Foster-Lyapunov functions via Laplace transforms of transition measures with applications to storage and risk processes

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Abstract

Let $\{X_t\}$ be a storage model with the release rule $r(x)$ and jumps U with rate β and let $s(x)$ be a real function such that $s(x) = \beta \frac{Ee^{s(x)U}-1}{r(x)}$. Using the function $f(x) = \int_0^x e^{\int_0^y s(u)du} dy$ or close modification of $f(\cdot)$ sufficient conditions for asymptotic behaviour of the storage and risk processes are developed.

Keywords: Lyapunov function, drift criteria

0 Introduction

The construction of the special class of Foster-Lyapunov functions via Laplace transforms of transition measures, presented in this paper, is motivated by two well-known results, concerning the behaviour of one-dimensional diffusion processes and the birth-death processes [1], [2], [4].

The assumption of the existence exponential moments of inputs involved in the first three sections is quite restrictive, but also leads to new somewhat sharper (than in [1], [4]) results. The last section deals with the relaxation of the existence of moments of inputs.

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1 Some preliminaries

Drift-criteria in [4], used in this paper, involve the concept of the truncation $\{X_t^m\}$ of $\{X_t\}$ and the concept of the extended generator A_m of the truncated process. Let $\{O_n : n \in Z^+\}$ be a fixed family of open precompact sets for which $O_n \uparrow R^+$ as $n \to \infty$. Let T^m be defined as the first-entrance time to O_m^c and defined X_t^m by

$$X_t^m = \begin{cases} X_t, & t < T^m \\ \zeta_m, & t \geq T^m, \end{cases}$$

where ζ_m is any fixed state in O_m^c.

Model I. Let $\{X_t\}$ be a continuous time Markov process on the state space $[0, \infty)$, satisfying the storage equation

$$X_t = X_0 + A_t - \int_0^t r(X_s)ds$$

where $\{A_t\}$ is an input process which we shall assume to be a compound Poisson process without drift term,

$$A_t = \sum_{i=0}^{N_t} U_i,$$

where $\{N_t\}, t \geq 0$ is a Poisson process with jump rate β and $\{U_i\}$ is a sequence of independent identically distributed random variables with

$$\mathbb{P}\{U \leq x\} = G(x)$$

and independent of $\{N_t\}$. (Here $U > 0$, i.e., $G(0) = 0$). The function $r : [0, \infty) \to [0, \infty), r(0) = 0$ is called the release rate for the system. The release rate being $r(x)$ at content x means that in between jumps, $\{X_t\}$ should satisfy the differential equation

$$d\dot{x} = -r(x)dt,$$

where $d\dot{x}$ means left derivative. We shall assume that r is strictly positive, left continuous and has strictly positive right limits everywhere on $[0, \infty)$. We also assume that

$$0 < \inf_{\epsilon < x < \epsilon^{-1}} r(x),$$
$$\sup_{0 < x < \epsilon^{-1}} r(x) < \infty$$

for any $\epsilon \in (0,1)$ and

$$\int_0^x r(y)^{-1} dy < \infty, \quad x > 0.$$

Let us suppose that there exists a real function $s(x)$ on C for some compact $C \subset [0,\infty)$, not identically zero which satisfies the equation

$$s(x) = \beta \frac{\mathbb{E}e^{s(x)U} - 1}{r(x)}.$$

Model II. In [4], a risk model $\{Y_t\}_t \geq 0$ with instantaneous increase $r(Y_s)$ between the (downward) jumps of a Poisson process is defined. The function r is continuous with

$$R(\infty) = \int_0^\infty \frac{dy}{r(y)} = \infty, \text{ and that } \{A_t, t \geq 0\}$$

and r are as in for Model I. It is shown in [3] that the generator of the risk process $\{Y_t\}$ is given by

$$\begin{aligned} AV(x) &= r(x)V'(x) - \beta \int_0^\alpha [V(x) - V(x-y)] H(dy) \\ &- \beta H(x,\infty)[V(x) - V(0)] \end{aligned}$$

with the domain containing those bounded continuous V for which $r(x)V'(x)$ is bounded and continuous, and has a finite limit as $x \to 0$. Define the function s on $x \in C^c$, which is not identically zero, which satisfies the equation

$$-\beta H(x,\infty) \frac{1 - e^{-xs(x)}}{s(x)} + r(x) - \beta \int_0^x \frac{1 - e^{-ys(x)}}{s(x)} H(dy) = 0.$$

2 Bounded inputs (Model I)

Suppose through this section that the input U is bounded with probability 1, i.e., there exists some bounded interval I such that $\mathbb{P}\{U \in I\} = 1$.

