# A NEW CONSISTENT DISCRETE-VELOCITY MODEL FOR THE BOLTZMANN EQUATION

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**Abstract.** This paper discusses the convergence of a new discrete-velocity model to the Boltzmann equation. First the consistency of the collision integral approximation is proved. Based on this we prove the convergence of solutions for a modified model to DiPerna-Lions solutions of the Boltzmann equation. As a test numerical example, the solutions to the discrete problems are compared to the exact solution of the Boltzmann equation in the space homogeneous case.

**Keywords:** the Boltzmann equation, DiPerna-Lions solutions, discrete-velocity models, quadrature formulas, consistency, convergence
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**Introduction.** The nonlinear Boltzmann equation describes the evolution of a gas which is seen as a collection of interacting particles. The model is physically relevant when the gas is rarefied and the binary collisions of particles prevail. The equation reads as follows:

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = Q(f,f), \quad (x,t) \in \mathcal{D} \subseteq \mathbb{R}^3 \times \mathbb{R}, \quad \xi \in \mathbb{R}^3,$$

where $f = f(\xi, x, t)$ is the distribution function of a gas, which depends on the velocity variable $\xi$, and space and time variables $x$, $t$. Also, $Q(f,f)$ is the quadratic integral collision operator which acts on $f$ as a function of $\xi$. We leave the details to the next section and refer to the books by Cercignani, Illner and Pulvirenti [12], and Truesdell and Muncaster [34] for the physical background, mathematical theory and applications of this equation.
Due to the complex structure of the collision operator and high dimension of the problem, the only way to obtain solutions of the Boltzmann equation in physically nontrivial situations is to use numerical computations. The most efficient methods for solving nonlinear kinetic problems are currently the *particle methods* with the two best known examples being Bird's method (Direct Simulation Monte Carlo or DSMC) [4] and the Nanbu scheme [25], later modified by Babovsky [2]. In these methods the random dynamics of $N$-particle systems is modelled (the number of particles $N$ is in practice several orders less than the actual number of molecules in a gas) and the averages of the random process realizations are taken as an approximation to the solution. The main drawback of particle methods is that the convergence of the averages is slow and the results are subject to random fluctuations. If, instead of the particle approach, traditional techniques of numerical analysis of partial differential equations are used, such as finite difference or finite element schemes, several difficulties arise. The major one is that the numerical complexity of calculating the collision integral grows faster with the number of approximation points than is the case for particle schemes (typically $O(N^2)$ vs. $O(N)$). Thus the application of these "deterministic" methods in the current stage of computing technology is restricted essentially to problems with symmetries where the dimension of the problem can be a priori reduced. On the positive side, the accuracy obtained by using these methods can be higher than for the particle methods, since the averaging step is not needed. It is therefore

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reasonable to study these approaches if one has in mind applications to problems for which the accuracy of the numerical solution is of critical importance (see papers [27] and [32] for examples of such studies).

In this paper we study one approach which can be characterized as deterministic, namely the *discrete-velocity models* (DVM). Such models were considered, starting from the works of Carleman [11] and Broadwell [8], as model equations with the same formal properties as the Boltzmann equation, which are however more accessible for numerical and in some cases analytical treatment. The dependence on the molecular velocity $\xi$ is discretized in these models in such a way that all or at least some of the invariants of the original equation are preserved, which means that correct limit solutions can be expected on the macroscopic level. The formal theory of such equations as well as their mathematical and numerical aspects are described in detail in lecture notes by Gatignol [15], Cabannes [10] and the book by Monaco and Preziosi [24]. However, for a long time the question of connection between DVM and the Boltzmann equation was left without a strict analysis; only in the 1980s dis the first works in that direction appear: [1, 33, 6, 16, 19]. While the numerical results suggested that DVM can be used to approximate the Boltzmann equation, there was until recently no strict mathematical proof of convergence for the models which retain the conservation properties of the Boltzmann equation.

The model introduced by Goldstein *et al.* [16] and Inamuro and Sturtevant [19] was later used by Buet [9] who developed an efficient numerical scheme for the Boltzmann equation, based on a combination of that model and a Monte-Carlo approach. He also gave a heuristic argument on why the convergence to the Boltzmann collision integral should be expected. The rigorous proof of the consistency of the model was given by Bobylev, Palczewski and Schneider [7] whose proof was based on very recent, detailed results from number theory. Another scheme of the same type, for which the consistency is also proved, was introduced by Rogier and Schneider [30] for a two-dimensional velocity space and generalized to three dimensions by Michel and Schneider [22]. According to the approach of papers [7], [30] and [22], the discrete-velocity model is considered as a quadrature formula for the five-dimensional integral in the collision operator, and the convergence of such a quadrature to the collision integral is proved.

The question of convergence of solutions of the DVM equations to solutions of the Boltzmann equation was studied by Mischler [23]. Using the weak formulation of the discrete equations, he generalized the weak $L^1$ theory of DiPerna and P. L. Lions to DVM and proved that the solutions (taken in the DiPerna-Lions sense) converge weakly in $L^1$ to a solution of the Boltzmann equation. The strong $L^1$ convergence was proved for a finite-volume scheme which conserves mass but not the momentum and energy. A result of strong $L^1$ convergence for the homogeneous Cauchy problem was also proven by Palczewski and Schneider [28].

It should be noted also that, while the numerical complexity of directly calculating the collision term using a DVM with $N$ velocities is $O(N^{2+\varepsilon})$, $\varepsilon > 0$, the efficiency of the scheme can be improved by using Monte Carlo techniques as in the method of Buet [9] or by using the solution symmetries, as Ohwada has done [26].

In this paper we propose and analyse a discrete-velocity model with a structure similar to the ones discussed above. This model is also conservative, satisfies an entropy condition and has only physical collision invariants. To obtain the model we follow the same scheme as Bobylev *et al.* [7], but use a different form of the collision term, which was first introduced by Carleman [11]. In this form, variables of integration are used so that the integration over a sphere in the collision term is replaced by the integration over planes in $\mathbb{R}^3$. This leads to a significant simplification in the structure of the quadrature formula and allows us to prove a consistency result, analogous to the one given by Bobylev *et al.* [7], while using simpler and more direct techniques. A model of this type with two-dimensional velocity space was previously considered by B. Wennberg and F. Golse [35]. We prove the error estimates, for the quadrature formula, with error bounds from $Ch^{1/4}$ to $C_\varepsilon h^{1-\varepsilon}$ for distribution functions from the class $C^m$, $m \geq 1$. The collision cross-sections should be smooth and satisfy specific conditions on the angular dependence. Based on these results we establish consistency in classes of continuous functions polynomially decaying at infinity. The modifications analogous to the ones done by Mischler [23] allow us to prove the weak $L^1$ convergence to a DiPerna-Lions solution of the Boltzmann equation. In the last section we present the results of numerical computations for the space homogeneous relaxation problem.

**1. The Boltzmann equation and discrete-velocity models.** The collision integral $Q(f,f)$ is given by the expression

$$Q(f,f) = \int\limits_{\mathbb{R}^3}\int\limits_{S^{(2)}} \left(f(\xi')\,g(\eta') - f(\xi)\,g(\eta)\right) B(\xi-\eta,\omega)\,d\omega\,d\eta, \quad \xi \in \mathbb{R}^3, \tag{1.1}$$

where

$$\xi' = \xi - \omega(\omega, \xi-\eta), \quad \eta' = \eta + \omega(\omega,\xi-\eta).$$

In the obvious splitting, $Q(f,f) = Q^+(f,f) - Q^-(f,f)$, $Q^+$ and $Q^+$ are usually referred to as the "gain" and the "loss" term, respectively.
The function $B(u,\omega)$ is of the form

$$B(u,\omega) = B_0\left(|u|, \frac{|(u,\omega)|}{|u|}\right), u \in \mathbb{R}^3, \omega \in S^{(2)}.$$

It contains the information about the binary interactions of particles and reflects the physical properties of the model. The condition $B(u,\cdot) \in L^1(S^{(2)})$, $u \in \mathbb{R}^3$ is usually assumed to obtain separately convergent integrals for the "gain" and "loss" parts of (1.1). If the particle interactions are modelled by inverse power forces with angular cut-off, then

$$B_0(r,x) = r^\gamma b(x), \tag{1.2}$$

where $\gamma \in (-3,1]$, and $b \in L^1([0,1])$. In the case of "hard sphere" molecules, $B_0(r,x) = rx$.
In discrete-velocity models it is assumed that the velocities of the particles belong to a finite set $V = \{\xi_i\}_{i=1}^N \subseteq \mathbb{R}^3$; thus the distribution function $f(\xi)$, $\xi \in \mathbb{R}^3$ is replaced by a finite-dimensional approximation $f_i$, $\xi_i \in V$, and the values of $f_i$ are determined from the following system of equations:

$$\frac{\partial f_i}{\partial t} + \xi_i \cdot \nabla f_i = Q_i(f,f), \quad (x,t) \in \mathcal{D} \subseteq \mathbb{R}^3 \times \mathbb{R},\ \xi_i \in V, \tag{1.3}$$

$$Q_i(f,f) = \sum_{jkl} A_{ij}^{kl}\,(f_k f_l - f_i f_j), \tag{1.4}$$

where $A_{ij}^{kl}$ are constant coefficients and the summation is taken over all indices corresponding to the discrete velocities in $V$. If the coefficients $A_{ij}^{kl}$ satisfy the conditions

$$A_{ij}^{kl} = A_{ji}^{lk}, \quad A_{ij}^{kl} = A_{kl}^{ij}, \tag{1.5}$$

then a symmetry property analogous to the one of the Boltzmann equation (cf. [12]) holds:

$$\sum_{i\in\mathbb{Z}^3} Q_i(f,f)\psi_i = -\frac{1}{4}\sum_{i,j,k,l} A_{ij}^{kl}\,(f_k f_l - f_i f_j)(\psi_k + \psi_l - \psi_i - \psi_j), \tag{1.6}$$

and if

$$A_{ij}^{kl} \neq 0 \;\text{ only if }\; \xi_i + \xi_j = \xi_k + \xi_l \;\text{ and }\; \xi_i^2 + \xi_j^2 = \xi_k^2 + \xi_l^2, \tag{1.7}$$

then the discrete analogues of the conservation laws and the entropy condition follow from (1.6):

$$\sum_i Q_i(f,f)\begin{pmatrix} 1 \\ \xi_i \\ |\xi_i|^2 \end{pmatrix} = 0; \qquad \sum_i Q_i(f,f)\,\log f_i \leq 0. \tag{1.8}$$

