Functional convergence in distribution of quadratic variations for a large class of Gaussian processes: application to a time deformation model

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Abstract

We are interested in the functional convergence in distribution of the process of quadratic variations taken along a regular partition for a large class of Gaussian processes indexed by $[0, 1]$, including the standard Wiener process as a particular case. This result is applied to the estimation of a time deformation that makes a non-stationary Gaussian process stationary.

keywords: estimation; quadratic variation process

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1 Introduction.

We are interested in the functional convergence in distribution of the quadratic variation process for a large class of Gaussian processes indexed by $[0, 1]$. This convergence result is obtained assuming smoothness of the covariance function outside the diagonal.

The quadratic variations are first introduced by Lévy (1940) who shows that if Z is the standard Wiener process on $[0, 1]$, then almost surely ($a.s.$) as $n \to \infty$

$$\sum_{k=1}^{2^n} [Z(k/2^n) - Z((k-1)/2^n)]^2 \longrightarrow 1. \quad (1)$$

Baxter (1956) and further Gladyshev (1961) generalise this result to a large class of Gaussian processes.

Guyon and León (1989) introduce an important generalisation of these variations for a Gaussian stationary non-differentiable process with covariance function $r(u) = 1 - u^\beta L(u)$, where $\beta \in]0, 2[$ and L is a slowly varying function at

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zero. Let H be a real function. The H-variation of process Z indexed by $[0,1]$ is defined by

$$\sum_{k=1}^{n} H\left(\frac{Z(k/n)-Z((k-1)/n)}{(2(r(0)-r(1/n)))^{1/2}}\right).$$

Guyon and León (1989) study the convergence in distribution of the H-variations, suitably normalised, for non-differentiable Gaussian processes.

The generalisation of these variations for Gaussian fields is studied in Guyon (1987) and León and Ortega (1989). Another generalisation for non-stationary Gaussian processes and quadratic variations along curves is done in Adler and Pyke (1993).

For Gaussian process Z with stationary increments, Istas and Lang (1997) define general quadratic variations, substituting a general discrete difference operator to the simple difference $Z(k/n)-Z((k-1)/n)$. They give conditions on the discrete difference and on the covariance function of Z that ensure the *a.s.* convergence and the asymptotic normality of these quadratic variations, suitably normalised. Then, they use these quadratic variations to estimate the Hölder index of the process.

For non-stationary Gaussian processes, with increments stationary or not, we give a general result concerning the functional asymptotic normality of the process of the quadratic variations which corresponds to the linear interpolation of the points $(p/n, V_n(p/n))$ with $V_n(p/n) = \sum_{k=1}^{p}[Z(k/n)-Z((k-1)/n)]^2$, $p = 1,2,\dots,n$.

We apply this result to the estimation of a time deformation for non-stationary models of the form

$$Z(x) = \delta(\Phi(x)), \quad x \in [0,1], \tag{2}$$

where δ is a stationary random process with known covariance and the deformation Φ is deterministic, bijective and continuously differentiable in $[0,1]$, as is its inverse. Model (2) appears first in Sampson and Guttorp (1992) and gives a class of non-stationary random fields. In the one-dimensional case, Perrin (1997) gives different methods for estimating Φ. Perrin and Senoussi (1998) exhibit a characterisation for processes satisfying (2). When the process Z under study is Gaussian, we show it is possible to construct a non-parametric estimator of Φ from one realisation of Z observed at discrete times k/n, $k = 0,1,\dots,n$, and give the asymptotic normality of this estimator as the number of observations n grows to ∞. Testing the stationarity of Z, *i.e.* testing if Φ is the identity or not, is also considered.

The paper is structured as follows. Section 2 sets up notations, assumptions and definitions and describes the quadratic variation process. Section 3 contains the related problem: the estimation of a time deformation. In section 4 we give our main result, theorem 4.1, dealing with the functional convergence in distribution of the process of the quadratic variations. This result is applied in section 5 to the related problem described in section 3.

2 The process of quadratic variations

Let $Z = \{Z(x), x \in [0,1]\}$ be a real-valued centred Gaussian random process with covariance function $r(x,y)$. Assume that

(A1) r is continuous in $[0,1]^2$ and its second derivatives are uniformly bounded for $x \neq y$.

Assumption **(A1)** is satisfied for a large class of processes including: *(i)* processes with independent increments such that $x \longmapsto r(x,x)$ is of class C^2; *(ii)* stationary processes with rational spectral densities. For other examples see Baxter (1956).

(A1) gives the following property dealing with the regularity of the sample paths of Z (*e.g.* Neveu (1980), p. 93).

Property 2.1 *For any constant $\gamma \in]0, 1/2[$, a.s.*

$$\lim_{h \to 0} h^{-\gamma} \sup_{|y-x| \leq h} |Z(y) - Z(x)| = 0$$

It follows that Z is continuous (in the sense that a.s. Z has continuous sample paths).

We denote using $r^{(m,m')}$ the m, m'-partial derivative of r with respect to x and y and set

$$\begin{aligned} D^-(x) = r^{(0,1)}(x, x^-) &= \lim_{y \nearrow x} r^{(0,1)}(x,y), \quad x \in]0,1], \\ D^+(x) = r^{(0,1)}(x, x^+) &= \lim_{y \searrow x} r^{(0,1)}(x,y), \quad x \in [0,1[. \end{aligned}$$

These limits exist because the second order derivatives of r are uniformly bounded. Then we have the following result.