The following result describes the asymptotic behavior of $\{X_t\}$ ([1], Ch. 3):

Proposition 2.1. *The process $\{X_t\}$ is either transient in the sense that*

$$\mathbb{P}_x\{X_t \to \infty\} = 1$$

for all x, or recurrent in the sense that for all $x \geq 0$, $u > 0$, $\tau(u) = \inf\{t \geq 0; X_t = u\}$,

$$\mathbb{P}_x\{\tau(u) < \infty\} = 1.$$

Let $\{Y_n\}$ be the content just before $(n+1)$th jump, i.e.,

$$Y_n = X_{\tau_n -}$$

where $\tau_0 = 0$ and

$$\tau_n = \inf\{t > \tau_{n-1} : X_{t-} \neq X_t\}.$$

Then $\{Y_n\}$ is a Markov chain and it is sufficient to state recurrence and transience for $\{Y_n\}$. [1]

Proposition 2.2. *(i) Either $\{Y_n\}$ is transient in the sense that*

$$\mathbb{P}_x\{Y_n \to \infty\} = 1$$

for all x, or $\{Y_n\}$ is recurrent in the sense that

$$\mathbb{P}_x\{Y_n \leq y \text{ } i.o.\} = 1$$

for all $x \geq 0$ and all $y > 0$; (ii) In the recurrent case, $\{Y_n\}$ is Harris recurrent.

Define the functions $\underline{s}(x), \bar{s}(x)$ by

$$\underline{s}(x) = \inf_{x \in y+I} s(y), \quad \bar{s}(x) = \sup_{x \in y+I} s(y)$$

Th. 2.1. *If*

(i)

$$\int_0^\infty e^{\int_0^x \underline{s}(u)du} dx = \infty,$$

then $\{Y_n\}$ and $\{X_t\}$ are Harris recurrent;

(ii)

$$\int_0^\infty e^{\int_0^x \bar{s}(u)du} dx < \infty,$$

then $\{Y_n\}$ and $\{X_t\}$ are transient;

Proof. (ii) Let

$$Af(x) = \beta \int_0^\infty [f(x+y) - f(x)]G(dy) - f'(x)r(x)$$

denote the generator of $\{X_t\}$, defined on all bounded differentiable functions with bounded $f'(x)r(x)$, [4].

Let us define the function $\bar{f}(x)$ by $\bar{f}(x) = \int_0^x e^{\int_0^y \bar{s}(u)du} dy$. An elementary application of Dynkin formula yields

$$\mathbb{E}_x \bar{f}(Y_1) - \bar{f}(x) \geq \mathbb{E}_x \int_0^{\tau_1} e^{\int_0^{X_s} \bar{s}(u)du} \left[\beta \frac{\mathbb{E}e^{s(X_s)U} - 1}{s(X_s)} - r(X_s) \right] ds = 0.$$

(i) The function $\bar{f}$ is norm-like and $\forall x \in O_m$, $m \in Z^+$, $x \notin C$-compact

$$A_m \underline{f}(x) \leq \left[\beta \frac{\mathbb{E}e^{s(x)U} - 1}{s(x)} - r(x) \right] = 0,$$

and the assumptions of Th. 3.3 in [4] are satisfied. □

Th. 2.2. *If $\underline{s} = \inf_{x \in C^c} s(x) > -\infty$, $\bar{s} = \sup_{x \in C^c} s(x) < \infty$ for some compact $C \subset [0, \infty]$, and*

(i)

$$\int_0^\infty e^{-\int_0^x \underline{s}(u)dy} dx < \infty,$$

then the process $\{X_t\}$ is Harris ergodic,

(ii) *for some $\lambda > 0$*

$$\int_0^\infty -e^{\int_0^x [\underline{s}(y) - \lambda]dy} dx < \infty,$$

then the process $\{X_t\}$ is exponentially ergodic.

Proof.

Let us test the function

(i)

$$f_\epsilon(x) = \int_0^x e^{\int_0^y [\underline{s}(u)\epsilon(u)]du} dy,$$

where $\epsilon(x)$ is defined by $\epsilon(x) = \epsilon e^{-\int_0^x \underline{s}(u)du}$. The assumptions of Th. 4.4 in [3] hold. Indeed,

$$A_m f_\epsilon(x) \le e^{-\int_0^x [\underline{s}(u)-\epsilon(u)]du} \left\{ \beta \frac{\mathbb{E} e^{[s(x)-\underline{\epsilon}(x)]U} - 1}{s(x) - \underline{\epsilon}(x)} - r(x) \right\} \le$$
$$e^{-\int_0^x [s(u)-\epsilon(u)]du} \beta \left[\frac{\mathbb{E} e^{[s(x)-\underline{\epsilon}(x)]U} - 1}{s(x) - \underline{\epsilon}(x)} - \frac{\mathbb{E} e^{s(x)U} - 1}{s(x)} \right] \le$$
$$\le -\frac{1}{2} e^{\int_0^x [\underline{s}(u)-\epsilon(u)]du} \cdot \epsilon(x) \mathbb{E} U^2 .$$

Since

$$0 < \lim_{x\to\infty} \epsilon(x) e^{\int_0^x [\underline{s}(u)-\epsilon(u)]du} < \infty$$

we get that there exists some $\gamma > 0$ such that $A_m f_\epsilon(x) \le -\gamma \, \forall x \in O_m$, $m \in Z_1^+, x \notin C$.