Collision invariants can be defined as those vectors $\psi_i$ for which

$$\psi_i + \psi_j = \psi_k + \psi_l \quad \text{when} \quad A_{ij}^{kl} \neq 0, \tag{1.9}$$

and it is true that $f_i$ is an equilibrium distribution if, and only if, $\log f_i$ is a collision invariant, see Gatignol [15]). According to (1.7) the linear combinations of 1, $\xi_i$ and $|\xi_i|^2$ are collision invariants. The differences from the classical Boltzmann equation can appear when there are other solutions to the system (1.9) as indeed is so for some models [15]. This also leads to the equations of hydrodynamical limit having a different structure than for the Boltzmann equation and may display itself in non-physical behavior of the solutions.
We deal with a specific situation when the collision invariants of the DVM are restricted to the physical ones, as in the models studied in [16, 19, 7] and [30, 22]. The velocity set $V$ is assumed to be part of the regular grid

$$\mathcal{Z}_h = h\mathbb{Z}^3 = \big\{ h(i_1, i_2, i_3) \,\big|\, i_{1..3} \in \mathbb{Z} \big\} \tag{1.10}$$

contained in a bounded set $\mathcal{B}_h \subseteq \mathbb{R}^3$. To simplify the presentation, we consider the consistency problem for the infinite model with $V = \mathcal{Z}_h$. We return to the question of choosing the set $\mathcal{B}_h \subseteq \mathbb{R}^3$ in section 6, where we address the problem of convergence for solutions of the discrete equations. The DVM discussed by Bobylev *et al.* [7] is obtained by applying a rectangle quadrature approximation to the outer integration over $\mathbb{R}^3$ in (1.1) and then approximating the inner integral over $S^(2)$, based on its representation as

$$\frac{2}{|\xi_i - \xi_j|^2}\int\limits_{\Sigma_{ij}} \Big( f\Big(\frac{\xi_i+\xi_j}{2} + \sigma\Big) f\Big(\frac{\xi_i+\xi_j}{2} - \sigma\Big) - f(\xi_i)\, f(\xi_j)\Big)\, \frac{B(\xi_i - \xi_j, \omega(\sigma))}{|\cos\theta(\sigma)|}\, d\sigma,$$

where $\Sigma_{ij}$ is the sphere of diameter $|\xi_i - \xi_j|$ passing through the points $\xi_i$ and $\xi_j$, and $\theta$ is the angle between $\xi_i - \xi_j$ and $\omega$. For this approximation, those points of $\mathcal{Z}_h$

which fall on the sphere $\Sigma_{ij}$ are taken with equal weights $\frac{|\Sigma_{ij}|}{|S_{ij}|}$, where $|S_{ij}|$ is the total number of such points. Thus, setting

$$(1.11)\quad A_{ij}^{kl} = h^3 \frac{2\pi}{|S_{ij}|} \frac{B(\xi_i - \xi_j, \omega_{ij}^{kl})}{|\cos\theta_{ij}^{kl}|} \chi_{ij}^{kl}, \quad \text{where} \quad \chi_{ij}^{kl} = \begin{cases} 1, \text{ if } & \begin{matrix} k+l = i+j \\ k^2+l^2 = i^2+j^2 \end{matrix} \\ 0 & \text{otherwise,} \end{cases}$$

one obtains a model of the type (1.4) with the properties (1.5) and (1.7), which thereby satisfies (1.8). It is known that this model has no other collision invariants except mass, momentum, and energy [9].

The consistency theorem for this model is given by Bobylev *et al.* [7]:

THEOREM 1.1. *Suppose $f$ is a continuous function on $\mathbb{R}^3$ such that*

$$\sup |f(\xi)|(1+\xi^2)^3 < +\infty,$$

*and $q(u,\omega) = B(u,\omega)/\cos\theta$ is continuous and satisfies $0 \le q(u,\omega) \le a + b|u|$ for $u \in \mathbb{R}^3$, $\omega \in S^{(2)}$. Then*

$$Q(f,f)(\xi_i) - Q_i(f,f) \longrightarrow_{h\to 0} 0$$

*uniformly with respect to $\xi_i$ on compact subsets of $\mathbb{R}^3$.*

*Suppose now that $f$ and $\widetilde{q}(u,w) = w^{-2}q(w, \frac{2u}{w})$ are smooth ($C^k$, $k \ge 6$) functions and $f$ has compact support. Then for sufficiently small $h$,*

$$|Q(f,f)(\xi_i) - Q_i(f,f)| \le C_\varepsilon h^{2/175-\varepsilon}$$

*with the same uniformity condition for $\xi_i$.*

The main difficulty of the proof is to establish the consistency of the approximation for the integrals over the spheres. The convergence of this approximation for continuous integrands is related to properties of the distribution of integer points on spheres of large integer radius; the proof here is based on recent results from number theory (see references in Bobylev *et al.* [7]). Note that the error estimate of the theorem indicates a very slow rate of convergence for $h \to 0$ although the practical convergence observed in numerical experiments is much faster.

In the approach of Schneider with Rogier [30] and with Michel [22] the same velocity grid $\mathcal{Z}_h$ is used, but the order of sphere and space integrations is reversed. First, a grid for calculating the integral over the sphere is defined as a central projection of the set

$$\mathcal{F}_N^d = \big\{(p_1,..,p_d) \in \mathbb{Z}^d \,\big|\, |p_i| \le N,\ \text{g.c.d.}(p_1,..,p_d) = 1\big\}$$

onto $S^{(2)}$. The weights for the quadrature formula over the sphere are obtained from a subdivision of the sphere into the set of cells $R(p)$ centered around the points $\frac{p}{|p|}$, $p \in \mathcal{F}_N^d$, and the following approximation is obtained for $Q(f,f)$:

$$(1.12)\quad Q(f,f)(\xi_i) \approx \sum_{p\in\mathcal{F}_N^d} |R(p)| \int_{\mathbb{R}^d} \big(f(\xi')\,f(\eta') - f(\xi_i)\,f(\eta)\big)\,B(\xi_i - \eta, \frac{p}{|p|})\,d\eta.$$

A quadrature formula of the type (1.4) is obtained by using only those values of $\eta$ in (1.12) for which $\xi'$ and $\eta'$ belong to $\mathcal{Z}_h$. From the collision geometry it follows that this set is the $d$-dimensional lattice

$$\big\{h(rp+m) \,\big|\, r \in \mathbb{Z},\ m \in \mathbb{Z}^d,\ p\cdot m = 0\big\}$$

with the volume of the fundamental cell equal to $h^d|p|^2$; thus, the inner integral is approximated as follows:

$$\int\limits_{\mathbb{R}^3} \big(f(\xi')\,f(\eta') - f(\xi_i)\,f(\eta)\big)\,B(\xi_i-\eta,\frac{p}{|p|})\,d\eta \approx h^d|p|^2\sum_{r\in\mathbb{Z}}\sum_{\substack{m\cdot p=0\\ m\in\mathbb{Z}^d}} B(h(rp+m),\frac{p}{|p|})$$

$$\times\big(f(\xi_i+hrp)\,f(\xi_i+hm)\ -f(\xi_i)\,f(\xi_i+h(rp+m))\big). \tag{1.13}$$

By combining expressions (1.12) and (1.13), the equations for the discrete model of the type (1.4) are obtained, with the coefficients $A_{ij}^{kl}$ given by

$$A_{ij}^{kl} = h^3|p_{ik}|^2\,|R(p_{ik})|\,B(\xi_i-\xi_j,\omega_{ij}^{kl})\,\chi_{ij}^{kl}\,\chi_{\{|k-i|\le N\}} \tag{1.14}$$

where $\chi_{ij}^{kl}$ is as in (1.11) and $p_{ik} = (i-k)/\text{g.c.d.}(i_1-k_1,...,i_d-k_d)$. From the definition of the coefficients $A_{ij}^{kl}$, it is clear that the relations (1.5), (1.7) hold. It is proved [30, 22], that the collision invariants reduce to the classical ones (and the proof is equivalent to the one given by Buet for his model [9]). The convergence result of the type of Theorem 1.1 is also obtained, with the convergence estimate $C_\varepsilon h^{1-\varepsilon}$ in the case of two dimensions, and $C_\varepsilon h^{6/11-\varepsilon}$ in three dimensions.

**2. A discrete-velocity model using Carleman's variables.** In this paper we introduce a discrete-velocity model based on Carleman's representation of the Boltzmann collision integral [11]. The principles for constructing this model are similar to those considered above. The main difference is that in Carleman's variables the integration over spheres in the collision term is replaced by integration over planes. Thus, after applying the rectangle formula for the integral over $\mathbb{R}^3$, the question of uniform distribution of integer points in the domain of integration becomes much simpler than for spheres.

The Carleman representation involves a change of variables:

$$(\eta,\,\omega)\ \mapsto\ \big(p=\xi-\omega(\omega,\xi-\eta),\ q=\eta+\omega(\omega,\xi-\eta)\big).$$

For fixed values of $\xi$ and $p$, $q$ runs twice over the plane $E_{\xi p}$, containing $\xi$ and orthogonal to $\xi-p$, when $\omega$ runs over the unit sphere. The functional determinant of the inverse transform is $|\xi-p|^{-2}$.

It is convenient for our purposes to modify these new variables as follows:

$$u = p-\xi, \quad w = q-\xi.$$

We also use the notation $E_u$ for the plane orthogonal to $u$:

$$E_u = \big\{w\in\mathbb{R}^3\ \big|\ (u,w)=0\big\}\,.$$

In the new variables the collision operator is transformed as follows:

$$Q(f,g)(\xi) = \int\limits_{\mathbb{R}^3}\int\limits_{E_u} (f(\xi+u)g(\xi+w) - f(\xi)g(\xi+u+w))\,\widetilde{B}(u,w)\,dw\,du, \tag{2.1}$$

where

$$\widetilde{B}(u,w) = 2\,|u|^{-2}B_0\left(\sqrt{u^2+w^2},\,\frac{|u|}{\sqrt{u^2+w^2}}\right). \tag{2.2}$$

To construct the discrete-velocity model we use the same velocity space $\mathcal{Z}_h$ defined by (1.10) as in the two models discussed above. Recall that the letters $i$, $j$, $k$, $l$ always denote vectors in $\mathbb{Z}^3$. Using them as the indices of velocity variables, we always mean multiplication by $h$: $\xi_i = hi$, etc. When considering functions on $\mathcal{Z}_h$, the lower index is used to denote the value at the corresponding point: $f_i = f(\xi_i) = f(hi)$, $i \in \mathbb{Z}^3$. To introduce a quadrature formula which uses the values of the distribution function only at the points of $\mathcal{Z}_h$ for approximating the integrals in (2.1), we follow the same formal scheme as Bobylev *et al.* [7] and Buet [9]. First we work formally, assuming that the collision kernel $\widetilde{B}$ in (2.1) is a smooth function. We leave the necessary modifications due to singularities of $\widetilde{B}$ to the end of this section. In our notations

$$\widetilde{F}(\xi, u, w) = (f(\xi+u)g(\xi+w) - f(\xi)g(\xi+u+w))\widetilde{B}(u,w)$$

and

$$\widetilde{G}(\xi, u) = \int_{E_u} \widetilde{F}(\xi, u, w)\, dw\, du.$$

The integrals over $\mathbb{R}^3$ are approximated by the three-dimensional rectangle formula:

$$\int_{\mathbb{R}^3} \widetilde{G}(\xi, u)\, du \approx h^3 \sum_{k\in\mathbb{Z}^3} \widetilde{G}(\xi, u_k). \tag{2.3}$$

