Some Ising model related results for certain subshifts of finite type

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Abstract

We generalise some results for the Ising model to certain subshifts of finite type. We obtain results concerning the global Markov property, nonuniqueness of Gibbs measures for asymmetric models on nonamenable graphs, and certain large deviation properties.

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1 Introduction

The Ising model is one of the most investigated examples of Markov random fields. It is well known that this model can have a Gibbs state multiplicity on $\mathbf{Z}^d$ for $d \geq 2$. For a general introduction to Gibbs states and the Ising model, see [9]. Several other properties for this model have been further looked into over the last few decades and we shall here focus our attention on three different results for the Ising model; these are the global Markov property for the plus measure ([10]), phase transition on nonamenable graphs with an external field ([16]) and certain large deviation properties ([22], [23]). We extend these results to the beach model and the iceberg model. These models are examples of subshifts of finite type for which there can exist more than one measure of maximal entropy if the parameter of the model is chosen large enough. In [12] it was shown that the phase transition property for the beach model has the same monotonicity property regarding the "temperature" as the Ising model. One implication of this property can for the Ising model be stated as the existence of a critical inverse temperature (in $[0, \infty]$) such that to the right of this value we have phase transition for the Ising model and to the left of this value we do not have phase transition. The iceberg model is also an example of a subshift of finite type with a possible Gibbs state multiplicity regarding measures of maximal entropy (see [5]). This model can be viewed as a lattice analogue of the Widom-Rowlinson model (introduced in [24]), and it has recently been shown in [3] that there exists a graph, for which the model on this graph, does not share the same monotonicity property regarding the "temperature" of the model as the Ising model and the beach model. The purpose of this paper is to extend some results for the Ising model to the beach model and to the iceberg model on general graphs. In the study of large deviation properties we will restrict our attention to the beach model on $\mathbf{Z}^d$.

In the most general setting of a graph that we will discuss, let $G = (V, E)$ be a countably infinite graph of bounded degree, where V is the set of nodes and E is the set of edges. Call two nodes x and y **nearest neighbours** if there is an edge between them and denote this relationship by $x \sim y$. If $W \subset V$ be a finite subset of V, let $\partial W := \{x \in V - W : \exists y \in W \text{ such that } x \sim y\}$ be the **boundary** of W. For two subsets $A_1, A_2 \in V$, we define the distance $d(A_1, A_2)$ as the length of the shortest path between A_1 and A_2. If A_1 or $A_2 = \emptyset$ then $d(A_1, A_2) := \infty$. The rest of the paper is organised as follows. In Section 2 we define the Ising model and present the results for the Ising model that we have extended to our subshift of finite type context. We also give the definitions for the beach model and the iceberg model and make some short remarks on known properties for these models. Then we prove the global

Markov property for our models in Section 3. In Section 4 we regard the asymmetric versions of the models defined on a nonamenable graph and in Section 5 we study large deviation properties.

2 The models

Let G be a countably infinite graph of bounded degree with vertex set V and edge set E. We consider a system on G whose symbol set on V is F, a finite set of at least two elements. A **configuration** on $A \subseteq V$ is a map $\eta_A : A \to F$. The values of such a configuration on a subset $B \subseteq A$ is given by $\eta_A(B)$. A measure μ on F^V is said to be a **Markov random field** if μ admits conditional probabilities such that for all finite $W \subset V$, all $\xi_W \in F^W$ and all $\eta_{V-W} \in F^{V-W}$ we have

$$\mu(\xi_W|\eta_{V-W}) = \mu(\xi_W|\eta_{V-W}(\partial W)). \tag{2.1}$$

In other words, the Markov random field property says that the conditional distribution of what we see on W given everything else depends only on the values on the boundary ∂W.

A consistent set of conditional distributions for all finite W and all boundary conditions η_{V-W} is called a **specification**, denoted by $\mathcal{Q}$. The specification $\mathcal{Q}$ is said to be Markovian if every element in $\mathcal{Q}$ is Markov (see (2.1)). If μ is a measure on F^V satisfying all conditional distributions in a Markovian $\mathcal{Q}$, we say that μ is a Gibbs measure for $\mathcal{Q}$. Such measures are automatically Markov random fields and the existence of Gibbs measures follows from standard compactness arguments (see [9]). The fundamental question is whether a Markovian specification allows the existence of more than one Gibbs measure. If this is the case, we say that we have a **phase transition**. One of the most studied models which can exhibit a phase transition is the Ising model. We give a short description of this model in the next example and then we quote the results for it that we have extended to the beach model and the iceberg model.

Example 2.1: The Ising model. The Ising model on a graph G is a certain random assignment of plus and minus ones on V. We will here define the specification for the Ising model. Let $F = \{-1, 1\}$ and let ν be a probability measure on $\mathcal{X} = \{-1, 1\}^V$ and let $\eta \in \mathcal{X}$ be a random element chosen according to ν. We say that ν is a **Gibbs state for the Ising model on** G **with external field** h **and coupling constant** J if, for all finite sets $W \subset V$, all $\omega' \in \{-1,1\}^W$ and ν-a.a. $\omega'' \in \{-1,1\}^{V \backslash W}$, we have

$$P(\eta(W) = \omega'|\eta(V\backslash W) = \omega'') = \frac{1}{Z^{\omega''}_{W,h,J}} e^{ha_W(\omega') + Jb_W(\omega',\omega'')}$$

where $a_W(\omega') = \sum_{v \in W} \omega'(v)$, $b_W(\omega', \omega'') = \sum_{u,v \in W \cup \partial W : u \sim v} \omega(u)\omega(v)$, $Z^{\omega''}_{W,h,J}$ is a normalisation constant and $\omega \in \{-1,1\}^V$ is defined by letting $\omega(v)$ be $\omega'(v)$ for $v \in W$ and $\omega''(v)$ for $v \in V - W$.

It is well known that this specification is consistent and hence the existence of a Gibbs measure follows. We now demonstrate a way to construct a Gibbs state for the Ising model. Let $(W_n) \uparrow V$ be a sequence of finite subsets converging to V in the sense that $W_1 \subseteq W_2 \subseteq W_3 \dots$ and $\cup_i W_i = V$. Fix a sequence $\delta_n \in \{-1,1\}^{V-W_n}$, and consider weak subsequential limits of the sequence $\{\nu^{\delta_n}_{W_n,h,J}\}$ where the probability measure $\nu^{\delta_n}_{W_n,h,J}$ is obtained by fixing δ_n on $V - W_n$ and choosing a configuration on W_n according to the specification with boundary condition δ_n. Any weak subsequential limit is a Gibbs state (see [9] p.67). We shall henceforth suppose that $J > 0$. This will yield some good monotonicity properties for the Ising model. For two probability measures ν and μ on F^V, we write $\nu \preceq \mu$ to indicate that ν is stochastically smaller than μ, which means that $\int f d\nu \leq \int f d\mu$ for all increasing functions f on F^V. Now for the Ising model, with (W_n) still fixed, consider the two sequences $\{\nu^+_{W_n,h,J}\}$ and $\{\nu^-_{W_n,h,J}\}$ of measures coresponding to $\delta_n \equiv +1$ and $\delta_n \equiv -1$ respectively. Standard monotonicity arguments (see [19] p.188) imply that $\nu^-_{W_n,h,J} \preceq \nu^\delta_{W_n,h,J} \preceq \nu^+_{W_n,h,J}$ for any $\delta \in \{-1,1\}^{V-W_n}$ and that the sequences $\{\nu^-_{W_n,h,J}\}$ and $\{\nu^+_{W_n,h,J}\}$ are stochastically increasing and decreasing in n respectively. Such arguments also imply that the weak limits $\nu^-_{h,J} := \lim_{n\to\infty} \nu^-_{W_n,h,J}$ and $\nu^+_{h,J} := \lim_{n\to\infty} \nu^+_{W_n,h,J}$ exist and are independent of the sequence (W_n). In addition one has $\nu^-_{h,J} \preceq \nu \preceq \nu^+_{h,J}$ for any Gibbs state ν with parameters h, J. We will call the measures $\nu^-_{h,J}$ and $\nu^+_{h,J}$ the minus and plus measures for the Ising model with parameters h, J respectively. The first theorem that we have extended to our models was proved for the Ising model by Goldstein ([10]). Before we quote this theorem we introduce notation that will prove useful for us in the next sections and give the definition of the global Markov property. A probability measure μ is said to have the **global Markov property** if μ admits conditional probabilities such that for all $W \subset V$, all bounded measurable functions f onF^W and all η_{V-W} we have

$$\mu(f|\eta_{V-W}) = \mu(f|\eta_{V-W}(\partial W)).$$

The difference between this definition and the local Markov property (see eq. (2.1)) is the possibility of having infinite W's.

Theorem 2.2: ([10]) *Let $G = \mathbf{Z}^d$. For the plus measure for the Ising model with $h = 0$, the global Markov property holds.*

This theorem is easy to extend to every countably infinite graph of bounded degree, using the same method as in [10]. Note also that the extension of the theorem to $h \neq 0$ follows from the same argument. Dobrushin ([6]) has given an example of a Gibbs measure for the Ising model on $\mathbf{Z}^3$ satisfying the local Markov property which does not satisfy the global Markov property. Choosing $h = 0$ and J large enough, this example consists of a convex combination

$$\frac{1}{2}\mu_-^+ + \frac{1}{2}\mu_+^-$$

of Gibbs measures, where μ_-^+ denotes the limiting nontranslation invariant measure on $\mathbf{Z}^3$ obtained by taking the limit of the measures $\mu_n^{-/+}$ on the standard boxes Λ_n where the boundary condition equals $+1$ if the third coordinate is > 0 and equals -1 otherwise and μ_+^- is defined analogously by reversing the sign of the boundary condition for the boxes Λ_n except for those points in $\partial\Lambda_n$ for which the third coordinate equals 0. This example of Dobrushin is not an extremal measure and in the paper [10] where Goldstein proves Theorem 2.2 above, he also conjectures that every extremal Gibbs measure on $\mathbf{Z}^d$ satisfies the global Markov property. This conjecture has been shown to be false by Israel ([15]). In Section 3 we modify the argument of Goldstein to show that the global Markov property holds for the plus measure both for the beach model and the iceberg model defined on any countably infinite graph of bounded degree. These models will be defined below. Note that Goldstein also has proved the global Markov property the plus measure for the Widom-Rowlinson model. This model is an example of having a continuous underlying structure ($\mathbf{R}^d$). The iceberg model is a lattice analogue of this model.