(ii) The assumptions of Th. 6.1 in [4] are satisfied for the norm-like function

$$f_\lambda(x) = \int_0^x e^{\int_0^y [\underline{s}(u)-\lambda]du} dy.$$

Indeed,

$$A_m f_\lambda(x) \le e^{\int_0^x [\underline{s}(u)-\lambda]du} \left\{ \beta \frac{\mathbb{E} e^{[s(x)-\lambda]U} - 1}{[s(x) - \lambda]} - r(x) \right\} \le$$
$$\le e^{\int_0^x [\underline{s}(u)-\lambda]du} \beta \frac{-\lambda \frac{1}{2} \mathbb{E} U^2}{f_\lambda(x)} \cdot f_\lambda(x) \le$$
$$\le -\frac{\lambda}{2} \beta \mathbb{E} U^2 f_\lambda(x) \cdot \frac{1}{\int_0^\infty e^{-\int_0^x [\underline{s}(y)-\lambda]dy} dx}.$$

□

Corollary 2.1. *Suppose $s(x) \uparrow 0$ and let $0 < \alpha < 1$. Then the process $\{X_t\}_{t\geq 0}$ is (i) recurrent if*

$$\int_0^\infty e^{\int_0^x \inf_{u\in y+I} \frac{2(1+\alpha)[r(u)/\beta - \mathbb{E}U]}{\mathbb{E}U^2} dy} dx = \infty,$$

(ii) transient

$$\int_0^\infty e^{\int_0^x \sup_{u\in y+I} \frac{2(1-\alpha)[r(u)/\beta - \mathbb{E}U]}{\mathbb{E}U^2} dy} dx < \infty.$$

□

Corollary 2.2. *Suppose $s(x) \downarrow 0, 0 < \alpha < 1$.*

Then the process $\{X_t\}_{t\geq 0}$ is Harris ergodic if

$$\int_0^\infty e^{-\int_0^x \inf_{u\in y+I} \frac{2(1-\alpha)[r(u)/\beta - \mathbb{E}U]}{\mathbb{E}U^2} dy} dx < \infty.$$

□

Corollary 2.3. *Suppose that the function $s(x)$ is bounded variation. Then the process $\{X_t\}_{t\geq 0}$ is*

(i) *recurrent if and only if*

$$\int_0^\infty e^{\int_0^x s(y)dy} dx = \infty,$$

(ii) *transient if and only if*

$$\int_0^\infty e^{\int_0^x s(y)dy} dx < \infty$$

(iii) *Harris ergodic if and only if*

$$\int_0^\infty e^{-\int_0^x s(y)dy} dx < \infty$$

(iv) *exponentially ergodic if and only if for some $\lambda > 0$*

$$\int_0^\infty e^{-\int_0^x [s(y)-\lambda] dy} dx < \infty.$$

□

3 General case: unbounded "input"

1) Model I: Conditions for recurrence and transience. The construction uses an well-defined auxiliary function $\underline{s} < s$ such that for some $\epsilon > 0$

$$\mathbb{E}\left\{ \int_x^{x+U} e^{\int_x^y \underline{s}(u)du} dy; U \geq \epsilon x \right\} \leq \mathbb{E}\left\{ \frac{e^{s(x)U} - 1}{s(x)} - \frac{e^{\underline{s}(x)U} - 1}{\underline{s}(x)} \right\} \tag{3.1}$$

Let us define

$$s_0(x) = \inf_{y \in [\frac{x}{1+\epsilon}, x]} s(y), \quad s^0(x) = \sup_{y \in (0,x]} s(y),$$
$$f_0(x) = \int_0^x e^{\int_0^y s_0(u)du} dy, \, f^0(x) = \int_0^x e^{\int_0^y s^0(u)du} dy.$$

Th. 3.1. *(i) Suppose that (3.1) holds $\forall x \in C^c$. If $f_0(\infty) = \infty$, then $\{X_t\}$ is Harris recurrent. (ii) If $f^0(\infty) < \infty$, then $\{X_t\}$ is transient.*

Proof. (i) The function $f_0(\cdot)$ is norm-like and

$$A_m f_0(x) \leq e^{\int_0^x s_0(u)du} \left\{ -\frac{1}{\beta} r(x) + \frac{\mathbb{E}e^{\underline{s}_0(x)U} - 1}{\underline{s}_0(x)} + \right.$$
$$\left. + \mathbb{E}\left[\int_x^{x+U} e^{\int_y^x \underline{s}(u)du} dy; U \geq \epsilon x \right] \right\} \leq 0$$

$\forall x \notin C$, $m \in \mathbb{Z}^+$, $x \in O_m$. The Harris recurrence of $\{X_t\}$ follows by Th. 3.3 in [4].

