EXTREMES AND UPCROSSING INTENSITIES FOR P-DIFFERENTIABLE STATIONARY PROCESSES

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Given a stationary differentiable in probability process $\{\xi(t)\}_{t\in\mathbb{R}}$ we express the asymptotic behaviour of the tail $\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}$ for large u through a certain functional of the conditional law $(\xi'(1)|\xi(1)>u)$. Under technical conditions this functional becomes the upcrossing intensity $\mu(u)$ of the level u by $\xi(t)$. However, by not making explicit use of $\mu(u)$ we avoid the often hard-to-verify technical conditions required in the calculus of crossings and to relate upcrossings to extremes.

We provide a useful criterion for verifying a standard condition of tightness-type used in the literature on extremes. This criterion is of independent value.

Although we do not use crossings theory, our approach has some impact on this subject. Thus we complement existing results due to e.g., Leadbetter (1966) and Marcus (1977) by providing a new and useful set of technical conditions which ensure the validity of Rice's formula $\mu(u)=\int_0^\infty z\, f_{\xi(1),\xi'(1)}(u,z)\;dz$.

As examples of application we study extremes of $\mathbb{R}^n$-valued Gaussian processes with strongly dependent component processes, and of totally skewed moving averages of α-stable motions. Further we prove Belayev's multi-dimensional version of Rice's formula for outcrossings through smooth surfaces of $\mathbb{R}^n$-valued α-stable processes.

Introduction. Given a differentiable stationary stochastic process $\{\xi(t)\}_{t\in\mathbb{R}}$ there is, under technical conditions, a direct relation between the asymptotic behaviour of $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$ for large u and that of the expected number of upcrossings $\mu(u)$ of the level u by $\{\xi(t)\}_{t\in[0,1]}$ [e.g., Leadbetter and Rootzén (1982), Leadbetter et al. (1983, Chapter 13), Albin (1990, Theorem 7), and Albin (1992)]. Under additional technical conditions one further has the explicit expression

$$\text{(0.1)}\qquad \text{Rice's formula:}\qquad \mu(u)=\int_0^\infty z\, f_{\xi(1),\xi'(1)}(u,z)\;dz$$

[e.g., Leadbetter (1966), Marcus (1977), and Leadbetter et al. (1983, Chapter 7)].

However, although very reasonable, the above mentioned two sets of "technical conditions" are quite forbidding, and have only been verified for Gaussian processes and processes closely related to them. Hence, although conceptually satisfying, the upcrossing-approach to extremes have neither led far outside 'Gaussian territory',

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nor generated many results on extremes that cannot be easier proved by other methods. See e.g., Adler and Samorodnitsky (1997) and Albin (1992) for outlines of the technical problems associated with the formula (0.1), and with relating the asymptotic behaviour of $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$ to that of $\mu(u)$, respectively.

In Section 3 we show that, under certain conditions on the functions q and w,

(0.2)
$$\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\sim\mathbf{P}\{\xi(1)>u\}+\frac{1}{x}\mathbf{P}\bigg\{\frac{q(u)\xi'(1)}{w(u)}\geq\frac{1}{x}\frac{\xi(1)-u}{w(u)}\,\bigg|\,\xi(1)>u\bigg\}\frac{\mathbf{P}\{\xi(1)>u\}}{q(u)}$$

as $u\uparrow\hat{u}\equiv\sup\big\{x\in\mathbb{R}:\mathbf{P}\{\xi(1)>x\}>0\big\}$ and $x\downarrow 0$ (in that order). Note that, assuming absolute-continuity, the right-hand side of (0.2) is equal to

$$\mathbf{P}\{\xi(1)>u\}+\int_0^\infty\bigg[\int_0^z f_{\xi(1),\xi'(1)}(u+xqy,z)\,dy\bigg]dz. \tag{0.3}$$

If we send $x\downarrow 0$ before $u\uparrow\hat{u}$, (0.1) shows that, under continuity assumptions, (0.3) behaves like $\mathbf{P}\{\xi(1)>u\}+\mu(u)$. However, technical conditions that justify this change of order between limits, and thus the link between extremes and upcrossings, can seldom be verified outside Gaussian contexts. In fact, since $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}\not\sim\mathbf{P}\{\xi(1)>u\}+\mu(u)$ for processes that cluster [e.g., Albin (1992, Section 2)], the "technical conditions" are not only technical! So while (0.2) is a reasonably general result, the link to $\mu(u)$ can only be proved in more special cases.

Non-upcrossing based approaches to local extremes rely on the conditions

$\boxed{1}$ weak convergence (as $u\uparrow\hat{u}$) of the finite dimensional distributions of the process $\big\{\big(w(u)^{-1}[\xi(1-q(u)t)-u]\;\big|\;\xi(1)>u\big)\big\}_{t\in\mathbb{R}}$;

$\boxed{2}$ excursions above high levels do not last too long (local independence); and

$\boxed{3}$ a certain type of tightness for the convergence in $\boxed{1}$.

It is rare that $\big(w^{-1}[\xi(1-qt)-u]\;\big|\;\xi(1)>u\big)$ can be handled sharp enough to verify $\boxed{1}$. Somewhat less seldom one can carry out the estimates needed to prove

Assumption 1. *There exist random variables* η' *and* η'' *with*

$$\eta''\leq 0\ \text{ a.s.}\quad\text{and}\quad\limsup_{u\uparrow\hat{u}}\mathbf{E}\bigg\{\bigg|\frac{q(u)^2\,\eta''}{w(u)}\bigg|^{\varrho}\,\bigg|\,\xi(1)>u\bigg\}<\infty\quad\text{for some }\varrho>1, \tag{0.4}$$

and such that

$$\lim_{u\uparrow\hat{u}}\mathbf{P}\bigg\{\bigg|\frac{\xi(1+q(u)t)-u}{w(u)\,t}-\frac{\xi(1)-u}{w(u)\,t}-\frac{q(u)t\eta'}{w(u)\,t}-\frac{q(u)^2t^2\eta''}{2\,w(u)\,t}\bigg|>\lambda\,\bigg|\,\xi(1)>u\bigg\}=0 \tag{0.5}$$

for $\lambda>0$ *and* $t\neq 0$.

Further there exists a random variable ζ' *such that, for each choice of* $\lambda>0$,

$$\text{(0.6)}\qquad \mathbf{P}\left\{\left|\frac{\xi(1+q(u)t)-u}{w(u)\,t}-\frac{\xi(1)-u}{w(u)\,t}-\frac{q(u)t\,\zeta'}{w(u)\,t}\right|>\lambda\,\middle|\,\xi(1)>u\right\}\leq C\,|t|^{\varrho}$$

for $u>\tilde{u}$ *and* $t\neq 0$, *for some constants* $C>0$, $\tilde{u}\in\mathbb{R}$ *and* $\varrho>1$.

Of course, one usually takes $\eta'=\zeta'=\xi'(1)$ in Assumption 1, where $\xi'(t)$ is a stochastic derivative of $\xi(t)$. The variable η'' is not always needed in (0.5) [so that (0.5) holds with $\eta''=0$], and η'' is not always choosen to be $\xi''(1)$ when needed.

An infinitely divisible process has representation $\xi(t)=\int_{x\in\text{some space}} f_t(x)\,dM(x)$ in law, where f is a deterministic function and M an independently scattered random measure. Taking $\xi'\equiv\int\frac{\partial}{\partial t}f_t(x)\big|_{t=1}dM(x)$ when $f_{(\cdot)}$ is smooth, we thus get

$$\frac{\xi(1-qt)-u}{w\,t}-\frac{\xi(1)-u}{w\,t}+\frac{qt\xi'}{w\,t}=\frac{q}{w}\int\left(\frac{f_{1-qt}(x)-f_1(x)}{q\,t}+\frac{\partial}{\partial t}f_t(x)\Big|_{t=1}\right)dM(x).$$

Hence we expect Assumption 1 to hold, and the verification can be suprisingly easy.

In Section 2 we prove that (0.6) implies the tightness-requirement $\boxed{3}$, and in Section 3 that (0.5) can replace the requirement about weak convergence $\boxed{1}$.

Usually condition $\boxed{2}$ cannot be derived from Assumption 1. However, by requiring that (0.5) holds for t-values $t=t(u)\to\infty$ as $u\uparrow\hat{u}$, it is possible to weaken the meaning of "too long" in $\boxed{2}$, and thus allow "quite long" local dependence. This option is crucial for our application to Gaussian processes in Section 5.

In Section 4 we show that a version of (0.6) can be used to evaluate the upcrossing intensity $\mu(u)$ through the relation [e.g., Leadbetter et al. (1983, Section 7.2)]

$$\text{(0.7)}\qquad \mu(u)=\lim_{s\downarrow 0}s^{-1}\,\mathbf{P}\big\{\xi(1-s)<u<\xi(1)\big\}.$$

Let $\{X(t)\}_{t\in\mathbb{R}}$ be a separable $n|1$-matrix-valued centered stationary and two times mean-square differentiable Gaussian process with covariance function $R(t)=\mathbf{E}\{X(s)X(s+t)^T\}$. In Section 5 we find the asymptotic behaviour of $\mathbf{P}\big\{\sup_{t\in[0,1]}X(t)^TX(t)>u\big\}$ as $u\to\infty$ under the additional assumption that

$$\liminf_{t\to 0}t^{-4}\inf_{\{x\in\mathbb{R}^n\,:\,\|x\|=1\}}x^T[I-R(t)R(t)^T]x>0:$$

This is an important generalization of results in the literature which are valid only when $(X'(1)\,|\,X(1))$ has a non-degenerate normal-distribution in $\mathbb{R}^n$, i.e.,

$$\liminf_{t\to 0}t^{-2}\inf_{\{x\in\mathbb{R}^n\,:\,\|x\|=1\}}x^T[I-R(t)R(t)^T]x>0.$$

Let $\{M(t)\}_{t\in\mathbb{R}}$ denote an α-stable Lévy motion with $\alpha>1$ that is totally skewed to the left. Consider the moving average $\xi(t)=\int_{-\infty}^{\infty} f(t-x)\,dM(x),\ t\in\mathbb{R},$ where f is a non-negative sufficiently smooth function in $\mathbb{L}^{\alpha}(\mathbb{R})$. In Section 6 we determine the asymptotic behaviour of $\mathbf{P}\bigl\{\sup_{t\in[0,1]}\xi(t)>u\bigr\}$ as $u\to\infty$. This result cannot be proven by traditional approaches to extremes since the asymptotic behaviour of conditional totally skewed α-stable distributions is not known (cf. $\boxed{1}$).

In Section 7 we prove a version of Rice's formula named after Belyaev (1968) for the outcrossing intensity through a smooth surface of an $\mathbb{R}^n$-valued stationary and $\mathbf{P}$-differentiable α-stable process $\{X(t)\}_{t\in\mathbb{R}}$ with independent component processes.

1. Preliminaries. All stochastic variables and processes that appear in this paper are assumed to be defined on a common complete probability space $(\Omega,\mathcal{F},\mathbf{P})$.

In the sequel $\{\xi(t)\}_{t\in\mathbb{R}}$ denotes an $\mathbb{R}$-valued strictly stationary stochastic process, and we assume that a separable and measurable version of $\xi(t)$ have been choosen. Such a version exists e.g., assuming $\mathbf{P}$-continuity almost everywhere [Doob (1953, Theorem II.2.6)], and it is to that version our results apply.

Let $G(x)\equiv\mathbf{P}\{\xi(1)\le x\}$ and $\hat{u}\equiv\sup\bigl\{x\in\mathbb{R}: G(x)<1\bigr\}$. We shall assume that G belongs to a domain of attraction of extremes. Consequently there exist functions $w:(-\infty,\hat{u})\to(0,\infty)$ and $F:(-\hat{x},\infty)\to(-\infty,1)$ such that

$$(1.1)\qquad \lim_{u\uparrow\hat{u}}\frac{1-G(u+xw(u))}{1-G(u)}=1-F(x)\quad\text{for } x\in(-\hat{x},\infty),\quad\text{for some } \hat{x}\in(0,\infty].$$

If G is Type II-attracted we can assume that $\hat{x}=-1$, $F(x)=1-(1+x)^{-\gamma}$ for some $\gamma>0$, and $w(u)=u$. Otherwise G is Type I- or Type III-attracted, and then we can take $\hat{x}=\infty$ and $F(x)=1-\mathrm{e}^{-x}$, and assume that $w(u)=o(u)$ with

$$(1.2)\qquad w(u+xw(u))/w(u)\to 1\qquad\text{locally uniformly for } x\in\mathbb{R}\ \text{ as } u\uparrow\hat{u}.$$

See e.g., Resnick (1987, Chapter 1) to learn more about the domains of attraction.

In general, the tail of G is the by far most important factor affecting the tail-behaviour of the distribution of $\sup_{t\in[0,1]}\xi(t)$. Virtually all marginal distributions G that occur in the study of stationary processes belong to a domain of attraction.

In most assumptions and theorems we assume that a function $q:(-\infty,\hat{u})\to(0,\infty)$ with $\mathfrak{Q}\equiv\limsup_{u\uparrow\hat{u}}q(u)<\infty$ have been specified. The first step when applying these results is to choose a suitable q. Inferences then depend on which assumptions hold for this choice of q. In particular, we will use the requirement

$$(1.3)\qquad \mathfrak{R}(\lambda)\equiv\limsup_{u\uparrow\hat{u}} q(u)/q(u-\lambda w(u)) \quad\text{satisfies}\quad \limsup_{\lambda\downarrow 0}\mathfrak{R}(\lambda)<\infty.$$

Sometimes we also need a second function $D:(-\infty,\hat{u})\to(0,\infty)$ such that

$$(1.4)\qquad D(u)\leq 1/q(u) \quad\text{and}\quad \limsup_{u\uparrow\hat{u}} D(u)=\mathfrak{Q}^{-1}\quad (=+\infty\ \text{ when }\ \mathfrak{Q}=0).$$

In order to link the asymptotic behaviour of $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$ to that of $(\xi'(1)|\xi(1)>u)$, we make intermediate use of the tail-behaviour of the sojourn time

$$L(u)\equiv L(1;u) \qquad\text{where}\qquad L(t;u)\equiv\int_0^t I_{(u,\hat{u})}(\xi(s))\,ds \quad\text{for}\quad t>0,$$

and its first moment $\mathbf{E}\{L(u)\}=\mathbf{P}\{\xi(1)>u\}=1-G(u)$. Our study of sojourns, in turn, crucially relies on the sequence of identities

$$\begin{aligned}(1.5)\quad \int_0^x \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy &= \frac{1}{\mathbf{E}\{L(u)\}}\int_0^1 \mathbf{P}\big\{L(s;u)/q\leq x,\,\xi(s)>u\big\}\,ds\\ &= \int_0^1 \mathbf{P}\bigg\{\int_0^{s/q} I_{(u,\hat{u})}\big(\xi(1-qt)\big)\,dt\leq x\,\bigg|\,\xi(1)>u\bigg\}\,ds\\ &= qx+\int_{qx}^1 \mathbf{P}\bigg\{\int_0^{s/q} I_{(u,\hat{u})}\big(\xi(1-qt)\big)\,dt\leq x\,\bigg|\,\xi(1)>u\bigg\}\,ds.\end{aligned}$$

In (1.5) the first equality is rather deep, and has long been used by S.M. Berman [e.g., Berman (1982, 1992)]: See e.g., Albin (1998, Eq. 3.1) for a proof. The second equality follows easily from stationarity, while the third is trivial.