Next we quote a theorem of Jonasson and Steif ([16]). For the Ising model on $\mathbf{Z}^d$ with $h \neq 0$, it is a well known result (see [7] p.151) that phase transition cannot occur, whilst on a binary tree with $h \neq 0$ phase transition can occur (see [9] p.250). The following definition gives a useful way of making a general distinction between graphs.

Definition 2.3: *A countably infinite graph of bounded degree is called* **nonamenable** *if*

$$\kappa(G) := \inf_{W \text{ finite } \subset G} \frac{|\partial W|}{|W|} > 0.$$

If $\kappa(G) = 0$ the graph is called **amenable** *($\kappa(G)$ is called the* **Cheeger constant***).*

One example of a class of amenable graphs is $\{\mathbf{Z}^d,\ d \geq 1\}$. A tree with uniform degree $d \geq 3$ is on the other hand nonamenable.

Definition 2.4: *An infinite graph $G = (V, E)$ is* **quasi-transitive** *if there exists a finite number of vertices $x_1, \ldots, x_k$ in V such that for any $x \in V$, there is an automorphism of G taking x to some x_i. If k kan be taken to 1, the graph is* **transitive**.

Theorem 2.5: ([16]) *Let G be countably infinite graph with maximum degree $d < \infty$. For the Ising model the following holds:*

a) If G is nonamenable, $h > 0$ and $J > (2\kappa(G))^{-1}(2h+1+\log(3(d+1)))$, then a phase transition occurs.

b) If G is amenable and quasitransitive, then phase transition does not occur for any $h > 0$ and $J \in [0, \infty)$.

We have extended part (a) of this result to both the beach model and the iceberg model. The fact that $h > 0$ for the Ising model corresponds to the case of an asymmetric beach model and asymmetric iceberg model. The notion of "asymmetric" for these models will be explained in Section 4.

The last result for the Ising model that we have extended, is the result of large deviations for the magnetisation on $\mathbf{Z}^d$ in a box Λ_n. The following results are well known from statistical mechanics. For a configuration $\eta : \mathbf{Z}^d \to \{-1, 1\}$, let

$$Y_A = \frac{1}{|A|} \sum_{x \in A} \eta(x)$$

be the magnetisation in a finite $A \subset \mathbf{Z}^d$. Denote by m^* the spontaneous magnetisation $\int Y_0 d\nu^+$ under the plus measure. It is not hard to show that $m^* > 0$ if and only if we have phase transition. The results that we have extended are the following:

Theorem 2.6: *Fix the dimension d. Let $h = 0$ and fix J. Let $m \in (m^*, 1)$. Then there exist positive constants A_1, A_2, c_1, c_2 such that*

$$A_1 e^{-c_1 |\Lambda_n|} \leq \nu^+_{0,J}(Y_{\Lambda_n} \geq m) \leq A_2 e^{-c_2 |\Lambda_n|}.$$

Theorem 2.7: *Fix the dimension d. Let $h = 0$ and fix J such that $m^* > 0$. Let $m \in (-m^*, m^*)$. Then there exist positive constants A, c such that*

$$\nu^+_{0,J}(Y_{\Lambda_n} \leq m) \geq A e^{-c|\partial \Lambda_n|}.$$

The two theorems above are regarded as "easy to prove" in statistical mechanics. The last theorem is a surface order lower bound large deviation property to the left of the expected value for the magnetisation. In this region, the Ising model has a clear distinction from the i.i.d. case where we always have volume order large deviation. The corresponding surface order large deviation upper bound has been of larger interest over the last decade. The first result concerning this matter was obtained by Schonmann [23]:

Theorem 2.8: *Fix the dimension d. Let $h = 0$ and $m \in (-m^*, m^*)$. There exists a J_1 such that $J > J_1$ implies that there exist positive constants A, c such that*

$$\nu^+_{0,J}(Y_{\Lambda_n} \leq m) \leq A e^{-c|\partial \Lambda_n|}.$$

Soon after Schonmann had shown this, Aizenman et. al ([1]) showed that for dimensions large enough there was a gap between the lowest value of J_1 that you could obtain from Schonmann's argument and the critical inverse temperature. Pisztora ([22]) has narrowed this gap by using FK-measures. When we extend this result to the beach model and the iceberg model, we will essentially follow arguments from Schonmann (with some suitable modifications). We have not tried to apply Pisztora's arguments on either the beach model or the iceberg model.

We are now going to give definitions of the beach model and the iceberg model on a countably infinite graph of bounded degree. Before we do this, we give the definition of a subshift of finite type. A configuration $\eta : A \to F$ is a **restriction** of a configuration $\zeta : B \to F$ if $A \subseteq B$ and ζ agrees with η on A. In this case ζ is called an **extension** of η. Note that if we choose $G = \mathbf{Z}^d$ then G acts on configurations by translation. For $y, x \in \mathbf{Z}^d$ we let $T_y(x) = x + y$, and for $A \subseteq \mathbf{Z}^d$ we set $T_yA = \{x + y : x \in A\}$. Further, if $\eta : A \to F$, we let $T_y\eta(x) = \eta(T_{-y}(x))$ for $x \in T_yA$.

Definition 2.9: *Let $\eta_i : A_i \to F$; $1 \leq i \leq K$ be a finite set S of configurations with $A_i \subset \mathbf{Z}^d$ finite for each $1 \leq i \leq K$. The* **subshift of finite type** *in d dimensions corresponding to S is the set $\mathcal{X} \subseteq F^{\mathbf{Z}^d}$ consisting of all configurations $\eta : \mathbf{Z}^d \to F$ such that for all $y \in \mathbf{Z}^d$ it is not the case that $T_y\eta$ is an extension of some η_i (The η_i's should be thought of as the disallowed finite configurations.)*

A configuration $\tilde{\eta} : A \to F$ is said to be **compatible** with $\mathcal{X}$ if $\exists \eta \in \mathcal{X}$ such that $\tilde{\eta}$ is a restriction of η.

Definition 2.10: *For a subshift of finite type $\mathcal{X}$ we say that $\mathcal{X}$ is* **strongly irreducible** *if there is an $r \geq 0$ such that whenever we have two finite compatible configurations $\eta_A : A \to F$ and $\eta_B : B \to F$ where the distance between A and B is greater than r, there exists an $\eta \in \mathcal{X}$ that is an extension of both η_A and η_B.*

We will after Theorem 2.15 make a comment on why we let our interest focus on strongly irreducible subshifts of finite type. Note that even if the definition above is some kind of restriction, many interesting systems satisfy it.

The next definition gives a measure of the degree of complexity of a subshift of finite type. Let $\Lambda_n = [-n, n]^d$ and $\mathcal{X}_n = \{\tilde{\eta} : \Lambda_n \to F \text{ with } \tilde{\eta} \text{ compatible}\}$. Further we let $N_n = |\mathcal{X}_n|$ where $|\cdot|$ is the cardinality. Let $\mathcal{X}(\tilde{\eta}) = \{\eta \in \mathcal{X} : \eta \text{ is an extension of } \tilde{\eta}\}$.

Definition 2.11: *The* **topological entropy** *of $\mathcal{X}$ is*

$$H(\mathcal{X}) := \lim_{n\to\infty} \frac{\log N_n}{|\Lambda_n|}.$$

Suppose that μ is a translation invariant measure on $\mathcal{X}$. Then the **measure theoretic entropy** *of μ is*

$$H(\mu) := \lim_{n\to\infty} -\frac{1}{|\Lambda_n|} \sum_{\tilde{\eta}\in\mathcal{X}_n} \mu(\mathcal{X}(\tilde{\eta})) \log \mu(\mathcal{X}(\tilde{\eta})).$$

Both of these limits exist by subadditivity and one can show that for any such μ we have $H(\mu) \leq H(\mathcal{X})$. It is in fact well known that $H(\mathcal{X}) = \sup_\mu H(\mu)$ where the supremum is taken over all translation invariant measures on $\mathcal{X}$. See [20] for an elementary proof. We say that μ is a **measure of maximal entropy** if $H(\mu) = H(\mathcal{X})$. The study of phase transitions for subshifts of finite type has (in [4], [5], [12]) concerned itself with the existence of more than one measure of maximal entropy. Note that the definition of a subshift of finite type cannot be generalised to a more general graph, since the definition uses the notion of translations. The next result is due to Burton and Steif [4] and is one of the main properties needed for a generalisation of the notion of measures of maximal entropy for subshifts of finite type to a countably infinite graph. It tells us that for translation invariant measures for subshifts of finite type on $\mathbf{Z}^d$, looking for measures of maximal entropy is the same as looking for measures with uniform conditional probabilities as defined below. Before we quote the result, we make the following definitions.

Definition 2.12: *Given $f, g \in F$, let $s = \{f, g\} \in F \times F$ denote a* **symbol pair**. *We say that a set $S \subseteq F \times F$ is* **symmetric** *if $\{f, g\} \in S$ implies that $\{g, f\} \in S$.*

Definition 2.13: *Let $G = (V, E)$ be a countably infinite graph and let $S \subseteq F \times F$ be a symmetric set of symbol pairs. The* **symmetric nearest neighbour system (SNNS)** *corresponding to S is the set $\mathcal{X} \subseteq F^V$ consisting of all configurations $\eta : V \to F$ such that for all $x, y \in V$ with $x \sim y$ it is not the case that $\{\eta(x), \eta(y)\} \in S$ (the set S should be thought of as the set of disallowed pairs).*

Definition 2.14: *Given an SNNS $\mathcal{X}$, a measure μ on $\mathcal{X}$ having the local Markov property is said to have the* **uniform conditional probability (u.c.p.)** *property if*

$$\forall \text{ finite } W\ \forall \text{ compatible } \eta_{\partial W},\ \mu(\cdot|\eta_{\partial W}) \text{ is uniform over all } \eta_W \text{ on } W$$

that together with $\eta_{\partial W}$ constitute a compatible configuration on $W \cup \partial W$.