Lemma 2.1 *Assume (A1). Then D^- and D^+ are continuous in $[0,1]$.*

Proof. We set $\Delta = \left\{(x,y) \in [0,1]^2, x \neq y\right\}$. First, we show that D^- is continuous in $]0,1]$. We have the inequality for all $x \in]0,1]$ and all $h > 0$

$$\begin{aligned} |D^-(x+h) - D^-(x)| &\leq |r^{(0,1)}(x+h, (x+h)^-) - r^{(0,1)}(x+h, x^-)| \\ &+ |r^{(0,1)}(x+h, x^-) - r^{(0,1)}(x, x^-)| \\ &\leq h \left(\sup_{(x,y) \in \Delta} |r^{(0,2)}(x,y)| + \sup_{(x,y) \in \Delta} |r^{(1,1)}(x,y)| \right) \end{aligned}$$

from which we deduce the continuity of D^- in $]0,1]$.

Then, it remains to prove that $\lim_{x \searrow 0} D^-(x)$ is finite. For that, we write

$$\lim_{x \searrow 0} D^-(x) = \lim_{x \searrow 0} \lim_{h \searrow 0} \left(r^{(0,1)}(x, x-h) - r^{(0,1)}(x, 0) + r^{(0,1)}(x, 0) \right).$$

We have $|r^{(0,1)}(x,x-h)-r^{(0,1)}(x,0)| \leq |x-h| \sup_{(x,y)\in\Delta} |r^{(0,2)}(x,y)|$ and $r^{(0,1)}(x,0)$ is piecewise continuous in $[0,1]$. Therefore, $\lim_{x\searrow 0} D^-(x) = r^{(0,1)}(0^+,0)$.

A similar treatment gives D^+ continuous in $[0,1[$ and $\lim_{x\nearrow 1} D^+(x) = r^{(0,1)}(1^-,1)$.

□

Let us now introduce the singularity function α of Z

$$\alpha(x) = D^-(x) - D^+(x), \quad x \in [0,1].$$

It follows directly from lemma 2.1 that α is uniformly continuous in $[0,1]$. Note that the existence of the first derivative of $r(x,y)$ at $x = y$ is not assumed. Indeed, the existence of this derivative would make $\alpha(x) = 0$ for all $x \in [0,1]$.

Let n be a positive integer. We set for $k = 1, 2, \dots, n$

$$\Delta Z_k = Z(k/n) - Z((k-1)/n).$$

Let $\Pi_n(1) = \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\right\}$ be the regular partition of $[0,1]$ at constant scale $1/n$. We denote by $\lfloor nx \rfloor$ the greatest integer smaller than or equal to nx. For $x \in [0,1]$, we define the quadratic variations $V_n(x)$ of Z along $\Pi_n(x) = \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots \leq \frac{\lfloor nx \rfloor}{n}\right\}$ as follows

$$V_n(x) = \sum_{k=1}^{\lfloor nx \rfloor} (\Delta Z_k)^2 .$$

When $\lfloor nx \rfloor = 0$, we set $\sum_{k=1}^{0} (\Delta Z_k)^2 = 0$. The process $V_n = \{V_n(x), x \in [0,1]\}$ is a random element of the space of functions that are right-continuous and have left-hand limits. The following definition allows us to consider a continuous version of V_n.

Definition 2.1 *The process of the quadratic variations of* Z, $v_n = \{v_n(x), x \in [0,1]\}$, *is defined by*

$$\begin{cases} v_n(x) &= V_n(x) + (nx - \lfloor nx \rfloor)\left(\Delta Z_{\lfloor nx \rfloor + 1}\right)^2, \quad x \in [0,1[, \\ v_n(1) &= V_n(1). \end{cases}$$

Thus, v_n corresponds to the linear spline with mesh $\Pi_n(1)$ that interpolates points $(p/n, V_n(p/n))$, $p = 1, 2, \dots, n$. From now on, we no longer distinguish the case $x \in [0,1[$ from the case $x = 1$ in the definition of v_n.

3 The estimation of a time deformation.

Let Z be a centred Gaussian process with correlation function r satisfying **(A1)**. Consider the problem of estimating the function $\Phi : [0,1] \longmapsto \mathbb{R}$ from one realisation of Z observed at discrete times k/n, $k = 0, 1, \ldots, n$, in the model

$$Z(x) = \delta(\Phi(x)), \quad x \in [0,1], \tag{3}$$

where δ is a stationary random process with known correlation and the deformation Φ satisfies the following assumption

(B) Φ is bijective and continuously differentiable in [0,1], as is its inverse.