Then the values of the integrand at the points of $\mathcal{Z}_h$ lying on the planes $E_{u_k}$ are used to calculate integrals over the planes. To obtain an expression for this last approximation, let us consider the set $\mathcal{L}_{u_k,h}$ which is the intersection of the discrete velocity space and the plane $E_{u_k}$ for a fixed $k \in \mathbb{Z}^3$, $k \neq 0$:

$$\mathcal{L}_{u_k,h} \stackrel{\text{def}}{=} E_{u_k} \cap \mathcal{Z}_h = \left\{ hl \in \mathbb{R}^3 \;\middle|\; (u_k, l) = 0,\ l \in \mathbb{Z}^3 \right\} = hL_k, \tag{2.4}$$

where

$$L_k = \left\{ l \in \mathbb{Z}^3 \;\middle|\; (k, l) = 0 \right\}, \tag{2.5}$$

which is the set of solutions of the linear Diophantine equation $(k,l) = 0$. This last set forms a lattice of rank two in $\mathbb{Z}^3$, i.e.,

$$L_k = \{e_1 m + e_2 n \mid e_1, e_2 \in \mathbb{Z}^3;\ m, n \in \mathbb{Z}\},$$

where the vectors $e_1$ and $e_2$, called the basis vectors of the lattice, are linearly independent over $\mathbb{R}$ as vectors of $\mathbb{R}^3$. Then the standard lattice rule can be used for calculation of the integrals over $E_{u_k}$:

$$\int_{E_{u_k}} \widetilde{F}(\xi, u_k, w)\, dw \;\approx\; \delta_{u_k} \sum_{w\in\mathcal{L}_{u_k,h}} \widetilde{F}(\xi, u_k, w) = h^2 \Delta_k \sum_{l\in L_k} \widetilde{F}(\xi, u_k, w_l), \tag{2.6}$$

where $\delta_{u_k}$ and $\Delta_k$ are the areas of fundamental cells of $\mathcal{L}_{u_k,h}$ and $L_k$, respectively (that is, the areas of the parallelograms spanned by the lattice basis vectors). We have $\delta_{u_k} = h^2\Delta_k$. Note that, though the basis of the lattice can be chosen in different ways, $\Delta_k$ (and $\delta_{u_k}$) does not depend on this choice [18], and we have the following explicit expression for $\Delta_k$ [31]:

$$\Delta_k = |k|/\mathfrak{g}(k). \tag{2.7}$$

Here $|k| = ({k_1}^2 + {k_2}^2 + {k_3}^2)^{1/2}$ and $\mathfrak{g}(k) = \text{g.c.d.}(k_1, k_2, k_3)$, the greatest common divisor of the components of $k$. This quadrature formula can be interpreted intuitively as follows: each basis of the integration lattice $\mathcal{L}_{k,h}$ defines a splitting of the plane $E_{u_k}$ into the set of equal parallelograms centered around the corresponding lattice points. They are all obtained by shifting the fundamental cell. Summing up the values of the integrand at the points of the lattice times the area of the cell gives the approximation (2.6).

Combining (2.3) and (2.6) we arrive at the following expressions for the discrete collision term for $\xi_i \in \mathcal{Z}_h$:

$$(2.8)\quad Q(f,g)(\xi_i) = h^5 \sum_{k\in\mathbb{Z}^3} \Delta_k \sum_{l\in L_k} \left(f(\xi_i + u_k)g(\xi_i + w_l) - f(\xi_i)g(\xi_i + u_k + w_l)\right)\widetilde{B}_{jk},$$

where $\widetilde{B}_{jk} = \widetilde{B}(u_k, w_l)$.

Using (2.8) it is easy to obtain the expression for the discrete collision term in the form (1.4). The definition of the coefficients $A_{ij}^{kl}$ is now

$$A_{ij}^{kl} = h^5 \Delta_{k-i} \widetilde{B}_{k-i,l-i} \chi_{ij}^{kl}, \tag{2.9}$$

where $\chi_{ij}^{kl}$ is as in (1.11). To obtain this we used the fact that

$$\left\{ \begin{array}{c} (k-i) \perp (l-i) \\ j = k + l - i \end{array} \right. \quad \text{if and only if} \quad \left\{ \begin{array}{c} k + l = i + j \\ k^2 + l^2 = i^2 + j^2. \end{array} \right.$$

From this form of the discrete collision operator, the symmetry conditions (1.5) can be observed easily. Clearly, (1.7) is also fulfilled. The proof of the statement that the collision invariants are reduced to the classical ones can be obtained by analogy with the two models discussed above by repeating the arguments of Rogier and Schneider [30] or Buet [9]. Thus, the discrete-velocity model obtained for the collision term satisfies all of the conservation properties discussed above. We summarize these properties in the following theorem.

Theorem 2.1. *The discrete-velocity model defined by (1.3), (1.4) with the velocity space (1.10) and coefficients defined by (2.9) satisfy the discrete mass, momentum, and energy conservation laws as well as the entropy property, expressed by the conditions (1.8). All of the solutions to the equation (1.9) are given by the linear combinations of* $1$, $\xi_i$ *and* $|\xi_i|^2$. *This means that there are no other collision invariants except the classical ones.*

To treat the problem of consistency of the discrete-velocity model described, we have to make some smoothness assumptions on the integrands, in particular the function $\widetilde{B}(u,w)$. As can be seen from (1.2) and (2.2), $\widetilde{B}(u,w)$ has a singularity at $u = 0$ even when the kernel function $B_0$ in the collision term is smooth. Thus, to include physically meaningful cases we have to admit singularities in $\widetilde{B}(u,w)$; accordingly we assume that for some $\beta \geq 0$

$$\left\| |u|^\beta \widetilde{B}(u,w) \right\|_{C^m} < +\infty \tag{2.10}$$

for an integer $m \geq 0$. Here $\|\cdot\|_{C^m}$ denotes the norm of $C^m(\mathcal{P})$, $\mathcal{P}$ being the domain of definition for $\widetilde{B}$:

$$\mathcal{P} = \left\{(u,w) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; u \perp w\right\}.$$

This assumption is fulfilled for any $m \geq 1$ with $\beta = 1$ for "hard sphere" interactions, since for these $\widetilde{B}(u,w) = 2|u|^{-1}$. For the collision kernels of the type (1.2) assumption (2.10) and the condition $\beta < 3$, which will be used in subsequent analysis, put specific restrictions on the values of $\gamma$ and the form of function $b(x)$, so that one can find the cases with both "hard" ($\gamma > 0$) and "soft" ($\gamma \leq 0$) interactions with the values of $\beta$ in the interval $1 < \beta < 3$.
In the formal construction of the model we need to assign some values to the coefficients $A_{ij}^{kl}$, for $k = l$ and $i = j$, which correspond to the points of singularity of the collision kernel $\widetilde{B}(u,w)$. We put $A_{ij}^{kl} = 0$ for these $i$, $j$, $k$, $l$. Since these coefficients correspond to "trivial" collisions which do not give any contribution to the collision term, all the steps of formal analysis of the DVM still hold and Theorem 2.1 is also true.
In the next three sections we give the consistency analysis for the discrete collision term of the model.

**3. A consistency result for the discrete-velocity model.** The key result of the paper is the following theorem.
THEOREM 3.1. *Assume that $f$, $g \in C^m(\mathbb{R}^3)$ with compact support ($m \geq 1$) and that the kernel $\widetilde{B}$ in the collision term satisfies (2.10) with $0 \leq \beta < 3$. Then, for sufficiently small $h$*

$$|Q(f,g)(\xi_i) - Q_h(f,g)(\xi_i)| \leq Ch^r, \quad \xi_i \in \mathcal{Z}_h,$$

*where*

$$r = \min\left(\frac{m}{m+3}, \beta, m/\left(m+3+\frac{\beta-m}{m+3-\beta}\right)\right),$$

*and the constant $C$ does not depend on $h$ and $\xi_i$.*
**Remark.** *We note that the third term in the expression for $r$ can be omitted when $m \geq 3$. In assumption (2.10), as well as in the formulation of Theorem 3.1, the condition of $C^m$ can be changed to Lipschitz continuous/Lipschitz differentiable.*
The consistency of the quadrature formula in the class of polynomially decaying continuous functions is then obtained as a corollary. As could be expected, no convergence estimate is available for that case. We consider the spaces of a.e. bounded functions $L_r^\infty$ with $r > 0$, which consist of those $f$ for which $\|f\|_r = \|\langle\xi\rangle^r f(\xi)\|_{L^\infty} < +\infty$, where $\langle\xi\rangle$ denotes the polynomial weight $(1+\xi^2)^{1/2}$.
COROLLARY 3.2. *Let the collision kernel be such that $|u|^2\widetilde{B}(u,w)$ is continuous and for some $\gamma$: $0 \leq \gamma \leq 2$, $\sup\limits_{\rho,x} \langle\rho\rangle^{-\gamma} B_0(\rho,x) < +\infty$. Assume that $f, g \in C \cap L_r^\infty(\mathbb{R}^3)$. Then for all $\varepsilon > 0$*

$$\|Q(f,g) - Q_h(f,g)\|_{r-\gamma-1-\varepsilon} \to 0 \text{ as } h \to 0. \tag{3.1}$$

From the assumptions of Theorem 3.1 it follows that the function

$$F(\xi,u,w) = (f(\xi+u)g(\xi+w) - f(\xi)g(\xi+u+w))|u|^\beta \widetilde{B}(u,w) \tag{3.2}$$

belongs to $C^m(\mathbb{R}^3 \times \mathcal{P})$. For $\xi$ in a compact set $K$, the support of $F^\pm(\xi,\cdot,\cdot)$ is uniformly bounded. For a fixed compact $K$, let us denote by $R$ and $L$ global bounds for the support and the Lipschitz constant of these functions:

$$R = \sup\left\{|u| + |w| \;\middle|\; F(\xi,u,w) \neq 0,\ (u,w) \in \mathcal{P},\ \xi \in K\right\}, \tag{3.3}$$

$$L = \sup_{\xi \in K} \operatorname{Lip}_{\mathcal{P}} \nabla^m F(\xi,\cdot,\cdot). \tag{3.4}$$

Set

$$G(\xi, u) = \int_{E_u} F(\xi, u, w)\, dw. \tag{3.5}$$

The first step in the proof of the consistency theorem is to consider separately the two possible error sources: space and plane discretizations. The difference in (3.1) is estimated as follows: ($\sum^\circ$ denotes summation over all points but zero)

$$\begin{aligned}
\big|Q\ (f,g)(\xi_i) - Q_h(f,g)(\xi_i)\big| &= \Big| \int\limits_{\mathbb{R}^3} |u|^{-\beta} \int\limits_{E_u} F(\xi_i, u, w)\, du\, dw \\
&- \ h^5 \sum_{k\in\mathbb{Z}^3}^{\circ} |u_k|^{-\beta} \Delta_k \sum_{l\in L_k} F(\xi_i, u_k, w_l)\Big| \\
&\le \Big| \int\limits_{\mathbb{R}^3} |u|^{-\beta} G(\xi_i, u)\, du - h^3 \sum_{k\in\mathbb{Z}^3}^{\circ} |u_k|^{-\beta} G(\xi_i, u_k)\Big| \\
&+ \Big| \sum_{k\in\mathbb{Z}^3}^{\circ} h^3 |u_k|^{-\beta} \Big( \int\limits_{E_{u_k}} F(\xi_i, u_k, w)\, dw - h^2 \Delta_k \sum_{l\in L_k} F(\xi_i, u_k, w_l)\Big)\Big| \\
&= \Big| \sum_{|k|\le R/h}^{\circ} \mathcal{S}_k(\xi_i, G, h)\Big| + \Big| \sum_{|k|\le R/h}^{\circ} h^3\, \mathcal{R}_k(\xi_i, F, h)\Big| = \mathcal{S} + \mathcal{R},
\end{aligned} \tag{3.6}$$

where

$$\mathcal{R}_k(\xi, F, h) = |u_k|^{-\beta} \Big( \int\limits_{E^R_{u_k}} F(\xi, u_k, w)\, dw \ - h^2 \Delta_k \sum_{l\in L^R_k} F(\xi, u_k, w_l)\Big),$$

$$\mathcal{S}_k(\xi, G, h) = \int\limits_{B_k} |u|^{-\beta} G(\xi, u)\, du \ - h^3 |u_k|^{-\beta} G(\xi, u_k).$$

Here $B_k$ is the cube with the side length $h$ and center $u_k$, and $L^R_k$ and $E^R_{u_k}$ are the intersections of $L_k$ and $E_{u_k}$ with the ball of radius $R$ centered at 0. Now, the term $\mathcal{R}$ represents the total error due to the discretization of the plane integrals, and $\mathcal{S}$ is the error of the three-dimensional rectangle formula for the function $G$. In the next two sections we give the bounds for each of these two terms.