(ii) An elementary application of the Dynkin's formula yields for the bounded function $f^0(\cdot)$ and the imbedded chain $\{Y_n\}_{n=0}^{\infty}$:

$$\mathbb{E}_x f^0(Y_1) - f^0(x) \geq \mathbb{E}_x \int_0^{\tau_1} e^{\int_0^{X_s} s^0(u)du} \left[\beta \frac{\mathbb{E}e^{s(X_s)U} - 1}{s(X_a s} - r(X_s) \right] ds = 0.$$

□

In the limiting case when $s(x) \uparrow 0$, the construction used in (3.1) can be simplified.

Corollary 3.1. *Let $s(x) \uparrow 0$ as $x \to \infty$, and $0 < \alpha < 1$.*

(i) *If for some* $0 < \epsilon < 1$

$$\mathbb{E}[U; U \geq \epsilon x] \leq \alpha \left[-\frac{r(x)}{\beta} + \mathbb{E}U \right] \tag{3.2}$$

and

$$\int_0^\infty e^{\int_0^x \inf_{[\frac{y}{1+\epsilon}, y] \ni u} \frac{2(\alpha+1)(\frac{r(u)}{\beta} - \mathbb{E}U)}{\mathbb{E}U^2} dy} dx = \infty$$

then $\{X_t\}_{t \geq 0}$ *is Harris recurrent.*

(ii) *If*

$$\int_0^\infty e^{\int_0^x \sup_{u \in (0,y]} \frac{2(1-\alpha)(\frac{1}{\beta} r(u) - \mathbb{E}U)}{\mathbb{E}U^2} dy} dx < \infty,$$

then $\{X_t\}_{t \geq 0}$ *is transient.* □

Example 3.1. The process $\{X_t\}$ is (i) Harris recurrent if for some $\theta < 1$ and all sufficiently large x

$$2x \left[\mathbb{E}U - \frac{r(x)}{\beta} \right] - \theta \mathbb{E}U^2 \leq 0,$$

and (ii) transient if for some $\theta > 1$

$$2x \left[\mathbb{E}U - \frac{r(x)}{\beta} \right] - \theta \mathbb{E}U^2 \geq 0.$$

□

2) Model II: Conditions for recurrence and transience: limiting case, $s(x) \uparrow 0$ as $x \to \infty$.

The construction uses an auxiliary function $\bar{s} > s$ such that for some $0 < \epsilon < 1$:

$$\begin{aligned} 2 \int_{\epsilon x}^\infty \int_{x-y}^x e^{-\int_z^x \bar{s}(u) du} dz H(dy) + H(x, \infty) \int_0^{(1-\epsilon)x} e^{-\int_y^x \bar{s}(u) du} dy \leq \\ \leq [\bar{s}(x) - s(x)] \left[\int_0^x y^2 H(dy) + x^2 H(x, \infty) \right]. \end{aligned} \tag{3.3}$$

Let us define for some $0 < \epsilon < 1$

$$s^0(x) = \sup_{y \in [x, \frac{x}{1-\epsilon}]} \bar{s}(y), \quad g^0(x) = \int_0^x e^{\int_0^y s^0(u)du} dy$$

$$s_0(x) = \inf_{y \in [x, \infty)} s(y), \quad g_0(x) = \int_0^x e^{\int_0^y s_0(u)du} dy.$$

Th. 3.2. *(i) Suppose that (3.3) holds $\forall x \in C^c$. If for some $0 < \alpha < 1$*

$$g^0(\infty) = \int_0^\infty e^{\int_0^x \sup_{u \in [y, \frac{y}{1-\epsilon}]} \frac{-2(1-\alpha)[\frac{r(u)}{\beta} - \int_0^u zH(dz) - uH(u,\infty)]}{\int_0^u z^2 H(dz) + u^2 H(u,\infty)} dy} dx < \infty,$$

then $\{Y_t\}$ is transient.
(ii) If

$$g_0(\infty) = \int_0^\infty e^{\int_0^x \inf_{[y,\infty) \ni u} \frac{-2(1+\alpha)[\frac{r(u)}{\beta} - \int_0^u zH(dz) - uH(u,\infty)]}{\int_0^u z^2 H(dz) + u^2 H(u,\infty)} dy} dx,$$

then $\{Y_t\}$ is Harris recurrent.

