the set $\{|x| < 1,\ x \cdot y < 0\}$, for large η and some $c = c(b,d)$. This implies that the weak$-L^1(\gamma)$ quasinorm of $L^{-b}\Pi_0(e^{\eta^2}\delta_y)$ is large, and the result follows. $\square$

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