Model (3) is equivalent to the following

$$r(x,y) = R(\Phi(y) - \Phi(x)) \tag{4}$$

where R is the correlation function of δ. Note that if (Φ, R) is a solution to (4), then for any $b > 0$ and $c \in \mathbb{R}$, $(\tilde{\Phi}, \tilde{R})$ with $\tilde{\Phi}(x) = b\Phi(x) + c$ and $\tilde{R}(u) = R(u/b)$ is a solution as well. Thus, without loss of generality we may impose that

$$\Phi(0) = 0 \text{ and } \Phi(1) = 1. \tag{5}$$

Consequently, the stationary correlation function R is uniquely determined as

$$R(u) = r(0, \Phi^{-1}(u)) \text{ and } R(-u) = R(u).$$

It follows from **(A1)** and **(B)** that the stationary correlation function $R(u)$ is continuous and differentiable for u different from 0, and that its derivative to the left and its derivative to the right at 0 exist and satisfy

$$\begin{aligned} R^{(1)}(0^-) &= D^-(x)/\Phi^{(1)}(x), \\ R^{(1)}(0^+) &= D^+(x)/\Phi^{(1)}(x), \end{aligned}$$

where $\Phi^{(1)}$ denotes the derivative of Φ and $R^{(1)}$ the derivative of R. Thus, the singularity function α satisfies the following relation

$$\alpha(x) = 2R^{(1)}(0^-)\Phi^{(1)}(x).$$

Finally, under conditions (5), we get for all $x \in [0,1]$

$$\Phi(x) = \frac{\int_0^x \alpha(u)du}{\int_0^1 \alpha(u)du}. \tag{6}$$

Therefore, the estimation of Φ requires an estimation of the primitive of α: $x \longmapsto \int_0^x \alpha(u)du$. Once an estimator of Φ will be built, we will give the functional asymptotic normality of this estimator suitably normalised as the number of observations n grows to ∞.

4 Functional convergence in distribution

Consider the Gaussian vector

$$W_{\lfloor nx \rfloor} = \left(Z\left(\frac{0}{n}\right), Z\left(\frac{1}{n}\right), \dots, Z\left(\frac{\lfloor nx \rfloor}{n}\right) \right)^t.$$

Its covariance matrix is

$$\Sigma_{\lfloor nx \rfloor} = \begin{pmatrix} r(0/n,0/n) & r(0/n,1/n) & \cdots & r(0/n,\lfloor nx \rfloor/n) \\ & r(1/n,1/n) & \cdots & r(1/n,\lfloor nx \rfloor/n) \\ & & \ddots & \vdots \\ & & & r(\lfloor nx \rfloor/n,\lfloor nx \rfloor/n) \end{pmatrix}.$$

Let $L_{\lfloor nx \rfloor}$ be a matrix with $\lfloor nx \rfloor$ rows and $\lfloor nx \rfloor + 1$ columns defined as follows

$$L_{\lfloor nx \rfloor} = \begin{pmatrix} -1 & +1 & 0 & \cdots & 0 \\ 0 & -1 & +1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -1 & +1 \end{pmatrix}.$$

The covariance matrix of the centred Gaussian vector $\left(\Delta Z_1, \Delta Z_2, \dots, \Delta Z_{\lfloor nx \rfloor}\right)^t$ is $L_{\lfloor nx \rfloor}\Sigma_{\lfloor nx \rfloor}L^t_{\lfloor nx \rfloor}$. We denote the eigenvalues of this matrix by $\lambda_{1,\lfloor nx \rfloor}, \lambda_{2,\lfloor nx \rfloor}, \dots, \lambda_{\lfloor nx \rfloor,\lfloor nx \rfloor}$ and $P_{\lfloor nx \rfloor} = \left((P_{\lfloor nx \rfloor})_{k,j}\right)$ is the orthogonal matrix such that $Diag(\lambda_{k,\lfloor nx \rfloor}) = P^t_{\lfloor nx \rfloor}L_{\lfloor nx \rfloor}\Sigma_{\lfloor nx \rfloor}L^t_{\lfloor nx \rfloor}P_{\lfloor nx \rfloor}$. Then the Gaussian variables defined by

$$\chi_{k,\lfloor nx \rfloor} = (\lambda_{k,\lfloor nx \rfloor})^{-1/2} \sum_{j=1}^{\lfloor nx \rfloor} (P_{\lfloor nx \rfloor})_{j,k}\Delta Z_j, \quad k = 1, 2, \dots, \lfloor nx \rfloor,$$

are independent reduced Gaussian variables so that

$$V_n(x) = \sum_{k=1}^{\lfloor nx \rfloor} (\Delta Z_k)^2 = \sum_{k=1}^{\lfloor nx \rfloor} \lambda_{k,\lfloor nx \rfloor}\chi^2_{k,\lfloor nx \rfloor}, \tag{7}$$

where the $\chi^2_{k,\lfloor nx \rfloor}$ are independent chi-square variables with one degree of freedom. The following theorem gives a uniform upper bound for $\lambda_{\lfloor nx \rfloor}$, the maximum of the eigenvalues $\left(\lambda_{k,\lfloor nx \rfloor}\right)_{k=1,2,\dots,\lfloor nx \rfloor}$.