Theorem 2.15: [4] *Let $G = \mathbf{Z}^d$ and let μ be a translation invariant probability measure for a strongly irreducible subshift of finite type that is also a symmetric nearest neighbour system. Then the following are equivalent:*

(i) μ is a measure of maximal entropy.

(ii) The conditional distribution of μ on any finite set W given the configuration on ∂W is μ-a.s. uniform over all configurations on W that togehter with the configuration on ∂W constitute a compatible configuration on $W \cup \partial W$.

When considering a probability measure for a SNNS on a general graph, the point is that the property (ii) generalises perfectly well to general graphs, but the property (i) does not. Therefore we shall say that a phase transition occurs on G for a SNNS if there exists more than one u.c.p. measure. A trivial example of such a phase transition is obtained by setting $F = \{0, 1\}$ and disallowing zeros and ones to sit next to each other. Then there exists a u.c.p. measure that with probability one gives the configuration the value 0 for all vertices in G and there exists a corresponding u.c.p. measure that gives the value 1 on all vertices in G. To prevent such trivial examples of phase transition we again assume that the SNNS is strongly irreducible. This property (see definition 2.10) is defined in the same way as for subshifts of finite type. Note that even in the case of $G = \mathbf{Z}^2$, there is not equivalence between phase transitions for measures of maximal entropy and phase transitions for u.c.p. measures. An example of this is the generalised hardcore model ([4]) where $F = \{0, 1, \ldots, N\}$ and positives are not allowed to sit next to each other. If N is chosen large enough for this model, then there is phase transition with respect to u.c.p. measures since then there exist two extreme u.c.p. measures that give different probabilities to the value 0 at the origin ([2]). This situation can almost be thought of as an "infinite checkerboard". Yet there is only one convex combination (each of probability 1/2) of those two extreme u.c.p. measures that is translation invariant and hence a measure of maximal entropy. Since translation invariance is a necessary condition in the definition (2.11) of measure theoretic entropy, the set of measures of maximal entropy is not the same as the set of u.c.p. measures. Since there is no generalisation for the measure theoretic entropy to general graphs, but there is an obvious generalisation of u.c.p. measures regarding this matter, we will study u.c.p. measures.

Definition 2.16: The beach model. Let M_1 and M_2 be positive integers such that $M_1 < M_2$ and let

$$F = F_1 \cup F_2 \cup F_3 \cup F_4$$

where

$$\begin{aligned} F_1 &= \{-M_2, -M_2+1, \ldots, -M_1-1\} \\ F_2 &= \{-M_1, -M_1+1, \ldots, -1\} \\ F_3 &= \{1, 2, \ldots, M_1\} \\ F_4 &= \{M_1+1, M_1+2, \ldots, M_2\}. \end{aligned}$$

Call a symbol $f \in F$

negative	if	$f \in F_1 \cup F_2$
positive	if	$f \in F_3 \cup F_4$
unprivileged	if	$f \in F_1 \cup F_4$
privileged	if	$f \in F_2 \cup F_3$

and consider the SNNS where a negative may not sit next to a positive unless both are privileged. At a first sight the beach model can be regarded as two-parametric, but so far the only parameter that has been paid attention to is the ratio $M := \frac{M_2}{M_1}$. This is because that the notion of phase transition for measures of maximal entropy is only dependent on M ([12]). Let μ be a measure having the u.c.p. property. Nonuniqueness on $\mathbf{Z}^d$ for $d \geq 2$ for measures of maximal entropy (or u.c.p. measures) for this model was first obtained in [4], where they studied the case $M_1 = 1$ and $M_2 = M$. Note that for rational M's it has been shown (see [12]) that on $\mathbf{Z}^d$ for $d \geq 2$, there exists an $M_c(d) > 1$ such that if $M < M_c(d)$ then we do not have phase transition and for $M > M_c(d)$ we have phase transition. Simulations done by Nelander ([21]) indicate that the critical value for $d = 2$ lies between 2.0 and 2.25. In this paper it is also conjectured that $M_c(d)$ is nonincreasing in d. Results on phase transition for the beach model on more general graphs than $\mathbf{Z}^d$ can be found in Häggström ([14]). Here he also introduces a version of the beach model that allows real-valued M's. Observe that we in this paper set $M_1 = 1$, which corresponds to using integer values of M. This is for the sake of simplicity. Our theorems are easy to extend to rational values of M.

Remark: There are now powerful random-cluster methods for the study of the beach model and the iceberg model. A reference to these tools is Häggström ([13]).

Example 2.17: The iceberg model. Let N_1 and N_2 be positive integers and let $N := \frac{N_2}{N_1}$, and define the symbol set $H = H_- \cup H_0 \cup H_+$ where

$$\begin{aligned} H_- &= \{-N_2, -N_2+1, \ldots, -1\} \\ H_0 &= \{0_1, 0_2, \ldots, 0_{N_1}\} \\ H_+ &= \{1, 2, \ldots, N_2\}. \end{aligned}$$

The Iceberg model in $d \geq 2$ dimensions was introduced in [5] and is the SNNS for which no element of H_- sits next to an element of H_+. This model can also be viewed as a variant of the Widom-Rowlinson model, which was first studied in [24]. Note that there is an important difference between the beach model and the iceberg model in that the occurence of phase transition is increasing in M on any graph G for the beach model ([13]), but not increasing in N for all G for the iceberg model ([3]).

We now give a short description of stochastic domination for these SNNS. We have already made a brief introduction on this subject for the Ising model. When we define stochastic domination for the beach model and the iceberg model we declare $\eta \preceq \delta$ if $\eta(x) \leq \delta(x)$ for all $x \in V$. One then makes the following definition (see [19]).

Definition 2.18: *If μ and ν are probability measures on F^V where F is a finite set contained in $\mathbf{R}$ and where V is countable (perhaps finite), we say $\mu \preceq \nu$ if there exists a probability measure m on $F^V \times F^V$ whose first and second marginals are μ and ν respectively and such that*

$$m\{(\eta, \delta) : \ \eta \preceq \delta\} = 1.$$

The following lemma, which is an analogue of Holley's theorem and can be proved by use of the coupling method (see [17]), is proved for $V = \mathbf{Z}^d$ in [4]. We omit the proof of our lemma since it is easily generalized for a countably infinite graph V.

Lemma 2.19: *Let S be a finite set contained in V. For the beach model or the iceberg model, let $\eta \preceq \delta$ be defined on ∂S and be compatible. Then $\mu(\cdot|\eta) \preceq \mu(\cdot|\delta)$ where $\mu(\cdot)$ denotes a probability measure that has the u.c.p. property.*

Using this theorem, one can define μ^+ and μ^- analogously to the Ising model. For example let μ^+ be obtained as the limit $\lim_{n\to\infty} \mu^+_{W_n}$ of measures $\mu^+_{W_n}$ which assigns all nodes in $V - W_n$ the maximal value on $V - W_n$ and which have the suitable distribution on W_n. Compactness and monotonicity then as usual imply existence. The following simple condition tells us whether there is phase transition for the beach model and the iceberg model respectively. This also covers the asymmetric case, but we defer the definition of the asymmetric beach model until Section 4. The proof is not hard to show and is omitted.

Theorem 2.20: *For $v \in V$ let $\{\eta(v) = +\}$ denote the event that $\eta(v)$ is positive. Then there is phase transition if and only if*

$$\exists v \in V \text{ such that } \mu^+(\eta(v) = +) \neq \mu^-(\eta(v) = +).$$

This means that we can simply show phase transition if we find a $v \in V$ such that $\mu^+(\eta(v) = +) \neq \mu^-(\eta(v) = +)$. The coupling method in the proof of Lemma 2.19 can also be used to prove the FKG-property. We say that a bounded function $g : \eta_W \to \mathbf{R}$ on a finite set $W \subset V$ is **increasing** if $\eta_W^{(1)} \preceq \eta_W^{(2)}$ implies $g(\eta_W^{(1)}) \leq g(\eta_W^{(2)})$ for all $\eta_W^{(1)}$ and all $\eta_W^{(2)}$ with $\eta_W^{(1)} \preceq \eta_W^{(2)}$.

Theorem 2.21 (FKG-property): Let W be finite and denote by $\mu_W(\cdot|\eta_W)$ the conditional u.c.p. measure on W given the compatible configuration $\eta_{\partial W}$, either for the beach model or for the iceberg model. Then for increasing $g_1, g_2 : \eta_W \to \mathbf{R}$ we have that

$$\mu_W(g_1 g_2|\eta_W) \geq \mu_W(g_1|\eta_W)\mu_W(g_2|\eta_W).$$

This can be proved by use of standard coupling arguments by letting

$$\mu_1(\eta_W) = \mu_W(\eta_W|\eta_{\partial W})$$

and

$$\mu_2 = \frac{\mu_W(\eta_W|\eta_{\partial W})g_1(\eta_W)}{\int g_1(\eta_W)d\mu_1(\eta_W)}.$$

Then, by standard coupling arguments one can show that $\mu_1 \preceq \mu_2$ and by applying these measures to g_2, the FKG-property follows.

Before we end this section we introduce notation that will prove useful for us in the next sections. Let $\mathcal{F}_V$ be the σ-algebra generated by the set $\mathcal{X}$ of compatible configurations on V and let $\mathcal{F}_A$ the sub σ-algebra generated by the set of all compatible configurations on $A \subset V$. Denote by $P(\mathcal{X}, \mathcal{F}_V)$ the set of probability measures on $(\mathcal{X}, \mathcal{F}_V)$. Given $\mu \in P(\mathcal{X}, \mathcal{F}_V)$ let $\mu(\cdot|\mathcal{F}_{\partial W})$ be the conditional measure on W given a compatible configuration on the boundary ∂W. For any function $f \in \mathcal{F}_V$, denote by $\mu(f)$ the expectation of f. With this notation the local Markov property can now be defined as:

$$\forall \text{ finite } W \subset V \; \forall \text{ bounded } f \in \mathcal{F}_W : \mu(f|\mathcal{F}_{V-W}) = \mu(f|\mathcal{F}_{\partial W}).$$

We will make use of this definition mainly in the next section.