**4. Approximation of the integrals over planes.** The first problem is to approximate the integrals over the planes $E_u$ using the lattice formula (2.6). For a fixed $k \in \mathbb{Z}^3$, and a basis for the integration lattice $\mathcal{L}_{k,h}$, the plane $E_{u_k}$ is split into a set of equal parallelograms $\{D_{k,l}\}_{l\in L_k}$, centered at the lattice points. We estimate the local approximation error on each parallelogram cell by using a standard approach based on the Bramble-Hilbert lemma (cf. [13]). Then, summation over all cells lying on the plane gives an estimate for the plane quadrature, which is subsequently used to estimate the total error given by the term $\mathcal{R}$ in (3.6).

The further analysis is based on the following classical lemma.

LEMMA 4.1. *Let $\Omega_0 \subseteq \mathbb{R}^n$ be an open bounded set with a Lipschitz boundary, such that $|\Omega_0| = 1$. Let $A$ be a linear invertible mapping on $\mathbb{R}^n$, and $\Omega = A(\Omega_0)$. Let $g \in W^m_p(\Omega)$, $m = 1, 2$, $p \in (n/m, \infty]$, and*

$$E(g) = \int_\Omega g(x)\, dx - |\Omega| g(0). \tag{4.1}$$

*Then*

$$|E(g)| \le C|\Omega|^{1/q}\|A\|^m |g|_{m,p,\Omega}. \tag{4.2}$$

*Here, $C$ does not depend on $\Omega$ and $g$, $\|A\|$ denotes the matrix norm of the mapping $A$, $q$ the conjugate exponent to $p$, and*

$$|g|_{m,p,\Omega} = \Big(\sum_{|\alpha|=m} \|\partial^\alpha g\|_p{}^p\Big)^{1/p}, \quad p<\infty, \qquad |g|_{m,\infty,\Omega} = \max_{|\alpha|=m} \|\partial^\alpha g\|_\infty. \tag{4.3}$$

The proof of the lemma for the case when $\Omega$ is a unit $n$-dimensional cube and $A$ is a scalar operator can be found in Raviart [29]. The generalization to the formulation above is then evident (cf. also [13]). To apply the result of the lemma to the integration lattice $\mathcal{L}_{k,h}$, take a square of the unit area $D_0$ as the reference domain $\Omega_0$ and the fundamental cell $D_{k,l}$, $l=0$ as $\Omega$; let $A_{k,h}\colon D_0 \to D_{k,l}$ be a linear bijection transforming $D_0$ into $D_{k,l}$. Then, with Lemma 4.1, the following bound for the error of plane integration is obtained.

LEMMA 4.2. *Let $K \subseteq \mathbb{R}^3$ be a compact set and let $F(\xi,u,w) \in C^1(\mathbb{R}^3 \times \mathcal{P})$ have compact support in $(u,w)$ uniformly with respect to $\xi \in K$. Let $R$ and $L$ be the constants defined by (3.3) and (3.4). Then*

$$|\mathcal{R}_k(\xi,F,h)| \le C_{R,L}|u_k|^{-\beta}\|A_{k,h}\|, \tag{4.4}$$

*where $C_{R,L}$ does not depend on $k$ or $h$ and is uniform with respect to $\xi$ in $K$.*

**Proof.** Using Lemma 4.1 we get the estimate

$$\begin{aligned}
|\mathcal{R}_k(\xi,F,h)| &= |u_k|^{-\beta}\Big|\int_{E^R_{u_k}} F(\xi,u_k,w)\,dw - h^2\Delta_k \sum_{l\in L^R_k} F(\xi,u_k,w_l)\Big| \\
&\le |u_k|^{-\beta}\sum_{l\in L^R_k}\Big|\int_{D_{k,l}} F(\xi,u_k,w) - h^2\Delta_k F(\xi,u_k,w_l)\Big| \\
\le C|u_k|^{-\beta}\|A_{k,h}\| &\sum_{l\in L^R_k} |D_{k,l}||F(\xi,u_k,\cdot)|_{1,\infty,D_{k,l}} \le CLR^2|u_k|^{-\beta}\|A_{k,h}\|,
\end{aligned}$$

which proves the lemma. □

**Remark.** *For functions of the class $W^m_p(\Omega)$ with $m \ge 3$, an estimate of the form (4.2) can be obtained for*

$$E_k(g) = \int_\Omega g(x)\,dx - |\Omega|g(0) - \sum_{2\le|j|\le m-1} d_j\,|\Omega|^{|j|}\int_\Omega \partial^j g(x)\,dx$$

*with a suitable choice of the coefficients $d_j$ (see [29]). Then, if the function $F$ in the formulation of Lemma 4.2 is of class $C^m$, the bound (4.4) can be changed to*

$$|\mathcal{R}_k(\xi,F,h)| \le C_{R,L}\|A_{k,h}\|^m. \tag{4.5}$$

Lemma 4.2 shows that quadrature formula (2.6) converges whenever the linear size of the cell, given by the norm of $A_{k,h}$, tends to zero as $h\to 0$. For lattice (2.5), the value of the norm can be expressed in terms of the length of the maximal basis vector.

If $\{e_1, e_2\}$ is a basis of $L_k$, that is $\{he_1, he_2\}$ is the one of $\mathcal{L}_{k,h}$, we have (taking the spherical norm for definiteness)

$$\|A_{k,h}\| = h \max(|e_1 + e_2|, |e_1 - e_2|) \leq 2h \max(|e_1|, |e_2|). \tag{4.6}$$

A bound for the norms of the basis vectors for the lattice defined by a linear Diophantine equation may be obtained using a rather elementary argument, which we recall in the next lemma.

LEMMA 4.3. *Let $a \in \mathbb{Z}^3$, $(a, x) = 0$ be a linear Diophantine equation. Then the basis of solutions $\{e_1, e_2\}$ of this equation can be chosen so that*

$$|e_1||e_2| \leq \sqrt{\tfrac{4}{3}}\,|a|. \tag{4.7}$$

**Proof.** Let $L_a$ be a lattice of solutions to $(a, x) = 0$. Then $det(L_a) = |a|/(a_1, a_2, a_3)$, where $(a_1, a_2, a_3)$ is the greatest common divisor of $a_1$, $a_2$, $a_3$. This can be easily checked directly using the expression of the solution with parameters obtained by the Euclid algorithm, or it can be obtained as a corollary of a more general fact: [31, Lemma 4C, Ch.1], Now the bound (4.7) follows from the inequality

$$|e_1||e_2| \leq \sqrt{\tfrac{4}{3}}\, \det L_a \tag{4.8}$$

which is known as Hermite's bound for the reduced basis of a lattice: [18, p.71]. □

**Remark.** *Since Lemma 4.3 deals only with two-dimensional lattices, the estimate (4.8) can be proved using simple geometrical arguments. Since $det(L_a)$ is the area of a basis parallelogram, $det(L_a) = |e_1||e_2| \sin(e_1, e_2)$; thus condition (4.8) is equivalent to $\sin(e_1, e_2) \geq \sqrt{3/4} = \sin(\pi/3)$. Then, if the angle between the vectors of the original basis is less than $\pi/3$ , we can also use $e_1$, $e_2 - ne_1$ as a basis for any integer $n$. For some $n$ the angle between $e_1$ and $e_2 - ne_1$ is then greater or equal $\pi/3$.*

The inequality (4.7) implies that, for the vectors of the reduced basis, the bound $|e_i| \leq \sqrt{\frac{4}{3}}\,|a|$ is valid. In fact, the constant in this estimate can be improved to give

$$|e_i| \leq |a|, \tag{4.9}$$

so that the norm of the linear mapping $A_{k,h}$ can be estimated as

$$\|A_{k,h}\| \leq 2h|k| = 2|u_k|. \tag{4.10}$$

We see that this bound does not imply convergence of the norm to zero when $h \to 0$. It is easy to verify that indeed there is no uniform convergence for all $k$. This is shown by taking the integer vectors $k$ of the form $(k_1, k_2, 0)$, with $k_1$, $k_2$ relatively prime and $|k| > 1/(2h)$. For such $k$, $\|A_{k,h}\| = 2|u_k| > 1$. Thus, the quadrature formulas over different planes do not converge uniformly. However, using the inequality (4.7) we can show that the fraction of points $u_k$, for which $\|A_{k,h}\|$ remains large, becomes small as $h$ tends to zero. We use this in the next lemma to prove that the error of the integral plane quadrature tends to zero.

LEMMA 4.4. *Let $K \subseteq \mathbb{R}^3$ be a compact set, and let $F(\xi, u, w) \in C^1(\mathbb{R}^3 \times \mathcal{P})$ have compact support in $(u, w)$ uniformly with respect to $\xi \in K$. Let $R$ and $L$ be the constants defined by (3.3) and (3.4). Then for sufficiently small $h$*

$$\Big| \sum_{k \in \mathbb{Z}^3} h^3 \mathcal{R}_k(\xi, F, h) \Big| \leq C_{R,L} h^r, \tag{4.11}$$

*with*

$$r = \min\Big(\frac{1}{4}, 1/\big(4 + \tfrac{2-\beta}{1+\beta}\big)\Big).$$

*This estimate is uniform with respect to $\xi$ on compact subsets of $\mathbb{R}^3$.*

**Proof.** Let us take a compact $K \subseteq \mathbb{R}^3$, $N = N(h) = R/h$ for $R$ defined by (3.3), and $\mathcal{A}_h = \big\{k \in \mathbb{Z}^3 \bigm| |k| \le N\big\}$. Then the sum over $\mathbb{Z}^3$ in (4.11) can be replaced by the sum over $\mathcal{A}_h$.

Lemma 4.3 offers a basis $\{e_1^{(k)}, e_2^{(k)}\}$ of integral solutions to $(k, x) = 0$, satisfying (4.7) and (4.9). Let $e_{\max}^{(k)}$ and $e_{\min}^{(k)}$ be the basis vectors with maximal and minimal Euclidean norms, respectively. Take a constant $\alpha \in (0,1)$ (to be fixed later).

If the inequality $|e_{\min}^{(k)}| \ge |k|^\alpha$ holds for a subset $\mathcal{B}_h = \mathcal{B}_h(\alpha)$ of $\mathcal{A}_h$, we can use (4.7) to get $|e_{\max}^{(k)}| \le \sqrt{\frac{4}{3}}|k|^{1-\alpha}$. Hence, with (4.9)

$$\|A_{k,h}\| \le 2h|e_{\max}^{(k)}| = C|u_k|^{1-\alpha}h^\alpha. \tag{4.12}$$

Then,

$$\begin{aligned}\sum_{k\in\mathcal{B}_h} h^3|\mathcal{R}_k(\xi, F, h)| &\le C_{R,L}h^\alpha \sum_{k\in\mathcal{B}_h} h^3|u_k|^{-\beta+1-\alpha}\\ &\le C_{R,L}h^\alpha \int\limits_{|u|\le R} |u|^{-\beta+1-\alpha}du = C_{R,L}h^\alpha.\end{aligned}$$

For the remaining part of the indices $k$ we have

$$\mathcal{A}_h \setminus \mathcal{B}_h = \big\{k \in \mathcal{A}_h \bigm| |e_{\min}^{(k)}| < |k|^\alpha\big\} \subseteq \big\{k \in \mathcal{A}_h \bigm| |e_{\min}^{(k)}| < N^\alpha\big\} \stackrel{\mathrm{def}}{=} \mathcal{D}_h.$$

Now, it is easy to estimate the number of points in $\mathcal{D}_h$, and hence in $\mathcal{A}_h \setminus \mathcal{B}_h$. Simple geometrical arguments show that

$$\Big|\big\{j \in \mathbb{Z}^3 \bigm| |j| \le n\big\}\Big| \le Cn^3, \tag{4.13}$$

$$\Big|E_{u_k} \cap \big\{j \in \mathbb{Z}^3 \bigm| |j| \le n\big\}\Big| \le Cn^2. \tag{4.14}$$

Using (4.13) and (4.14) we conclude that the number of different vectors $e_{\min}^{(k)}$ satisfying the inequality $|e_{\min}^{(k)}| < N^\alpha$ is estimated by $CN^{3\alpha}$, and the number of vectors $k \in \mathcal{A}_h$ on each plane $\big\{k \in \mathbb{Z}^3 \bigm| (e_{\min}^{(k)}, k) = 0\big\}$ is estimated by $CN^2$. This yields

$$\big|\mathcal{D}_h\big| \le CN^{2+3\alpha}$$

and if $\beta \le 1$

$$\begin{aligned}\Big|\sum_{k\in\mathcal{D}_h} h^3\mathcal{R}_k(\xi, F, h)\Big| &\le C_{R,L}\sum_{k\in\mathcal{D}_h} h^3|u_k|^{-\beta+1}\\ &\le C_{R,L}h^3R^{-\beta+1}\left(\frac{R}{h}\right)^{2+3\alpha} = C_{R,L}h^{1-3\alpha}.\end{aligned}$$

By letting $\alpha = \frac{1}{4}$ we obtain the estimate $C(F)h^{1/4}$ for both parts of $\sum\limits_{j\in\mathcal{A}_h} h^3\mathcal{R}_k(\xi, F, h)$.