Lemma 4.1 *Assume (A1). Then*

$$\sup_{x\in[0,1]} \lambda_{\lfloor nx \rfloor} = O(1/n).$$

Proof. For $(j,k) \in [1,2,\dots,n]^2$, let $a_{j,k} = E(\Delta Z_j \Delta Z_k)$, then

$$a_{j,k} = r\left(\frac{j}{n}, \frac{k}{n}\right) + r\left(\frac{j-1}{n}, \frac{k-1}{n}\right) - r\left(\frac{j}{n}, \frac{k-1}{n}\right) - r\left(\frac{j-1}{n}, \frac{k}{n}\right). \tag{8}$$

To give an upper bound for $\lambda_{\lfloor nx \rfloor}$, first see that $\left(\lambda_{k,\lfloor nx \rfloor}\right)_{k=1,2,\ldots,\lfloor nx \rfloor}$ are the eigenvalues of the matrix $(a_{j,k})_{1 \le j,k \le \lfloor nx \rfloor}$ and then use the following inequality (*e.g.* Horn and Johnson, p. 33).

$$\lambda_{\lfloor nx \rfloor} \le \max_{1 \le k \le \lfloor nx \rfloor} \sum_{j=1}^{\lfloor nx \rfloor} |a_{j,k}| \le \max_{1 \le k \le n} \sum_{j=1}^{n} |a_{j,k}| = O(1/n).$$

Let A be a bound for the three quantities $|r^{(2,0)}(x,y)|$, $|r^{(1,1)}(x,y)|$ and $|r^{(0,2)}(x,y)|$ in the range $0 \le x \ne y \le 1$. Using for $r(x,y)$ a Taylor series expansion with remainder, it can easily be shown that $j \ne k$ implies

$$|a_{j,k}| \le \frac{3A}{n^2}. \tag{9}$$

Also for $k = 1,2,\ldots,n$

$$a_{k,k} = \frac{1}{n}\left(D^-\left(\frac{k}{n}\right) - D^+\left(\frac{k}{n}\right)\right) + O(1/n^2) = \frac{1}{n}\alpha\left(\frac{k}{n}\right) + O(1/n^2) \tag{10}$$

where $O(1/n^2)$ is independent of k. The function α being uniformly bounded in $[0,1]$, we have

$$|a_{k,k}| = O(1/n), \quad k = 1,2,\ldots,n, \tag{11}$$

where $O(1/n)$ is independent of k.

□

Define the assumption for the singularity function α

(A2) α has a bounded first derivative in $[0,1]$.

For instance, assumption **(A2)** is satisfied for: *(i)* processes with independent increments such that $x \longmapsto r(x,x)$ is of class C^2; *(ii)* stationary processes with rational spectral densities.

We know from Baxter (1956) that $V_n(x)$ is a consistent estimator of $\int_0^x \alpha(u)du$. The following lemma gives an upper bound for the bias of $V_n(x)$.

Lemma 4.2 *Assume (A1)-(A2). Then, the following holds*

$$\sup_{x \in [0,1]} \left| E(V_n(x)) - \int_0^x \alpha(u)du \right| = O(1/n).$$

Proof. We have by definition of $V_n(x)$

$$E(V_n(x)) = \sum_{k=1}^{\lfloor nx \rfloor} a_{k,k}. \tag{12}$$

It follows from (10) that $\sum_{k=1}^{\lfloor nx\rfloor} a_{k,k} = \frac{1}{n}\sum_{k=1}^{\lfloor nx\rfloor}\alpha\left(\frac{k}{n}\right) + O(1/n)$. We have

$$\left|\frac{1}{n}\sum_{k=1}^{\lfloor nx\rfloor}\alpha\left(\frac{k}{n}\right) - \int_0^x \alpha(u)du\right| \leq \sum_{k=1}^{\lfloor nx\rfloor}\int_{\frac{(k-1)}{n}}^{\frac{k}{n}}\left|\alpha\left(\frac{k}{n}\right)-\alpha(u)\right|du + \int_{\frac{\lfloor nx\rfloor}{n}}^{x}|\alpha(u)|du.$$

Since α is continuous (lemma 2.1) and has a bounded first derivative in $[0,1]$ (from **(A2)**) we have the estimates $\sup_{x\in[0,1]}\int_{\frac{\lfloor nx\rfloor}{n}}^{x}|\alpha(u)|du = O(1/n)$ and $\sup_{x\in[0,1]}\sum_{k=1}^{\lfloor nx\rfloor}\int_{\frac{(k-1)}{n}}^{\frac{k}{n}}\left|\alpha\left(\frac{k}{n}\right)-\alpha(u)\right|du = O(1/n)$.

□

Set for all $x\in[0,1]$

$$\begin{cases} T_n(x) &= \sqrt{n}\left(V_n(x)-E(V_n(x))\right) \\ T(x) &= \int_0^x \alpha(u)dW(u) \end{cases}$$

and consider the centred Gaussian process $T=\{T(x), x\in[0,1]\}$ with covariance function $E\left(T(x)T(y)\right) = 2\int_0^{x\wedge y}\alpha^2(u)du$. Before giving our main theorem, we first establish two lemmas.