3 The global Markov property

As we have seen, many systems have the local Markov property. Another property is the global Markov property which for an SNNS can be defined as follows:

Definition 3.1: *A measure μ on $\mathcal{X}$ is said to have the* **global Markov property (GMP)** *if*

$$\forall A \subset V \quad \forall \text{ bounded } f \in \mathcal{F}_A : \; \mu(f|\mathcal{F}_{V-A}) = \mu(f|\mathcal{F}_{\partial A}).$$

This definition is different from the definition of the local Markov property in the sense that for the local Markov property A is required to be finite.

To prove this property for the beach model on any countably infinite graph of bounded degree, we will need a characterisation of the graph. Given $r \in \mathbf{N}$, we say that an infinite connected locally finite graph is r**-separated** if

$$\forall A \subset V \; \exists (W_n) \uparrow V \;\; \ni \;\; \forall n \; d((W_n - A) - \partial A, \partial(A \cap W_n) \cap (V - W_n)) \geq r$$

where (W_n) is a sequence of finite subsets of V.

Lemma 3.2: *Any countably infinite graph of bounded degree is r-separated for any $r \in \mathbf{N}$.*

The proof of the lemma above is given at the end of this section.

Theorem 3.3: *For the beach model on an (r=3)-separated graph and for the iceberg model on an (r=2)-separated graph, the GMP holds for μ^+.*

Before we prove this theorem we introduce the concept of maximal extensions:

Definition 3.4: *Let η_W be a compatible configuration on $W \subseteq V$. We say that $\eta_V : V \to F \in \mathcal{F}_V$ is a* **maximal** *extension of η_W if η_V is an extension of η_W such that for all extensions $\tilde{\eta}_V : V \to F \in \mathcal{F}_V$ of η_W satisfies $\eta_V(x) \geq \tilde{\eta}_V(x) \quad \forall x \in V$. We denote the restriction of such a maximal extension to $B \subseteq V$ by $\eta^+_B|\eta_W$.*

Such an element exists for the beach model and for the iceberg model since if σ and α are in $\mathcal{X}$ and $\sigma_W = \alpha_W = \eta_W$, then $\sigma \vee \alpha \in \mathcal{X}$ where $\sigma \vee \alpha(x) = \max\{\sigma(x), \; \alpha(x)\}$. To see this, suppose $\sigma \vee \alpha \notin \mathcal{X}$. Then there exists a pair $x, y \in V$ such that $x \sim y$ and the symbol pair for $\sigma \vee \alpha$ on x, y is one of the disallowed symbol pairs. From the definition of the beach model (and the iceberg model respectively) it now follows that either $\sigma(x, y)$ or $\alpha(x, y)$ is a disallowed pair, yielding that either σ or α is not compatible. Uniqueness is obvious from the definition.

The maximal extension of a configuration η_W on W is closely related to the value at each point in W for both the beach model and the iceberg model:

Lemma 3.5:

$$\forall W \quad \forall \eta_W \in \mathcal{F}_W \quad \forall x \in V - W: \quad \eta_x^+ | \eta_W = \min_{u \in W} \{\eta_x^+ | \eta_u\}.$$

We prove this lemma for the beach model. The arguments can easily be changed to be valid for the iceberg model.

Proof of Lemma 3.5: Fix W and choose an $x \in V - W$. We prove this by showing that

(i) $\eta_x^+ | \eta_W \leq \min_{u \in W} \{\eta_x^+ | \eta_u\}$.

(ii) $\eta_x^+ | \eta_W \geq \min_{u \in W} \{\eta_x^+ | \eta_u\}$.

(i) Since $\eta_{V-W}^+ | \eta_W$ together with η_W is an extension of η_u to V, it follows that

$$\eta_{V-W}^+ | \eta_W \leq \eta_{V-W}^+ | \eta_u \quad \forall u \in W.$$

Hence $\eta_x^+ | \eta_W \leq \min_{u \in W} \{\eta_x^+ | \eta_u\}$.

(ii) It is clear that $\eta_x^+ | \eta_W$ can only equal -1,1 or M. If it equals -1, it is neighbouring some point $u \in W$ such that $\eta_u \in (-M, -M+1, \ldots, -2)$. Then $\min_{u \in W} \{\eta_x^+ | \eta_u\} = -1$. If $\eta_x^+ | \eta_W = +1$, then there exists either a point $u \in W$ with $d(u,x) = 1$ having $\eta_u = -1$ or a point $u \in W$ with $d(u,x) = 2$ having $\eta_u \in (-M, -M+1, \ldots, -2)$ which gives the conclusion that $\min_{u \in W} \{\eta_x^+ | \eta_u\} \leq 1$. If it equals M then the inequality is trivial. This ends the proof of (ii).

$\square$

Proof of Theorem 3.3: We prove this theorem for the beach model. The proof for the iceberg model is analogous. Fix $A \subset V$ and choose a sequence $(W_n) \uparrow V$ such that $\forall n \; d((W_n - A) - \partial A, \partial(A \cap W_n) \cap (V - W_n)) \geq 3$. Let $\mu_{W_n,A}^+$ denote the measure on W_n where $\eta_{W_n - A}$ is chosen according to μ^+, the boundary configuration $\eta_{\partial W_n}$ is chosen to be $\eta_{\partial W_n}^+ | \eta_{W_n - A}$ from Lemma 3.5 and $\eta_{W_n \cap A}$ is chosen according to the u.c.p.-property given the boundary condition $\eta_{\partial(W_n \cap A)}$ on $\partial(W_n \cap A)$. Note that the boundary condition on $\partial(W_n \cap A)$ is a restriction of the union of $\eta_{W_n - A}$ and $\eta_{\partial W_n}^+ | \eta_{W_n - A}$ since $\partial(W_n \cap A) = (\partial(W_n \cap A) \cap (W_n - A)) \cup (\partial(W_n \cap A) \cap \partial W_n)$. Hence $\eta_{\partial(W_n \cap A)}$ is uniquely determined by $\eta_{W_n - A}$ since $\eta_{\partial W_n}^+ | \eta_{W_n - A}$ is uniquely determined by $\eta_{W_n - A}$. Let $\mu_{W_n \cap A}^+(\cdot | \eta_{W_n - A})$ denote the conditional probability measure given the above construction on $W_n \cap A$ given $\mathcal{F}_{W_n - A}$. It is clear that for $f \in \mathcal{F}_{W_n \cap A}, g \in \mathcal{F}_{W_n - A}$,

$$\mu^+_{W_n,A}(gf) = \int \mu^+(d\eta) g(\eta) \mu^+_{W_n \cap A}(f | \eta_{W_n - A}). \tag{3.1}$$

We have the following lemma which we will prove later:

Lemma 3.6: *For the beach model on any 3-separated graph and for the iceberg model on any 2-separated graph, for all $A \subseteq V$ there exists a sequence $(W_n) \uparrow V$ such that*

$$\forall n \;\; \mu^+_{W_n \cap A}(\cdot | \eta_{W_n - A}) \in \mathcal{F}_{W_n \cap \partial A}.$$

From domination arguments we have that $\mu^+|_{W_n} \preceq \mu^+_{W_n,A} \preceq \mu^+_{W_n}$ on W_n. Hence

$$\lim_{n \to \infty} \mu^+_{W_n,A} = \mu^+. \tag{3.2}$$

That this is a sufficient condition for the GMP for μ^+ follows from an argument similar to Goldstein ([10]): We are going to show that

$$\forall f \in \cup_m \mathcal{F}_{W_m \cap A} \;\; \exists f_\infty \in \mathcal{F}_{\partial A} \quad \ni \forall g \in \cup_n \mathcal{F}_{W_n - A} \;\; \mu^+(gf) = \mu^+(g f_\infty). \tag{3.3}$$

?Using monotone class arguments? for functions gives that this will be enough to show the GMP. Fix m and let $f \in \mathcal{F}_{W_m \cap A}$. Let $\eta : V \to F \in \mathcal{F}_V$ and let $\eta_{W_n - A}$ be the restriction of η to $W_n - A$. Let

$$f_n(\eta) = \mu^+_{W_n \cap A}(f | \eta_{W_n - A})$$

for $n \geq m$. Fix $n \geq m$ and let $g \in \mathcal{F}_{W_n - A}$. It follows from (3.1) and (3.2) that (3.3) will be satisfied by

$$f_\infty = \lim_{n \to \infty} f_n(\eta), \tag{3.4}$$

and $f_\infty \in \mathcal{F}_{\partial A}$ since $f_n \in \mathcal{F}_{W_n \cap \partial A}$ from Lemma 3.6, provided that this limit exists (see Lemma 3.5 p.31 [25]). Lemma 3.5 gives that the sequence of measures $\mu^+_{W_n \cap A}(f | \eta_{W_n - A})|_{W_m \cap A}$ is nonincreasing in n and hence the existence of the limit (3.4) is easily established using monotonicity and compactness arguments. This completes the proof.

□

Next we prove Lemma 3.6 for the beach model. The proof can easily be adapted to the iceberg model.

