A generalization of Littlewood’s Tauberian theorem for the Laplace transform

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0 Introduction

In this paper we will consider a Tauberian problem of the following kind:

Suppose that $\alpha$ and $\beta$ are two functions of bounded variation on any finite interval $[0,T], T > 0,$ and suppose that $\alpha(0) = \beta(0) = 0$. If furthermore

\begin{equation}
\int_0^\infty e^{-st}d\alpha(t) \sim \int_0^\infty e^{-st}d\beta(t), \quad s \to 0+,
\end{equation}

where $\beta$ satisfies some kind of regularity condition and $\alpha$ satisfies a Tauberian condition, then

\begin{equation}
\alpha(t) \sim \beta(t), \quad t \to \infty.
\end{equation}

Here (0.2) means that

\[ \alpha(t) = \beta(t) + o(\beta(t)), \quad t \to \infty \]

with a corresponding interpretation for (0.1).

The method used to prove this consists in making a certain substitution in (0.1) thereby making a convolution transform out of the Laplace transform. We then prove that the span of translates of this convolution kernel is dense in a certain weighted space, a result which might be of some independent interest. The solution of the problem depends on this. Our result is a generalisation of a classical Tauberian theorem due to Littlewood [4], but sometimes also referred to as Hardy and Littlewood’s [1] or Karamata’s [2] theorem.

As formulated above this kind of problems has been treated by Korenblum [3], Mohr [5] and Wagner [7, 8] but with stronger conditions than ours.
1 Preliminaries

Suppose that $\alpha$ is of local bounded variation on the positive real line with $\alpha(0) = 0$, and suppose that

\[(1.1) \quad F(s) = \int_0^\infty e^{-st}d\alpha(t)\]

exists for all $s > 0$.

After a partial integration in (1.1) we see that

\[(1.2) \quad F(s) = s\int_0^\infty e^{-st}\alpha(t)dt,\]

where the integral is absolutely convergent for all $s > 0$ (cf. e.g. [9, p. 41]). In (1.2) we make the substitutions

$s = \exp(-x)$ and $t = \exp u$

and finally obtain

\[(1.3) \quad F(s) = \int_{-\infty}^\infty K(x-u)\phi(u)du = K*\phi(x),\]

where

\[(1.4) \quad K(x) = \exp(-\exp(-x) - x)\]

and

\[(1.5) \quad \phi(u) = \alpha(\exp u).\]

We also note that the Laplace transform (1.1) exists for all $s > 0$ only if

$$\limsup_{t \to \infty} t^{-1} \log |\alpha(t)| = 0,$$

(cf. e.g. [9, p. 43]).

For the function $\phi$ of (1.5) this implies that

\[(1.6) \quad |\phi(x)| = \exp(o(1)e^x), \ x \to \infty.\]

This fact leads us to introduce the following classes of functions:

**DEFINITION.** For any positive and non-increasing function $p$ we let $L^1_e(p)$ consist of all measurable functions $H$ such that

$$\|H\|_p^1 = \int_{-\infty}^\infty |H(-x)|\exp(p(x)e^x)dx < \infty$$

We also let $L^\infty_e(p)$ denote the space of all measurable functions $\phi$ such that

$$\|\phi\|_p^\infty = \operatorname{ess sup}_{-\infty < x < \infty} |\phi(x)|\exp(-p(x)e^x) < \infty.$$
We see that $L^1_c(p)$ is a Banach space under this norm with $L^\infty_c(p)$ as its dual space, which means that any bounded linear functional on $L^1_c(p)$ is of the form

$$K \to \int_{-\infty}^{\infty} K(-x)\phi(x)dx$$

for some function $\phi \in L^\infty_c(p)$. (cf. e.g. [6 p. 136])

Every measurable function which satisfies (1.6) also belongs to $L^\infty_c(p)$ for some $p$ such that $p(x) = o(1), x \to \infty$. We then perhaps have to change the function somewhat for negative values of its argument, which have no influence on our problem.

For convenience we let $C$ stand for positive constants not necessarily the same in any two places.

2 Tauberian theorems

The following theorem is stronger than necessary for our applications to the Laplace transform. Perhaps it is of some interest in its own.

**THEOREM 1** The span of translates of the Laplace kernel $K$ is dense in $L^1_c(p)$ if $p(x) = o(1), x \to \infty$.

**PROOF.** Take $\phi \in L^\infty_c(p)$ and suppose that

$$(2.1) \quad K * \phi(x) = 0, \quad -\infty < x < \infty.$$ 

Now let

$$\alpha(t) = \phi(\log t), \quad 0 < t < \infty.$$ 

By making appropriate substitutions in (2.1) we see that

$$\int_0^{\infty} e^{-st}\alpha(t)dt = 0, \quad 0 < s < \infty.$$ 

By the uniqueness of the Laplace transform (cf. [9 p. 62-63]) we obtain that

$$\alpha(t) = 0, \quad a.e.$$ 

Thus (2.1) implies that

$$(2.2) \quad \phi(x) = 0, \quad a.e.$$ 

It now follows from the Hahn-Banach theorem that the span of translates of $K$ is dense in $L^1_c(p)$. The arguments for this is as follows:
Let $M$ be the span of translates of $K$. Then $M$ is a linear subspace in $L^1_c(p)$ such that all functionals
\[ \phi \to \phi(H) = \int_{-\infty}^{\infty} H(-x)\phi(x)dx = 0 \]
for all functions $H$ in $M$.