Convention. Given functions h_1 and h_2 we write $h_1(u)\preceq h_2(u)$ when $\limsup_{u\uparrow\hat{u}}(h_1(u)-h_2(u))\leq 0$ and $h_1(u)\succeq h_2(u)$ when $\liminf_{u\uparrow\hat{u}}(h_1(u)-h_2(u))\geq 0$.

2. First bounds on extremes. Tightness. In Propositions 1 and 2 below we use (a strong version of) $\boxed{2}$ and $\boxed{3}$, respectively, to derive upper and lower bounds for the tail $\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}$. When combined these bounds show that

$$\mathbf{P}\big\{\sup_{t\in[0,1]}\xi(t)>u\big\}\quad\text{asymptotically behaves like}\quad \mathbf{E}\{L(u)\}/q(u)\quad\text{as}\quad u\uparrow\hat{u},$$

except for a multiplicative factor bounded away from 0 and ∞. In Theorem 1 we further prove that (0.6) implies condition $\boxed{3}$ in the shape of Assumption 3 below.

Our lower bound rely on the following requirement:

Assumption 2. *We have*

$$\lim_{d\to\infty}\limsup_{u\uparrow\hat{u}}\int_{d\wedge(1/q(u))}^{1/q(u)}\mathbf{P}\big\{\xi(1-q(u)t)>u\,\big|\,\xi(1)>u\big\}\,dt=0.$$

Assumption 2 requires that if $\xi(1)>u$, then $\xi(t)$ have not spent too much time above the level u prior to $t=1$. Obviously, Assumption 2 is void when $\mathfrak{Q}>0$.

As will be shown in Section 3, Assumption 2 can be relaxed if (0.5) is assumed to hold also for t-values $t=t(u)\to\infty$ as $u\uparrow\hat{u}$. This gives an opportunity to allow stronger local dependence than is usual in local theories of extremes.

Proposition 1. *If Assumption 2 holds, then we have*

$$\liminf_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}>0.$$

Proof. Clearly we have

$$\text{(2.1)}\quad \liminf_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\geq\liminf_{u\uparrow\hat{u}}\frac{1}{x}\int_0^x\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy$$
$$=\frac{1}{x}\bigg(1-\limsup_{u\uparrow\hat{u}}\int_x^\infty\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\bigg)$$

for each $x>0$. Given an $\varepsilon\in(0,1)$, (1.5) and Assumption 2 further show that

$$\text{(2.2)}\quad \limsup_{u\uparrow\hat{u}}\int_x^\infty\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy$$
$$\leq\limsup_{u\uparrow\hat{u}}\int_0^1\mathbf{P}\bigg\{\int_0^{d\wedge(s/q)}I_{(u,\hat{u})}\big(\xi(1-qt)\big)\,dt>(1-\varepsilon)x\,\bigg|\,\xi(1)>u\bigg\}\,ds$$
$$+\limsup_{u\uparrow\hat{u}}\int_0^1\mathbf{P}\bigg\{\int_{d\wedge(s/q)}^{s/q}I_{(u,\hat{u})}\big(\xi(1-qt)\big)\,dt>\varepsilon x\,\bigg|\,\xi(1)>u\bigg\}\,ds$$
$$\leq 0+\frac{1}{\varepsilon x}\limsup_{u\uparrow\hat{u}}\int_{s=0}^{s=1}\int_{t=d\wedge(s/q)}^{t=s/q}\mathbf{P}\big\{\xi(1-qt)>u\,\big|\,\xi(1)>u\big\}\,dtds$$
$$\leq 0+\frac{1}{\varepsilon x}\limsup_{u\uparrow\hat{u}}\int_{t=d\wedge(1/q)}^{t=1/q}\mathbf{P}\big\{\xi(1-qt)>u\,\big|\,\xi(1)>u\big\}\,dt$$
$$\leq 0+\frac{1}{\varepsilon x}\varepsilon^2\quad\text{for } x\geq d/(1-\varepsilon)\ \text{ and }\ d\geq d_0,\ \text{ for some }\ d_0\geq 1.$$

Choosing $x=d_0/(1-\varepsilon)$ and inserting in (2.1) we therefore obtain

$$\liminf_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\geq\frac{1-\varepsilon}{d_0}\bigg(1-\frac{\varepsilon(1-\varepsilon)}{d_0}\bigg)>0.\quad\square$$

Of course, an upper bound on extremes require an assumption of tightness-type:

Assumption 3. *We have*

$$\lim_{a\downarrow 0}\limsup_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\bigg\{\sup_{t\in[0,1]}\xi(t)>u,\ \max_{\{k\in\mathbb{Z}\,:\,0\leq aq(u)k\leq 1\}}\xi(akq(u))\leq u\bigg\}=0.$$

Assumption 3 were first used by Leadbetter and Rootzén (1982), and it plays a central role in the theory for both local and global extremes [e.g., Leadbetter and Rootzén (1982), Leadbetter et al. (1983, Chapter 13), and Albin (1990)].

Proposition 2. *If Assumption 3 holds, then we have*

$$\limsup_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}<\infty.$$

Proof. Clearly we have

$$\begin{aligned}&\limsup_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\\ &\quad\leq\limsup_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\bigg\{\sup_{t\in[0,1]}\xi(t)>u,\ \max_{0\leq aqk\leq 1}\xi(akq)\leq u\bigg\}\\ &\qquad+\limsup_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\max_{0\leq aqk\leq 1}\xi(akq)>u\Big\}.\end{aligned}$$

Here the first term on the right-hand side is finite (for $a>0$ sufficiently small) by Assumption 3, while the second term is bounded by

$$\limsup_{u\uparrow\hat{u}}\frac{q(u)}{\mathbf{E}\{L(u)\}}\Big(\big(1+[1/(aq)]\big)\big(1-G(u)\big)\Big)\ \leq\ \Omega+a^{-1}\ <\ \infty.\quad\Box$$

Assumption 3 is usually very difficult to verify. In fact, no useful direct criteria for its verification, which do not require knowledge of asymptotic behaviour of $\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}$, have been suggested previously.

To prove that (0.6) implies Assumption 3 we recall that if $\{\xi(t)\}_{t\in[a,b]}$ is separable and $\mathbf{P}$-continuous, then every dense subset of $[a,b]$ is a separant for $\xi(t)$.

Theorem 1. *Suppose that* (1.1) *and* (1.3) *hold. If in addition* (0.6) *holds and* $\xi(t)$ *is* $\mathbf{P}$*-continuous, then Assumption 3 holds.*

Proof. Letting $\Lambda_n\equiv\sum_{k=1}^{n}2^{-k}$ and $E_n\equiv\bigcup_{k=0}^{2^n}\{\xi(aqk2^{-n})>u+\Lambda_n\lambda aw\}$ we have

(2.3) $$\mathbf{P}\bigg\{\xi(0)\leq u,\ \sup_{t\in[0,aq]}\xi(t)>u+\lambda aw,\ \xi(aq)\leq u\bigg\}$$

$$
\begin{aligned}
&= \mathbf{P}\Big\{E_0^c,\ \bigcup_{n=1}^{\infty}\bigcup_{k=0}^{2^n}\{\xi(aqk2^{-n}) > u+\lambda aw\}\Big\} \\
&\le \mathbf{P}\Big\{E_0^c,\ \textstyle\bigcup_{n=1}^{\infty} E_n\Big\} \\
&\le \mathbf{P}\Big\{\textstyle\bigcup_{n=1}^{\infty}(E_n \setminus E_{n-1})\Big\} \\
&\le \sum_{n=1}^{\infty}\sum_{k=0}^{2^{n-1}-1}\mathbf{P}\Big\{\xi\big(aq(2k)2^{-n}\big) \le u+\Lambda_{n-1}\lambda aw,\ \xi\big(aq(2k+1)2^{-n}\big) > u+\Lambda_n\lambda aw, \\
&\qquad\qquad \xi\big(aq(2k+2)2^{-n}\big) \le u+\Lambda_{n-1}\lambda aw\Big\} \\
&\le \sum_{n=1}^{\infty}\sum_{k=0}^{2^{n-1}-1}\mathbf{P}\Big\{\frac{\xi(1-aq2^{-n})-u}{w} - \frac{\xi(1)-u}{w} + \frac{aq2^{-n}\zeta'}{w} < -2^{-n}\lambda a,\ \xi(1) > u\Big\} \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{2^{n-1}-1}\mathbf{P}\Big\{\frac{\xi(1+aq2^{-n})-u}{w} - \frac{\xi(1)-u}{w} - \frac{aq2^{-n}\zeta'}{w} < -2^{-n}\lambda a,\ \xi(1) > u\Big\} \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{2^{n-1}-1}\mathbf{P}\Big\{\frac{aq2^{-n}\zeta'}{w} > 0,\ -\frac{aq2^{-n}\zeta'}{w} > 0\Big\} \\
&\le 2\sum_{n=1}^{\infty}\sum_{k=0}^{2^{n-1}-1} C\,(2^{-n}a)^{\varrho}\,\mathbf{P}\{\xi(1) > u\} \\
&= o(a)\,\mathbf{P}\{\xi(1) > u\} \qquad \text{as } \ a\downarrow 0.
\end{aligned}
$$

Consequently it follows that

$$
\begin{aligned}
(2.4)\qquad &\limsup_{u\uparrow\hat{u}} \frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t) > u+\lambda aw,\ \max_{0\le aqk\le 1}\xi(akq) \le u\Big\} \\
&\le \limsup_{u\uparrow\hat{u}} \frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{0\,\le\, t\,\le\,([1/(aq)]+1)aq}\xi(t) > u+\lambda aw,\ \max_{0\le k\le[1/(aq)]+1}\xi(akq) \le u\Big\} \\
&\le \limsup_{u\uparrow\hat{u}} \frac{q(u)}{\mathbf{E}\{L(u)\}}\,([1/(aq)]+1)\,\mathbf{P}\Big\{\xi(0)\le u,\ \sup_{t\in[0,aq]}\xi(t) > u+\lambda aw,\ \xi(aq)\le u\Big\} \\
&\le \limsup_{u\uparrow\hat{u}}\,(a^{-1}+q)\ o(a) \\
&\to 0 \qquad \text{as } \ a\downarrow 0.
\end{aligned}
$$

Hence it is sufficient to prove that

$$
\Delta(\lambda) \equiv \limsup_{a\downarrow 0}\limsup_{u\uparrow\hat{u}} \frac{q(u)}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{u < \sup_{t\in[0,1]}\xi(t) \le u+\lambda aw\Big\} \to 0 \qquad \text{as } \ \lambda\downarrow 0.
$$

Put $\tilde{u}\equiv u-\lambda aw$, $\tilde{w}\equiv w(\tilde{u})$ and $\tilde{q}\equiv q(\tilde{u})$ for $a,\lambda > 0$. Then we have [cf. (1.2)]

$$
u = \tilde{u}+\lambda aw \ge \tilde{u}+\tfrac{1}{2}\lambda a\tilde{w} \qquad \text{and} \qquad \overline{u} = u+\lambda aw \le \tilde{u}+2\lambda a\tilde{w}
$$

for u sufficiently large. Combining (2.4) with (1.1) and (1.3) we therefore obtain

$$
\begin{aligned}
&\Delta(\lambda)\\
&\leq \limsup_{a\downarrow 0}\limsup_{u\uparrow\hat{u}} \frac{\mathfrak{R}(\lambda a)\,(1-F(-\lambda a))\,\tilde{q}}{\mathbf{E}\{L(\tilde{u})\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>\tilde{u}+\tfrac{1}{2}\lambda a\tilde{w},\ \max_{0\leq a\tilde{q}k\leq 1}\xi(ak\tilde{q})\leq\tilde{u}\Big\}\\
&\quad+\limsup_{a\downarrow 0}\limsup_{u\uparrow\hat{u}} \frac{\mathfrak{R}(\lambda a)\,(1-F(-\lambda a))\,\tilde{q}}{\mathbf{E}\{L(\tilde{u})\}}\mathbf{P}\Big\{\tilde{u}<\max_{0\leq a\tilde{q}k\leq 1}\xi(ak\tilde{q})\leq\tilde{u}+2\lambda a\tilde{w}\Big\}\\
&\leq 0\ +\limsup_{a\downarrow 0}\limsup_{u\uparrow\hat{u}} \frac{\mathfrak{R}(\lambda a)\,(1-F(-\lambda a))}{a\,(1-G(u))}\mathbf{P}\{u<\xi(0)\leq u+2\lambda a w\}\\
&\leq \Big(\limsup_{a\downarrow 0}\mathfrak{R}(a)\Big)\Big(\limsup_{a\downarrow 0}\frac{F(2\lambda a)}{a}\Big)\\
&\to 0 \qquad \text{as } \lambda\downarrow 0. \quad \square
\end{aligned}
$$

3. Extremes of P-differentiable processes. We are now prepared to prove (0.2). As mentioned in the introduction, we want the option to employ a stronger version of (0.5) [(3.1) below] in order to be able to relax Assumption 2 [to (3.2)]:

Theorem 2. *Assume that there exist random variables* η' *and* η'' *such that*

(3.1)
$$\lim_{u\uparrow\hat{u}}\int_0^{D(u)}\mathbf{P}\bigg\{\bigg|\frac{\xi(1+q(u)t)-u}{w(u)\,t}-\frac{\xi(1)-u}{w(u)\,t}-\frac{q(u)t\eta'}{w(u)\,t}-\frac{q(u)^2t^2\eta''}{2\,w(u)\,t}\bigg|>\lambda\,\bigg|\,\xi(1)>u\bigg\}=0$$

for each $\lambda>0$ *[where the function* $D(u)$ *satisfies* (1.4)*]. Further assume that*

$$\limsup_{u\uparrow\hat{u}}\int_{D(u)}^{1/q(u)}\mathbf{P}\{\xi(1-q(u)t)>u\,|\,\xi(1)>u\}\,dt=0. \tag{3.2}$$

If in addition (0.4), (1.1), *and Assumption 3 hold, then we have*

$$\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat{u}}\bigg(q(u)+\frac{1}{x}\mathbf{P}\bigg\{\frac{q(u)\,\eta'}{w(u)}\geq\frac{1}{x}\,\frac{\xi(1)-u}{w(u)}\,\bigg|\,\xi(1)>u\bigg\}\bigg)\ <\ \infty.$$

As $x\downarrow 0$ *we further have*

$$\limsup_{u\uparrow\hat{u}}\bigg|\frac{q(u)}{\mathbf{E}\{L(u)\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}-q(u)-\frac{1}{x}\mathbf{P}\bigg\{\frac{q(u)\,\eta'}{w(u)}\geq\frac{1}{x}\,\frac{\xi(1)-u}{w(u)}\,\bigg|\,\xi(1)>u\bigg\}\bigg|\to 0.$$