Proof of Lemma 3.6: Fix $A \subset V$ and choose a sequence $W_n \uparrow V$ such that $\forall n \; d((W_n - A) - \partial A, \partial(W_n \cap A) \cap (V - W_n)) \geq 3$. Fix n and fix a compatible $\eta_{W_n - A}$. Let $\eta^{(1)} : \partial(W_n \cap A) \to F$ be the restriction of the union of $\eta_{W_n - A}$ and $\eta^+_{\partial W_n} | \eta_{W_n - A}$ to $\partial(W_n \cap A)$. Let $\eta^{(2)} : \partial(W_n \cap A) \to F$ be the restriction of the union of $\eta_{W_n \cap \partial A}$ and $\eta^+_{\partial W_n} | \eta_{W_n \cap \partial A}$ to $\partial(W_n \cap A)$ where $\eta_{W_n \cap \partial A}$ is the restriction of $\eta_{W_n - A}$ to $W_n \cap \partial A$. The lemma follows once we have proven that $\eta^{(1)} \equiv \eta^{(2)}$. It is clear that $\eta^{(1)} = \eta^{(2)}$ on $W_n \cap \partial(W_n \cap A)$. Hence we have to show that $\eta^{(1)} = \eta^{(2)}$ on $\partial W_n \cap \partial(W_n \cap A)$. Fix $x \in \partial W_n \cap \partial(W_n \cap A)$. From Lemma 3.5 we have that $\eta^{(1)}_x = \min_{u \in W_n - A} \{\eta^+_x | \eta_u\}$ and that $\eta^{(2)}_x = \min_{u \in W_n \cap \partial A} \{\eta^+_x | \eta_u\}$ which implies $\eta^{(1)}_x \leq \eta^{(2)}_x$. We have that

$$\eta^{(1)}_x = \min_{u \in W_n - A} \{\eta^+_x | \eta_u\} = \min_{u \in W_n - A, \;\; d(u,x) \leq 2} \{\eta^+_x | \eta_u\}$$

since $\eta^+_x | \eta_u = M$ if $d(x, u) \geq 3$. We will be done with the proof if we show that $\{u \in W_n - A, \;\; d(u, x) \leq 2\} \subseteq W_n \cap \partial A$ since then $\eta^{(1)}_x \geq \eta^{(2)}_x$. Since the graph is 3-separated we have that

$$d((W_n - A) - \partial A, \partial(W_n \cap A) \cap (V - W_n)) \geq 3 \Rightarrow$$

$$d((W_n - A) - \partial A, x) \geq 3 \;\; \text{since } x \in \partial W_n \cap \partial(W_n \cap A) \Rightarrow$$

$$\{u \in W_n - A, \;\; d(u, x) \leq 2\} \subseteq W_n \cap \partial A$$

since if $u \in W_n - A$ and $u \notin W_n \cap \partial A$, then $u \in (W_n - A) - \partial A$.

□

Proof of Lemma 3.2: Fix an $r \in \mathbf{N}$. Fix $A \subset V$ and choose a sequence $(A_n) \uparrow A$ of finite subsets of A converging to A. Let $(V_n) \uparrow V$ be an increasing sequence of finite subsets converging to V. For $n \geq 1$ define

$$B_n = \{x \in \partial A; \; d(x, A_n) = 1\}$$

$$C_n = \{x \in (V_n - A) - \partial A; \; d(x, A - A_n) \geq r\}.$$

Clearly $(B_n) \uparrow \partial A$ as n tends to infinity. Since every vertex in the graph has bounded degree, it also follows that

$$\forall x \in (V - A) - \partial A \; \exists n_x \text{ such that } d(x, A - A_n) \geq r \; \forall n \geq n_x.$$

Using that $\forall x \in (V - A) - \partial A \; \exists n'_x$ such that $x \in (V_n - A) - \partial A \; \forall n \geq n'_x$, we have that

$$\forall x \in (V - A) - \partial A \; \exists n''_x \text{ such that } x \in C_n \; \forall n \geq n''_x.$$

Hence, since (C_n) is increasing, $(C_n) \uparrow (V - A) - \partial A$. Now let

$$W_n = A_n \cup B_n \cup C_n. \tag{3.5}$$

Since $(A_n) \uparrow A$, $(B_n) \uparrow \partial A$ and $(C_n) \uparrow (V - A) - \partial A$, we have that $(W_n) \uparrow V$. We are now going to show that $d((W_n - A) - \partial A,\ \partial(W_n \cap A) \cap (V - W_n)) \geq r$ by showing that $\partial(W_n \cap A) \cap (V - W_n) \subseteq A - A_n$ and using that $(W_n - A) - \partial A = C_n$. Let $x \in \partial(W_n \cap A) \cap (V - W_n)$. We know from (3.5) that $W_n \cap A = A_n$. This gives that $x \in V - A_n$ and $d(x, A_n) = 1$. Supposing that $x \in V - A$ will give a contradiction, since if $x \in V - A$, then by using $d(x, A_n) = 1$ we have that $x \in \partial A$ and that $x \in B_n$ which would imply $x \in W_n$. Hence $x \in A$ and since $x \notin A_n$ it follows that $x \in A - A_n$.

□

4 Phase transition on nonamenable graphs

In this section we study asymmetric versions of the beach model and the iceberg model. First we prove a phase transition result for the asymmetric beach model. Then we make a remark on the fact that the proof can easily be adapted to prove the analogous result for the iceberg model.

The asymmetric beach model was introduced by Burton and Steif in [5], where it was shown that this model on $\mathbf{Z}^d$ has a unique measure of maximal entropy when M is chosen sufficiently large. This is believed to hold for all M for the asymmetric beach model. Earlier they showed ([4]) that the symmetric beach model ($k = 0$) on $\mathbf{Z}^d$ exhibits a phase transition when M is chosen large enough. Jonasson and Steif ([16]) have generalized the result of uniqueness of Gibbs states for the Ising model on $\mathbf{Z}^d$ with an external field to the corresponding result on a quasitransitive amenable graph. They have also shown that the Ising model on any nonamenable graph exhibits a phase transition when the inverse temperature is chosen sufficiently large. Here we show the corresponding result for the asymmetric beach model on nonamenable graphs. Recall definition 2.3 that a countably infinite graph of bounded degree is nonamenable if and only if

$$\kappa(G) := \inf_{W \text{ finite } \subset G} \frac{|\partial W|}{|W|} > 0.$$

Definition 4.1: The asymmetric beach model *Let $k \in \mathbf{N}$ and let $F = \{-M+k, -M+k+1, \ldots, -1, +1, +2, \ldots, M\}$ and let $\mathcal{X}$ be the SNNS for which no nearest neighbours have opposite sign unless both have absolute value equal to 1.*

This definition differs slightly from the definition of the symmetric beach model ($k = 0$) in that $k > 0$. On a nonamenable graph we have the following theorem:

Theorem 4.2: *Fix $k \in \mathbf{N}$. For the asymmetric beach model on a nonamenable graph with maximum degree d, there exists an $M'(G, k)$ such that we have phase transition whenever $M > M'(G, k)$.*

Proof of Theorem 4.2: We will need the following lemma due to Kesten ([18]).

Lemma 4.3 *Let G be an infinite graph with maximum degree d and let $\mathcal{C}_m$ be the set of connected sets with m vertices containing a fixed vertex x. Then $|\mathcal{C}_m| \le (e(d+1))^m$.*

Fix x and let $\mathcal{C}_m$ be as in the above lemma. For $C \in \cup_{m=1}^{\infty} \mathcal{C}_m$, let $A_C = \{\eta(\partial C) \equiv -1, \eta(\partial(C^c)) \equiv +1\}$, i.e. the event that the outer boundary of C

equals -1 and the inner boundary of C equals $+1$. Let $\{\eta(x) = +\}$ denote the event that $\eta(x)$ is positive. For a sequence $(W_n) \uparrow V$, where the W_n's are finite connected subsets of V containing x, let μ_n be short notation for μ_{W_n} defined in Section 1. Now for any $n \in \{1, 2, \dots\}$, we have

$$\mu_n^-(\eta(x) = +) \le \mu_n^-(\bigcup_{m=1}^{\infty} \bigcup_{C \in \mathcal{C}_m} A_C) \le \sum_{m=1}^{\infty} \sum_{C \in \mathcal{C}_m} \mu_n^-(A_C). \tag{4.1}$$

If we show that for each $C \in \cup_m \mathcal{C}_m$,

$$\mu_n^-(A_C) \le \left(\frac{2}{(M-k)^{\kappa(G)/d}} \right)^{|C|}, \tag{4.2}$$

then by inserting this into (4.1) and using Lemma 4.3, we get

$$\mu_n^-(\eta(x) = +) \le \sum_{m=1}^{\infty} \left(\frac{2e(d+1)}{(M-k)^{\kappa(G)/d}} \right)^m$$

which is less than $1/2$ uniformly in n if $M - k > (6e(d+1))^{d/\kappa(G)}$.

To prove (4.2), fix n and fix $C \in \cup_m \mathcal{C}_m$. Define Ω_C to be the set of compatible configurations on $C \cup \partial C$ which equal -1 on ∂C and suppose that $\eta \in A_C$ is a compatible configuration on $C \cup \partial C$. Note that $\eta \in A_C \Rightarrow \eta \in \Omega_C$. Define the subgraph $C_{plus}(\eta)$ of C as

$$C_{plus}(\eta) = \{v \in C : \eta(v) = +\}.$$

Now let $\tilde{\eta} \in F^{C \cup \partial C}$ be the configuration obtained from η by reversing the sign of any node in C_{plus}. Note that $\tilde{\eta}(x)$ will be negative for all x in C. This yields that $\tilde{\eta} \in \Omega_C$. Also note that the set $\{\eta' \in \Omega_C | \tilde{\eta}' = \tilde{\eta}\}$ has cardinality at most $2^{|C|}$. This is because from $\tilde{\eta}$ one can recover η up to the sign for each node in C.

Given $\eta \in \Omega_C$ such that $\eta \in A_C$, define the class $B(\tilde{\eta})$ to be

$$\{\rho \in \Omega_C : \rho(x) = \tilde{\eta}(x) \;\; \forall x \in C - \partial(C^c), \;\; \rho(x) < 0 \;\; \forall x \in \partial(C^c)\}.$$

Clearly

$$B(\tilde{\eta}) \subseteq \Omega_C, \;\; |B(\tilde{\eta})| = (M-k)^{|\partial(C^c)|}, \;\; \text{and} \;\; B(\tilde{\eta}) \cap B(\tilde{\eta}') = \emptyset$$

for $\tilde{\eta} \neq \tilde{\eta}'$, since if $\tilde{\eta} \neq \tilde{\eta}'$ they must differ at some point in $C - \partial(C^c)$ as they are $\equiv -1$ on $\partial(C^c)$.