Proof. (i) the bounded function $g^0(\cdot)$ is subharmonic on $x \in C^c$ and the imbedded chain $\{X_n\}_{n=0}^\infty$. Obviously, using (3.3), we get

$$Ag^0(x) = e^{\int_0^x s^0(u)du} \beta \Big\{ \int_0^x \frac{1 - e^{-ys(x)}}{s(x)} H(dy) + H(x,\infty) \frac{1 - e^{-xs(x)}}{s(x)} -$$
$$- \int_0^{\epsilon x} \int_{x-y}^x e^{-\int_z^x s^0(u)du} dz H(dy) - H(x,\infty) \int_0^{x(1-\epsilon)} e^{-\int_y^x s^0(u)du} dy \Big\} \geq$$
$$\geq e^{\int_0^x s^0(u)du} \beta \Big\{ \frac{r(x)}{\beta} - \int_0^x \int_{x-y}^x e^{-\int_z^x \bar{s}(x)du} H(dy) - \int_\epsilon^x x \int_{x-y}^x e^{-\int_z^x s^0(u)du} dz H(dy)$$
$$- H(x,\infty) \int_0^x e^{-\int_y^x \bar{s}(x)du} dy - H(x,\infty) \int_0^{x(1-\epsilon)} e^{-\int_y^x \bar{s}(u)du} dy \Big\} \geq 0,$$

and

$$\mathbb{E} g^0(X_1) - g^0(x) \geq 0.$$

(ii) The norm-like function $g_0(\cdot)$ is superharmonic on $x \in C^c \cap O_m$ $\forall m \in \mathbb{Z}^+$. Indeed, $\forall x \in C^c \cap Om, m \in \mathbb{Z}^+$

$$A_m g_0(\cdot) \leq e^{\int_0^x s_0(u)du} \Big\{ r(x) - \beta \int_0^x \frac{1 - e^{-ys(x)}}{s(x)} H(dy) - \beta H(x,\infty) \frac{1 - e^{-xs(x)}}{s(x)} \Big\} = 0$$

□

3) Model I: ergodicity and exponential ergodicity.
(i) *Ergodicity*: limiting case, $s(x) \downarrow 0$ as $x \to \infty$. Then

$$s(x) \sim \frac{2[\frac{1}{\beta}r(x) - \mathbb{E}U]}{\mathbb{E}U^2}, \quad \text{and we take}$$

$$\underline{s}(x) = \frac{2\theta[\frac{1}{\beta}r(x) - \mathbb{E}U]}{\mathbb{E}U^2}, s_0(x) = \inf_{y \in (0,x]} \underline{s}(y)$$

for some $0 < \epsilon < 1, 0 < \theta < 1$.

Th. 3.3. *If*

$$\int_0^\infty e^{-\int_0^x s_0(u)du} dx < \infty$$

then $\{X_t\}_{t \geq 0}$ *is Harris ergodic.*

Proof. Let us define the functions $\epsilon(x), f_\epsilon(x)$ as

$$\epsilon(x) = e^{-\int_0^x s_0(u)du},$$
$$f_\epsilon(x) = \int_0^x e^{\int_0^y [s_0[u) - \epsilon(u)]du} dy$$

then, testing the function $f_\epsilon(x)$ in (CD2) in [4], and estimating in the same way as in Th. 2.2 (i), we get that $\forall x \in C^c \cap O_m$ and $m \in \mathbb{Z}^+$

$$A_m f_\epsilon(x) \leq -\frac{1}{2}\mathbb{E}U^2 \cdot \epsilon(x) \cdot e^{\int_0^x [s_0(u) - \epsilon(u)]du}.$$

As $\displaystyle\int_0^\infty e^{-\int_0^x s_0(u)du} dx < \infty$,

$$A_m f_\epsilon(x) < -\gamma, \quad \forall x \in C^c \cap O_m, m \in \mathbb{Z}^+$$

and the assumptions of Th. 4.1 in [4] are satisfied. □

(ii) *Exponential ergodicity*: Limiting case, $s(x) \downarrow \lambda, \lambda > 0$.

Th. 3.4. *The process* $\{X_t\}$ *is exponentiallty ergodic if*

$$\int_0^\infty e^{-\int_0^x [s_0(u) - \lambda]du} dx < \infty.$$

Proof. The proof is similar to the proof of Th. 2.2 (i). ☐

Example 3.2. Suppose that

$$\frac{2[r(x)/\beta - \mathbb{E}U]}{\mathbb{E}U^2} \geq \frac{\theta}{x},\ \theta > 1$$

Then the process $\{X_t\}$ is Harris ergodic.