Proof. Choosing constants $\varepsilon,\lambda>0$, (1.5) and Markov's inequality show that

$$\frac{1}{x}\int_0^x\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy-q \tag{3.3}$$

$$
\begin{aligned}
&= \frac{1}{x}\int_{qx}^{1} \mathbf{P}\Big\{ \int_0^{s/q} I_{(0,\infty)}\Big(\frac{\xi(1-qt)-u}{w}\Big)\, dt \le x \,\Big|\, \xi(1)>u \Big\}\, ds \\
&\le \frac{1}{x}\int_{qx}^{1} \mathbf{P}\Big\{ \int_0^{D\wedge(s/q)} I_{(\lambda\varepsilon,\infty)}\Big(\frac{\xi(1)-u}{w} - \frac{qt\,\eta'}{w} + \frac{q^2t^2\eta''}{2\,w}\Big)\, dt \le (1+\varepsilon)x \,\Big|\, \xi(1)>u \Big\}\, ds \\
&\quad + \frac{1}{x}\int_0^1 \mathbf{P}\Big\{ \int_0^{D} I_{(\lambda\varepsilon,\infty)}\Big(\frac{\xi(1)-u}{w} - \frac{qt\eta'}{w} + \frac{q^2t^2\eta''}{2\,w} - \frac{\xi(1-qt)-u}{w}\Big)\, dt \ge \varepsilon x \,\Big|\, \xi(1)>u \Big\}\, ds \\
&\preceq \frac{1}{x}\mathbf{P}\Big\{ \frac{\xi(1)-u}{w} - \frac{q\,(1+\varepsilon)x\,\eta'}{w} + \frac{q^2\,(1+\varepsilon)^2x^2\,\eta''}{2\,w} \le \lambda\varepsilon \,\Big|\, \xi(1)>u \Big\} \\
&\quad + \frac{1}{x}\int_{\{s\in[qx,1]\,:\,(1+\varepsilon)x\,\ge\, D\wedge(s/q)\}} ds \\
&\quad + \frac{1}{x}\mathbf{P}\Big\{ \frac{\xi(1)-u}{w} \le \lambda\varepsilon \,\Big|\, \xi(1)>u \Big\} \\
&\quad + \frac{1}{x}\,\frac{1}{\varepsilon x}\int_0^{D} \mathbf{P}\Big\{ \frac{\xi(1)-u}{w} - \frac{qt\,\eta'}{w} + \frac{q^2t^2\,\eta''}{2\,w} - \frac{\xi(1-qt)-u}{w} \ge \lambda\varepsilon \,\Big|\, \xi(1)>u \Big\}\, dt
\end{aligned}
$$

for $0<x<\mathfrak{Q}^{-1}$. By (0.4) and (1.1), the first term on the right-hand side is

$$
\begin{aligned}
&\le \frac{1}{x}\mathbf{P}\Big\{ \frac{q\,\eta'}{w} \ge \frac{1-\delta}{(1+\varepsilon)^2x}\,\frac{\xi(1)-u}{w} \,\Big|\, \xi(1)>u \Big\} \\
&\qquad + \frac{1}{x}\mathbf{P}\Big\{ \frac{1}{(1+\varepsilon)^2x}\,\frac{\xi(1)-u}{w} - \frac{x^{(3\varrho-1)/(2\varrho)}}{(1+\varepsilon)x} \le \frac{1-\delta}{(1+\varepsilon)^2x}\,\frac{\xi(1)-u}{w} \,\Big|\, \xi(1)>u \Big\} \\
&\qquad + \frac{1}{x}\mathbf{P}\Big\{ \frac{q^2(1+\varepsilon)^2x^2\,\eta''}{2\,w} \le -x^{(3\varrho-1)/(2\varrho)} \,\Big|\, \xi(1)>u \Big\} \\
&\qquad + \frac{1}{x}\mathbf{P}\Big\{ \frac{1}{(1+\varepsilon)x}\,\frac{\xi(1)-u}{w} - \frac{\lambda\varepsilon}{(1+\varepsilon)x} \le \frac{1}{(1+\varepsilon)^2x}\,\frac{\xi(1)-u}{w} \,\Big|\, \xi(1)>u \Big\} \\
&\preceq \frac{1}{x}\mathbf{P}\Big\{ \frac{q\,\eta'}{w} \ge \frac{1-\delta}{(1+\varepsilon)^2x}\,\frac{\xi(1)-u}{w} \,\Big|\, \xi(1)>u \Big\} \\
&\qquad + \frac{F\big((1+\varepsilon)x^{(3\varrho-1)/(2\varrho)}/\delta\big)}{x} \\
&\qquad + \frac{1}{x}\,\frac{(1+\varepsilon)^{2\varrho}x^{2\varrho}}{x^{(3\varrho-1)/2}}\,\mathbf{E}\Big\{ \Big|\frac{q^2\,\eta''}{2\,w}\Big|^{\varrho} \,\Big|\, \xi(1)>u \Big\} \\
&\qquad + \frac{F(\lambda(1+\varepsilon))}{x} \qquad \text{for } \delta\in(0,1).
\end{aligned}
$$

Further the second term in (3.3) is $\preceq \mathfrak{Q}\varepsilon$ for $D>(1+\varepsilon)x$, the third $\sim F(\lambda\varepsilon)/x$ by (1.1), and the fourth $\preceq 0$ by (3.1). Taking $\tilde{x} = (1+\varepsilon)^2x/(1-\delta)$, (3.3) thus give

$$
\frac{1}{\tilde{x}}\int_0^{\tilde{x}} \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\, dy - q - \frac{1}{\tilde{x}}\mathbf{P}\Big\{ \frac{q\eta'}{w} \ge \frac{1}{\tilde{x}}\frac{\xi(1)-u}{w} \,\Big|\, \xi(1)>u \Big\}
$$

$$
\begin{aligned}
&\preceq \frac{1-\delta}{(1+\varepsilon)^2}\bigg(\frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy - q - \frac{1}{x}\mathbf{P}\bigg\{\frac{q\eta'}{w}\geq\frac{1-\delta}{(1+\varepsilon)^2x}\,\frac{\xi(1)-u}{w}\,\bigg|\,\xi(1)>u\bigg\}\bigg)\\
&\quad + \frac{1-\delta}{(1+\varepsilon)^2}\,\frac{1}{x}\int_x^{\tilde{x}} \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy\\
&\preceq \frac{1-\delta}{(1+\varepsilon)^2}\;\frac{F\big((1+\varepsilon)x^{(3\varrho-1)/(2\varrho)}/\delta\big)}{x}\\
&\quad + \frac{1-\delta}{(1+\varepsilon)^2}\,(1+\varepsilon)^{2\varrho}x^{(\varrho-1)/2}\;\limsup_{u\uparrow\hat{u}}\mathbf{E}\bigg\{\bigg|\frac{q^2\,\eta''}{2\,w}\bigg|^{\varrho}\,\bigg|\,\xi(1)>u\bigg\}\\
&\quad + \frac{1-\delta}{(1+\varepsilon)^2}\;\frac{F(\lambda(1+\varepsilon))}{x}\\
&\quad + \frac{1-\delta}{(1+\varepsilon)^2}\,\mathfrak{Q}\varepsilon\\
&\quad + \frac{1-\delta}{(1+\varepsilon)^2}\;\frac{F(\lambda\varepsilon)}{x}\\
&\quad + \frac{1-\delta}{(1+\varepsilon)^2}\;\frac{(1+\varepsilon)^2x-(1-\delta)x}{(1-\delta)x}\;\limsup_{u\uparrow\hat{u}}\frac{q}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\\
&\to 0 \qquad \text{as}\;\; \varepsilon\downarrow 0,\;\; \lambda\downarrow 0,\;\; x\downarrow 0\;\; \text{and}\;\; \delta\downarrow 0\quad \text{(in that order)},
\end{aligned}
$$

where we used Proposition 2 in the last step. It follows that

$$
\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat{u}}\bigg(\frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy - q - \frac{1}{x}\mathbf{P}\bigg\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\,\bigg|\,\xi(1)>u\bigg\}\bigg)\leq 0. \tag{3.4}
$$

To proceed we note that (3.2) combines with a version of the argument (2.2) to show that, given an $\varepsilon>0$, there exists an $u_0=u_0(\varepsilon)<\hat{u}$ such that

$$
\int_0^1\bigg[\int_{D\wedge(s/q)}^{s/q}\mathbf{P}\{\xi(1-qt)>u\mid\xi(1)>u\}\,dt\bigg]ds\leq\varepsilon^2\qquad\text{for}\;\; u\in[u_0,\hat{u}).
$$

Through a by now familiar way of reasoning we therefore obtain

$$
\begin{aligned}
&\frac{1}{x}\int_0^x \frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy - q\\
&\geq \frac{1}{x}\int_{qx}^1\mathbf{P}\bigg\{\int_0^{D\wedge(s/q)} I_{(0,\infty)}\bigg(\frac{\xi(1-qt)-u}{w}\bigg)\,dt\leq(1-\varepsilon)x\,\bigg|\,\xi(1)>u\bigg\}\,ds\\
&\quad - \frac{1}{x}\int_0^1\mathbf{P}\bigg\{\int_{D\wedge(s/q)}^{s/q} I_{(u,\hat{u})}\big(\xi(1-qt)\big)\,dt\geq\varepsilon x\,\bigg|\,\xi(1)>u\bigg\}\,ds\\
&\geq \frac{1-qx}{x}\,\mathbf{P}\bigg\{\int_0^D I_{(0,\infty)}\bigg(\frac{\xi(1-qt)-u}{w}\bigg)\,dt\leq(1-\varepsilon)x\,\bigg|\,\xi(1)>u\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{x}\,\frac{1}{\varepsilon x}\int_0^1\bigg[\int_{D\wedge(s/q)}^{s/q}\mathbf{P}\big\{\xi(1-qt)>u\;\big|\;\xi(1)>u\big\}\,dt\bigg]ds\\
&\succeq\frac{1-qx}{x}\,\mathbf{P}\bigg\{\int_0^D I_{(-\lambda\varepsilon,\infty)}\bigg(\frac{\xi(1)-u}{w}-\frac{qt\,\eta'}{w}+\frac{q^2t^2\,\eta''}{2\,w}\bigg)\,dt\le(1-2\varepsilon)x\;\bigg|\;\xi(1)>u\bigg\}\\
&\quad-\frac{1}{x}\,\mathbf{P}\bigg\{\int_0^D I_{(\lambda\varepsilon,\infty)}\bigg(\frac{\xi(1-qt)-u}{w}-\frac{\xi(1)-u}{w}+\frac{qt\,\eta'}{w}-\frac{q^2t^2\,\eta''}{2\,w}\bigg)\,dt\ge\varepsilon x\;\bigg|\;\xi(1)>u\bigg\}\\
&\quad-\frac{1}{x}\,\frac{1}{\varepsilon x}\,\varepsilon^2\\
&\ge\frac{1-qx}{x}\,\mathbf{P}\bigg\{\frac{\xi(1)-u}{w}-\frac{q\,(1-2\varepsilon)x\,\eta'}{w}\le-\lambda\varepsilon\;\bigg|\;\xi(1)>u\bigg\}\\
&\quad-\frac{1}{x}\,\frac{1}{\varepsilon x}\int_0^D\mathbf{P}\bigg\{\frac{\xi(1-qt)-u}{w}-\frac{\xi(1)-u}{w}+\frac{qt\,\eta'}{w}-\frac{q^2t^2\,\eta''}{2\,w}\ge\lambda\varepsilon\;\bigg|\;\xi(1)>u\bigg\}\,dt\\
&\quad-\frac{\varepsilon}{x^2}\\
&\succeq\frac{1-qx}{x}\,\mathbf{P}\bigg\{\frac{q\,\eta'}{w}\ge\frac{1}{(1-2\varepsilon)^2x}\,\frac{\xi(1)-u}{w}\;\bigg|\;\xi(1)>u\bigg\}\\
&\quad-\frac{1}{x}\,\mathbf{P}\bigg\{\frac{1}{(1-2\varepsilon)x}\,\frac{\xi(1)-u}{w}+\frac{\lambda\varepsilon}{(1-2\varepsilon)x}\ge\frac{1}{(1-2\varepsilon)^2x}\,\frac{\xi(1)-u}{w}\;\bigg|\;\xi(1)>u\bigg\}\\
&\quad-\frac{\varepsilon}{x^2}\\
&\succeq\frac{1-qx}{x}\,\mathbf{P}\bigg\{\frac{q\,\eta'}{w}\ge\frac{1}{(1-2\varepsilon)^2x}\,\frac{\xi(1)-u}{w}\;\bigg|\;\xi(1)>u\bigg\}-\frac{F(\frac{1}{2}\lambda(1-2\varepsilon))}{x}-\frac{\varepsilon}{x^2}
\end{aligned}
$$

for $0<x<\mathfrak{Q}^{-1}$ and $0<\varepsilon<\frac{1}{2}$. Consequently we have

$$
\begin{aligned}
&\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}-q-\frac{1}{(1-2\varepsilon)^2x}\mathbf{P}\bigg\{\frac{q\eta'}{w}\ge\frac{1}{(1-2\varepsilon)^2x}\,\frac{\xi(1)-u}{w}\bigg|\xi(1)>u\bigg\}\\
&\quad\succeq\liminf_{u\uparrow\hat{u}}\frac{1}{(1-qx)\,(1-2\varepsilon)^2}\bigg(\frac{1}{x}\int_0^x\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy-q\\
&\qquad\qquad-\frac{1-qx}{x}\,\mathbf{P}\bigg\{\frac{q\eta'}{w}\ge\frac{1}{(1-2\varepsilon)^2x}\,\frac{\xi(1)-u}{w}\;\bigg|\;\xi(1)>u\bigg\}\bigg)\\
&\qquad-\limsup_{u\uparrow\hat{u}}\bigg(\frac{1}{(1-qx)\,(1-2\varepsilon)^2}-1\bigg)\,\frac{q}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\\
&\quad\ge-\frac{F(\frac{1}{2}\lambda(1-2\varepsilon))}{(1-\mathfrak{Q}x)\,(1-2\varepsilon)^2\,x}-\frac{\varepsilon}{(1-\mathfrak{Q}x)\,(1-2\varepsilon)^2\,x^2}\\
&\qquad-\frac{\mathfrak{Q}x\,(1-2\varepsilon)^2+4\varepsilon(1-\varepsilon)}{(1-\mathfrak{Q}x)\,(1-2\varepsilon)^2}\;\limsup_{u\uparrow\hat{u}}\frac{q}{\mathbf{E}\{L(u)\}}\,\mathbf{P}\Big\{\sup_{t\in[0,1]}\xi(t)>u\Big\}\\
&\quad\to 0\qquad\text{as }\;\varepsilon\downarrow 0,\;\;\lambda\downarrow 0\;\;\text{and}\;\;x\downarrow 0\quad\text{(in that order)},
\end{aligned}
$$

where we used Proposition 2 in the last step. Hence we conclude that

(3.5)
$$\liminf_{x\downarrow 0}\liminf_{u\uparrow\hat u}\biggl(\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\Bigl\{\sup_{t\in[0,1]}\xi(t)>u\Bigr\}-q-\frac{1}{x}\mathbf{P}\Bigl\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\Bigm|\xi(1)>u\Bigr\}\biggr)\geq 0,$$

which in particular (again using Proposition 2) implies that

$$\begin{aligned}(3.6)\qquad &\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\biggl(q+\frac{1}{x}\mathbf{P}\Bigl\{\frac{q\,\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\Bigm|\xi(1)>u\Bigr\}\biggr)\\ &\leq\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\biggl(q+\frac{1}{x}\mathbf{P}\Bigl\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\Bigm|\xi(1)>u\Bigr\}-\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\Bigl\{\sup_{t\in[0,1]}\xi(t)>u\Bigr\}\biggr)\\ &\qquad+\limsup_{u\uparrow\hat u}\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\Bigl\{\sup_{t\in[0,1]}\xi(t)>u\Bigr\}\\ &<\infty.\end{aligned}$$