Lemma 4.3 *Assume (A1). Then, for any $p\in\mathbb{N}^\star$ and whenever $x_1, x_2, \dots,$ x_p all lie in $[0,1]$, $(T_n(x_1), T_n(x_2), \dots, T_n(x_p))$ converges in distribution to the finite-dimensional Gaussian variable $(T(x_1), T(x_2), \dots, T(x_p))$.*

Proof. *(i)* First, we show that, for all $x\in[0,1]$, $T_n(x)$ converges in distribution to $T(x)$. *(ii)* Then, we show that, for all $(x,y)\in[0,1]^2$, the 2-dimensional variable $(T_n(x), T_n(y))$ converges in distribution to $(T(x), T(y))$ by using a generalisation of Cramér-Wold theorem. *(iii)* Finally, we conclude that the convergence in distribution still holds for any finite-dimensional variable.
(i) Due to (7) we have for all $x\in[0,1]$

$$T_n(x) = \sqrt{n}\left(V_n(x)-E(V_n(x))\right) = \sqrt{n}\sum_{k=1}^{\lfloor nx\rfloor}\lambda_{k,\lfloor nx\rfloor}(\chi^2_{k,\lfloor nx\rfloor}-1).$$

First we show that the variance of $T_n(x)$ converges to $2\int_0^x \alpha^2(x)dx$ as $n\to\infty$. We have $Var(T_n(x)) = nVar(V_n(x)) = n(E(V_n^2(x)) - (EV_n(x))^2)$. Recalling the definition (8) of $a_{j,k}$, $(j,k)\in[1,2,\dots,n]^2$, we have

$$\begin{aligned} E\left(V_n(x)\right) &= \sum_{k=1}^{\lfloor nx\rfloor} a_{k,k}, \\ E\left(V_n^2(x)\right) &= 3\sum_{k=1}^{\lfloor nx\rfloor} a_{k,k}^2 + 2\sum_{k=1}^{\lfloor nx\rfloor}\sum_{j>k}\left(a_{k,k}a_{j,j} + 2a_{j,k}^2\right). \end{aligned}$$

the second equality coming from, for $(\xi_1, \xi_2, \xi_3, \xi_4)^t$ a centred Gaussian vector

$$E(\xi_1\xi_2\xi_3\xi_4) = E(\xi_1\xi_2)E(\xi_3\xi_4) + E(\xi_1\xi_3)E(\xi_2\xi_4) + E(\xi_1\xi_4)E(\xi_2\xi_3). \tag{13}$$

Therefore,

$$Var(T_n(x)) = 2n \sum_{k=1}^{\lfloor nx \rfloor} \sum_{j=1}^{\lfloor nx \rfloor} a_{j,k}^2 = 2n \sum_{k=1}^{\lfloor nx \rfloor} a_{k,k}^2 + 4n \sum_{k=1}^{\lfloor nx \rfloor} \sum_{j>k} a_{j,k}^2. \tag{14}$$

Using the estimates of $a_{k,k}$ in (10) and $a_{j,k}$ in (9), the second term on the right-hand side of (14) converges to 0 as $n \to \infty$ and

$$2n \sum_{k=1}^{\lfloor nx \rfloor} a_{k,k}^2 = \frac{2}{n} \sum_{k=1}^{\lfloor nx \rfloor} \alpha^2 \left(\frac{k}{n} \right) + o(1).$$

Since α is Riemann integrable in $[0,1]$, $2n \sum_{k=1}^{\lfloor nx \rfloor} a_{k,k}^2$ converges to $2 \int_0^x \alpha^2(x)dx$ as $n \to \infty$.

Then we show that the variables $X_{k,\lfloor nx \rfloor} = \sqrt{n}\lambda_{k,\lfloor nx \rfloor} \left(\chi_{k,\lfloor nx \rfloor}^2 - 1 \right)$ satisfy the conditions of the Lyapounov central limit theorem (theorem 27.3 in Billingsley (1995)). Indeed, for each n the variables $X_{k,\lfloor nx \rfloor}$ are independent, have finite variance and are centred. Moreover

$$E(|X_{k,\lfloor nx \rfloor}|^3) \leq 15 n^{3/2} \lambda_{k,\lfloor nx \rfloor}^3, \quad k = 1, 2, \ldots, \lfloor nx \rfloor.$$

We know that $s_n(x) = \sqrt{Var(T_n(x))}$ converges to $\sqrt{2 \int_0^x \alpha^2(x)dx}$ as $n \to \infty$ and, from lemma 4.1, that the maximum $\lambda_{\lfloor nx \rfloor}$ of the eigenvalues $\lambda_{k,\lfloor nx \rfloor}$ is a $O(1/n)$. Therefore, $\sum_{k=1}^{\lfloor nx \rfloor} E(|X_{k,\lfloor nx \rfloor}|^3) = O(1/\sqrt{n})$ and $\lim_{n\to\infty} \sum_{k=1}^{\lfloor nx \rfloor} \frac{E(|X_{k,\lfloor nx \rfloor}^3|)}{s_n^3(x)} = 0$. Thus, $T_n(x)/\sqrt{Var(T_n(x))}$ converges in distribution to a reduced Gaussian variable.

(ii) Consider now the 2-dimensional variable $(T_n(x), T_n(y))$. We now show that $(T_n(x), T_n(y))$ converges in distribution to $(T(x), T(y))$.