If $\beta > 1$, then $|u_k|^{-\beta+1}$ is unbounded on $\mathcal{D}_h$, and the contributions of "large" and "small" $k$ should be considered separately. For this purpose let us take some $\sigma \in (0,1)$, and split the above sum in the following way:

$$\sum_{k\in\mathcal{D}_h} h^3|u_k|^{-\beta+1} \le \sum_{k\le N^{1-\sigma}} h^3|u_k|^{-\beta+1} + \sum_{\substack{N^{1-\sigma}\le k\le N\\ k\in\mathcal{D}_h}} h^3|u_k|^{-\beta+1}$$

$$\le C\int_{|u|\le R^{1-\sigma}h^{\sigma}} |u|^{-\beta+1}\,du + h^3\left(\frac{R}{h}\right)^{2+3\alpha}\max_{N^{1-\sigma}\le k\le N}|u_k|^{-\beta+1}$$

$$\le C_R\big(h^{\sigma(4-\beta)} + h^{1-3\alpha-\sigma(1-\beta)}\big).$$

Thus, if $\sigma = \alpha/(4-\beta)$, and $\alpha = 1/\big(4+\frac{\beta-1}{4-\beta}\big)$, we find

$$\Big|\sum_{k\in\mathcal{D}_h} h^3\mathcal{R}_k(\xi,F,h)\Big| \le C_{R,L}h^{\alpha}.$$

By combining the estimates for $\mathcal{B}_h$ and $\mathcal{D}_h$, we complete the proof of the lemma. □
**Remark.** *For functions from the class $C^m$, the estimate of Lemma 4.4 can be improved. By applying inequality (4.5) instead of (4.4), we find that for $m = 2$ the value*

$$r = \min\left(\frac{2}{5}, 2/\big(5+\tfrac{\beta-2}{5-\beta}\big)\right),$$

*is obtained, and for $m \ge 3$,*

$$r = \frac{m}{m+3}.$$

**5. Approximation of the integrals over $\mathbb{R}^3$ and proofs of the consistency results .** This section treats the approximation error of the quadrature formula over $\mathbb{R}^3$, which is given by the term $\mathcal{S}$ in (3.6). Let $F(\xi,u,w)$ and $G(\xi,u)$ be defined by (3.2) and (3.5), $F \in C^1(\mathbb{R}^3\times\mathcal{P})$, and for a compact $K \subseteq \mathbb{R}^3$, let $R$ and $L$ be the constants defined by (3.3) and (3.4). Then $suppG(\xi,\cdot) \subseteq B(0,R)$, and $G(\xi,\cdot)$ is a $C^1$-function outside a neighborhood of origin, uniformly bounded for all $\xi \in K$. Thus the only difference between the present situation and the one covered in Lemma 4.2 is that now the integrand $|u|^{-\beta}G(\xi,u)$ (recall that $Q(f,g) = \int_{\mathbb{R}^3}|u|^{-\beta}G(\xi,u)\,du$) has a singularity at $u = 0$. Thus the aim of the first lemma is to give a Lipschitz condition for the function $G$.
LEMMA 5.1. *Let $K \subseteq \mathbb{R}^3$ be a fixed compact set; then for $u,u' \ne 0$ and $\xi \in K$*

$$\big||u|^{-\beta}G(\xi,u) - |u'|^{-\beta}G(\xi,u')\big| \le C_{R,L}|u|^{-1}(|u|^{-\beta}+|u'|^{-\beta})|u-u'|.$$

**Proof.** First, we prove that $G$ satisfies an analogous inequality obtained by choosing $\beta = 0$. Let us fix two nonzero vectors $u$ and $u' \in \mathbb{R}^3$, and denote by $A_{u\to u'}$ the rotation map, with respect to the line orthogonal to both $u$ and $u'$, which transforms $E_u$ to $E_{u'}$. Then $G(\xi,u') = \int_{E_u} F(\xi,u',A_{u\to u'}w)\,dw$ and, from the Lipschitz condition for $F$, it follows that

$$\big|G(\xi,u) - G(\xi,u')\big| \le L\int_{E_u\cap\{|u|\le R\}} \big(|u-u'| + |w - A_{u\to u'}w|\big)\,dw'.$$

To estimate $w - A_{u\to u'}w$, we note that for all vectors $x$ in the unit ball $B(0,1)$

$$|x - A_{u\to u'}x| \le \left|\frac{u}{|u|} - A_{u\to u'}\frac{u}{|u|}\right| = \left|\frac{u}{|u|} - \frac{u'}{|u'|}\right| \le 2\frac{|u-u'|}{|u|}$$

and, hence, $|w - A_{u\to u'}w| \le 2\frac{|u-u'|}{|u|}|w|$. Then

$$\big|G(\xi,u) - G(\xi,u')\big| \le L|u-u'| \int\limits_{E_u\cap\{|u|\le R\}} dw + 2L\frac{|u-u'|}{|u|} \int\limits_{E_u\cap\{|u|\le R\}} |w|\,dw \le C_{R,L}|u|^{-1}|u-u'|.$$

Now, the estimate of the lemma is obtained using the uniform boundedness of $G$ from the elementary inequality

$$\frac{|x^{-\beta} - y^{-\beta}|}{|x-y|} = x^{-\beta-1}\frac{|1-\left(\frac{y}{x}\right)^{-\beta}|}{|1-\frac{y}{x}|} \le Cx^{-\beta-1}\left(1+\left(\frac{y}{x}\right)^{-\beta}\right) = Cx^{-1}(y^{-\beta}+x^{-\beta})$$

which holds for all $x$, $y > 0$, $x \neq y$. □

LEMMA 5.2. *Let $K \subseteq \mathbb{R}^3$ be a compact set with $R$ and $L$ as the constants given by (3.3) and (3.4). Then*

$$\Big|\sum_{k\in\mathbb{Z}^3} \mathcal{S}_k(\xi,G,h)\Big| \le C_{R,L}h^r,$$

*where $r = \min(1, 3-\beta)$, and the bound is uniform with respect to $\xi$ in $K$.*

**Proof.** Using Lemma 5.1, the Lipschitz constant of $|u|^{-\beta}G(\xi,u)$ on each cube $B_k$ (that is $\big\||u|^{-\beta}G(\xi,u)\big|_{1,\infty,B_k}$) is estimated by $C_{R,L}\max\limits_{B_k}|u|^{-1-\beta}$, so that by applying Lemma 4.1 we obtain

$$\begin{aligned}\Big|\sum_{k\in\mathbb{Z}^3} \mathcal{S}_k(\xi,G,h)\Big| &\le \sum_{|k|\le R/h}\Big|\int\limits_{B_k}|u|^{-\beta}G(\xi,u)\,du - h^3G(\xi,u_k)\Big| \\ &\le \int\limits_{B_0}|u|^{-\beta}G(\xi,u)\,du + C_{R,L}h\sum_{0<|k|\le R/h} h^3\max_{B_k}|u|^{-1-\beta}.\end{aligned}$$

The integral over $B_0$ is bounded as $C\int_{B_0}|u|^{-\beta}\,du = C_1h^{3-\beta}$. Standard arguments show that the last sum is bounded by $C_2h^{r-1}$ so that the estimate of the lemma follows. □

The proof of Theorem 3.1 is now achieved by combining the estimates of Lemmas 4.4 and 5.2. To prove Corollary 3.2, we state the stability result for the discrete collision operator $Q_h$ in the norms of $L_r^\infty$. The desired result can then be obtained by approximation. For the original Boltzmann collision operator with the collision kernel satisfying the conditions of Corollary 3.2, the estimate $\|Q(f,g)\|_{r-\gamma} \le C\|f\|_r\|g\|_r$ is known [11, 21]. We prove an analogous estimate for $Q_h$.

LEMMA 5.3. *Let $B$ satisfy the conditions of Corollary 3.2, and let* $\lambda = \sup\limits_{\rho,x}\langle\rho\rangle^{-\gamma}B_0(\rho,x)$ .
*Then for all $f$, $g \in C\cap L_r^\infty(\mathbb{R}^3)$ with $r > 3$*

$$\|Q_h(f,g)\|_{r-1-\gamma} \le C\,\lambda\,\|f\|_r\|g\|_r$$

*where $C$ depends only on $r$.*

**Remark.** *The result of the lemma indicates that at infinity the discrete collision term may decay slower than the Boltzmann collision integral. The reason is that the size of the cells of the quadrature formula for planes can be on the order of the relative velocity. Thus the result of Corollary 3.2 involves lower order polynomial norms than might be expected.*

**Proof.** The kernel $\widetilde{B}$ is estimated according to (2.2) as follows:

$$(5.1)\qquad \widetilde{B}(u,w) \le 2\lambda |u|^{-2}\big\langle \sqrt{u^2+w^2}\big\rangle^{\gamma} \le 2\lambda |u|^{-2}\big(\langle u\rangle^{\gamma} + \langle \xi\rangle^{\gamma} + \langle \xi + w\rangle^{\gamma}\big).$$

Due to the homogeneity of $Q_h$, it suffices to estimate the expression $\frac{1}{\lambda}Q_h(\langle\cdot\rangle^{-r},\langle\cdot\rangle^{-r})$. The "gain" and "loss" terms $Q_h^+$ and $Q_h^-$ can be dealt with separately; we give the proofs only for the $Q_h^+$ term, since the estimates for $Q_h^-$ are obtained in the same way. It is sufficient to consider only the terms corresponding to the first term in estimate (5.1), the other two being similar. Thus for the $Q_h^+$-term we need to estimate the expression

$$(5.2)\qquad I_1 = h^3 \sum_{\xi'\in\mathcal{Z}_h} \langle \xi'\rangle^{-r} |\xi'-\xi|^{-2} \langle \xi'-\xi\rangle^{\gamma}\, \delta_{\xi'-\xi} \sum_{\eta'\in \xi+\mathcal{L}_{\xi'-\xi}} \langle \eta'\rangle^{-r}.$$

For any monotonously decreasing function $\varphi(x)$, $x\in[0,\infty)$,

$$h\sum_{n=0}^{\infty}\varphi(hn) \le h\varphi(0) + \int_0^{\infty}\varphi(x)\,dx.$$