Example 3.3. Arbitrary release and exponential input,

$$\mathbb{P}\{U > x\} = e^{-\delta x}, \quad \text{then } s(x) = \delta - \frac{\beta}{r(x)}.$$

☐

The process $\{X_t\}$ is (i) Harris recurrent if for some $0 < \epsilon < 1$

$$\int_0^\infty e^{\int_0^x \inf_{u \in [u \in y/1+\epsilon, y]}[\delta - \frac{\beta}{r(u)} dy]} dx = \infty,$$

(ii) transient if

$$\int_0^\infty e^{\int_0^x \sup_{u \in [0,y]}[\delta - \frac{\beta}{r(u)}] dy} dx < \infty$$

(iii) ergodic, if

$$\int_0^\infty e^{-\int_0^x \inf_{u \in [0,y]}[\delta - \frac{\beta}{r(u)}] dy} dx < \infty.$$

4) Model II: ergodicity and exponential ergodicity. Let us consider the limiting case when $s(x) \downarrow 0$ as $x \to \infty$. Let $\underline{s}$ be an auxiliary function $\underline{s} < s$ such that $\underline{s} \downarrow 0$ and for some $0 < \epsilon < 1$:

$$\int_{\epsilon x}^\infty y H(dy) \leq \frac{1}{2}[s(x) - \underline{s}(x)] \int_0^x y^2 H(dy). \tag{3.4}$$

Let

$$s_0(x) = \inf_{y \in [x, \frac{x}{1-\epsilon}]} \bar{s}(y) \text{ and } g_\epsilon(x) = \int_0^x e^{\int_0^y [s_0(u) - \epsilon(u)] du} dy,$$
$$\epsilon(x) = e^{\int_0^x s_0(u) du}.$$

Let

$$s_0(x) = \inf_{y \in [x, \frac{x}{1-\epsilon}]} \bar{s}(y) \text{ and } g_\epsilon(x) = \int_0^x e^{\int_0^y [s_0(u) - \epsilon(u)] du} dy,$$
$$\epsilon(x) = e^{-\int_0^x s_0(u) du}.$$

Th. 3.5. *Suppose that the inequality (3.4) holds for the function* $\underline{s}$ *and some* $0 < \epsilon < 1$. *Assume, in addition, that*

$$e^{-\int_0^x s_0(u)du} = O^*\left(e^{-\int_0^{x/1-\epsilon} s_0(u)du}\right).$$

If $\int_0^\infty e^{-\int_0^x s_0(u)du}dx < \infty$, *then* $\{Y_t\}$ *is Harris ergodic.*

Th. 3.6. *Let* $\underline{s}(x) \downarrow \lambda$ *as* $x \to \infty$ *for some* $\lambda > 0$. *Suppose that (3.4) holds true for the function* $\underline{s}$ *and some* $0 < \epsilon < 1$. *If*

$$\int_0^\infty e^{-\int_0^x [s_0(u)-\lambda]du}dx < \infty$$

then $\{Y_t\}$ *is exponentially ergodic.*

Proof. Quite similar to Th. 2.2 (i). □

Example 3.4. Suppose that

$$2\frac{\int_0^x yH(dy) + xH(x,\infty) - \frac{r(x)}{\beta}}{\int_0^x y^2H(dy) + x^2H(x,\infty)} > \frac{\theta}{x}, \quad \theta > 1$$

then $\{Y_t\}$ is Harris ergodic.

4 General case: infinite moments

1) Model I: Conditions for recurrence and transience.
Suppose that for some $s > 0$

$$\mathbb{E}\int_x^{x+U} \frac{\beta}{r(y)}e^{s(y-x)}dy < \infty$$

and let us define the function $s(x)$ by

$$\mathbb{E}\int_x^{x+U} \frac{\beta}{r(y)}e^{s(x)(y-x)}dy = 1,$$

and consider an auxiliary function $\underline{s}(x) \leq s(x)$ such that for some $0 < \alpha < 1$ the following inequality holds:

$$\begin{aligned} &\mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{\int_x^y \underline{s}(u)du} dy; U > \alpha x\Big] \leq \\ &\leq \mathbb{E}\Big\{\int_x^{x+U} \frac{\beta}{r(y)}[e^{s(x)(y-x)} - e^{\underline{s}(x)(y-x)}]dy\Big\}. \end{aligned} \tag{4.1}$$

We consider the limiting case when $s(x) \uparrow 0$, as $x \to \infty$, so we get

$$s(x) \sim \frac{1 - \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} dy\Big]}{\mathbb{E}\Big[\int_x^{x+U} \frac{\beta(y-x)dy}{r(y)}\Big]} \quad \text{and } \underline{s}(x) = \gamma s(x),$$

where $\gamma > 1$. Let us define the function $s_0(x)$ by

$$s_0(x) = \inf_{y \in [\frac{x}{1+\alpha}, x]} \underline{s}(y)$$

for some $0 < \alpha < 1$.