According to the Cramér-Wold theorem (*e.g.* Billingsley (1968)), it is equivalent to show that, for any real coefficients θ_1 and θ_2, the linear combination $\theta_1 T_n(x) + \theta_2 T_n(y)$ converges in distribution to $\theta_1 T(x) + \theta_2 T(y)$. The lemma 6 in Lang and Azaïs says that the Cramér-Wold theorem remains true when we restrict to $\theta_1 \geq 0$ and $\theta_2 \geq 0$. Suppose that $x \leq y$ and let θ_1 and θ_2 are positive. Then

$$\theta_1 T_n(x) + \theta_2 T_n(y) = \sqrt{n} \sum_{k=1}^{\lfloor ny \rfloor} \left(Y_k^2 - E\left(Y_k^2\right) \right)$$

where $Y_k = \sqrt{(\theta_1 + \theta_2)}\Delta Z_k$ if $k \leq \lfloor nx \rfloor$ and $Y_k = \sqrt{\theta_2}\Delta Z_k$ if $\lfloor nx \rfloor + 1 \leq k \leq \lfloor ny \rfloor$. The covariance matrix $(c_{j,k})$ of the centred Gaussian vector $(Y_1, Y_2, \ldots,$

$Y_{\lfloor ny \rfloor})^t$ is

$$\left\{ \begin{array}{lcll} c_{j,k} & = & (\theta_1+\theta_2)a_{j,k} & \text{if } \ j,k \leq \lfloor nx \rfloor \\ c_{j,k} & = & \sqrt{(\theta_1+\theta_2)}\sqrt{\theta_2}a_{j,k} & \text{if } \ j \leq \lfloor nx \rfloor \text{ and } k \geq 1+\lfloor nx \rfloor \\ c_{j,k} & = & \theta_2 a_{j,k} & \text{if } \ j,k \geq \lfloor nx \rfloor + 1 \end{array} \right. \tag{15}$$

As for (7), we have

$$\theta_1 T_n(x) + \theta_2 T_n(y) = \sqrt{n} \sum_{k=1}^{\lfloor ny \rfloor} \tau_{k,\lfloor ny \rfloor} \left(\chi^2_{k,\lfloor ny \rfloor} - 1 \right)$$

where $\tau_{1,\lfloor ny \rfloor}, \tau_{2,\lfloor ny \rfloor}, \dots, \tau_{\lfloor ny \rfloor,\lfloor ny \rfloor}$ are the eigenvalues of the covariance matrix $(c_{j,k})$. The maximum $\tau_{\lfloor ny \rfloor}$ of these eigenvalues satisfies

$$\tau_{\lfloor ny \rfloor} \leq \max_{1 \leq k \leq \lfloor nx \rfloor} \sum_{j=1}^{\lfloor nx \rfloor} |c_{j,k}|$$

It follows from (9), (11) and (15) that $\tau_{\lfloor ny \rfloor} = O(1/n)$. By a similar treatment as the one used in the previous point *(i)*, we show that $\dfrac{\theta_1 T_n(x) + \theta_2 T_n(y)}{\sqrt{Var(\theta_1 T_n(x) + \theta_2 T_n(y))}}$ converges in distribution to a reduced Gaussian variable.

It remains to identify the limit of $Var(\theta_1 T_n(x) + \theta_2 T_n(y))$ as $n \to \infty$. First $\theta_1^2 Var(T_n(x)) + \theta_2^2 Var(T_n(y))$ converges to $2\theta_1^2 \int_0^x \alpha^2(u)du + 2\theta_2^2 \int_0^y \alpha^2(u)du$ as $n \to \infty$; then we have the decomposition

$$Cov(T_n(x), T_n(y)) = Var(T_n(x)) + E\left(T_n(x)(T_n(y) - T_n(x))\right).$$

Using (13), we have

$$\begin{aligned} E\left(T_n(x)(T_n(y) - T_n(x))\right) &= n \left(\sum_{k=1}^{\lfloor nx \rfloor} \sum_{j=\lfloor nx \rfloor+1}^{\lfloor ny \rfloor} \left(E((\Delta Z_k)^2 (\Delta Z_j)^2) - a_{j,j} a_{k,k} \right) \right) \\ &= 2n \left(\sum_{k=1}^{\lfloor nx \rfloor} \sum_{j=\lfloor nx \rfloor+1}^{\lfloor ny \rfloor} a_{j,k}^2 \right) \end{aligned}$$

Making use of the estimate in (9), we obtain that $|E\left(T_n(x)(T_n(y) - T_n(x))\right)| = O(1/n)$ and $Var(\theta_1 T_n(x) + \theta_2 T_n(y))$ converges to $Var(\theta_1 T(x) + \theta_2 T(y))$ as $n \to \infty$.

(iii) Finally, to conclude that $(T_n(x_1), T_n(x_2), \dots, T_n(x_p))$ converges in distribution to $(T(x_1), T(x_2), \dots, T(x_p))$ we apply the following treatment

- first note that any linear combination $\theta_1 T_n(x_1) + \theta_2 T_n(x_2) + \cdots + \theta_p T_n(x_p)$ can be expressed as the difference of two positive linear combinations of quadratic variations;
- note also that any positive linear combination of quadratic variations is still a quadratic variation;

- therefore, any linear combination of quadratic variations reduces to the difference of two quadratic variations;
- consequently, we apply the lemma 6 of Lang and Azaïs to prove the convergence in distribution of this difference as we did in the point *(ii)*.

□

Set for $m = 2$ and 4, $M_m = E(\chi^2_{k,\lfloor nx \rfloor} - 1)^m$.