Using (1.5), (3.4) and (3.6) we readily deduce that

$$\begin{aligned}(3.7)\qquad &\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\Bigl\{L(u)/q\leq x,\ \max_{0<aqk\leq 1}\xi(akq)>u\Bigr\}\\ &\leq\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\frac{q}{\mathbf{E}\{L(u)\}}\sum_{k=1}^{[1/(aq)]}\mathbf{P}\bigl\{L(akq;u)/q\leq x,\ \xi(akq)>u\bigr\}\\ &\leq\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\frac{1}{a\,\mathbf{E}\{L(u)\}}\sum_{k=1}^{[1/(aq)]}\int_{a(k-1)q}^{akq}\mathbf{P}\bigl\{L(t;u)/q\leq x,\ \xi(t)>u\bigr\}\,dt\\ &\leq\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\frac{1}{a\,\mathbf{E}\{L(u)\}}\int_0^1\mathbf{P}\bigl\{L(t;u)/q\leq x,\ \xi(t)>u\bigr\}\,dt\\ &\leq\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\frac{x}{a}\biggl(\frac{1}{x}\int_0^x\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy-q-\frac{1}{x}\mathbf{P}\Bigl\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\Bigm|\xi(1)>u\Bigr\}\biggr)\\ &\qquad+\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\frac{x}{a}\biggl(q+\frac{1}{x}\mathbf{P}\Bigl\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\Bigm|\xi(1)>u\Bigr\}\biggr)\\ &\leq 0.\end{aligned}$$

Further a calculation similar to (3.3)-(3.4) combines with (3.6) to reveal that

$$\begin{aligned}(3.8)\qquad &\limsup_{u\uparrow\hat u}\mathbf{P}\bigl\{L(u)/q\leq x\bigm|\xi(1)>u\bigr\}\\ &=\limsup_{u\uparrow\hat u}\mathbf{P}\biggl\{\int_0^{1/q}I_{(0,\infty)}\Bigl(\frac{\xi(1-qt)-u}{w}\Bigr)\,dt\leq x\biggm|\xi(1)>u\biggr\}\\ &\leq\limsup_{u\uparrow\hat u}\mathbf{P}\biggl\{\int_0^{d\wedge(1/q)}I_{(\lambda\varepsilon,\infty)}\Bigl(\frac{\xi(1)-u}{w}-\frac{qt\,\eta'}{w}\Bigr)dt\leq(1+\varepsilon)x\biggm|\xi(1)>u\biggr\}\end{aligned}$$

$$+\limsup_{u\uparrow\hat u}\mathbf{P}\left\{\int_0^d I_{(\lambda\varepsilon,\infty)}\left(\frac{\xi(1)-u}{w}-\frac{qt\eta'}{w}-\frac{\xi(1-qt)-u}{w}\right)dt\geq\varepsilon x\ \middle|\ \xi(1)>u\right\}$$

$$\leq\limsup_{u\uparrow\hat u}\mathbf{P}\left\{\frac{\xi(1)-u}{w}-\frac{q\,(1+\varepsilon)x\,\eta'}{w}\leq\lambda\varepsilon\ \middle|\ \xi(1)>u\right\}+F(\lambda\varepsilon)$$

$$+\limsup_{u\uparrow\hat u}\frac{1}{\varepsilon x}\int_0^d\mathbf{P}\left\{\frac{\xi(1)-u}{w}-\frac{qt\,\eta'}{w}-\frac{\xi(1-qt)-u}{w}\geq\lambda\varepsilon\ \middle|\ \xi(1)>u\right\}dt$$

$$\leq\limsup_{u\uparrow\hat u}\mathbf{P}\left\{\frac{q\,\eta'}{w}\geq\frac{1}{(1+\varepsilon)^2x}\,\frac{\xi(1)-u}{w}\ \middle|\ \xi(1)>u\right\}+F(\lambda(1+\varepsilon))+F(\lambda\varepsilon)$$

$\to 0$ as $\varepsilon\downarrow 0$, $\lambda\downarrow 0$ and $x\downarrow 0$ (in that order).

In view of the obvious fact that

$$\mathbf{P}\{\sup_{t\in[0,1]}\xi(t)>u\}$$

$$\leq\frac{1}{x}\int_0^x\mathbf{P}\left\{\{L(u)/q>y\}\cup\left\{\max_{0\leq aqk\leq 1}\xi(akq)>u\right\}\cup\left\{\sup_{t\in[0,1]}\xi(t)>u\right\}\right\}dy$$

$$\leq\frac{1}{x}\int_0^x\mathbf{P}\{L(u)/q>y\}\,dy$$

$$+\mathbf{P}\left\{L(u)/q\leq x,\ \max_{0\leq aqk\leq 1}\xi(akq)>u\right\}$$

$$+\mathbf{P}\left\{\sup_{t\in[0,1]}\xi(t)>u,\ \max_{0\leq aqk\leq 1}\xi(akq)\leq u\right\},$$

Assumption 3 combines with (3.4) and (3.7)-(3.8) to show that
(3.9)

$$\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\left(\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\left\{\sup_{t\in[0,1]}\xi(t)>u\right\}-q-\frac{1}{x}\mathbf{P}\left\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\ \middle|\ \xi(1)>u\right\}\right)$$

$$\leq\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\left(\frac{1}{x}\int_0^x\frac{\mathbf{P}\{L(u)/q>y\}}{\mathbf{E}\{L(u)/q\}}\,dy-q-\frac{1}{x}\mathbf{P}\left\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\ \middle|\ \xi(1)>u\right\}\right)$$

$$+\limsup_{a\downarrow 0}\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\left\{L(u)/q\leq x,\ \max_{0<aqk\leq 1}\xi(akq)>u\right\}$$

$$+\limsup_{x\downarrow 0}\limsup_{u\uparrow\hat u}\ q\,\mathbf{P}\{L(u)/q\leq x\mid\xi(1)>u\}$$

$$+\limsup_{a\downarrow 0}\limsup_{u\uparrow\hat u}\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\left\{\sup_{t\in[0,1]}\xi(t)>u,\ \max_{0\leq aqk\leq 1}\xi(akq)\leq u\right\}$$

≤ 0.

The theorem now follows from the inequalities (3.5)-(3.6) and (3.9). $\square$

In Theorem 2 the absence of Assumption 2 disallowes use of Proposition 1. However, if we do make Assumption 2, then a stronger limit theorem can be proved:

Corollary 1. *Suppose that* (1.1) *and* (1.3) *hold. If in addition Assumptions 1 and 2 hold, and* $\xi(t)$ *is* **P**-*continuous, then the conclusion of Theorem 2 holds with*

$$\liminf_{x\downarrow 0}\ \liminf_{u\uparrow\hat{u}}\left(q(u)+\frac{1}{x}\mathbf{P}\left\{\frac{q(u)\,\eta'}{w(u)}\geq\frac{1}{x}\,\frac{\xi(1)-u}{w(u)}\;\middle|\;\xi(1)>u\right\}\right)>0.$$

Proof. By Theorem 1, (0.6) implies Assumption 3. Moreover (0.5) implies that (3.1) holds for some function $D(u)\to\mathfrak{Q}^{-1}$ (sufficiently slowly) as $u\uparrow\hat{u}$. Since Assumption 2 implies that (3.2) holds for any function $D(u)\to\mathfrak{Q}^{-1}$, it follows that the hypothesis of Theorem 2 holds. Further (3.9) and Proposition 1 show that

$$\liminf_{x\downarrow 0}\liminf_{u\uparrow\hat{u}}\left(q+\frac{1}{x}\mathbf{P}\left\{\frac{q\,\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\;\middle|\;\xi(1)>u\right\}\right)$$
$$\geq\liminf_{x\downarrow 0}\liminf_{u\uparrow\hat{u}}\left(q+\frac{1}{x}\mathbf{P}\left\{\frac{q\eta'}{w}\geq\frac{1}{x}\frac{\xi(1)-u}{w}\;\middle|\;\xi(1)>u\right\}-\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\left\{\sup_{t\in[0,1]}\xi(t)>u\right\}\right)$$
$$+\liminf_{u\uparrow\hat{u}}\frac{q}{\mathbf{E}\{L(u)\}}\mathbf{P}\left\{\sup_{t\in[0,1]}\xi(t)>u\right\}$$
$$>0.\quad\square$$

4. Rice's formula for P-differentiable stationary processes. In Theorem 3 below we derive two expressions for the upcrossing intensity of the process $\{\xi(t)\}_{t\in\mathbb{R}}$ in terms of the joint distribution of $\xi(1)$ and its stochastic derivative $\xi'(1)$. Our result is a useful complement and alternative to existing results in the literature.

Theorem 3. *Let* $\{\xi(t)\}_{t\in\mathbb{R}}$ *be a stationary process with a.s. continuous sample paths. Choose an open set* $U\subseteq\mathbb{R}$ *and assume that*

$\xi(1)$ *has a density function that is essentially bounded on* U.

(i) *Assume that there exist a random variable* $\xi'(1)$ *and a constant* $\varrho>1$ *such that*

$$\lim_{s\downarrow 0}\ \operatorname*{ess\,sup}_{y\in U}\mathbf{E}\left\{\left|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\right|^{\varrho}\;\middle|\;\xi(1)=y\right\}f_{\xi(1)}(y)=0,\tag{4.1}$$

and such that

$$\lim_{s\downarrow 0}\mathbf{E}\left\{\left|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\right|\right\}=0.\tag{4.2}$$

Then the upcrossing intensity $\mu(u)$ *of a level* u *by* $\xi(t)$ *is given by*

$$\mu(u)=\lim_{s\downarrow 0}s^{-1}\mathbf{P}\{\xi(1)-s\xi'(1)<u<\xi(1)\}\quad\textit{for }u\in U.\tag{4.3}$$

(ii) *Assume that there exist a random variable* $\xi'(1)$ *and a constant* $\varrho>1$ *such that*

$$(4.4)\qquad \lim_{s\downarrow 0}\ \operatorname*{ess\,sup}_{y>u} \mathbf{E}\biggl\{\Bigl|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\Bigr|^{\varrho}\ \Bigl|\ \xi(1)=y\biggr\}\, f_{\xi(1)}(y) = 0 \quad \textit{for } u\in U.$$

Then the upcrossing intensity $\mu(u)$ *is given by* (4.3)*.*

(iii) *Assume that* (4.3) *holds and that* $\mathbf{E}\{[\xi'(1)^{+}]\}<\infty$*. Then we have*

$$\mu(u)\begin{cases} \geq \lim\limits_{\varepsilon\downarrow 0}\displaystyle\int_0^{\infty} \operatorname*{ess\,inf}_{u<y<u+\varepsilon} \mathbf{P}\bigl\{\xi'(1)>x \mid \xi(1)=y\bigr\}\, f_{\xi(1)}(y)\ dx \\ \leq \lim\limits_{\varepsilon\downarrow 0}\displaystyle\int_0^{\infty} \operatorname*{ess\,sup}_{u<y<u+\varepsilon} \mathbf{P}\bigl\{\xi'(1)>x \mid \xi(1)=y\bigr\}\, f_{\xi(1)}(y)\ dx \end{cases} \quad \textit{for } u\in U.$$

(iv) *Assume that* (4.3) *holds, that there exists a constant* $\varrho>1$ *such that*

$$(4.5)\qquad \operatorname*{ess\,sup}_{y\in U} \mathbf{E}\bigl\{[\xi'(1)^{+}]^{\varrho} \bigm| \xi(1)=y\bigr\}\, f_{\xi(1)}(y) < \infty,$$

and that $\mathbf{E}\{[\xi'(1)^{+}]\}<\infty$*. Then* $\mu(u)$ *is finite and satisfies*

$$(4.6)\qquad \mu(u)\begin{cases} \geq \displaystyle\int_0^{\infty} \lim_{\varepsilon\downarrow 0}\ \operatorname*{ess\,inf}_{u<y<u+\varepsilon} \mathbf{P}\bigl\{\xi'(1)>x \mid \xi(1)=y\bigr\}\, f_{\xi(1)}(y)\ dx \\ \leq \displaystyle\int_0^{\infty} \lim_{\varepsilon\downarrow 0}\ \operatorname*{ess\,sup}_{u<y<u+\varepsilon} \mathbf{P}\bigl\{\xi'(1)>x \mid \xi(1)=y\bigr\}\, f_{\xi(1)}(y)\ dx \end{cases} \quad \textit{for } u\in U.$$

(v) *Assume that* (4.3) *holds, that there exists a constant* $\varrho>1$ *such that*

$$(4.7)\qquad \operatorname*{ess\,sup}_{y>u} \mathbf{E}\bigl\{[\xi'(1)^{+}]^{\varrho} \bigm| \xi(1)=y\bigr\}\, f_{\xi(1)}(y) < \infty \qquad \textit{for } u\in U.$$

Then $\mu(u)$ *is finite and satisfies* (4.6)*.*

(vi) *Assume that* (4.6) *holds and that* $(\xi(1),\xi'(1))$ *has a density such that*

$$\int_x^{\infty} f_{\xi(1),\xi'(1)}(u,y)\ dy \qquad \textit{is a continuous function of } u\in U \quad \textit{for almost all } x>0.$$

Then $\mu(u)$ *is given by Rice's formula* (0.1) *for each* $u\in U$*.*

(vii) *Assume that* (4.6) *holds, that*

$$\xi(1) \quad \textit{has a density that is continuous and bounded away from zero on } U,$$

and that

$$(\xi(1),\xi'(1)) \quad \textit{has a density } f_{\xi(1),\xi'(1)}(u,y) \textit{ that is a continuous function of } u\in U$$

for each $y>0$*. Then* $\mu(u)$ *is given by Rice's formula* (0.1) *for each* $u\in U$*.*

Proof of (i). Using (0.7) together with (4.2) and (4.3) we obtain

$$(4.8) \qquad \mu(u)$$

$$= \liminf_{s\downarrow 0} s^{-1}\,\mathbf{P}\{\xi(1-s)<u<\xi(1)\}$$

$$\leq \liminf_{s\downarrow 0} \frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>(1-\lambda)\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\}$$

$$+\limsup_{s\downarrow 0} s^{-1}\,\mathbf{P}\{u<\xi(1)\leq u+\delta s\}$$

$$+\limsup_{s\downarrow 0} \frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)\leq(1-\lambda)\frac{\xi(1)-u}{s},\ \xi(1-s)<u,\ u+\delta s<\xi(1)\leq u+\varepsilon\Big\}$$

$$+\limsup_{s\downarrow 0} \frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)\leq(1-\lambda)\frac{\xi(1)-u}{s},\ \xi(1-s)<u,\ \xi(1)>u+\varepsilon\Big\}$$

$$\leq \frac{1}{1-\lambda}\liminf_{s\downarrow 0} \frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\}$$

$$+\,\delta\,\limsup_{s\downarrow 0}\ \operatorname*{ess\,sup}_{u<y<u+\delta s}\ f_{\xi(1)}(y)$$

$$+\limsup_{s\downarrow 0}\int_{\delta}^{\varepsilon/s}\mathbf{P}\bigg\{\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}>\lambda x\ \bigg|\ \xi(1)=u+xs\bigg\}\,f_{\xi(1)}(u+xs)\ dx$$

$$+\limsup_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\bigg\{\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}>\frac{\lambda\varepsilon}{s}\bigg\}$$

$$\leq \frac{1}{1-\lambda}\liminf_{s\downarrow 0} \frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\}$$

$$+\,\delta\,\limsup_{s\downarrow 0}\ \operatorname*{ess\,sup}_{u<y<u+\delta s}\ f_{\xi(1)}(y)$$

$$+\limsup_{s\downarrow 0}\ \operatorname*{ess\,sup}_{u<y<u+\varepsilon}\ \mathbf{E}\bigg\{\bigg|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\bigg|^{\varrho}\ \bigg|\ \xi(1)=y\bigg\}\,f_{\xi(1)}(y)\int_{\delta}^{\varepsilon/s}\frac{dx}{(\lambda x)^{\varrho}}$$

$$+\limsup_{s\downarrow 0}\frac{1}{\lambda\varepsilon}\,\mathbf{E}\bigg\{\bigg|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\bigg|\bigg\}$$

$$\to \liminf_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\} \qquad \text{as}\quad \lambda\downarrow 0,\quad \varepsilon\downarrow 0\quad \text{and}\quad \delta\downarrow 0.$$