Since $s(x) < 0$ the inequality (4.1) can be rewritten as

$$\mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} dy; U > \alpha x\Big] \leq [s(x) - \underline{s}(x)] \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)}(y-x)dy\Big]. \tag{4.2}$$

Th. 4.1. *Suppose that (4.2) holds true for the well-defined function $\underline{s}(x)$ and some $0 < \alpha < 1$. Then the process $\{X_t\}$ is Harris recurrent*

$$\int_0^{\infty} \frac{\beta}{r(x)} e^{\int_0^x s_0(u)du} dx = \infty.$$

Proof. The function $f_0(\cdot)$ is superharmonic on $C^c \cap O_m, m \in Z^+$. Indeed, for all $m \in \mathbb{Z}^+, x \in C^c \cap O_m$

$$
\begin{aligned}
A_m f_0(x) \le\; & e^{\int_0^x s_0(u)du} \Big\{ \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{\int_x^y s_0(u)du} dy; U \le \alpha x \Big] + \\
& + \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{\int_x^y s_0(u)du} dy; U > \alpha x \Big] - 1 \Big\} \le \\
& \le e^{\int_0^x s_0(u)du} \Big\{ \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{\underline{s}(x)(y-x)} dy \Big] - 1 + \\
& + \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{\int_x^y s_0(u)du} dy; U > \alpha x \Big] \Big\} \le \\
& \le e^{\int_0^x s_0(u)du} \Big\{ \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{s(x)(y-x)} dy \Big] - 1 \Big\} = 0.
\end{aligned}
$$

Then the Harris recurrence of $\{X_t\}$ follows by (CD1) in [4]. $\square$

Let us define

$$\hat{s}(x) = \sup_{y \in (0,x]} s(y), \hat{f}(x) = \int_0^x e^{\int_0^y \hat{s}(u)du} dy.$$

Th. 4.2. *The process $\{X_t\}$ is transient if*

$$\int_0^\infty \frac{\beta}{r(x)} e^{\int_0^x \hat{s}(u)du} dx < \infty.$$

Proof. Indeed, the bounded function $\hat{f}(\cdot)$ is subharmonic for the imbedded chain $\{Y_n\}$:

$$A\hat{f}(x) \ge e^{\int_0^x \hat{s}(u)du} \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{\int_x^u \hat{s}(u)du} dy - 1 \Big] = 0.$$

$\square$

Model I: *Positive and exponential recurrence.*
The limiting case $s(x) \downarrow 0$ as $x \to \infty$. Let $s_0(x) = \inf_{y \in (0,x]} s(x)$.

Th. 4.3. *If* $\int_0^\infty \frac{\beta}{r(x)} e^{-\int_0^x s_0(y)dy} dx < \infty.$

Proof. Let $\epsilon(x) = e^{-\int_0^x s_0(u)du} \frac{\beta}{r(x)}$.

$$f_\epsilon(x) = \int_0^x \frac{\beta}{r(y)} e^{\int_0^y [s_0(u)-\epsilon(u)]du}.$$

Then, testing the function $f_\epsilon(x)$ in (CD2) in [4], we get that $\forall m \in \mathbb{Z}^+, x \in C^c \cap O_m$

$$A_m f_\epsilon(x) \leq e^{\int_0^x [s_0(u)-\epsilon(u)]du} \Big\{ \mathbb{E}\Big[\int_x^{x-U} \frac{\beta}{r(y)} e^{\int_x^y [s_0(u)-\epsilon(u)]du} dy; U \leq \alpha x \Big]$$
$$+ \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{\int_x^y s_0(u)-\epsilon(u)]du} dy : U > \alpha x \Big] - 1 \Big\}.$$

Using (4.1), we obtain that

$$A_m f_\epsilon(x) \leq e^{\int_0^x [s_0(u)-\epsilon(u)]du} \mathbb{E}\Big[\int_x^{x+U} \frac{\beta}{r(y)} e^{s(x)-\epsilon(x)](y-x)} dy -$$
$$- \int_x^{x+U} \frac{\beta}{r(y)} e^{s(x)(y-x)} dy \Big].$$

As $\displaystyle\int_0^\infty \frac{\beta}{r(x)} e^{-\int_0^x s_0(y)dy} dx < \infty$, we get that

$$A_m f_\epsilon(x) \leq -\epsilon(x) \cdot e^{\int_0^x [s_0(u)-\epsilon(u)]du} \mathbb{E} \int_x^{x+U} \frac{\beta(y-x)}{r(y)} dy$$

and by assumption of the theorem $\forall m \in \mathbb{Z}^+ \quad x \in C^c \cap O_m$

$$A_m f_\epsilon(x) \leq -\gamma.$$

□

Th. 4.4. *Let $s(x) > 0, s(x) \downarrow \lambda, \lambda > 0$ as $x \to \infty$. The process is exponentially ergodic if*

$$\int_0^\infty \frac{\beta}{r(x)} e^{-\int_0^x [s_0(y)-\lambda_0]dy} dx < \infty.$$