Lemma 4.4 *Assume (A1). Then for* $0 \leq x_1 \leq x \leq x_2 \leq 1$

$$E\left(|T_n(x) - T_n(x_1)|^2 |T_n(x_2) - T_n(x)|^2\right) \leq B(x_2 - x_1)^2.$$

where B is a positive constant.

Proof. Using the Cauchy-Schwarz inequality we have

$$E\left(|T_n(x) - T_n(x_1)|^2 |T_n(x_2) - T_n(x)|^2\right) \leq \sqrt{E\left(|T_n(x) - T_n(x_1)|^4\right)}\sqrt{E\left(|T_n(x_2) - T_n(x)|^4\right)}.$$

Then it is sufficient to prove that $E\left(|T_n(y) - T_n(x)|^4\right) \leq B(y-x)^2$, for $0 \leq x \leq y \leq 1$. We have

$$T_n(y) - T_n(x) = \sqrt{n} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor ny \rfloor} (\Delta Z_k)^2 = \sqrt{n} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor ny \rfloor} \lambda_{k,\lfloor ny \rfloor}(\chi^2_{k,\lfloor ny \rfloor} - 1)$$

where $\lambda_{\lfloor nx \rfloor+1,\lfloor ny \rfloor}, \lambda_{\lfloor nx \rfloor+2,\lfloor ny \rfloor}, \cdots, \lambda_{\lfloor ny \rfloor,\lfloor ny \rfloor}$ are the eigenvalues of the covariance matrix of the Gaussian vector $(\Delta Z_{\lfloor nx \rfloor+1}, \Delta Z_{\lfloor nx \rfloor+2}, \cdots, \Delta Z_{\lfloor ny \rfloor})^t$. Thus

$$\begin{aligned} E\left(|T_n(y) - T_n(x)|^4\right) &= n^2 \left(M_4 \sum_{k=\lfloor nx \rfloor+1}^{\lfloor ny \rfloor} \lambda^4_{k,\lfloor ny \rfloor} + 6M_2^2 \sum_{k=1}^{\lfloor nx \rfloor+1}\sum_{j>k} \lambda^2_{k,\lfloor ny \rfloor}\lambda^2_{k',\lfloor ny \rfloor} \right) \\ &\leq 3n^2 M_4 \left(\sum_{k=\lfloor nx \rfloor+1}^{\lfloor ny \rfloor} \lambda^2_{k,\lfloor ny \rfloor} \right)^2 . \end{aligned}$$

We can show as we showed lemma 4.1 that the maximum of the eigenvalues $\lambda_{k,\lfloor ny \rfloor}$ is a $O(1/n)$. Thus

$$E\left(|T_n(y) - T_n(x)|^4\right) \leq B(y-x)^2.$$

□

Here is our main theorem.

Theorem 4.1 *Assume (A1)-(A2). Then* $\left\{\sqrt{n}(v_n(x) - \int_0^x \alpha(u)du), x \in [0,1]\right\}$ *converges in distribution in* $C([0,1])$ *to the Gaussian process* $\left\{\int_0^x \sqrt{2}\alpha(u)dW(u), x \in [0,1]\right\}$ *as* $n \to \infty$.

Proof. For any $x \in [0,1]$ we have the decomposition

$$\sqrt{n}\left(v_n(x) - \int_0^x \alpha(u)du\right)$$
$$= \sqrt{n}\,(v_n(x) - E(v_n(x))) + \sqrt{n}\left(E(v_n(x)) - \int_0^x \alpha(u)du\right). \quad (16)$$

The second term on the right-hand side of (16) can be decomposed as follows

$$\begin{aligned}\sqrt{n}\left(E(v_n(x)) - \int_0^x \alpha(u)du\right) &= \sqrt{n}\left(E(V_n(x)) - \int_0^x \alpha(u)du\right)\\ &+ \sqrt{n}\,(nx - \lfloor nx\rfloor)\, E\left(\Delta Z_{\lfloor nx\rfloor+1}\right)^2. \quad (17)\end{aligned}$$

According to lemma 4.2, the first term on the right-hand side of (17) converges uniformly in $[0,1]$ to 0. We have $\sup_{x\in[0,1]} (nx - \lfloor nx\rfloor) \le 1$; for $x = 1$, $(nx - \lfloor nx\rfloor) = 0$, and for any $x \in [0,1[$ there is one $k \in [1,2,\dots,n]$ such that $\lfloor nx\rfloor + 1 = k$ and $E\left(\Delta Z_{\lfloor nx\rfloor+1}\right)^2 = a_{k,k}$ as defined by (8). We showed that $|a_{k,k}| = O(1/n)$ uniformly in k (*cf.* (11)). So the second term on the right-hand side of (17) converges to 0 uniformly in $[0,1]$.