The formula (4.3) follows using this estimate together with its converse

$$(4.9) \qquad \mu(u)$$

$$\geq \limsup_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>(1+\lambda)\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\}$$

$$
\begin{aligned}
&- \limsup_{s\downarrow 0} s^{-1}\,\mathbf{P}\{u<\xi(1)\le u+\delta s\}\\
&- \limsup_{s\downarrow 0} \frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>(1+\lambda)\frac{\xi(1)-u}{s},\ \xi(1-s)\ge u,\ u+\delta s<\xi(1)\le u+\varepsilon\Big\}\\
&- \limsup_{s\downarrow 0} \frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>(1+\lambda)\frac{\xi(1)-u}{s},\ \xi(1-s)\ge u,\ \xi(1)>u+\varepsilon\Big\}\\
\ge{}& \frac{1}{1+\lambda}\limsup_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\}\\
&- \delta\,\limsup_{s\downarrow 0}\ \operatorname*{ess\,sup}_{u<y<u+\delta s} f_{\xi(1)}(y)\\
&- \limsup_{s\downarrow 0}\int_\delta^{\varepsilon/s}\mathbf{P}\bigg\{\frac{\xi(1-s)-\xi(1)+s\xi'(1)}{s}>\lambda x\ \bigg|\ \xi(1)=u+xs\bigg\}\,f_{\xi(1)}(u+xs)\;dx\\
&- \limsup_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\bigg\{\frac{\xi(1-s)-\xi(1)+s\xi'(1)}{s}>\frac{\lambda\varepsilon}{s}\bigg\}\\
\to{}& \limsup_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\}\qquad\text{as}\quad\lambda\downarrow 0,\quad\varepsilon\downarrow 0\quad\text{and}\quad\delta\downarrow 0.\quad\square
\end{aligned}
$$

Proof of (ii). When (4.4) holds we do not need (4.3) in (4.8), because the estimates that used (4.2) and (4.3) can be replaced with the estimate

$$
\begin{aligned}
&\limsup_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)\le(1-\lambda)\frac{\xi(1)-u}{s},\ \xi(1-s)<u,\ \xi(1)>u+\delta s\Big\}\\
&\quad\le \limsup_{s\downarrow 0}\int_\delta^{\infty}\mathbf{P}\bigg\{\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}>\lambda x\ \bigg|\ \xi(1)=u+xs\bigg\}\,f_{\xi(1)}(u+xs)\;dx\\
&\quad\le \limsup_{s\downarrow 0}\ \operatorname*{ess\,sup}_{y>u}\mathbf{E}\bigg\{\bigg|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\bigg|^{\varrho}\ \bigg|\ \xi(1)=y\bigg\}\,f_{\xi(1)}(y)\int_\delta^{\infty}\frac{dx}{(\lambda x)^{\varrho}}\\
&\quad=0
\end{aligned}
$$

In (4.9) we make similar use of the fact that [under (4.4)]

$$
\begin{aligned}
&\limsup_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>(1+\lambda)\frac{\xi(1)-u}{s},\ \xi(1-s)\ge u,\ \xi(1)>u+\delta s\Big\}\\
&\quad\le \limsup_{s\downarrow 0}\int_\delta^{\infty}\mathbf{P}\bigg\{\frac{\xi(1-s)-\xi(1)+s\xi'(1)}{s}>\lambda x\ \bigg|\ \xi(1)=u+xs\bigg\}\,f_{\xi(1)}(u+xs)\;dx\\
&\quad=0.\quad\square
\end{aligned}
$$

Proof of (iv). Since $\mathbf{E}\{[\xi'(1)^+]\}<\infty$ there exists a sequence $\{s_n\}_{n=1}^{\infty}$ such that

$$
\mathbf{P}\{\xi'(1)>[s_n\sqrt{|\ln(s_n)|}\,]^{-1}\}<s_n/\sqrt{|\ln(s_n)|}\quad\text{for } n\ge 1\quad\text{and}\quad s_n\downarrow 0\ \text{ as } n\to\infty.
$$

Using (4.3) and (4.5) we therefore obtain

$$(4.10)\quad \mu(u)=\liminf_{s\downarrow 0}\frac{1}{s}\,\mathbf{P}\Big\{\xi'(1)>\frac{\xi(1)-u}{s},\ \xi(1)>u\Big\}$$

$$\le \limsup_{n\to\infty}\int_0^{[s_n\sqrt{|\ln(s_n)|}\,]^{-1}}\mathbf{P}\big\{\xi'(1)>x\,\big|\,\xi(1)=u+xs_n\big\}\,f_{\xi(1)}(u+xs_n)\;dx$$

$$+\limsup_{n\to\infty}s_n^{-1}\,\mathbf{P}\big\{\xi'(1)>[s_n\sqrt{|\ln(s_n)|}\,]^{-1}\big\}$$

$$\le\int_0^\infty \operatorname*{ess\,sup}_{u<y<u+\varepsilon}\mathbf{P}\big\{\xi'(1)>x\,\big|\,\xi(1)=y\big\}\,f_{\xi(1)}(y)\;dx$$

$$+\limsup_{n\to\infty}1/\sqrt{|\ln(s_n)|}\qquad\text{for each}\quad\varepsilon>0.$$

On the other hand (4.3) alone yields

$$\mu(u)\ge\lim_{\Delta\to\infty}\limsup_{s\downarrow 0}\int_0^\Delta\mathbf{P}\big\{\xi'(1)>x\;\big|\;\xi(1)=u+xs\big\}\,f_{\xi(1)}(u+xs)\;dx$$

$$\ge\lim_{\Delta\to\infty}\int_0^\Delta\operatorname*{ess\,inf}_{u<y<u+\varepsilon}\mathbf{P}\big\{\xi'(1)>x\;\big|\;\xi(1)=y\big\}\,f_{\xi(1)}(y)\;dx\qquad\text{for each}\quad\varepsilon>0.\quad\square$$

Proof of (iv). When (4.5) holds the estimate (4.10) can be sharpened to

$$(4.11)\quad\mu(u)\le\limsup_{n\to\infty}\int_0^\Delta\mathbf{P}\big\{\xi'(1)>x\,\big|\,\xi(1)=u+xs_n\big\}\,f_{\xi(1)}(u+xs_n)\;dx$$

$$+\limsup_{n\to\infty}\int_\Delta^{[s_n\sqrt{|\ln(s_n)|}\,]^{-1}}\frac{\mathbf{E}\{[\xi'(1)^+]^\varrho|\xi(1)=u+xs_n\}}{x^\varrho}\,f_{\xi(1)}(u+xs_n)\,dx$$

$$+\limsup_{n\to\infty}s_n^{-1}\,\mathbf{P}\big\{\xi'(1)>[s_n\sqrt{|\ln(s_n)|}\,]^{-1}\big\}$$

$$\le\int_0^\Delta\lim_{\varepsilon\downarrow 0}\operatorname*{ess\,sup}_{u<y<u+\varepsilon}\mathbf{P}\big\{\xi'(1)>x\,\big|\,\xi(1)=y\big\}\,f_{\xi(1)}(y)\;dx$$

$$+\limsup_{n\to\infty}\operatorname*{ess\,sup}_{y\in U}\mathbf{E}\big\{[\xi'(1)^+]^\varrho\;\big|\;\xi(1)=y\big\}\,f_{\xi(1)}(y)\int_\Delta^{[s_n\sqrt{|\ln(s_n)|}\,]^{-1}}\frac{dx}{x^\varrho}$$

$$+\limsup_{n\to\infty}1/\sqrt{|\ln(s_n)|}$$

$$\begin{cases}\to\displaystyle\int_0^\infty\lim_{\varepsilon\downarrow 0}\operatorname*{ess\,sup}_{u<y<u+\varepsilon}\mathbf{P}\big\{\xi'(1)>x\;\big|\;\xi(1)=y\big\}f_{\xi(1)}(y)\;dx & \text{as }\ \Delta\to 0\\ <\infty\qquad\text{for each}\quad\Delta>0\end{cases}.\quad\square$$

Proof of (v). When (4.7) holds, one can replace the estimates of $s_n^{-1}\,\mathbf{P}\big\{\xi'(1)>[s_n\sqrt{|\ln(s_n)|}\,]^{-1}\big\}$ in (4.11) [that uses $\mathbf{E}\{[\xi'(1)^+]\}<\infty$] with the estimates

$$\limsup_{s\downarrow 0}\int_{[s\sqrt{|\ln(s)|}\,]^{-1}}^\infty\mathbf{P}\big\{\xi'(1)>x\;\big|\;\xi(1)=u+xs\big\}\,f_{\xi(1)}(u+xs)\;dx$$

$$\le \limsup_{s\downarrow 0}\ \operatorname*{ess\,sup}_{y>u} \mathbf{E}\big\{[\xi'(1)^+]^\varrho \,\big|\, \xi(1)=y\big\}\, f_{\xi(1)}(y) \int_{[s\sqrt{|\ln(s)|}\,]^{-1}}^{\infty} \frac{dx}{x^\varrho}$$

$$= 0. \quad \square$$

Proof of (vi) and (vii). The statement (vi) follows from (4.6). The theorem of Scheffé (1947) shows that the hypothesis of (vi) holds when that of (vii) does. $\square$

Example. Let $\{\xi(t)\}_{t\in\mathbb{R}}$ be a mean-square differentiable standardized and stationary Gaussian process with covariance function r. Then we have

$$\mathbf{E}\bigg\{\Big|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\Big|^\varrho \,\bigg|\, \xi(1)=y\bigg\} f_{\xi(1)}(y)$$

$$\le 2^\varrho \bigg[\mathbf{E}\bigg\{\Big|\frac{r(s)\xi(1)-\xi(1-s)-s\xi'(1)}{s}\Big|^\varrho\bigg\} + \Big|\frac{1-r(s)}{s}\Big|^\varrho |y|^\varrho\bigg] f_{\xi(1)}(y)$$

[since $r(s)\xi(1)-\xi(1-s)-s\xi'(1)$ is independent of $\xi(1)$], so that (4.4) holds [since $r'(0)=0$]. Further (4.7) holds [since $\xi'(1)$ is independent of $\xi(1)$]. Hence Theorem 3 (ii), (v) and (vii) show that Rices's formula holds when $\xi'(1)$ is non-degenerate.

5. Extremes of Gaussian processes in $\mathbb{R}^n$ with strongly dependent components. Let $\{X(t)\}_{t\in\mathbb{R}}$ be a separable $n|1$-matrix-valued centered stationary and two times mean-square differentiable Gaussian process with covariance function

(5.1)

$$\mathbb{R}\ni t \curvearrowright R(t) = \mathbf{E}\{X(s)X(s+t)^T\} = I+r't+\tfrac{1}{2}r''t^2+\tfrac{1}{6}r'''t^3+\tfrac{1}{24}r^{(iv)}t^4+o(t^4) \in \mathbb{R}_{n|n}$$

as $t\to 0$ (where I is the identity in $\mathbb{R}_{n|n}$): It is no loss to assume that $R(0)$ is diagonal, since this can be achieved by a rotation that does not affect the statement of Theorem 4 below. Further, writing $\mathcal{P}$ for the projection on the subspace of $\mathbb{R}_{n|1}$ spanned by eigenvectors of $R(0)$ with maximal eigenvalue, the probability that a local extrema of $X(t)$ is not generated by $\mathcal{P}X(t)$ is asymptotically neglible [e.g., Albin (1992, Section 5)]. Hence it is sufficient to study $\mathcal{P}X(t)$, and assuming (without loss by a scaling argument) unit maximal eigenvalue, we arrive at $R(0)=I$.

To ensure that $\{X(t)\}_{t\in[0,1]}$ does not have periodic components we require that

$$\mathbf{E}\{X(t)X(t)^T|X(0)\} = I-R(t)R(t)^T \quad \text{is non-singular.}$$

Writing $S_n\equiv\big\{z\in\mathbb{R}_{n|1} : z^Tz=1\big\}$, this requirement becomes

(5.2) $\qquad \inf_{x\in S_n} x^T[I-R(t)R(t)^T]x > 0 \qquad$ for each choice of $t\in(0,1]$.

The behaviour of local extremes of the χ^2-process $\xi(t) \equiv X(t)^T X(t)$ will then depend on the behaviour of $I - R(t)R(t)^T$ as $t \to 0$.

Sharpe (1978) and Lindgren (1980) studied extremes of χ^2-processes with independent component processes. Lindgren (1989) and Albin (1992, Section 5) gave extensions to the general 'non-degenerate' case when (5.1) and (5.2) hold and

$$\text{(5.3)} \qquad \liminf_{t\to 0} t^{-2} \inf_{x\in S_n} x^T [I - R(t)R(t)^T] x > 0.$$

Using (5.1) in an easy calculation, (5.3) is seen to be equivalent with the requirement that the covariance matrix $r'r' - r''$ of $(X'(1) \,|\, X(1))$ is non-singular.

We shall treat χ^2-processes which do not have to be 'non-degenerate' in the sense (5.3). More specifically, we study local extremes of the process $\{\xi(t)\}_{t\in[0,1]}$, satisfying (5.1) and (5.2), under the very weak additional assumption that

$$\text{(5.4)} \qquad \liminf_{t\to 0} t^{-4} \inf_{x\in S_n} x^T [I - R(t)R(t)^T] x > 0.$$

Now let $\kappa^n(\cdot)$ denote the $(n-1)$-dimensional Hausdorff measure over $\mathbb{R}_{n|1}$.

Theorem 4. *Consider a centered and stationary Gaussian process* $\{X(t)\}_{t\in\mathbb{R}}$ *in* $\mathbb{R}_{n|1}$, $n \geq 2$, *satisfying* (5.1)-(5.2) *and* (5.4). *Then we have*

$$\text{(5.5)} \quad \lim_{u\to\infty} \frac{1}{\sqrt{u}\,\mathbf{P}\{X(1)^T X(1) > u\}} \mathbf{P}\Big\{ \sup_{t\in[0,1]} X(t)^T X(t) > u \Big\}$$
$$= \int_{z\in S_n} \frac{1}{\sqrt{2\pi}} \sqrt{z^T [r'r' - r''] z}\, \frac{d\kappa^n(z)}{\kappa^n(S_n)}.$$

Proof. The density g and distribution function G of $\xi(t) \equiv X(t)^T X(t)$ satisfy

$$\text{(5.6)} \quad \begin{cases} g(x) = 2^{-n/2}\,\Gamma(\tfrac{n}{2})^{-1}\, x^{(n-2)/2}\, \mathrm{e}^{-x/2} & \text{for } x > 0 \\ 1 - G(u) \sim 2^{-(n-2)/2}\,\Gamma(\tfrac{n}{2})^{-1}\, u^{(n-2)/2}\, \mathrm{e}^{-x/2} & \text{as } u \to \infty \end{cases}.$$

Consequently (1.1) holds with $\hat{u} = \infty$, $F(x) = 1 - \mathrm{e}^{-x}$ and $w(u) = 2$.

To verify Assumption 1 we take $\eta' = \zeta' = 2X(1)^T X'(1)$ and $\eta'' = 4X(1)^T (r'' - r'r')X(1)$. Since r' is skew-symmetric, so that $X(1)^T r' X(1) = 0$, we then obtain

$$\text{(5.7)} \qquad \xi(1+t) - \xi(1) - t\eta'$$
$$= \|X(1+t) - R(t)^T X(1)\|^2 + 2X(1)^T R(t) \big[X(1+t) - R(t)^T X(1) - tX'(1) - tr'X(1)\big]$$
$$- 2X(1)^T [I - R(t)]\,[tX'(1) + tr'X(1)]$$

$$- 2X(1)^T[I - R(t)R(t)^T]X(1).$$

Here the fact that $\mathbf{E}\{X(1+t)X'(1)^T\} = R'(-t) = -R'(t)^T$ implies that

(5.8) $\quad X(1+t) - R(t)^T X(1)$ and $X'(1) + r'X(1)$ are independent of $X(1)$.