We therefore obtain

$$\mu_n^-(A_C) = \mu_n^-(\eta(\partial(C^c)) \equiv +1|\eta(\partial C) \equiv -1)\mu_n^-(\eta(\partial C) \equiv -1)$$

$$\leq \mu_n^-(\eta(\partial(C^c)) \equiv +1|\eta(\partial C) \equiv -1) \leq \frac{|\{\tilde{\xi}|\xi \in A_C\}|2^{|C|}}{|\{\tilde{\xi}|\xi \in A_C\}|(M-k)^{|\partial(C^c)|}}$$

$$= \frac{2^{|C|}}{(M-k)^{|\partial(C^c)|}} \leq \left(\frac{2}{(M-k)^{\kappa(G)/d}}\right)^{|C|}$$

uniformly in n, where we in the last inequality used that $|\partial(C^c)| \geq |\partial C|/d$. Hence (4.2) is shown.

A similar argument gives that $M > (6e(d+1))^{d/\kappa(G)}$ is a sufficient condition to show that $\mu_n^+(\eta(x) = -) < 1/2$ uniformly in n. Clearly this condition is valid if $M - k > (6e(d+1))^{d/\kappa(G)}$. From this we can choose $M'(G,k)$ to be $k + (6e(d+1))^{d/\kappa(G)}$. This finishes the proof of Theorem 4.2.

□

Remark: Note that one can show phase transition for the symmetric beach model on a nonamenable graph using the same method as above with $k = 0$, but the point is that it is generally harder to get phase transition in asymmetric situations.

Remark: An asymmetric definiton of the iceberg model is easily obtained by fixing $k \in \mathbf{N}$ and letting $H_- = \{-N+k, -N+k+1, \dots, 1\}$, $H_0 = \{0\}$ and $H^+ = \{1, 2, \dots, N\}$. The proof above is then easily generalised to be valid for the asymmetric iceberg model by letting Ω_C in the proof of (3.2) above be the set of compatible configurations on $C \cup \partial C$ which are in $\{-1, \dots, -N_2\}$ on ∂C and which equal 0 on $\partial(C^c)$ and choosing $N'(G,k) = k + (6e(d+1))^{d/\kappa(G)}$.

5 Large deviation results

In this section we study large deviation results for the symmetric beach model on $\mathbf{Z}^d$. We show some volume order and surface order large deviation properties for the magnetization in a box Λ_n, where (Λ_n) denotes the increasing sequence of boxes

$$\Lambda_n = \{x \in \mathbf{Z}^d : -n \leq x_i \leq n,\ i = 1, \dots, d\}$$

converging to $\mathbf{Z}^d$. Let $sgn : \mathbf{Z}\backslash\{0\} \to \{-1, 1\}$ be the sign function. We define the **spontaneous magnetization** for the plus state for the beach model as

$$m^*(M) = \int sgn(\eta_0) d\mu^+(\eta).$$

It is not hard to show that $m^* > 0$ if and only if we have phase transition. The magnetization $Y_A(\eta)$ in a finite $A \subset \mathbf{Z}^d$ is defined as

$$Y_A(\eta) = \frac{1}{|A|} \sum_{x \in A} sgn(\eta_x)$$

where $|A|$ denote the number of vertices in A. Since μ^+ can easily be shown to be extremal, we have that μ^+ has a trivial tail σ-field (see Theorem 7.7.a. p. 118, [9]). The translation invariance and the triviality of the tail σ-field for μ^+ imply that μ^+ is ergodic (Proposition 14.9. p. 293, [9]). Since sgn is a bounded function we have that Y_A converges μ^+ almost surely to m^* as $A \uparrow V$ (Theorem 14.A.8. p.306, [9] and the following remark on p.307).

Remark: Note that the process $\{sgn(\eta_x),\ x \in \mathbf{Z}^d\}$ under μ^+ is distributed according to the "site-centred ferromagnet", a model introduced by Häggström in [12].

Let $Y^+_{\Lambda_n}$ denote the magnetization on Λ_n where the boundary condition on $\partial(\Lambda_n^c)$ is chosen to be M.

Lemma 5.1: *Fix M. $Y^+_{\Lambda_n}$ converges to m^* $(= m^*(M))$ in probability.*

We now state the volume order large deviation theorem, which holds to the right of m^*.

Warning: throughout this section A and c below represent generic constants that may change value for different expressions.

Theorem 5.2: *Fix M and let $m \in (m^*, 1]$. There exist positive constants A_1, A_2, c_1, c_2 such that*

$$A_1 e^{-c_1|\Lambda_n|} \leq \mu^+(Y_{\Lambda_n} \geq m) \leq A_2 e^{-c_2|\Lambda_n|}.$$

This theorem corresponds to volume order large deviation for the Ising model to the right of the expected value for the magnetisation in a box Λ_n. The first inequality is easy to prove once we have established the following lemma.

Lemma 5.3: *Fix M and d. There exists an $\alpha > 0$ such that*

$$\forall \text{ finite } A \subset \mathbf{Z}^d \ \ \forall\, \eta_A \in \mathcal{F}_A \ \ \mu^+(\eta_A) \geq \alpha^{|A|}.$$

Proposition 5.4: *Fix M such that $m^* > 0$. Let $m \in (-m^*, m^*)$. There exist positive constants A, c such that*

$$\mu^+(Y_{\Lambda_n} \leq m) \geq Ae^{-c|\partial\Lambda_n|}.$$

This proposition corresponds to a surface order large deviation lower bound to the left of the expected value of the magnetisation for the Ising model in a box Λ_n (see e.g. [23]). This surface order large deviation property is different from what one obtains for i.i.d. processes, since deviations are always of volume order in that case. We are now about to introduce a sufficient condition for an upper bound of surface order large deviation to the left of the expected value for the magnetisation in a box Λ_n for the beach model. The condition used by Schonmann in [23] for the Ising model is similar. Note that Pisztora ([22]) has sharpened Schonmann's surface order large deviation result by using FK-measures but we have not tried to apply Pisztora's arguments on the beach model.

Definition 5.5: *$C \subseteq \mathbf{Z}^d$ is an* **enclosing set** *of $B \subseteq \mathbf{Z}^d$ if*

$$\begin{array}{ll} (i) & B \subseteq C \\ (ii) & C \text{ is finite} \\ (iii) & C \text{ and } C^c \text{ are each connected.} \end{array}$$

Let $\mathcal{H}$ be the set of subsets S of $\mathbf{Z}^d$ such that $x_d = 0$ for all $x \in S$. For a finite $S \in \mathcal{H}$ consider the event

$$E_S = \{\text{there exists an enclosing set } C \text{ of } S \text{ such that } \eta_{\partial(C^c)} \equiv +1\}.$$

Definition 5.6: *Given $\eta \in \mathcal{X}$, we say $K \subseteq \mathbf{Z}^d$ is a* **positive cluster** *(with respect to η) if K is connected, $\eta(x) \geq 1$ for all $x \in K$ and K is maximal with* *respect to these two properties. A* **negative cluster** *is defined analogously. A* **cluster** *is either a positive or a negative cluster.*

Denote by E^-_{absent} the event that there is absense of negative infinite clusters. The sufficient condition that we have is the existence of $A, c > 0$ such that

$$\mu^+(E_S) \leq Ae^{-c||S||}, \quad \text{for any finite } S \in \mathcal{H}, \tag{5.1}$$

provided that $\mu^+(E^-_{\text{absent}})$ and where

$$||S|| = \inf\{|R| : R \subset \mathbf{Z}^d, \ R \text{ is connected}, \ S \subseteq R\}.$$

For each dimension $d \geq 2$ Burton and Steif ([4]) have shown that if $M > 4e28^d$, then the probability of the occurence of an enclosing set of the origin with inner boundary of cardinality l with only plus or minus ones on the inner boundary surrounding a fixed point decreases exponentially with l. They have also shown that in this case there are no negative infinite clusters $a.s.$ It follows that for $M > 4e28^d$ (5.1) holds. Our main result is

Theorem 5.7: *Let $m \in [-1, m^*)$. If (5.1) holds then there exist positive constants A, c such that*

$$\mu^+(Y_{\Lambda_n} \leq m) \leq Ae^{-c|\partial\Lambda_n|}.$$

Remark: For the Ising model Aizenman et. al. ([1]) have shown that for dimensions large enough, there is a gap between the critical inverse temperature β_c and $\beta_1 = \inf\{\beta :$ The corresponding condition to (5.1) for the Ising model is valid$\}$. In the proof of that theorem they use the fact that there exists a constant $K < \infty$ such that $\beta_c(d) \leq K/d$ for all d. Their method of showing this gap cannot be used for the beach model (yet) since it has not even been shown that $M_c(d)$ is decreasing in d for the beach model, although one would suspect this ([21]).

Let

$$T_n = \{x \in \Lambda_n, \ x_d = 0\}$$

be a $d-1$ -dimensional cross section of Λ_n. Set also $T_{n,k} = \{x \in \Lambda_n : \ x_d = k\}$. To prove Theorem 5.7 we will need the following lemma.