Accordingly, the inner sum over the lattice points in (5.2) can be estimated as follows:

$$\delta_{\xi'-\xi}\sum_{\eta'\in\xi+\mathcal{L}_{\xi'-\xi}} \langle\eta'\rangle^{-r} \le 2\,\delta_{\xi'-\xi} \max_{\xi+E_{\xi'-\xi}} \langle\eta'\rangle^{-r}$$

$$+\,2\,\max(|e_1|,|e_2|) \sup_{l\subseteq \xi+E_{\xi'-\xi}} \int_l \langle\eta'\rangle^{-r}\,d\eta' \;+\; \int_{\xi+E_{\xi'-\xi}} \langle\eta'\rangle^{-r}\,d\eta'$$

$$(5.3)\qquad \le\; 2h|\xi'-\xi|\langle\eta'_{\max}\rangle^{-r} + c_1|\xi'-\xi|\langle\eta'_{\max}\rangle^{-r+1} + c_2\langle\eta'_{\max}\rangle^{-r+2},$$

where the supremum is taken over all lines $l$ on the plane $\xi+E_{\xi'-\xi}$ passing through the point $\xi$, and $\eta'_{\max}$ is the point where $\langle\eta'\rangle^{-r}$ attains its maximum on the plane $\xi+E_{\xi'-\xi}$. From the collision geometry, $|\eta'_{\max}| = |\xi|\cos\theta$, where $\theta$ is the angle between $\xi$ and $\xi'-\xi$. Then, for any smooth function $\psi$

$$\Big|\int_{\mathcal{C}_i}\psi(\xi)\,d\xi - h^3\psi(\xi_i)\Big| \le Ch\int_{\mathcal{C}_i}|\nabla\psi(\xi)|\,d\xi.$$

We apply this estimate for the expression obtained by substitution of estimate (5.3) into (5.2) and, for simplicity, deal only with the second term in (5.3), since the same arguments can be used for the remaining two terms. For $\mathcal{E}(\xi,\xi') = \langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\langle|\xi|\cos\theta\rangle^{-r+1}$,

$$\Big|h^3\sum_{\xi'\in\mathcal{Z}_h}\mathcal{E}(\xi,\xi') - \int_{\mathbb{R}^3}\mathcal{E}(\xi,\xi')\,d\xi'\Big| \le Ch\int_{\mathbb{R}^3}|\nabla_{\xi'}\mathcal{E}(\xi,\xi')|\,d\xi.$$

Thus, to estimate the sum over $\xi'\in\mathcal{Z}_h$ one needs only the bounds for $\int\mathcal{E}(\xi,\xi')\,d\xi'$ and $\int|\nabla_{\xi'}\mathcal{E}(\xi,\xi')|\,d\xi'$. We split the integration over $\xi'$ into two parts:

$$\int_{\mathbb{R}^3}\mathcal{E}(\xi,\xi')\,d\xi' = \int_{\{|\cos\theta|>\frac12\}} + \int_{\{|\cos\theta|\le\frac12\}}.$$

For the first integral on the right hand side, we have

$$\int_{\{|\cos\theta|>\frac{1}{2}\}} \langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\langle|\xi|\cos\theta\rangle^{-r+1}\,d\xi'$$
$$\le 2^{(r-1)/2}\langle\xi\rangle^{-r+1}\int_{\mathbb{R}^3}\langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\,d\xi' \le C_r\langle\xi\rangle^{-r+\gamma}.$$

For the second one, using the polar coordinates $\xi'-\xi=\rho\omega$, we get

$$\int_{\{|\cos\theta|\le\frac{1}{2}\}} \langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\langle|\xi|\cos\theta\rangle^{-r+1}\,d\xi'$$
$$\le 4\pi 2^{(r-1)/2}\int_0^\infty \rho\langle\rho\rangle^{\gamma}(|\xi|+\rho)^{-r}d\rho\int_0^{\frac{1}{2}}\langle|\xi|x\rangle^{-r+1}dx$$
$$\le \frac{C_r}{|\xi|}\int_0^\infty \rho\langle\rho\rangle^{\gamma}(|\xi|+\rho)^{-r}d\rho \le C_r\langle\xi\rangle^{-r+\gamma+1}.$$

We get the estimate for the integral of the gradient norm with the same exponent by using the same arguments. The estimate of the lemma then follows. □

Now Corollary 3.2 can be proved by the following argument: given $f$ and $g \in C\cap L_r^\infty(\mathbb{R}^3)$, we can take sequences of smooth functions with compact support $f_n$ and $g_n$ such that $\|f_n-f\|_{r-\varepsilon}$ and $\|g_n-g\|_{r-\varepsilon}\to 0$, and a sequence $\widetilde{B}^n$ such that $|u|^2\widetilde{B}^n(u,w)$ is smooth and $\left\||u|^2\langle\sqrt{|u|^2+|w|^2}\rangle^{-\gamma}(\widetilde{B}^n-\widetilde{B})\right\|_{L^\infty}\to 0$. Then

$$\|Q_h(f,g)-Q(f,g)\| \le \|Q_h(f,g)-Q_h(f_n,g_n)\|$$
$$+\|Q_h(f_n,g_n)-Q(f_n,g_n)\|+\|Q(f_n,g_n)-Q(f,g)\|,$$

where $\|\cdot\|$ is the norm $\|\cdot\|_{r-\gamma-1-\varepsilon}$. The second term on the left side converges to 0 for every fixed $n$, and the other two converge to 0 as $n\to\infty$ uniformly in $h$, which proves the statement of Corollary 3.2.

## 6. Convergence to the solutions of the Boltzmann equation.

The consistency results of Theorem 3.1 and Corollary 3.2 indicate the formal correspondence between the Boltzmann equation and the equations of DVMs. Some additional steps should be taken to justify the use of a numerical algorithm based on the DVM equations. First we need to make more precise the way the distribution function is approximated by the solution of the discrete problem. For this we use the finite-volume approximation, following S. Mischler [23]. The distribution function is approximated by a piecewise constant function

$$f(\xi,x,t)\approx f_h(\xi,x,t)=\sum_i f_i(x,t)\chi_{\mathcal{C}_i}(\xi),$$

where $\mathcal{C}_i$ are cubic cells with the side length $h$ centered at $\xi_i$. The values $f_i$ at the nodes (or cells) are determined from a system of equations of the form (1.3). This system can be rewritten in terms of the function $f_h(\xi,x,t)$ as the Boltzmann-like equation:

$$\frac{\partial f_h}{\partial t}+v_h(\xi)\cdot\nabla_x f_h=\iiint\left[f_h(\xi')f_h(\eta')-f_h(\xi)f_h(\eta)\right]B_h(\xi,\eta,\xi',\eta')\,d\eta\,d\xi'\,d\eta', \tag{6.1}$$

where $v_h(\xi)=\sum_i \xi_i\chi_{\mathcal{C}_i}(\xi)$, and the kernel $B_h$ is given by

$$B_h(\xi,\eta,\xi',\eta')=\frac{1}{h^9}\sum_{ijkl}A_{ij}^{kl}\chi_{\mathcal{C}_i\times\mathcal{C}_j\times\mathcal{C}_k\times\mathcal{C}_l}(\xi,\eta,\xi',\eta'). \tag{6.2}$$

The transition to the finite velocity space can now be made by taking a positive constant $R$ (which in a subsequent analysis is made dependent on $n$) and defining a new collision kernel $B_n^R$ as

$$B_n^R(\xi,\eta,\xi',\eta') = B_{\frac{1}{n}}(\xi,\eta,\xi',\eta')\,\chi_{\{|\xi|<R,\ |\eta|<R,\ |\xi'|<R,\ |\eta'|<R\}}, \tag{6.3}$$

which yields a finite number of velocities (or collision coefficients $A_{ij}^{kl}$) for every $n$.
Now the convergence of the solutions to the Cauchy problem for equation (6.1) to solutions of the Boltzmann equation can be studied. In the space homogeneous case the result of Palczewski and Schneider [28] can be adopted without any changes, since their proof uses only the consistency result analogous to the one proved in the previous sections. In the space inhomogeneous case we meet difficulties that are partly the same as for the model considered in Mischler [23] but there are some new ones as well. First, the existence of long time solutions to system (6.1) (with the kernel $B_h$ as well as $B_n^R$) is not known. Second, for the model considered, it is difficult to prove the properties required in Mischler's convergence proof. To overcome these difficulties we use the approximation scheme used by DiPerna and Lions [14] for proving existence of solutions to the Boltzmann equation.
We consider the Cauchy problem for the Boltzmann equation with given $L^1$ initial data $f^0$:

$$\frac{\partial f}{\partial t} + \xi\cdot\nabla_x f = Q(f,f), \tag{6.4}$$

$$f(\xi,x,0) = f^0(\xi,x), \tag{6.5}$$

where the solution $f$ is to be understood in the DiPerna-Lions sense, see [14]. We also introduce for $\delta>0$ an auxiliary Cauchy problem

$$\frac{\partial f_\delta}{\partial t} + \xi\cdot\nabla_x f_\delta = \widetilde{Q}_\delta(f_\delta,f_\delta) = \left(1+\delta\textstyle\int f_\delta\,d\xi\right)^{-1} Q_\delta(f_\delta,f_\delta), \tag{6.6}$$

$$f_\delta(\xi,x,0) = f_\delta^0(\xi,x), \tag{6.7}$$

where $Q_\delta$ is a collision operator of the form (1.1) with the kernel

$$B_\delta(\xi,\eta,\omega) = \min\left\{\frac{1}{\delta}, B(\xi-\eta,\omega)\,\chi_{\{\delta\le|\xi-\xi'|\le|\xi-\eta|\le 1/\delta\}}\right\} * \mu_\delta(\xi,\eta,\omega).$$

The initial data is $f_\delta^0 = f^0\chi_{\{|\xi|<1/\delta\}} * \mu_\delta(\xi,x) + \delta\exp\left(-|x|^2-|\xi|^2\right)$, where $\mu_\delta$ denotes $C^\infty$ mollifier with the support in a $\frac{\delta}{4}$ neighborhood of 0. To approximate the solutions of (6.6)–(6.7) one might use a DVM approximation of type (6.1) with the kernel $B_{\delta n}$ obtained from $B_\delta$ according to (2.9), (6.2) and (6.3). However this leads to technical difficulties in proving convergence; the reason is that, for such a choice of the kernel, we cannot control the $L^1$ norm of the collision terms $Q_{\delta n}$ uniformly in $n$. Thus we have to introduce an additional truncation into the DVM to get the needed properties. This is done by defining the kernel

$$B_{\delta\alpha n}^R(\xi,\eta,\xi',\eta') = B_{\delta n}^R(\xi,\eta,\xi',\eta')\chi_{\{|S_{\xi\eta}|<\alpha n\}}(\xi,\eta),$$

where $S_{\xi\eta}$ denotes the set of points of the discrete velocity space falling on the sphere which has the vector $\xi-\eta$ as the diameter; $\alpha$ is a large positive parameter. Since this kernel satisfies the necessary symmetry properties, the conservation of the (discrete) mass, momentum and energy still holds. We can now define the discrete collision operator $Q^R_{\delta\alpha n}$ as in (6.1) and formulate the approximate initial-value problem:

$$\frac{\partial F}{\partial t} + v_n(\xi)\cdot\nabla_x F = \widetilde{Q}^R_{\delta n\alpha}(F,F) = \left(1+\delta\textstyle\int F\,d\xi\right)^{-1} Q^R_{\delta n\alpha}(F,F), \tag{6.8}$$

$$F(\xi,x,0) = f^0_{\delta n}(\xi,x), \tag{6.9}$$

where $F$ denotes the approximate solution $f_{\delta n\alpha}$ to shorten notations. The initial data $f^0_{\delta n}$ is a piecewise constant in $\xi$ approximation of $f^0_\delta$:

$$f^0_{\delta n} = \sum_{\xi_i\in V_n} \chi_{\mathcal{C}_i}(\xi) \int_{\mathcal{C}_i} f^0_\delta(\eta,x)\,d\eta. \tag{6.10}$$

Our aim now is to prove that the parameters $\delta$, $\alpha$ and $R$ can be chosen in such a way that the solutions to the discrete problems converge to solutions of the Boltzmann equation.