In the notation $\mathbf{Var}\{X\} \equiv \mathbf{E}\{XX^T\}$, (5.1) implies that

$$(5.9) \qquad \begin{cases} \mathbf{Var}\big\{X(1+t) - R(t)^T X(1)\big\} = (r'r' - r'')t^2 + o(t^2) \\ \mathbf{Var}\big\{[I - R(t)][tX'(1) + tr'X(1)]\big\} = (r')^T(r'r' - r'')r't^4 + o(t^4) \end{cases}.$$

Using (5.1) together with the elementary fact that $r^{(iii)}$ is skew-symmetric we get

(5.11) $\quad \mathbf{Var}\big\{R(t)[X(1+t) - R(t)^T X(1) - tX'(1) - tr'X(1)]\big\}$

$\le 2\mathbf{Var}\big\{R(t)[X(1+t) - X(1) - tX'(1)]\big\} + 2\mathbf{Var}\big\{R(t)[(I - R(t)^T)X(1) - tr'X(1)]\big\}$

$= \frac{1}{4}(r^{(iv)} + r''r'')t^4 + o(t^4).$

Noting the easily established fact that $r''r' = r'r''$ we similarly obtain

$$(5.10) \qquad X(1)^T[I - R(t)R(t)^T]X(1) = X(1)^T\big[(r'r' - r'')t^2 + O(t^4)\big]X(1).$$

Putting $q(u) \equiv (1 \vee u)^{-1/2}$ and using (5.7)-(5.11) we conclude that

(5.12) $\quad \mathbf{P}\left\{\left|\frac{\xi(1+qt) - u}{wt} - \frac{\xi(1) - u}{wt} - \frac{qt\,\zeta'}{wt}\right| > \lambda \,\middle|\, \xi(1) > u\right\}$

$\le \sum_{i=1}^{n} \mathbf{P}\left\{\left|\Big(X(1+qt) - R(qt)^T X(1)\Big)_i\right| > \sqrt{\frac{\lambda t}{2n}}\right\}$

$\quad + \mathbf{P}\left\{\|X(1)\| > \sqrt{\frac{\lambda u}{4t}}\right\} \Big/ \mathbf{P}\{\|X(1)\|^2 > u\}$

$\quad + \sum_{i=1}^{n} \mathbf{P}\left\{\left|\Big(R(qt)[X(1+qt) - R(qt)^T X(1) - qtX'(1) - qtr'X(1)]\Big)_i\right| > \sqrt{\frac{\lambda t^3}{4nu}}\right\}$

$\quad + \sum_{i=1}^{n} \mathbf{P}\left\{\left|\Big([I - R(qt)][qtX'(1) + qtr'X(1)]\Big)_i\right| > \sqrt{\frac{\lambda t^3}{4nu}}\right\}$

$\quad + \mathbf{P}\left\{\left|X(1)^T[I - R(qt)^T R(qt)]X(1)\right| > \frac{\lambda t}{4}\right\} \Big/ \mathbf{P}\{\|X(1)\|^2 > u\}$

$\le K_1 \exp\{-K_2(u/t)\}$

$$\leq K_3\,|t|^2 \qquad \text{for} \quad u \geq u_0 \quad \text{and} \quad |t| \leq t_0,$$

for some constants $K_1, K_2, K_3, u_0, t_0 > 0$. Hence (0.6) holds with $C = K_3 \wedge t_0^{-2}$. By application of Theorem 1 it thus follows that Assumption 3 holds.

Choosing $D(u) \equiv (1 \vee u)^{\frac{5}{16}}$ and using (5.7)-(5.11) as in (5.12) we obtain

$$\mathbf{P}\left\{\left|\frac{\xi(1+qt)-u}{w\,t} - \frac{\xi(1)-u}{w\,t} - \frac{qt\,\eta'}{w\,t} - \frac{q^2t^2\,\eta''}{2\,w\,t}\right| > \lambda \,\middle|\, \xi(1) > u\right\}$$

$$\leq \sum_{i=1}^{n} \mathbf{P}\left\{\left|\Big(X(1+qt) - R(qt)^T X(1)\Big)_i\right| > \sqrt{\frac{\lambda t}{2n}}\right\}$$

$$+\,\mathbf{P}\{\|X(1)\| > \sqrt{2u}\} \,/\, \mathbf{P}\{\|X(1)\|^2 > u\}$$

$$+\sum_{i=1}^{n} \mathbf{P}\left\{\left|\Big(R(qt)\,[X(1+qt) - R(qt)^T X(1) - qt\,X'(1) - qt\,r'X(1)]\Big)_i\right| > \frac{\lambda t}{4\sqrt{2nu}}\right\}$$

$$+\sum_{i=1}^{n} \mathbf{P}\left\{\left|\Big([I - R(qt)]\,[qt\,X'(1) + qt\,r'X(1)]\Big)_i\right| > \frac{\lambda t}{4\sqrt{2nu}}\right\}$$

$$+\,\mathbf{P}\left\{\left|X(1)^T\Big(I - R(qt)R(qt)^T + [r'' - r'r']q^2t^2\Big)X(1)\right| > \frac{\lambda t}{4}\right\}\Big/\mathbf{P}\{\|X(1)\|^2 > u\}$$

$$\leq K_4\, \exp\{-K_5 u^{3/8}\} \qquad \text{for} \quad |t| \leq D(u) \quad \text{and} \quad u \quad \text{sufficiently large},$$

for some constants $K_4, K_5 > 0$. It follows immediately that (3.1) holds.

Now assume that $r'r' - r'' = 0$, so that the right-hand side of (5.5) is zero. Then (5.9) and (5.10) together with (5.12) show that (0.6) holds for some $q(u) = o(u^{-1/2})$ that is non-increasing [and thus satisfies (1.3)]. Hence Theorem 1 and Proposition 2 imply that also the left hand side of (5.5) is zero. We shall therefore in the sequel without loss assume that $r'r' - r'' \neq 0$.

In order to see how (5.5) follows from Theorem 2, we note that

$$f_{\xi(1),\xi'(1)}(y,z) = \int_{x \in S_n} \frac{1}{\sqrt{8\pi\, x^T[r'r' - r'']x}}\, \exp\left\{-\frac{z^2}{8y\, x^T[r'r' - r'']x}\right\} \frac{d\kappa^n(x)}{\kappa^n(S_n)}\; g(y).$$

[See the proof of Theorem 6 in Section 7 for an explanation of this formula.] Sending $u \to \infty$, and using (5.6), we therefore readily obtain

$$\int_0^\infty \left[\int_0^z f_{\xi(1),\xi'(1)}(u + xqy, z)\, dy\right] dz \sim \int_{z \in S_n} \frac{2}{\sqrt{2\pi}} \sqrt{u\, z^T[r'r' - r'']z}\; \frac{d\kappa^n(z)}{\kappa^n(S_n)}\; g(u)$$

$$\sim \int_{z \in S_n} \frac{1}{\sqrt{2\pi}} \sqrt{z^T[r'r' - r'']z}\; \frac{d\kappa^n(z)}{\kappa^n(S_n)}\; \frac{1 - G(u)}{q(u)}$$

for $x>0$. In view of (0.3), (5.5) thus follows from the conclusion of Theorem 2.

In order to verify the hypothesis of Theorem 2 it remains to prove (3.2). To that end we note that [cf. (5.7)]

$$
\begin{aligned}
(5.13) \qquad \xi(1+t)-\xi(1) = \big\|X(1+t)-R(t)^T X(1)\big\|^2 \\
+\,2\,X(1)^T R(t)\,[X(1+t)-R(t)^T X(1)] \\
-\,2\,X(1)^T\,[I-R(t)R(t)^T]\,X(1).
\end{aligned}
$$

Conditional on the value of $X(1)$, we here have [recall (5.8)]

$$
\begin{aligned}
(5.14) \quad & \mathbf{Var}\Big\{X(1)^T R(t)\,[X(1+t)-R(t)^T X(1)]\Big\} \\
&= X(1)^T R(t)\,[I-R(t)^T R(t)]\,R(t)^T X(1) \\
&= X(1)^T\Big([I-R(t)R(t)^T]-[I-R(t)R(t)^T]\,[I-R(t)R(t)^T]^T\Big)X(1) \\
&\leq X(1)^T\,[I-R(t)R(t)^T]\,X(1).
\end{aligned}
$$

Since [by (5.4)] $x^T[I-R(t)R(t)^T]x \geq K_6 t^4$ for $x\in S_n$ and $|t|\leq 1$, for some constant $K_6>0$, we have $X(1)^T[I–R(qt)R(qt)^T]X(1) \geq K_6 u^{1/4}$ when $\|X(1)\|^2>u$ and $t\in[D(u),q(u)]$. Using this estimate together with (5.13) and (5.14) we get

$$
\begin{aligned}
&\mathbf{P}\{\xi(1-q(u)t)>u\,|\,\xi(1)>u\} \\
&\quad\leq 2\,\mathbf{P}\Big\{\xi(1-q(u)t)>\xi(1)>u\,\Big|\,\xi(1)>u\Big\} \\
&\quad\leq 2\,\mathbf{P}\Big\{\big\|X(1+t)-R(t)^T X(1)\big\|^2 > X(1)^T\,[I-R(t)R(t)^T]\,X(1)\,\Big|\,\|X(1)\|^2>u\Big\} \\
&\qquad+2\,\mathbf{P}\Big\{\mathrm{N}(0,1)>\sqrt{X(1)^T\,[I-R(t)R(t)^T]\,X(1)}\,\Big|\,\|X(1)\|^2>u\Big\} \\
&\quad\leq 2\sum_{i=1}^{n}\mathbf{P}\bigg\{\Big|\Big(X(1+t)-R(t)^T X(1)\Big)_i\Big|>\sqrt{\frac{K_6 u^{1/4}}{2n}}\bigg\} \\
&\qquad+2\,\mathbf{P}\big\{\mathrm{N}(0,1)>\sqrt{K_6 u^{1/4}}\big\}.
\end{aligned}
$$

It follows that (3.2) holds. □

6. Extremes of totally skewed moving averages of α-stable motion. Given an $\alpha\in(1,2]$ we write $Z\in S_\alpha(\sigma,\beta)$ when Z is a strictly α-stable random variable with characteristic function

$$
(6.1) \qquad \mathbf{E}\big\{\exp[i\theta Z]\big\} = \exp\big\{-|\theta|^\alpha\sigma^\alpha[1-i\beta\tan(\tfrac{\pi\alpha}{2})\mathrm{sign}(\theta)]\big\} \qquad \text{for} \quad \theta\in\mathbb{R}.
$$

Here the scale $\sigma=\sigma_Z\geq 0$ and the skewness $\beta=\beta_Z\in[-1,1]$ are parameters. Further we let $\{M(t)\}_{t\in\mathbb{R}}$ denote an α-stable motion that is totally skewed to the left, that is, $M(t)$ denotes a Lévy-process that satisfies $M(t)\in S_\alpha(|t|^{1/\alpha},-1)$.

For a function $g:\mathbb{R}\to\mathbb{R}$ we write $g^{\langle\alpha\rangle}\equiv\operatorname{sign}(g)\,|g|^\alpha$. Further we define

$$\langle g\rangle\equiv\int_{-\infty}^{\infty}g(x)\,dx\quad\text{for } g\in\mathbb{L}^1(\mathbb{R})\qquad\text{and}\qquad\|g\|_\alpha\equiv\langle|g|^\alpha\rangle^{1/\alpha}\quad\text{for } g\in\mathbb{L}^\alpha(\mathbb{R}).$$

It is well known that [e.g., Samorodnitsky and Taqqu (1994, Proposition 3.4.1)]

$$\text{(6.2)}\qquad\int_{\mathbb{R}}g\,dM\;\in\;S_\alpha\big(\|g\|_\alpha,-\langle g^{\langle\alpha\rangle}\rangle/\|g\|_\alpha^\alpha\big)\qquad\text{for}\quad g\in\mathbb{L}^\alpha(\mathbb{R}).$$

Given a non-negative function $f\in\mathbb{L}^\alpha(\mathbb{R})$ that satisfies (6.4)-(6.8) below, we shall study the behaviour of local extremes for the moving average process

$$\text{(6.3)}\qquad\xi(t)\;=\;\text{separable version of}\quad\int_{-\infty}^{\infty}f(t-x)\,dM(x)\qquad\text{for } t\in\mathbb{R}.$$

Note that by (6.2) we have $\xi(t)\in S_\alpha\big(\|f\|_\alpha,-1\big)$, so that $\{\xi(t)\}_{t\in\mathbb{R}}$ is a stationary α-stable process that is totally skewed to the left [i.e., $\beta_{\xi(t)}=-1$ for each $t\in\mathbb{R}$].

Example. When $\alpha=2$ the process $\xi(t)$ is centered Gaussian with covariance function $r_\xi(\tau)=\mathbf{E}\{\xi(t)\xi(t+\tau)\}=\int_{-\infty}^{\infty}f(\tau+x)f(x)\,dx$. Conversly a centered Gaussian process $\xi(t)$ such that $r_\xi\in\mathbb{L}^1(\mathbb{R})$ has the representation (6.3) in law where $f(x)=_{\mathbb{L}^2(\mathbb{R})}\int_{-\infty}^{\infty}\mathrm{e}^{ix\theta}\sqrt{R(\theta)}\,d\theta$ and $R(\theta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-i\theta\tau}r_\xi(\tau)\,d\tau$ is the spectral density.

In Albin (1993) we determined the behaviour of local extremes $\mathbf{P}\big\{\sup_{t\in[0,1]}X(t)>u\big\}$ as $u\to\infty$ for the totally skewed α-stable motion itself $X(t)=M(t)$, and for the Ornstein-Uhlenbeck process $X(t)=\mathrm{e}^{-t/\alpha}M(\mathrm{e}^t)=_d\int_{-\infty}^{t}\mathrm{e}^{-(t-x)/\alpha}\,dM(x)$. These processes are easy to deal with since multivariate probability estimates can be reduced to univariate ones using independence of increments for $M(t)$.

In Albin (1998^+) we derived upper and lower bounds (that are not asymptotically sharp) for extremes of a quite general totally skewed α-stable process.

Here we will determine the exact tail-behaviour of local extremes for the process $\xi(t)$ in (6.3) by means of verifying that it satisfies the hypothesis of Corollary 1.

Note that traditional approaches to local extremes [e.g., Albin (1990, Section 2)] rely on weak convergence as outlined in $\boxed{1}$. They cannot be applied to the process $\xi(t)$ because the behaviour of conditional totally skewed α-stable distributions is not known. [There is also an approach by Cramér (1965, 1966) based on the first and second moment of the number of upcrossings: This approach cannot be used because nothing is known about such a second moment for an α-stable process.]

Remark 1. Extremes for α-stable processes $\{X(t)\}_{t\in K}$ that are not totally skewed [so that $\beta_{X(t)} > -1$ for some $t \in K$] are well understood: By the works of de Acosta (1977, 1980), Samorodnitsky (1988), and Rosinski and Samorodnitsky (1993)

$$\text{the limit} \qquad \lim_{u\to\infty} u^\alpha \mathbf{P}\big\{\sup_{t\in K} X(t) > u\big\} \equiv L \qquad \text{exists.}$$

Further $L > 0$ iff. K is non-empty, while $L < \infty$ iff. $\{X(t)\}_{t\in K}$ is a.s. bounded.