Proof. The proof is quite similar to the proof of Th. 4.3. □

Model II: Recurrence and transience. Let us define the function $s(x)$ by

$$\int_0^x \frac{\beta}{r(y)} e^{s(x)(y-x)} H(dy) + H(x,\infty) \int_0^x \frac{\beta}{r(y)} e^{s(x)(y-x)} dy = 1.$$

Limiting case $s(x) \downarrow 0$ as $x \to \infty$. We get

$$s(x) \sim \frac{1 - \int_0^x \frac{\beta}{r(y)} H(dy) - H(x,\infty) \int_0^x \frac{\beta}{r(y)} dy}{\int^x \frac{\beta}{r(y)} (y-x) H(dy) + H(x,\infty) \int_0^x \frac{\beta}{r(y)} (y-x) dy}$$

and let us define

$$s_0(x) = \inf_{y \in [x,\infty)} s(y).$$

Th. 4.5. *If*

$$\int_0^\infty e^{\int_0^x s_0(y)dy} dx = \infty,$$

then $\{Y_t\}_{t\geq 0}$ *is Harris recurrent.*

Proof. The function $g_0(x) = \int_0^x e^{\int_0^y s_0(u)dy} dy$ is superharmonic on $C^c \cap O_m \quad \forall m \in \mathbb{Z}^+$. Indeed, $forall m \in \mathbb{Z}^+, \forall x \in C^c \cap O_m$

$$A_m g_0(x) \leq e^{\int_0^x s_0(u)du} \Bigg[\int_0^x \frac{\beta}{r(y)} e^{s(x)(y-x)} H(dy) + H(x,\infty) \int_0^x \frac{\beta}{r(y)} e^{s(x)(y-x)} dy - 1 \Bigg] = 0$$

□

In the results, concerning transience, we need the following construction involved, a well-defined auxiliary function $\bar{s}(x) \geq s(x)$, such that for some $0 < \epsilon < 1$ and some $\gamma > 0$

$$\int_{\epsilon x}^\infty \int_{x-y}^x \frac{\beta}{r(z)} e^{\gamma(x-z)} dz H(dy) \leq \leq [\bar{s}(x) - s(x)] \Bigg[\int_0^x \frac{\beta}{r(y)} (y-x) H(dy) + H(x,\infty) \int_0^\infty \frac{\beta}{r(y)} (y-x) dy \tag{4.3}$$

Let

$$s^0(x) = \sup_{y \in [x, \frac{x}{1-\epsilon}]} s(y).$$

Th. 4.6. *If*

$$\int_0^\infty \frac{\beta}{r(x)} e^{\int_0^x s^0(u)du} dx < \infty,$$

then the process $\{Y_t\}_{t\geq 0}$ *is transient.*

Proof. Similar to the proof of Th. 3.4. □

Proofs of the following results are quite similar to the proofs of Th. 4.3, Th. 4.4.

Modell II: *ergodicity and exponential ergodicity*

Let $s(x) \downarrow 0$ as $x \to \infty$, and suppose that $\exists 0 < \epsilon < 1$ such that

$$\begin{aligned} 2\int_{\epsilon x}^x \int_{x-y}^x \frac{\beta}{r(z)} dz H(dy) \leq [s(x) - \underline{s}(x)] \cdot \\ \cdot \left[\int_0^x \frac{\beta}{r(y)}(y-x)H(dy) + H(x,\infty)\int_0^x \frac{\beta}{r(y)}(y-x)dy \right] \end{aligned} \tag{4.4}$$

for some auxiliary function $\underline{s}(x) \leq s(x), \underline{s}(x) \downarrow 0$ as $x \to \infty$. Let $s_0(x) = \inf_{y\in[x,\frac{x}{1-\epsilon}]} \underline{s}(y)$.

Th. 4.7. *Suppose that the inequality (4.4) holds for the function* $\underline{s}(x)$ *and some* $0 < \epsilon < 1$. *If, in addition,*

$$e^{-\int_0^x s_0(u)du} = O^*\left(e^{-\int_0^{x/1-\epsilon} s_0(u)du}\right) \quad \text{and} \quad \int_0^\infty \frac{\beta}{r(x)} e^{-\int_0^\infty s_0(u)du} < \infty. \tag{4.5}$$

then the process $\{Y_t\}_{t\geq 0}$ *is Harris ergodic.*

Th. 4.8. *Let* $s(x) \downarrow \lambda$ *as* $x \to \infty$, $\lambda > 0$. *Assume that* $\exists$ *an auxiliary function* $\underline{s}(x) \leq s(x)$ *such that* $\underline{s}(x) \downarrow \lambda$ *as* $x \to \infty$ *and (4.4) holds true. If*

$$\int_0^\infty \frac{\beta}{r(x)} e^{-\int_0^x [s_0(u)-\lambda]du} dx < \infty,$$

then $\{Y_t\}_{t\geq 0}$ *is exponentially ergodic.*

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$\forall m \in \mathbb{Z}^+, x \in C^c \cap O_m$