Lemma 5.8: *If (5.1) holds and $m < m^*$, then there exist positive constants A, c such that*

$$\mu^+(X_{T_n} \leq m) \leq A\exp(-cn^{d-1}).$$

Proof of Lemma 5.1: We have from Lemma 2.19 that $Y^+_{\Lambda_n} \succeq Y_{\Lambda_n}$. Hence $\forall \epsilon > 0$

$$P(Y^+_{\Lambda_n} \leq m^* - \epsilon) \to 0 \text{ as } n \to \infty \tag{5.2}$$

since Y_{Λ_n} goes to m^* *a.s.* Now we show that $\forall \epsilon > 0$, $P(Y^+_{\Lambda_n} \geq m^* + \epsilon) \to 0$ as $n \to \infty$ by showing that for large n the expectation of $Y^+_{\Lambda_n}$ is close to m^*. We use a well known method from statistical mechanics. Let $A \subseteq \Lambda_n$ and denote by $\mu^+_{\Lambda_n}(Y_A)$ the expectation of Y_A given plus boundary conditions on $\partial \Lambda_n$. Note that Y_A can be expressed as $\frac{1}{|A|}\sum_{x \in A} Y_x$. Choose $\epsilon > 0$. Clearly we can choose k such that $\mu^+_{\Lambda_k}(Y_0) \leq m^* + \frac{\epsilon}{2}$. Now choose $n_\epsilon > k$ such that $\frac{|\Lambda_n - \Lambda_{n-k}|}{|\Lambda_n|} \leq \frac{\epsilon}{2}$ $\forall n \geq n_\epsilon$. We now have that

$$\mu^+_{\Lambda_n}(Y_{\Lambda_n}) = \mu^+_{\Lambda_n}(\frac{1}{|\Lambda_n|}\sum_{x \in \Lambda_n} Y_x) = \frac{1}{|\Lambda_n|}\mu^+_{\Lambda_n}(\sum_{x \in \Lambda_{n-k}} Y_x \ + \sum_{x \in \Lambda_n - \Lambda_{n-k}} Y_x) \leq$$

$$\frac{1}{|\Lambda_n|}\mu^+_{\Lambda_n}(\sum_{x \in \Lambda_{n-k}} Y_x \ + \sum_{x \in \Lambda_n - \Lambda_{n-k}} 1) \ = \ \frac{1}{|\Lambda_n|}(\sum_{x \in \Lambda_{n-k}} \mu^+_{\Lambda_n}(Y_x) \ + \ |\Lambda_n - \Lambda_{n-k}|) \leq$$

$$\frac{1}{|\Lambda_n|}((m^* + \frac{\epsilon}{2})|\Lambda_{n-k}| + |\Lambda_n - \Lambda_{n-k}|) \leq m^* + \epsilon \ \ \forall n \geq n_\epsilon.$$

Hence $\forall \epsilon > 0$ $\exists n_\epsilon$ such that $\mu^+_{\Lambda_n}(Y_{\Lambda_n}) \leq m^* + \epsilon$ $\forall n \geq n_\epsilon$. This gives that $\limsup_n \mu^+_{\Lambda_n}(Y_{\Lambda_n}) \leq m^*$. This gives together with (5.2) and the boundedness of Y_{Λ_n} that $Y^+_{\Lambda_n}$ converges to m^* in probability.

□

Proof of Theorem 5.2 The first inequality is easily seen from the fact that

$$\mu^+(Y_{\Lambda_n} \geq m) \geq \mu^+(\eta_{\Lambda_n} \equiv M) \geq \alpha^{|\Lambda_n|}$$

from Lemma 5.3. Here we can choose $A_1 = 1$ and $c_1 = \log(1/\alpha)$ to get $\mu^+(Y_{\Lambda_n} \geq m) \geq A_1 e^{-c_1|\Lambda_n|}$. For the proof of the other inequality we again use a method from statistical mechanics. Let $\epsilon = m - m^*$. From Lemma 5.1 we can choose k such that $P(Y^+_{\Lambda_k} \geq m^* + \epsilon/2) \leq 3^{-2/\epsilon}$. For $l \in \mathbf{N}$, let $i = (i_1, i_2, \ldots, i_d) \in \{-l, \ldots, l\}^d$ and define the disjoint regions

$$B_i = \{x \in \mathbf{Z}^d;\ -k \leq x_j - i_j(2k+1) \leq k,\ j = 1, \ldots, d\}.$$

Note that for n such that $(2n+1) = (2k+1)(2l+1)$ we have that $\Lambda_n = \cup_{i \in \{-l,\ldots,l\}^d} B_i$. Let

$$\zeta_i = 1 \text{ if } Y_{B_i} \geq m^* + \frac{\epsilon}{2}$$

$$\zeta_i = 0 \text{ if } Y_{B_i} < m^* + \frac{\epsilon}{2}.$$

Let F be the event that $\frac{1}{(2l+1)^d}\sum \zeta_i \geq \frac{\epsilon}{2}$ where we sum over $i \in \{-l,\ldots,l\}^d$. Let E be the event that $\eta_{\partial(B_i^c)} = +M$ for all $i \in \{-l,\ldots,l\}^d$. Using the FKG-property and the fact that F^c implies that $Y_{\cup_i B_i} \leq (1-\frac{\epsilon}{2})(m^*+\frac{\epsilon}{2})+\frac{\epsilon}{2} < m^*+\epsilon$ where the union is over $i \in \{-l,\ldots,l\}^d$, we have for n such that $(2n+1) = (2k+1)(2l+1)$ that

$$\mu^+(Y_{\Lambda_n} \geq m^*+\epsilon) \leq \mu^+(Y_{\Lambda_n} \geq m^*+\epsilon|E) \leq \mu^+(F|E).$$

Since conditional on E, the variables (ζ_i) are i.i.d. 0-1 random variables, there are at most $2^{(2l+1)^d}$ ways of assigning values to the variables (ζ_i) to indicate the event F. Using that on E the variables (ζ_i) are i.i.d. we have that

$$\mu^+(F|E) \leq 2^{(2l+1)^d}(P(Y_{B_i}^+ \geq m^*+\frac{\epsilon}{2}))^{\lfloor (2l+1)^d\epsilon/2\rfloor} \leq$$

$$\leq 2^{(2l+1)^d}3^{2/\epsilon}3^{-(2l+1)^d} = A_2 e^{-c_2|\Lambda_n|}.$$

The extension for all n is routine.

□

Proof of Lemma 5.3: Let $\alpha = (\frac{1}{2M})^{(1+2d+2d^2)}$. Fix A and let $k = |A|$. Let $\eta_A \in \mathcal{F}_A$. For a finite $W \subset \mathbf{Z}^d$, let

$$C_W = \{y \in \mathbf{Z}^d : \quad d(y,W) \leq 2\}.$$

We have for $x \in \mathbf{Z}^d$ that $|C_x| = 1+2d+2d(1+\frac{2d-2}{2}) = 1+2d+2d^2$ and for $x \in A$

$$\bigcup_{x\in A} C_x = C_A.$$

We have that $|C_A| \leq (1+2d+2d^2)k$ and that $|\{\eta_{C_A} \in \mathcal{F}_{C_A}\}| \leq (2M)^{|C_A|}$. Note that if $x \in \partial C_A$ then $d(x,A) = 3$. This yields that every compatible $\eta_{\partial C_A}$ is compatible with every η_A by irreducibility which gives that

$$\mu^+(\eta_A|\mathcal{F}_{\partial C_A}) \geq \frac{1}{(2M)^{|C_A|}},$$

where we use that for any compatible configuration on ∂C_A there exists at least one ccompatible configuration on C_A which restricted to A equals η_A and agrees with the configuration on ∂C_A. Hence

$$\mu^+(\eta_A) = \sum_{\eta_{\partial C_A}\in\mathcal{F}_{\partial C_A}} \mu^+(\eta_{\partial C_A})\mu^+(\eta_A|\eta_{\partial C_A}) \geq \sum_{\eta_{\partial C_A}\in\mathcal{F}_{\partial C_A}} \mu^+(\eta_{\partial C_A})\frac{1}{(2M)^{|C_A|}} \geq$$

$$\geq \sum_{\eta_{\partial C_A} \in \mathcal{F}_{\partial C_A}} \mu^+(\eta_{\partial C_A})(\frac{1}{(2M)^{1+2d+2d^2}})^k = \alpha^k.$$

□

Proof of Proposition 5.4: Fix n and let E_n be the event that $\eta_{\partial(\Lambda_n)} \equiv -M$. Lemma 5.3 give us that

$$\mu^+(Y_{\Lambda_n} \leq m) \geq \mu^+(E_n)\mu^+(Y_{\Lambda_n} \leq m|E_n) \geq \alpha^{|\partial(\Lambda_n)|}\mu^+(Y_{\Lambda_n} \leq m|E_n).$$

However $\mu^+(Y_{\Lambda_n} \leq m|E_n)$ tends to 1 as n tends to infinity by applying Lemma 5.1 to $X^-_{\Lambda_n}$ (which is defined analogously to $X^+_{\Lambda_n}$) and using the local Markov property. This finishes the proof.

□

Next we will prove Lemma 5.8 in the case $d = 2$. The method used for $d = 2$ can easily be generalized for larger dimensions. We will make some remarks on the case of larger dimensions after the proof of Lemma 5.9.