In order to verify the hypothesis of Corollary 1 we have to assume that

$$\text{(6.4)} \qquad \begin{cases} f & \text{is absolutely continuous with derivative} \quad f' \in \mathbb{L}^\alpha(\mathbb{R}) \\ f' & \text{is absolutely continuous with derivative} \quad f'' \in \mathbb{L}^\alpha(\mathbb{R}) \end{cases}.$$

Further we require that f 'obeys its Taylor expansion' in the sence(s) that

$$\text{(6.5)} \qquad \lim_{t\to 0} \Big\langle t^{-2}\left[f(t+\cdot) - f(\cdot) - t\, f'(\cdot)\right]^2 f(\cdot)^{\alpha-2}\Big\rangle \qquad = 0$$

$$\text{(6.6)} \qquad \lim_{t\to 0} \Big\| t^{-2}\left[f(t+\cdot) - f(\cdot) - t\, f'(\cdot) - \tfrac{1}{2}t^2 f''(\cdot)\right]\Big\|_\alpha = 0 .$$

To ensure that Taylor expansions of $\xi(t)$ are totally skewed we impose the condition

$$\text{(6.7)} \qquad \sup_{(t,x)\in(0,\delta]\times\mathbb{R}} \big|f(t+x) - f(x) - t\, f'(x)\big| \big/ \left[t^2 f(x)\right] < \infty \qquad \text{for some} \quad \delta > 0.$$

Finally non-degeneracy is guaranteed by requiring (recall Minkowski's inequality)

$$\text{(6.8)} \qquad \inf_{s,t\in[0,1]} \big(2\,\|f\|_\alpha - \|f(t+\cdot) + f(s+\cdot)\|_\alpha\big) > 0.$$

The conditions (6.4)-(6.8) may seem restrictive, and can be tedious to actually verify. Nevertheless, they do in fact hold for virtually every non-pathological two times differentiable function f in $\mathbb{L}^\alpha(\mathbb{R})$ that is locally bounded away from zero.

Example. The functions $f(x) = \mathrm{e}^{-x^2}$ and $f(x) = (1+x^2)^{-1}$ satisfy (6.4)-(6.8).

Theorem 5. *Consider the process $\xi(t)$ given by (6.3) where $f \in \mathbb{L}^\alpha(\mathbb{R})$ is non-negative with $\|f\|_\alpha > 0$. If (6.4)-(6.8) hold, then there exists a constant $C_\alpha > 0$ (that depends on α only) such that the hypothesis of Corollary 1 holds with*

$$\eta' = \int_{\mathbb{R}} f'(t-\cdot)\, dM, \qquad w(u) = C_\alpha (1\vee u)^{-1/(\alpha-1)} \qquad \textit{and} \qquad q(u) = (1\vee u)^{-\alpha/[2(\alpha-1)]}.$$

Proof. According to e.g., Samorodnitsky and Taqqu (1994, p. 17) we have

$$\text{(6.9)} \qquad \mathbf{P}\big\{S_\alpha(\sigma,-1) > u\big\} \sim A_\alpha \Big(\frac{u}{\sigma}\Big)^{-\alpha/2(\alpha-1)} \exp\Big\{-B_\alpha\Big(\frac{u}{\sigma}\Big)^{\alpha/(\alpha-1)}\Big\} \quad \text{as} \quad u\to\infty$$

for some constants $A_\alpha, B_\alpha > 0$. Thus (1.1) holds with $\hat{u} = \infty$, $F(x) = 1 - \mathrm{e}^{-x}$ and w as defined above. Further (6.6) combines with Hölder's inequality to show that

$$\lim_{t\to 0} \left\langle t^{-2}\left[f(t+\cdot) - f(\cdot) - t\,f'(\cdot) - \tfrac{1}{2}t^2 f''(\cdot)\right] f(\cdot)^{\alpha-1}\right\rangle = 0. \tag{6.10}$$

Consider an α-stable process $\{X(t)\}_{t\in T}$ with constant scale $\sigma_{X(t)} = \sigma_0 > 0$ and skewness $\beta_{X(t)} = -1$ for $t \in T$. By Albin (1998$^+$, the equation following Eq. 5.8) there exist constants $K_1, K_2, u_1 > 0$ (that depend on α and σ_0 only) such that

$$\mathbf{P}\{X(s) > u \,|\, X(t) > u\} \leq K_1 \exp\bigl\{-K_2 u^{\alpha/(\alpha-1)} \rho_X(s,t)\bigr\} \quad \text{for } s,t \in T \text{ and } u \geq u_1,$$

where $\rho_X(s,t) = 2\sigma_0 - \sigma_{X(t)+X(s)}$. Further, by Albin (1998$^+$, Eq. 10.8) (6.5) and (6.10) imply that there are constants $K_3(\alpha, f) > 1$ and $\Delta_1(\alpha, f) > 0$ such that

$$K_3^{-1}(t-s)^2 \;\leq\; \rho_\xi(s,t) \;\leq\; K_3\,(t-s)^2 \qquad \text{for} \quad s,t \in [0,1] \quad \text{with} \quad |t-s| \leq \Delta_1.$$

However, by (6.8) and the continuity of $\rho_\xi(s,t)$ [which is an easy consequence of (6.6)], there must then also exist a constant $K_4 \geq K_3$ such that

$$K_4^{-1}(t-s)^2 \;\leq\; \rho_\xi(s,t) \;\leq\; K_4\,(t-s)^2 \qquad \text{for} \quad s,t \in [0,1].$$

Adding things up we therefore conclude that

$$\mathbf{P}\bigl\{\xi(1-qt) > u \,\big|\, \xi(1) > u\bigr\} \leq K_1 \exp\bigl\{-K_2 K_4^{-1} u^{\alpha/(\alpha-1)} (qt)^2\bigr\} = K_1 \exp\bigl\{-K_2 K_4^{-1} t^2\bigr\}$$

for $u \geq 1 \vee u_1$, which of course ensures that Assumption 2 holds.

To check Assumption 1 we take $\Delta_2 > 0$ and note that $|1+x|^\alpha \leq 1 + \alpha x + K_5 |x|^\alpha$ for $x \in \mathbb{R}$, for some constant $K_5 > 0$. For the scale $\Sigma(\pm, q, t)$ of the variables

$$\xi(1) \,\pm\, q\,\bigl[\xi(1-qt) - \xi(1) + qt\xi'\bigr] \big/ (qt)^2 \,\in\, S_\alpha\bigl(\Sigma(\pm,q,t), -1\bigr), \qquad |t| \leq \Delta_2,$$

[where the skewness really is -1 for u large by (6.7)], we therefore have

$$\begin{aligned}
\Sigma(\pm,q,t)^\alpha &\leq \|f(1-\cdot)\|_\alpha^\alpha \,\pm\, \alpha\,q \left\langle \left[\frac{f(1-qt-\cdot) - f(1-\cdot) + qt\,f'(1-\cdot)}{(qt)^2}\right] f(1-\cdot)^{\alpha-1}\right\rangle \\
&\qquad + K_5\, q^\alpha \left\| \frac{f(1-qt-\cdot) - f(1-\cdot) + qt\,f'(1-\cdot)}{(qt)^2} \right\|_\alpha^\alpha \\
&\leq \|f\|_\alpha^\alpha \,\pm\, \alpha\,q \left\langle \left[\frac{f(-qt+\cdot) - f(\cdot) + qt\,f'(\cdot) - \frac{1}{2}(qt)^2 f''(\cdot)}{(qt)^2}\right] f(\cdot)^{\alpha-1}\right\rangle \\
&\qquad + \alpha\,q\,\langle \tfrac{1}{2} f'' f^{\alpha-1}\rangle
\end{aligned}$$

$$+\ K_5\, 2^\alpha\, q^\alpha \left\| \frac{f(-qt+\cdot) - f(\cdot) + qt\, f'(\cdot) - \frac{1}{2}(qt)^2 f''(\cdot)}{(qt)^2} \right\|_\alpha^\alpha$$

$$+\ K_5\, 2^\alpha\, q^\alpha\, \|\tfrac{1}{2} f''\|_\alpha^\alpha$$

$$= \|f\|_\alpha^\alpha + O(q) \qquad \text{uniformly for} \quad |t| \le \mathit{\Delta}_2$$

[where we used (6.6) and (6.10)]. Noting that $u\, O(q) = O(w/q)$ we thus conclude

$$\mathbf{P}\left\{ \pm\left(\frac{\xi(1-qt)-u}{w\,|t|} - \frac{\xi(1)-u}{w\,|t|} + \frac{qt\,\xi'}{w\,|t|} \right) > \lambda \,\middle|\, \xi(1) > u \right\}$$

$$\le \frac{1}{\mathbf{P}\{\xi(1)>u\}}\, \mathbf{P}\left\{ \xi(1) \pm q\, \frac{\xi(1-qt) - \xi(1) + qt\,\xi'}{(qt)^2} > u + \frac{\lambda\, w}{q\,|t|} \right\}$$

$$\le \frac{1}{\mathbf{P}\{\xi(1)>u\}}\, \mathbf{P}\left\{ S_\alpha\big(\|f\|_\alpha, -1\big) > \left(u + \frac{\lambda\, w}{q\,|t|} \right) \big(1 - O(q)\big) \right\}$$

$$\le \frac{1}{\mathbf{P}\{\xi(1)>u\}}\, \mathbf{P}\left\{ S_\alpha\big(\|f\|_\alpha, -1\big) > u + \frac{\lambda\, w}{2\,q\,|t|} \right\} \quad \text{for} \quad |t| \le \mathit{\Delta}_3 \quad \text{and} \quad u \text{ large,}$$

for some $\mathit{\Delta}_3 \in (0, \mathit{\Delta}_2]$. Now Assumption 1 follows readily from (6.9). □

7. Crossing intensities for α-stable processes in $\mathbb{R}^n$. Let $\{M_1(t)\}_{t\in\mathbb{R}}, \ldots, \{M_n(t)\}_{t\in\mathbb{R}}$ be independent α-stable Lévy motions with $\alpha \in (0,2) \setminus \{1\}$ and skewness $\beta = -1$ (cf. Section 6). Consider the $\mathbb{R}^n$-valued strictly α-stable process $\{X(t)\}_{t\in\mathbb{R}}$ with independent component processes given by

(7.1) $\quad X_i(t) = \text{separable version of} \quad \int_{-\infty}^{\infty} f_i(t;x)\, dM_i(x) \quad \text{where} \quad f_i(t;\cdot) \in \mathbb{L}^\alpha(\mathbb{R})$:

Each non-pathological (separable in probability) strictly α-stable process in $\mathbb{R}$ has this representation in law [e.g., Samorodnitsky and Taqqu (1994, Theorem 13.2.1)].

We assume that $X(t)$ is stationary, which [cf. (6.2)] means that the integrals

$$\int_{-\infty}^{\infty} \left| \sum_{j=1}^{m} \theta_j f_i(t_j+h;x) \right|^\alpha dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \left(\sum_{j=1}^{m} \theta_j f_i(t_j+h;x) \right)^{\langle\alpha\rangle} dx$$

do not depend on $h \in \mathbb{R}$ for $t_1, \ldots, t_m, \theta_1, \ldots, \theta_m \in \mathbb{R}$, $m \in \mathbb{N}$ and $i = 1, \ldots, n$.

Rice's formula for $\mathbb{R}$-valued α-stable processes have been proved in subsequently greater generality by Marcus (1989), Adler et al. (1993), Michna and Rychlik (1995), Adler and Samorodnitsky (1997), and Albin and Leadbetter (1998$^+$).

We will consider the multi-dimensional case with outcrossing of $X(t)$ through the boundary of an open and bounded set $A \subseteq \mathbb{R}^n$ that is starshaped with center at $a \in A$. This means that for each $x \in A$ we have $a + \lambda(x-a) \in A$ for $\lambda \in [0,1]$.

Example. An open nonempty and bounded convex set A is starshaped, and each $a \in A$ works as center for A.

We shall require that the starshaped set A is smooth in the sense that

(7.2) the map $r_A(x) \equiv \sup\{\lambda > 0 : \lambda^{-1} \bullet x \notin A\}$ is continuously differentiable,

where $\gamma \bullet x \equiv a + \gamma(x - a)$: The treatment is restricted to smooth starshaped sets in order to avoid uninteresting measure geometric technicalities: See Remark 4 for a discussion of how outcrossings through more general surfaces can be dealt with.

Remark 2. Lindgren (1980, Section 4) proved asymptotic Poisson character for the visits of Gaussian processes in $\mathbb{R}^n$ outside large starshaped sets.

Following Albin and Leadbetter (1998$^+$, Section 6) we assume that there exists a power $\nu \in [\alpha, 2) \cap (1, \alpha + 1)$ such that

$$(7.3)\quad \lim_{s \downarrow 0} \Big\langle \big|s^{-1}\big[f_i(1;\cdot) - f_i(1-s;\cdot) - s f_i'(1,\cdot)\big]\big|^{\nu} |f_i(1;\cdot)|^{\alpha-\nu} \Big\rangle = 0 \quad \text{for } i = 1, \ldots, n,$$

for some maps $f_1'(1;\cdot), \ldots, f_n'(1;\cdot) \in \mathbb{L}^{\alpha}(\mathbb{R})$ that satisfy

$$(7.4)\qquad \big\langle |f_i'(1,\cdot)|^{\nu} |f_i'(1,\cdot)|^{\alpha-\nu} \big\rangle < \infty \qquad \text{for } i = 1, \ldots, n.$$

In the case when $\alpha < 1$ we need an additional assumption:

$$(7.5)\qquad \lim_{\varepsilon \downarrow 0} \big\| \sup\nolimits_{|s|,|t| \leq \varepsilon} |f_i(1-s;\cdot) - f_i(1-t;\cdot)| \big\|_{\alpha} = 0 \qquad \text{for } i = 1, \ldots, n, \quad \text{if } \alpha < 1.$$

When $\alpha > 1$ we may take $\nu = \alpha$: This makes (7.4) void while (7.3) reduces to

$$\lim\nolimits_{s \downarrow 0} \big\| s^{-1}\big[f_i(1;\cdot) - f_i(1-s;\cdot) - s f_i'(1,\cdot)\big] \big\|_{\alpha} = 0 \qquad \text{for } i = 1, \ldots, n.$$

To ensure that $(X(1), X'(1))$ is absolutely continuous we use the requirement

$$(7.6)\qquad \int_{\{x \in \mathbb{R} : \gamma f_i(1;x) + \lambda f_i'(1;x) \neq 0\}} dx = 0 \quad \Rightarrow \quad \gamma = \lambda = 0 \quad \text{for } i = 1, \ldots, n:$$

By Samorodnitsky and Taqqu (1994, Lemma 5.1.1), (7.6) implies that the characteristic function of $(X_i(1), X_i'(1))$ is integrable, so that there exists a continuous and bounded density. [In fact, (7.6) is equivalent with absolute continuity.]

Example. For a moving average we have $f_i(t;\cdot) = f_i(t - \cdot)$, so that (7.3)-(7.6) hold when each f_i is continuously differentiable with non-empty compact support.

Let $\kappa^n(\cdot)$ denote the (n–1)-dimensional Hausdorff measure over $\mathbb{R}^n$ and $n(A;x) = \nabla r_A(x)/|\nabla r_A(x)|$ a normal to the boundary $\partial A = \{x \in \mathbb{R}^n : r_A(x) = 1\}$ of A.