Theorem 6.1. *Let the initial condition $f^0(\xi,x)$ in (6.4)–(6.5) be a function from $L^1(\mathbb{R}^3\times\mathbb{R}^3)$ and satisfy the condition*

$$\int f^0(\xi,x)\big(1+|\xi|^2+|x|^2+\log f^0(\xi,x)\big)\,dx\,d\xi \le C_0.$$

*Then there are sequences $\delta_n\to 0$, $\alpha_n\to\infty$ and $R_n\to\infty$, such that the sequence $f_{\delta_n\alpha_n n}(\xi,x,t) = F(\xi,x,t)$ of solutions to the problem (6.8)–(6.9) with the initial data (6.10), converges weakly in $L^1$ as $n\to\infty$ up to extraction of a subsequence, to $f(\xi,x,t)$ which is a solution to (6.4)–(6.5).*

**Proof:** We prove that for any fixed $\delta>0$, $T>0$, and for all $\varepsilon>0$ we can take $\alpha>0$, $R>0$ such that

$$\sup_{t\in[0,T]} \|f_\delta - F\|_{L^1_{x\xi}} < \varepsilon, \tag{6.11}$$

for an infinite subsequence of $n$. From this and the result of DiPerna and Lions [14], which says that $f_\delta\to f$ weakly in $L^1$, the statement of the theorem follows by a diagonalization argument.

From the mild form of the equations,

$$f_\delta(\xi,x,t) - F(\xi,x,t) = (f^0_\delta(x,t) - f^0_{\delta n}(x,t)) +$$
$$\int_0^t \Big(\widetilde{Q}(f_\delta,f_\delta)(\xi,x-\xi s,s) - \widetilde{Q}^R_{\delta\alpha n}(F,F)(\xi,x-v_n(\xi)s,s)\Big)\,ds.$$

The difference in the integral can be represented as

$$\begin{aligned}
&\int_0^t \Big(\widetilde{Q}_\delta(f_\delta,f_\delta)(\xi,x-\xi s,s) - \widetilde{Q}_\delta(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s)\Big)\,ds \\
&+\int_0^t \Big(\widetilde{Q}_\delta(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s) - \widetilde{Q}_{\delta\alpha n}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s)\Big)\,ds \\
&+\int_0^t \Big(\widetilde{Q}_{\delta\alpha n}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s) - \widetilde{Q}^R_{\delta\alpha n}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s)\Big)\,ds \\
&+\int_0^t \Big(\widetilde{Q}^R_{\delta\alpha n}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s) - \widetilde{Q}^R_{\delta\alpha n}(F,F)(\xi,x-v_n(\xi)s,s)\Big)\,ds.
\end{aligned} \tag{6.12}$$

Here $\widetilde{Q}_{\delta\alpha n}$ denotes the collision operator of the infinite DVM corresponding to $R=\infty$. The last integral is estimated as $\frac{6\alpha}{\delta^3}\int_0^t \|f_\delta - F\|_{L^1_{x\xi}} ds$. Indeed we have, denoting by $\overline{f},\ \overline{g}$ the mean values of $|f|$, $|g|$ over the cubes $\mathcal{C}_i$,

$$
\begin{aligned}
\|Q^{R+}_{\delta\alpha n}(f,g)\|_{L^1_\xi} &\le \frac{1}{n^3}\sum_{\xi\in\mathcal{Z}_h}\frac{1}{n^3}\sum_{\xi'\in\mathcal{Z}_h}\overline{f}(\xi')\ \delta_{\xi'-\xi}\sum_{\eta'\in\xi+\mathcal{L}_{\xi'-\xi}}\overline{g}(\eta')\chi_{\{|S_{\xi'\eta'}|<\alpha n\}}\widetilde{B}_\delta(\xi'-\xi,\eta'-\xi)\\
&= \frac{1}{n^3}\sum_{\xi'\in\mathcal{Z}_h}\overline{f}(\xi')\ \frac{1}{n^3}\sum_{\eta'\in\mathcal{Z}_h}\overline{g}(\eta')\chi_{\{|S_{\xi'\eta'}|<\alpha n\}}\sum_{\xi\in S_{\xi'\eta'}}\delta_{\xi'-\xi}\widetilde{B}_\delta(\xi'-\xi,\eta'-\xi)\\
&\le \frac{1}{n^3}\sum_{\xi'\in\mathcal{Z}_h}\overline{f}(\xi')\ \frac{1}{n^3}\sum_{\eta'\in\mathcal{Z}_h}\overline{g}(\eta')\chi_{\{|S_{\xi'\eta'}|<\alpha n\}}\frac{1}{\delta^2}\frac{1}{n}|S_{\xi'\eta'}| \le \frac{\alpha}{\delta^2}\|f\|_{L^1_\xi}\|g\|_{L^1_\xi} \qquad (6.13)
\end{aligned}
$$

and the same estimate for $Q^{R-}_{\delta\alpha n}$. This gives the Lipschitz property for $\widetilde{Q}^R_{\delta\alpha n}$, uniformly with respect to $n$:

$$
\|\widetilde{Q}^R_{\delta\alpha n}(f,f)-\widetilde{Q}^R_{\delta\alpha n}(g,g)\|_{L^1_{x\xi}} \le \frac{6\alpha}{\delta^3}\|f-g\|_{L^1_{x\xi}},
$$

and thus proves the needed estimate. The remaining three terms in the expression (6.12) contain only the limit solution $f_\delta$ which according to [14] satisfies

$$
\int\sum_{|\alpha|\le m}\left|\partial^\alpha f_\delta(\xi,x,t)\right|\left(1+|x|^k+|\xi|^k\right)dx\,d\xi \le C_\delta(T,m,k),\quad t\in[0,T], \qquad (6.14)
$$

for all $m\ge 0,\ k\ge 0$. Thus the first integral in (6.12) converges to 0 as $n\to\infty$ by continuity. To estimate the third term in (6.12) we use

$$
\begin{aligned}
&\widetilde{Q}_{\delta\alpha n}(f_\delta,f_\delta)-\widetilde{Q}^R_{\delta\alpha n}(f_\delta,f_\delta) = \left(1+\delta\textstyle\int f_\delta\,d\xi\right)^{-1}\\
&\Big(Q_{\delta\alpha n}(f_\delta\chi_{\{|\xi|>R\}},f_\delta)+Q_{\delta\alpha n}(f_\delta,f_\delta\chi_{\{|\xi|>R\}})\Big).
\end{aligned}
$$

Thanks to the $L^\infty_r$ estimate of Lemma 5.3 and an $L^\infty_r$ bound implied by (6.14), this difference converges to 0 in $L^\infty_r$ for any $r>3$ as $R\to\infty$ uniformly with respect to $n$ in any time interval $[0,T]$. For the second integral in (6.12) we have according to Lemma 5.3 the sequence $(Q_{\delta\alpha n})_{n=0}^\infty$ of bounded bilinear forms on $C\cap L^\infty_r$. By Alaoglu's theorem there is a subsequence still denoted by $(Q_{\delta\alpha n})$ and a bounded bilinear form $Q_{\delta\alpha}$ such that

$$
Q_{\delta\alpha n}(f_\delta,f_\delta)\longrightarrow Q_{\delta\alpha}(f_\delta,f_\delta),\ n\to\infty,
$$

pointwise in $x,\xi$ and, due to the uniform bounds, in $L^1_{x\xi}$. On the other hand,

$$
Q_{\delta\alpha n}(f_\delta,f_\delta)\longrightarrow Q_{\delta n}(f_\delta,f_\delta),\ \alpha\to\infty,
$$

uniformly in $n$. To obtain this we denote by $\varphi_n(\xi,\eta)$ the piecewise constant function with values $\frac{1}{n}|S_{\xi\eta}|$; then from (6.13) follows

$$
\begin{aligned}
&\|Q_{\delta\alpha n}(f_\delta,f_\delta)-Q_{\delta n}(f_\delta,f_\delta)\|_{L^1_\xi}\\
&\le \frac{2}{\delta^2}\int\!\!\int|f_\delta(\xi')||f_\delta(\eta')|\,\varphi_n(\xi',\eta')\chi_{\{|\varphi_n|>\alpha\}}d\xi'\,d\eta'\\
&\le \frac{2}{\delta^2}\|f_\delta\|_{L^1_\xi}\|f_\delta\|_{L^\infty_r}\sup_{|\xi'|\le R}\int_{|\eta'|\le R}\varphi_n(\xi',\eta')\chi_{\{|\varphi_n|>\alpha\}}d\eta'. \qquad (6.15)
\end{aligned}
$$

For the sequence of functions $\varphi_n$ the following holds:

$$(6.16) \qquad \int\limits_{|\eta'|\le R} \varphi_n^2(\xi',\eta')d\eta' = \frac{1}{n^3}\sum_{|\eta'|\le R}\Big(\frac{1}{n}|S_{\xi\eta}|\Big)^2 \le \frac{2}{n^5}\sum_{m=1}^{(Rn)^2}(\mathfrak{r}_3(m))^3 \le C,$$

where $\mathfrak{r}_3(m)$ denotes the number of integer points on the sphere of radius $\sqrt{m}$ and the constant $C$ does not depend on $n$. This estimate follows from the Gauss formula for $\mathfrak{r}_3(m)$ [17, Theorem 2, Chapter 4] and the estimates in Barban [3, Theorem 2]. The condition (6.16) implies the necessary weak $L^1$ compactness of $(\varphi_n)$ and thus the right-hand side of (6.15) tends to 0 as $\alpha \to \infty$ uniformly in $n$. Since the consistency result of Corollary 3.2 says that $Q_{\delta n}(f_\delta, f_\delta) \to Q_\delta(f_\delta, f_\delta)$ as $n \to \infty$, we obtain $Q_{\delta\alpha} \to Q_\delta(f_\delta, f_\delta)$ as $\alpha \to \infty$. Now for the second integral in (6.12) we have

$$\begin{aligned}
&\int_0^t \Big(\widetilde{Q}_\delta(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s) - \widetilde{Q}_{\delta\alpha n}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s)\Big)\,ds\\
&= \int_0^t \Big(\widetilde{Q}_\delta(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s) - \widetilde{Q}_{\delta\alpha}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s)\Big)\,ds\\
&+ \int_0^t \Big(\widetilde{Q}_{\delta\alpha}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s) - \widetilde{Q}^{\infty}_{\delta\alpha n}(f_\delta,f_\delta)(\xi,x-v_n(\xi)s,s)\Big)\,ds\\
&= \varepsilon_1(\alpha) + \varepsilon_2(\alpha,n),
\end{aligned}$$

where $\varepsilon_1(\alpha) \to 0$, $\alpha \to \infty$, and $\varepsilon_2(\alpha, n) \to 0$, $n \to \infty$ for any fixed $\alpha$.
It follows now, from combining the estimates for all four terms in (6.12) and the convergence of the initial data, that

$$\|f_\delta - F\|_{L^1_{x\xi}}(t) \le \frac{\alpha}{\delta^3}\int_0^t \|f_\delta - F\|_{L^1_{x\xi}}ds + \varepsilon_1(\alpha) + \varepsilon_2(\alpha,n) + \varepsilon_3(n) + \varepsilon_4(R,n),$$

where $\varepsilon_3(n) \to 0$ as $n \to \infty$ and $\varepsilon_4(R, n) \to 0$ as $n \to \infty$ for fixed $R$. Thus (6.11) follows by Gronwall's inequality. This completes the proof of the theorem. □