Proof of Lemma 5.8: Set $\epsilon = m^* - m$. By the triviality of the tail σ-field for μ^+, it is possible to choose an $L \in \mathbf{N}$ such that

$$\mu^+(Y_{T_L} \leq m^* - \epsilon/3) \leq 3^{-3/\epsilon}/2.$$

For $N \in \{1, 2, \ldots\}$ to be chosen later, define the regions

$$R_i = \{x \in \mathbf{Z}^2 : -L \leq x_1 - (i-1)N(2L+1) \leq L,\ x_2 = 0\},$$

$$S_j = \bigcup_{i=1}^{j} R_i \text{ for } j \in \mathbf{N}.$$

Consider the random variables

$$\zeta_i = 1 \text{ if } Y_{R_i} > m^* - \epsilon/3,$$
$$\zeta_i = 0 \text{ otherwise}$$

$$Z_j = j^{-1} \sum_{i=1}^{j} \zeta_i.$$

By Lemma 5.9 below and our choice of L it is possible to choose $N \in \{1, 3, 5, \ldots\}$ large enough that for any k and $i_1 < i_2 < \cdots < i_k$,

$$\mu^+(\zeta_{i_k} = 0|\zeta_{i_1} = \cdots = \zeta_{i_{k-1}} = 0) \leq 3^{-3/\epsilon}.$$

Then

$$\mu^+(\zeta_{i_1} = \cdots = \zeta_{i_k} = 0) \le (3^{-3/\epsilon})^k. \tag{5.3}$$

On the other hand, by translation invariance (see [4]), for n such that $2n+1 = (2L+1)Nj$, $j \in \{1, 3, 5, \ldots\}$,

$$\begin{aligned}\mu^+(Y_{T_n} \le m^* - \epsilon) &\le N\mu^+(Y_{S_j} \le m^* - \epsilon)\\ &\le N\mu^+(Z_j \le 1 - \epsilon/3) \end{aligned}\tag{5.4}$$

where we used the fact that if $Z_j > 1-\epsilon/3$ then $Y_{S_j} \ge (1-\epsilon/3)(m^*-\epsilon/3) - \epsilon/3 > m^* - \epsilon$. Since there are not more than 2^j choices of describing random variable Z_j, it follows from (5.3) that

$$\mu^+(Z_j \le 1 - \epsilon/3) \le 2^j(3^{-3/\epsilon})^{\lfloor j\epsilon/3 \rfloor} \le 3^{3/\epsilon}(2/3)^j = Ae^{-cn}$$

for some positive A and c. Together with (5.4) this finishes the proof when n is such that $2n+1 = (2L+1)Nj$. Note that L is a function of ϵ and that N is a function of (L, ϵ) and hence a function of ϵ. Hence N in eq. (5.4) can be incorporated in A above. The extension for all n is routine.

□

Next we will prove Lemma 5.9 for the case $d = 2$.

Lemma 5.9: *Let R_i, $i \in \mathbf{N}$ and S_j, $j \in \{1, 2, \ldots, i\}$ be as in the proof of Lemma 5.8. Assume that (5.1) holds. Then, for fixed L, given $\epsilon > 0$, it is possible to find N such that*

$$|\mu^+(F|E) - \mu^+(F)| \le \epsilon,$$

for any pair of negative events E and F which depend respectively on $\{\eta_x : x \in S_j\}$ and $\{\eta_x : x \in R_{j+1}\}$ for some j.

Definition 5.10: *For $\eta \in \mathcal{X}$ we say that an enclosing set A of B has an M-**boundary** if $A \supseteq B$, $\eta_{\partial A}(x) \in \{2, \ldots, M\}$ $\forall x \in \partial A$ and $\eta_{\partial(A^c)} \equiv +1$.*

Proof of Lemma 5.9: Set

$$\bar{E} = \{\eta_x = -M \ \forall x \in S_j\}.$$

By use of the FKG-property and Lemma 2.19,

$$\mu^+(F) \le \mu^+(F|E) \le \mu^+(F|\bar{E}). \tag{5.5}$$

Given $x \in R_{j+1}$, let $G_j(x)$ be the event that every R_i, $i = 1, \ldots, j$ is contained in an enclosing set of R_i having an M-boundary and the innermost (to be defined below) collection of such enclosing sets also contains the point $x \in R_{j+1}$. By the innermost collection $\mathcal{G}$ of such subsets, not necessarily containing x, we mean the collection of enclosing sets with the prescribed property for which the number of nodes in such a collection is minimized. That existence implies uniqueness of such a collection follows from the fact that if $C_1 = \cup_{i=1}^{j} A_{B_i}^{(1)}$ and $C_2 = \cup_{i=1}^{j} A_{B_i}^{(2)}$ are two such innermost collections with $C_1 \neq C_2$ and each A_{B_i} is an enclosing set of B_i with an M-boundary, then $C_3 := \cup_{i=1}^{j}(A_{B_i}^{(1)} \cap A_{B_i}^{(2)})$ consists of enclosing sets with M-boundaries having less nodes than C_1 and C_2. That $A_{B_i}^{(1)} \cap A_{B_i}^{(2)}$ is an enclosing set of B_i follows from the fact that $\partial(A_{B_i}^{(1)} \cap A_{B_i}^{(2)}) \subseteq \partial A_{B_i}^{(1)} \cup \partial A_{B_i}^{(2)}$ and that $(A_{B_i}^{(1)} \cap A_{B_i}^{(2)})^c$ necessarily is connected. Decompose $G_j(x)$ into the disjoint events $(k = 1, \dots, j)$

$$\begin{aligned} G_{j,k}(x) = \{ & G_j(x) \text{ occurs and the element of } \mathcal{G} \text{ which contains} \\ & \text{x, also contains } R_k \text{ but does not contain} \\ & R_i \text{ with } 1 \leq i < k\}. \end{aligned}$$

Write also

$$H_{r,s} = \{\eta_x = -M \ \forall x \in S_s \backslash S_{r-1}\}$$

where $s \in \{0, \dots, j\}$ and $r \in \{1, \dots, s\}$. Now

$$\mu^+(G_{j,k}(x)|\bar{E}) = \frac{\mu^+(G_{j,k}(x) \cap H_{1,k-1} \cap H_{k,j})}{\mu^+(H_{k,j} \cap H_{1,k-1})}. \tag{5.6}$$

By the FKG-property

$$\mu^+(H_{k,j} \cap H_{1,k-1}) \geq \mu^+(H_{k,j})\mu^+(H_{1,k-1}) \geq \alpha^{(2L+1)(j-k+1)}\mu^+(H_{1,k-1}) \tag{5.7}$$

where α is from Lemma 5.3. On the other hand for any C such that C is the innermost enclosing set enclosing both R_k and x and C does not contain R_i for $1 \leq i < k$,

$$\begin{aligned} & \mu^+(C \text{ has an } M\text{−boundary} \cap H_{k,j} \cap H_{1,k-1}) \\ & \leq \mu^+(C \text{ has an } M\text{−boundary } \cap H_{1,k-1}) \\ & = \mu^+(C \text{ has an } M\text{−boundary}) \cdot \\ & \quad \mu^+(H_{1,k-1}|C \text{ has an } M\text{−boundary}) \\ & \leq \mu^+(C \text{ has an } M\text{−boundary})\mu^+(H_{1,k-1}). \end{aligned}$$

In the last inequality we used the fact that C does not contain $R_i, i = 1, \dots, k-1$; therefore by the Markov property of μ^+ and by the FKG-property this inequality follows. Indeed for the increasing event $\{C$ has an $M-$boundary and the decreasing event $H_{1,k-1}$, the FKG-property (Theorem 2.20) implies the last inequality. Observe that it is possible that $\partial C \cap (\cup_{i=1}^{k-1} R_i) \neq \emptyset$ and that the last inequality holds trivially in this case. Summing over C gives

$$\begin{aligned}&\mu^+(G_{j,k}(x) \cap H_{1,k-1} \cap H_{k,j}) \\ &\leq \mu^+(\text{there is an enclosing set of } R_k \text{ and } x \text{ having an} \\ &\quad M-\text{boundary})\mu^+(H_{1,k-1}).\end{aligned} \tag{5.8}$$

From (5.6), (5.7), (5.8), and (5.1),

$$\begin{aligned}\mu^+(G_{j,k}(x)|\bar{E}) &\leq A\exp[-c(j-k+1)N(2L+1)] \cdot \alpha^{-(2L+1)(j-k+1)} \\ &= A\exp[(-cN - \log\alpha)(2L+1)(j-k+1)].\end{aligned}$$

Hence

$$\mu^+(G_j(x)|\bar{E}) \leq \sum_{k=1}^{j} \mu^+(G_{j,k}(x)|\bar{E}) \leq \sum_{r=1}^{\infty} A\exp[(-cN - \log\alpha)(2L+1)r].$$

From these inequalities it is clear that there exists N_0 such that if $N > N_0$ then

$$\mu^+(G_j(x)|\bar{E}) \leq \epsilon/(2L+1). \tag{5.9}$$

Set

$$G_j = \bigcup_{x \in R_{j+1}} G_j(x).$$

Then by (5.9), for $N > N_0$

$$\mu^+(G_j|\bar{E}) \leq \epsilon. \tag{5.10}$$

Now

$$\begin{aligned}\mu^+(F|\bar{E}) &= \mu^+(F|\bar{E} \cap G_j)\mu^+(G_j|\bar{E}) \\ &\quad + \mu^+(F|\bar{E} \cap (G_j^c))\mu^+((G_j^c)|\bar{E}).\end{aligned} \tag{5.11}$$

But on $\bar{E} \cap (G_j^c)$, R_{j+1} is isolated from S_j by enclosing sets of $R_i, i = 1, \ldots, j$ having M-boundaries. Therefore by the Markov and FKG properties

$$\mu^+(F|\bar{E} \cap (G_j^c)) \leq \mu^+(F).$$

So by (5.10) and (5.11)

$$\mu^+(F|\bar{E}) \leq \mu^+(F) + \epsilon$$

and using (5.5) the proof is complete.

□

In $d > 2$ one can prove Lemma 5.8 in an analogous way using the following definitions of R_i and S_j, $i, j \in \mathbf{N}^{d-1}$:

$$R_i = \{x \in \mathbf{Z}^d :\ 1 \le x_r - (i_r - 1)N(2L+1) \le L,\ r = 1, \dots, d-1;\ x_d = 0\},$$

$$S_j = \bigcup_{1 \le i_r \le j,\ r=1,\dots,d-1} R_i.$$

Lemma 5.9 can then be stated with E depending on $\{\eta_x :\ x \in S_j \backslash R_i\}$ for some j and i and F depending on $\{\eta_x :\ x \in R_i\}$ where i is such that $R_i \subset S_j$. Note that when one proves the analogue of (5.9) there is an extra polynomial factor in r, due to the number of sets R_k at distance r from R_i.

Proof of Theorem 5.7: By Lemma 5.8 and translation invariance

$$\mu^+(X_{\Lambda_n} \le m) \le \mu^+(\exists k \in [-n, n] \cap \mathbf{Z} \text{ such that } X_{T_{n,k}} \le m)$$

$$\le (2n+1)\mu^+(X_{T_n} \le m) \le Ae^{-c|\partial\Lambda_n|},$$

if $m < m^*$. This completes the proof.

□

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