Theorem 6. *Consider a stationary process $\{X(t)\}_{t\in\mathbb{R}}$ in $\mathbb{R}^n$, $n \geq 2$, given by* (7.1)*. Assume that the maps $f_1(t;\cdot),\ldots,f_n(t;\cdot) \in \mathbb{L}^\alpha(\mathbb{R})$ satisfy* (7.3)-(7.6)*, and let $X'(1) = \big(\int_{\mathbb{R}} f_1'(1;\cdot)\,dM_1,\ldots,\int_{\mathbb{R}} f_n'(1;\cdot)\,dM_n\big)$. Then the outcrossing intensity of $X(t)$ through the boundary ∂A of a starshaped set A satisfying* (7.2) *is given by*

Belyaev's formula:

$$\mu(\partial A) = \int_{\partial A} \mathbf{E}\Big\{ \big(n(A;x)\cdot X'(1)\big)^+ \,\Big|\, X(1)=x \Big\}\, f_{X(1)}(x)\, d\kappa^n(x)$$

$$= \int_{(x,y)\,\in\,\partial A\times\mathbb{R}^n} \big(n(A;x)\cdot y\big)^+ f_{X(1),X'(1)}(x,y)\, d\kappa^n(x)dy.$$

Remark 3. The statment of Theorem 6 follows from Belyaev (1968, Theorem 2) in the particular case when the component processes $X_i(t)$ are continuously differentiable a.s. with a derivative $X_i'(t)$ such that

(7.7) $$\lim_{t\downarrow 0} t^{-1}\mathbf{P}\big\{\sup_{|s|\leq t} |X_i'(1+s)-X_i'(1)|>\varepsilon\big\} = 0 \quad \text{for each } \varepsilon>0.$$

However, (in a way due to "heavy tails") (7.7) is a very restrictive requirement for an α-stable process. In fact, existing sufficient criteria in the literature for $X_i(t)$ just to possess continuously differentiable paths a.s. require that

(7.8) $$\lim_{s\to 0} s^{-1}\,\|f_i'(1+s;\cdot)-f_i'(1;\cdot)\|_\alpha^\alpha = 0$$

[e.g., Samorodnitsky and Taqqu (1984, Section 12.2)]. Further it is an elementary exercise to see that (7.8) is necessary for (7.7). Obviously, our requirement (7.3) is much weaker than (7.8), which thus in turn is only a necessary requirement for Belyaev's theorem to apply. [Is is known that sample paths of $X_i(t)$, $t\in\mathbb{R}$, cannot be more regular than the function $f_i(t;x)$, $t\in\mathbb{R}$; cf. Rosiński (1986).]

Proof of Theorem 6. Clearly $\mu(\partial A)$ equals the upcrossing intensity of the level 1 by the process $\xi(t)=r_A(X(t))$. It follows from well-known criteria for continuity of α-stable processes that (7.3) and (7.5) imply a.s. continuity of the processes $X_i(t)$ in the cases when $\alpha>1$ and $\alpha<1$, respectively. [See Albin and Leadbetter (1998^+, Section 6) for a detailed argument.] Hence $\xi(t)$ is continuous a.s.

Now define $\xi'(1) = X'(1)\cdot\nabla r_A(X(t))$. By the coarea formula [see e.g., Federer

(1969, Theorem 3.2.12), or (the less immense) Federer (1978, Section 3)] we have

$$\begin{aligned}
&\mathbf{P}\{\xi(1)\le u,\ \xi'(1)\le z\}\\
&= \int_{\{x\,:\,r_A(x)\le u\}}\int_{\{y\,:\,y\cdot\nabla r_A(x)\le z\}} f_{X(1),X'(1)}(x,y)\,dxdy\\
&= \int_{\hat u=0}^{\hat u=u}\int_{\hat z=0}^{\hat z=z}\int_{\{x\,:\,r_A(x)=\hat u\}}\int_{\{y\,:\,y\cdot\nabla r_A(x)=\hat z\}} f_{X(1),X'(1)}(x,y)\,\frac{d\kappa^n(x)\,d\kappa^n(y)}{|\nabla r_A(x)|^2}\,d\hat u d\hat z.
\end{aligned}$$

Since $r_A(\lambda\bullet x)=\lambda r_A(x)$ and $\nabla r_A(\lambda\bullet x)=\nabla r_A(x)$ for $\lambda>0$, it follows that

$$\begin{aligned}
f_{\xi(1),\xi'(1)}(u,z) &= \int_{\{x\,:\,r_A(x)=u\}}\int_{\{y\,:\,y\cdot\nabla r_A(x)=z\}} f_{X(1),X'(1)}(x,y)\,\frac{d\kappa^n(x)\,d\kappa^n(y)}{|\nabla r_A(x)|^2}\\
&= (uz)^{n-1}\int_{x\in\partial A}\int_{\{y\,:\,y\cdot\nabla r_A(x)=1\}} f_{X(1),X'(1)}(u\bullet x,\,zy)\,\frac{d\kappa^n(x)\,d\kappa^n(y)}{|\nabla r_A(x)|^2}.
\end{aligned}$$

Hence continuity of $f_{X(1),X'(1)}$ [cf. (7.6)] implies that of $f_{\xi(1),\xi'(1)}$. Since $f_{X(1)}$ is continuous and locally bounded away from zero, we similarly conclude that

$$f_{\xi(1)}(u) = u^{n-1}\int_{x\in\partial A} f_{X(1)}(u\bullet x)\,\frac{d\kappa^n(x)}{|\nabla r_A(x)|}$$

is continuous and locally bounded away from zero.

Choose a power $\varrho\in(1,\nu)$. According to Albin and Leadbetter [1998$^+$, Corollary 2 (i)] there exists a constant $K_{\alpha,\nu}>0$ (that depends on α and ν only) such that

$$\text{(7.9)}\qquad \mathbf{E}\{|\textstyle\int_{\mathbb{R}} g_i\,dM_i|^\varrho\,|\,X_i(1)=x_i\} \le \dfrac{K_{\alpha,\nu}\,\langle|g_i|^\nu|f_i(1;\cdot)|^{\alpha-\nu}\rangle^{\varrho/\nu}}{[(\nu-\varrho)\,\|f_i(1;\cdot)\|_\alpha^{\alpha+1-\nu}\,f_{X_i(1)}(x_i)]^{\varrho/\nu}}.$$

Also note that $R_A\equiv\sup_{x\in\mathbb{R}^n}|\nabla r_A(x)|<\infty$ [since $\nabla r_A(\lambda\bullet x)=\nabla r_A(x)$], so that

$$\text{(7.10)}\quad r_A(x+y) = r_A(x)+y\cdot\nabla r_A(x)+|y|\sigma(x,y)\quad\text{where}\quad \limsup_{|y|\downarrow 0}\sup_{x\in\mathbb{R}}\sigma(x,y)=0.$$

In order to deduce Belyaev's formula from Theorem 3 (ii), (v) and (vii), it remains to prove (4.4) and (4.7): Given a $\delta_1>0$ we can by (7.10) find a $\delta_2>0$ such that $\sigma(x,y)\le\delta_1$ when $|y|\le\delta_1$. Consequently (7.9) and (7.10) show that

$$\begin{aligned}
&\mathbf{E}\Big\{\Big|\frac{\xi(1)-\xi(1-s)-s\xi'(1)}{s}\Big|^\varrho\,\Big|\,\xi(1)=y\Big\}\,f_{\xi(1)}(y)\\
&= \int_{r_A(x)=y}\mathbf{E}\Big\{\Big|\frac{r_A(X(1))-r_A(X(1-s))-sX'(1)\cdot\nabla r_A(X(1))}{s}\Big|^\varrho\,\Big|\,X(1)=x\Big\}\\
&\qquad\qquad\times f_{X(1)}(x)\,\frac{d\kappa^n(x)}{|\nabla r_A(x)|}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{r_A(x)=y} \mathbf{E}\bigg\{ \Big| \frac{[X(1)-X(1-s)-s\,X'(1)]\cdot \nabla r_A(X(1))}{s} \\
&\quad + \frac{|X(1)-X(1-s)|\,\sigma(X(1),X(1-s)-X(1))}{s} \Big|^{\varrho} \,\Big|\, X(1)=x \bigg\}\, f_{X(1)}(x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|} \\
&\le (2nR_A)^{\varrho} \sum_{i=1}^{n} \int_{r_A(x)=y} \mathbf{E}\bigg\{ \Big| \frac{X_i(1)-X_i(1-s)-s\,X_i'(1)}{s} \Big|^{\varrho} \,\Big|\, X_i(1)=x_i \bigg\} \\
&\qquad\qquad \times f_{X(1)}(x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|} \\
&\quad + (2n\delta_1)^{\varrho} \sum_{i=1}^{n} \int_{r_A(x)=y} \mathbf{E}\bigg\{ \Big| \frac{X_i(1)-X_i(1-s)}{s} \Big|^{\varrho} \,\Big|\, X_i(1)=x_i \bigg\}\, f_{X(1)}(x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|} \\
&\quad + \bigg(\frac{2\sqrt{n}}{\delta_2}\bigg)^{\varrho} \sum_{i=1}^{n} \int_{r_A(x)=y} \mathbf{E}\big\{ |X_i(1)-X_i(1-s)| \,\big|\, X_i(1)=x_i \big\}\, f_{X(1)}(x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|} \\
&\le (2nR_A)^{\varrho} \sum_{i=1}^{n} \int_{\partial A} \frac{K_{\alpha,\nu} \big\langle |s^{-1}[f_i(1;\cdot)-f_i(1-s;\cdot)-s f_i'(1;\cdot)]|^{\nu}\, |f_i(1;\cdot)|^{\alpha-\nu} \big\rangle^{\varrho/\nu}\, y^{n-1}}{\big[(\nu-\varrho)\, \|f_i(1;\cdot)\|_{\alpha}^{\alpha+1-\nu} f_{X_i(1)}((y\bullet x)_i)\big]^{\varrho/\nu}} \\
&\qquad\qquad \times f_{X(1)}(y\bullet x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|} \\
&\quad + (2n\delta_1)^{\varrho} \sum_{i=1}^{n} \int_{\partial A} \frac{K_{\alpha,\nu} \big\langle |s^{-1}[f_i(1;\cdot)-f_i(1-s;\cdot)|^{\nu}\, |f_i(1;\cdot)|^{\alpha-\nu} \big\rangle^{\varrho/\nu}\, y^{n-1}}{\big[(\nu-\varrho)\, \|f_i(1;\cdot)\|_{\alpha}^{\alpha+1-\nu} f_{X_i(1)}((y\bullet x)_i)\big]^{\varrho/\nu}} \\
&\qquad\qquad \times f_{X(1)}(y\bullet x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|} \\
&\quad + \bigg(\frac{2\sqrt{n}}{\delta_2}\bigg)^{\varrho} \sum_{i=1}^{n} \int_{\partial A} \frac{K_{\alpha,\nu} \big\langle |f_i(1;\cdot)-f_i(1-s;\cdot)|^{\nu}\, |f_i(1;\cdot)|^{\alpha-\nu} \big\rangle^{\varrho/\nu}\, y^{n-1}}{\big[(\nu-\varrho)\, \|f_i(1;\cdot)\|_{\alpha}^{\alpha+1-\nu}\, f_{X_i(1)}((y\bullet x)_i)\big]^{\varrho/\nu}} \\
&\qquad\qquad \times f_{X(1)}(y\bullet x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|}.
\end{aligned}
$$

Using (7.3) and (7.4), (4.4) now follows sending $\delta_1 \downarrow 0$ and noting that by basic properties of α-stable densities [e.g., Albin and Leadbetter (1998^+, Equation 4.15)]

$$
y^{n-1} f_{X_i(1)}((y\bullet x)_i)^{1-\varrho/\nu} \prod_{j\neq i} f_{X_j(1)}((y\bullet x)_j) \qquad \text{is bounded for } \; y>0 \; \text{ and } \; x\in\partial A.
$$

The fact that (4.7) holds follows in a similar way using (7.4) and observing that

$$
\begin{aligned}
&\mathbf{E}\big\{ |\xi'(1)|^{\varrho} \,\big|\, \xi(1)=y \big\}\, f_{\xi(1)}(y) \\
&\le (nR_A)^{\varrho} \sum_{i=1}^{n} \int_{\partial A} \frac{K_{\alpha,\nu} \big\langle |f_i'(1;\cdot)|^{\nu}\, |f_i(1;\cdot)|^{\alpha-\nu} \big\rangle^{\varrho/\nu}\, y^{n-1}}{[(\nu-\varrho)\, \|f_i(1;\cdot)\|_{\alpha}^{\alpha+1-\nu} f_{X_i(1)}((y\bullet x)_i)]^{\varrho/\nu}}\, f_{X(1)}(y\bullet x)\, \frac{d\kappa^n(x)}{|\nabla r_A(x)|}. \;\square
\end{aligned}
$$

Remark 4. Consider a level surface $S=\{x:g(x)=1\}$ of a function $g(x)$ which is $\mathbb{C}^1$ in an open neighborhood of S. For the process $\xi(t)=g(X(t))$ we have

$$f_{\xi(1),\xi'(1)}(u,z)=\int_{\{x\,:\,g(x)=u\}}\int_{\{y\,:\,y\cdot\nabla g(x)=z\}} f_{X(1),X'(1)}(x,y)\,\frac{d\kappa^n(x)\,d\kappa^n(y)}{|\nabla g(x)|^2}$$

and

$$f_{\xi(1)}(u)=\int_{\{x\,:\,g(x)=u\}} f_{X(1)}(x)\,\frac{d\kappa^n(x)}{|\nabla g(x)|}$$

for u in an open neighborhood U of 1 and $z\in\mathbb{R}$, provided that

$$\int_{\{x\,:\,g(x)=u\}}\frac{d\kappa^n(x)}{|\nabla g(x)|}<\infty \quad \text{for } u\in U. \tag{7.11}$$

If g is extended to a suitable Lipschitz $\mathbb{C}^1$-function on $\mathbb{R}^n$ [satisfying a global version of (7.11)], then the proof of Theorem 6 works as before (replacing r_A with g), except for the proofs of continuity for $f_{\xi(1),\xi'(1)}(u,z)$ and $f_{\xi(1)}(u)$:

Assume that the functions

$$\frac{f_{X(1),X'(1)}(x,y)\,\nabla g(x)}{|\nabla g(x)|^2} \quad \text{and} \quad \frac{f_{X(1)}(x)\,\nabla g(x)}{|\nabla g(x)|^2} \quad \text{are Lipschitz in } x \tag{7.12}$$

for x in a neighborhood of S. Then the Gauss-Green formula [e.g., Federer (1969, Theorem 4.5.6), or Federer (1978, Section 4)] shows that

$$f_{\xi(1),\xi'(1)}(u,z)-f_{\xi(1),\xi'(1)}(v,z)=\int_{\{x\,:\,u<g(x)\le v\}}\mathrm{div}_x\frac{f_{X(1),X'(1)}(x,y)\,\nabla g(x)}{|\nabla g(x)|^2}\,dx$$

for $u<v$ in a neighborhood of 1. It follows that $f_{\xi(1),\xi'(1)}(u,z)$ is continuous in u. Similarly continuity of $f_{\xi(1)}(u)$ follows using the fact that

$$f_{\xi(1)}(u)-f_{\xi(1)}(v)=\int_{\{x\,:\,u<g(x)\le v\}}\mathrm{div}\frac{f_{X(1)}(x)\,\nabla g(x)}{|\nabla g(x)|^2}\,dx.$$

By application of Theorem 3 (v), (7.11) and (7.12) thus imply that

$$\mu(S)=\int_{(x,y)\,\in\,S\times\mathbb{R}^n}\Big(\tfrac{\nabla g(x)}{|\nabla g(x)|}\cdot y\Big)^+ f_{X(1),X'(1)}(x,y)\;d\kappa^n(x)\,dy.$$

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