It remains to study the convergence in distribution of $\sqrt{n}\,(v_n(x) - E(v_n(x)))$. We have

$$\begin{aligned}\sqrt{n}\,(v_n(x) - E(v_n(x))) &= \sqrt{n}\,(V_n(x) - E(V_n(x)))\\ &+ \sqrt{n}\,(nx - \lfloor nx\rfloor)\left(\Delta Z_{\lfloor nx\rfloor+1}\right)^2\\ &- \sqrt{n}\,(nx - \lfloor nx\rfloor)\, E\left(\Delta Z_{\lfloor nx\rfloor+1}\right)^2. \quad (18)\end{aligned}$$

As previously, the third term on the right hand side of (18) converges to 0 uniformly in $[0,1]$. From property 2.1 it follows that *a.s.* the second term on the right hand side of (18) converges to 0 uniformly in $[0,1]$. Finally, it follows from lemma 4.3, lemma 4.4 and theorem 15.6 in Billingsley (1968) that $\left\{\sqrt{n}\,(V_n(x) - E(V_n(x)))\,, x \in [0,1]\right\}$ converges in distribution to the Gaussian process $\left\{\int_0^x \sqrt{2}\alpha(u)dW(u), x \in [0,1]\right\}$.

□

5 Application to the estimation of a deformation model

We come back to the statistical problem related to the quadratic variations and described in section 3. We want to estimate the deformation Φ in the model (3)-(4) and defined by (6). An estimator of Φ is

$$\hat{\Phi}_n(x) = \frac{v_n(x)}{v_n(1)}.$$

Theorem 5.1 *Assume* ***(A1)***, ***(A2)*** *and* ***(B)***. *Then a.s.*

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |\hat{\Phi}_n(x) - \Phi(x)| = 0.$$

Proof. Since $\left(\hat{\Phi}_n\right)_{n\geq 1}$ is a sequence of increasing functions in $C([0,1])$, it suffices to show the pointwise *a.s.* convergence instead of the uniform *a.s.* convergence. Thus, we must show that *a.s.* for all $x \in [0,1]$

$$\lim_{n\to\infty} v_n(x) = \int_0^x \alpha(u)du.$$

For all $x \in [0,1]$ we have

$$v_n(x) = V_n(x) + (nx - \lfloor nx \rfloor)\left(\Delta Z_{\lfloor nx\rfloor+1}\right)^2. \tag{19}$$

It follows from property 2.1 that *a.s.* the second term on the right-hand side of (19) converges *a.s.* to 0. It remains to study

$$V_n(x) = E(V_n(x)) + V_n(x) - E(V_n(x)). \tag{20}$$

By a similar treatment as the one used in the proof of lemma 4.4, we can show that

$$E\left(V_n(x) - E(V_n(x))\right)^4 \leq 3M_4 \left(\sum_{k=1}^{\lfloor nx\rfloor} \lambda_{k,\lfloor nx\rfloor}^2\right)^2.$$

Thus, it follows from lemma 4.1 that $E(V_n(x) - E(V_n(x)))^4 = O(1/n^2)$. Using Markov inequality and Borel-Cantelli lemma, we obtain that *a.s.* $V_n(x) - E(V_n(x))$ converges to 0. As $E(V_n(x))$ converges to $\int_0^x \alpha(u)du$ (from lemma 4.2), *a.s.* the left-hand side of (20) converges to $\int_0^x \alpha(u)du$.

□

Hereafter, we prove the functional convergence in distribution.

Corollary 5.1 *Assume **(A1)-(A2)** and **(B)**. Then*

$$\left\{\sqrt{n}(\hat{\Phi}_n(x) - \Phi(x)), x \in [0,1]\right\}$$

converges in distribution in $C([0,1])$ *to the Gaussian process*

$$\left\{\frac{\sqrt{2}\int_0^x \alpha(u)dW(u)}{\int_0^1 \alpha(u)du} - \Phi(x)\frac{\sqrt{2}\int_0^1 \alpha(u)dW(u)}{\int_0^1 \alpha(u)du}, x \in [0,1]\right\}$$

as $n \to \infty$.

Proof. For all $x \in [0,1]$ we have the following decomposition

$$\hat{\Phi}_n(x) - \Phi(x) = \frac{v_n(x) - \int_0^x \alpha(u)du}{\int_0^1 \alpha(u)du} - \hat{\Phi}_n(x)\frac{v_n(1) - \int_0^1 \alpha(u)du}{\int_0^1 \alpha(u)du}.$$

The result follows directly from theorems 4.1 and 5.1.

□

We can now propose a test of stationarity for Gaussian processes satisfying model (3)-(4), that is $\Phi(x) = x$ against $\Phi(x) \neq x$. In this case, $\Big\{\sqrt{n}(\hat{\Phi}_n(x) - x)$, $x \in [0,1]\Big\}$ converges in distribution in $C([0,1])$ to the Brownian bridge $\Big\{\sqrt{2}(W(x) - xW(1))$,
$x \in [0,1]\Big\}$ as $n \to \infty$. Thus, $\sqrt{n} \sup_{x\in[0,1]} |\hat{\Phi}_n(x) - x|$ converges in distribution to the Kolmogorov distribution $\sqrt{2}D$ where $D = \sup_{x\in[0,1]} |W(x) - xW(1)|$. Recall that $P(D \leq y) = 1 + \sum_{k=1}^{\infty} \exp(-2k^2y^2)$ for all $y > 0$ (*e.g.* Dacunha-Castelle and Duflo (1986)). Therefore, we reject stationary hypothesis at the level of significance a if $\sqrt{n} \sup_{x\in[0,1]} |\hat{\Phi}_n(x) - x| \geq \sqrt{2}Q_{1-a}$ where Q_{1-a} is the quantile of order $1 - a$ of D.

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