**7. Numerical results. Spatially homogeneous relaxation.** The accuracy of the collision integral approximation, as well as its computational cost, is illustrated by a comparison with the exact solution of the space homogeneous Boltzmann equation for Maxwellian molecules, which was obtained by A. Bobylev [5] and, independently, by M. Krook and B. Wu [20]. This solution has the form

$$(7.1) \qquad f(\xi,t) = (2\pi\tau)^{-\frac{3}{2}}\Big(1 + \frac{1-\tau}{\tau}\big(\frac{\xi^2}{2\tau} - \frac{3}{2}\big)\Big)\exp\Big(-\frac{\xi^2}{2\tau}\Big),$$

$$\tau(t) = 1 - \theta\exp(-\lambda t)$$

with $\theta \in [0, \frac{2}{5}]$, and $\lambda = \frac{\pi}{2}\int_{-1}^{1} g(z)(1-z^2)\,dz$ for $g(z)$ such that $B(v,\omega) = \frac{(v,\omega)}{|v|}g\big(\frac{(v,\omega)}{|v|}\big)$. We choose the collision kernel such that $\widetilde{B}(u,w) = |u|^{-1}(u^2+w^2)^{-\frac{1}{2}}$; then $g(z) \equiv \frac{1}{2}$ and $\lambda = \frac{\pi}{3}$. We also take $\theta = \frac{2}{5}$ so that $f(0,0) = 0$. For the numerical scheme we take the points of the regular grid $\mathcal{Z}_h$ in a cube of suitable size (determined by the parameters of the distribution function), and include in the model only those collisions for which the postcollisional velocities remain in the fixed cube. For the parameters of the solution (7.1) chosen as above, we take the cube with the side length 7.0 centered at the origin as the domain of computation. The size of the cube is kept fixed in the time-dependent problem. We denote by $N$ the number of velocities in each direction; the total number of velocities is thus $N^3$. To increase the numerical efficiency, we follow J. Schneider [22] in using the symmetry properties of the DVM coefficients in the numerical algorithm. All computations were performed on *Sun Dual UltraSparc 1700 / 167 MHz.*

In the first test we compare the exact value of the collision operator computed on $f(\xi, 0)$ and the values obtained by using discrete-velocity models. We measure the difference between the exact and approximate data by calculating relative maximum error

$$e_\infty(f, f_h) = \frac{2 \max\limits_i |f(\xi_i) - f_i|}{\max\limits_i |f(\xi_i)| + \max\limits_i |f_i|}.$$

The results are presented in the Table 7.1.

TABLE 7.1

*Relative errors of the collision term approximation and computation times depending on the number of velocities in each direction*

| Number of velocities | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 |
|---|---|---|---|---|---|---|---|---|
| $e_\infty(Q, Q_h)$ | 0.044 | 0.035 | 0.033 | 0.028 | 0.024 | 0.021 | 0.019 | 0.017 |
| CPU time, sec. | 0.38 | 1.96 | 7.25 | 23.01 | 66.43 | 135.47 | 286.94 | 638.13 |

The results of the computations show that the dependence between the step in the velocity space $h$ and the error of approximation is close to linear for the interval of $h$ chosen.

The second problem we address is the time relaxation in a space homogeneous gas. In the numerical scheme, the discrete-velocity approximation (1.4) of the collision term is combined with the forward Euler time-stepping method. The time step is chosen so as to make the time integration error neglectable in comparision with the error of the discrete-velocity approximation. We compare numerical solutions with the exact one given in (7.1); this is done by computing the relative mean error in the solution

$$e_1(f, f_h) = \frac{2 \sum\limits_i |f(\xi_i) - f_i|}{\sum\limits_i |f(\xi_i)| + \sum\limits_i |f_i|}.$$

as a function of time, and we also compute a fourth-order moment

$$m_h^{(2,2,0)}(t) = h^3 \sum_i f_i(t) \xi_x^2 \xi_y^2,$$

which is compared with the exact value of the corresponding moment for solution (7.1):

$$m^{(2,2,0)}(t) = \int_{\mathbb{R}^3} \xi_{ix}^2 \xi_{iy}^2 f(\xi, t)\, d\xi.$$

The results for the calculation of the moment are shown on Fig. 7.1. The relative error in calculating the moment is below $2.5 \cdot 10^{-3}$ for $N = 10$ and decreases with increasing $N$. The Fig. 7.2 presents the mean error in the distribution function in dependence on time.

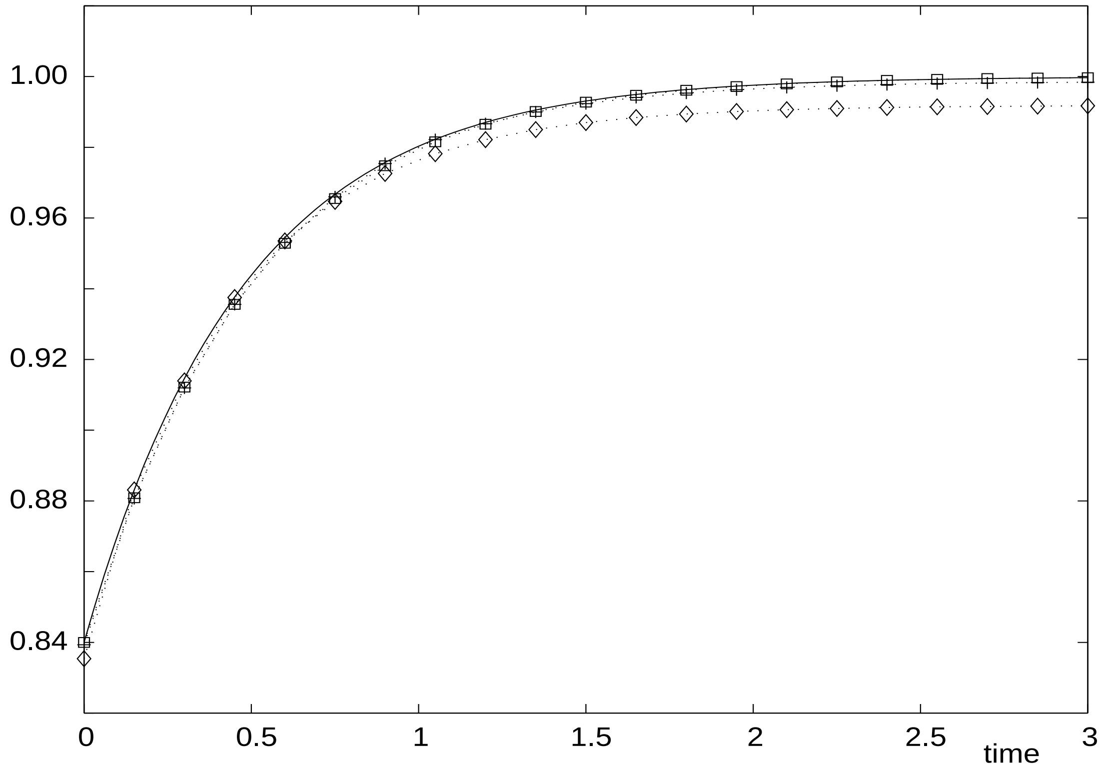


FIG. 7.1. *Fourth-order moment for the Bobylev-Krook-Wu solution, computational vs. exact. Solid line: exact; diamonds: $N = 6$; crosses: $N = 8$; squares: $N = 10$.*

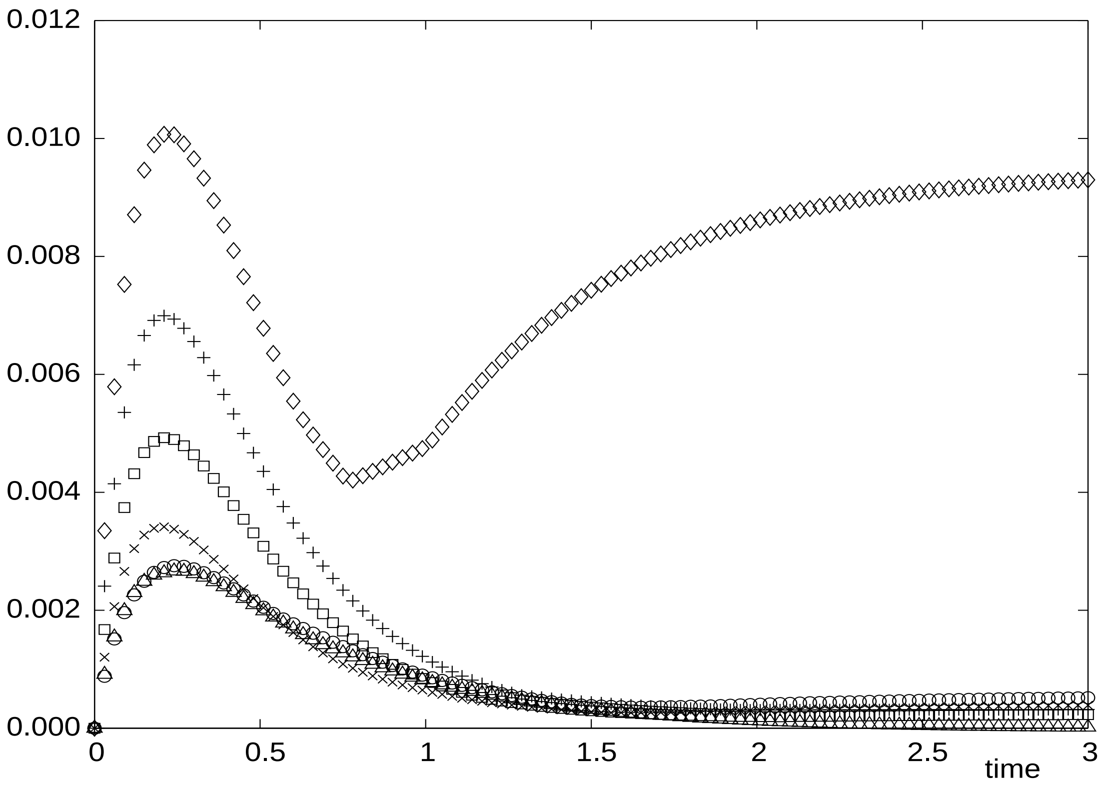


FIG. 7.2. *The relative mean error in the distribution function. Diamonds: $N = 6$; crosses: $N = 8$; squares: $N = 10$; $\times$-marks: $N = 12$; circles: $N = 14$; triangles: $N = 16$.*

**8. Conclusions.** The main results of the paper are Theorem 3.1 and Corollary 3.2 which give the consistency result for the discrete-velocity model introduced here and Theorem 6.1 which formulates the convergence result for a modified model. The consistency proof uses the Carleman representation (2.1), thus avoiding the difficulties in justifying the sphere integral approximation. However, analysing convergence to $L^1$ solutions still requires the properties of the model in the standard reperesentation, including the approximation of integrals over spheres. Thus the estimate (6.16) is one of the key points in the proof of Theorem 6.1.

Some generalizations of the approach of this paper are possible, of course, including choosing other quadrature formulas for the integrals over $\mathbb{R}^3$, planes and spheres. Another important aspect is the development of an efficient numerical scheme based on the discrete-velocity model. This is planned as the subject of future work, in which the numerical computations for the space inhomogeneous problems will also be included.


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