Suppose now that $M$ isn’t dense in $L^1_c(p)$. Then there must exist a function $K_0 \in L^1_c(p)$ and a function $\phi \in L^\infty_c(p)$ such that
\[ \int_{-\infty}^{\infty} K_0(-x)\phi(x)dx \neq 0. \quad (\text{cf. e.g. [6 p. 114]}) \]
By (2.2) this is impossible.

**THEOREM 2** Let $\alpha$ and $\beta$ be two real-valued functions of bounded variation on any finite interval $[0, T], T > 0$ with $\alpha(0) = \beta(0) = 0$. Let $\beta(t)$ be positive for $t > 0$ and let
\[ \int_0^\infty e^{-st}\alpha(t) \sim \int_0^\infty e^{-st}\beta(t), \quad s \to 0+. \]
Suppose that there exist positive numbers $\gamma$ and $m$ and that for any $\delta > 0$ we can find a number $T$ such that
\[ \gamma \beta(t) \leq \beta(rt) \leq (1 + \delta)r^m \beta(t) \quad \text{when } r \geq 1 \text{ and } t \geq T. \]
Furthermore, suppose that
\[ \lim_{\lambda \to 1+} \lim_{x \to \infty} \inf_{x \leq \lambda x} \left( \frac{\alpha(t) - \alpha(x)}{\beta(x)} \right) = 0. \]
Then
\[ \alpha(t) \sim \beta(t), \quad t \to \infty. \]

**PROOF.** By the method in Section 1 we transform (2.3) into
\[ K \ast \phi(x) \sim K \ast \psi(x), \quad x \to \infty. \]
Here $K$ and $\phi$ are defined by (1.4) and (1.5) respectively and
\[ \psi(x) = \beta(\exp x). \]
By (2.4) we see that for any $\delta > 0$ there exist a constant $X$ such that
\[ \gamma \psi(x) \leq \psi(x + y) \leq (1 + \delta)\exp(my)\psi(x) \quad \text{when } x \geq X, y \geq 0. \]
Condition (2.5) is equivalent to

$$\lim_{h \to 0^+} \lim_{x \to -\infty} \inf_{x \leq y \leq x + h} \left( \frac{\phi(y) - \phi(x)}{\psi(x)} \right) = 0.$$  

Now as a first step we prove that

$$\phi(x) = O(\psi(x)), \quad x \to \infty.$$  

To do this, we combine (2.6), (2.8) and (2.9) with the fact that $\phi(x)$ and $\psi(x)$ are bounded on any interval $-\infty < x \leq X$.

From (2.9) we easily see that

$$\phi(y) - \phi(x) \geq -C(1 + \psi(x)), \quad x \leq y \leq x + 2.$$  

Now suppose that $x \leq y \leq x + 1$ and write

$$K \ast \phi(y + 1) - K \ast \phi(x) = \int_{-\infty}^{\infty} K(-u)(\phi(u + y + 1) - \phi(u + x)) du =$$

$$= \int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{\infty}.$$  

By (2.10)

$$\phi(u + y + 1) - \phi(u + x) \geq -C(1 + \psi(u + x)),$$

and hence

$$\int_{-\infty}^{-1} K(-u)(\phi(u + y + 1) - \phi(u + x)) du + \int_{0}^{\infty} K(-u)(\phi(u + y + 1) - \phi(u + x)) du \geq$$

$$\geq -C\left( \int_{-\infty}^{-1} K(-u)(1 + \psi(u + x)) du + \int_{0}^{\infty} K(-u)(1 + \psi(u + x)) du \right) \geq$$

$$\geq -C\left( \int_{-\infty}^{\infty} K(-u)(1 + \psi(u + x)) du \right) =$$

$$= -C(1 + K \ast \psi(x)).$$

This inequality we combine with (2.11), (2.6) and the right-hand side of (2.8) and see that

$$\int_{-1}^{0} K(-u)(\phi(u + y + 1) - \phi(u + x)) du \leq$$

$$\leq K \ast \phi(y + 1) - K \ast \phi(x) + C(1 + K \ast \psi(x)) \leq$$

$$\leq C(1 + \psi(x)).$$

For $-1 \leq u \leq 0$ we write

$$\phi(u + y + 1) - \phi(u + x) = \phi(u + y + 1) - \phi(y) + \phi(x) - \phi(u + x) + \phi(y) - \phi(x) \geq$$

$$\geq -C(1 + \psi(y)) - C(1 + \psi(u + x)) + \phi(y) - \phi(x) \geq -C(1 + \psi(y)) + \phi(y) - \phi(x).$$
Here the last inequality follows from the left-hand side of (2.8).

From this we obtain that
\[ \int_{-1}^{0} K(-u)(\phi(u+y+1)-\phi(u+x))du \geq (-C(1+\psi(x))+\phi(y)-\phi(x)) \int_{-1}^{0} K(-u)du, \]
which implies that
\[ \phi(y) - \phi(x) \leq C(1 + \psi(x)) + \int_{-1}^{0} K(-u)(\phi(u+y+1) - \phi(u+x))du. \]

Combining this inequality with (2.12) we see that
\[ \phi(y) - \phi(x) \leq C(1 + \psi(x)), \quad x \leq y \leq x + 1, \]
which together with (2.10) gives that
\[ |\phi(y) - \phi(x)| \leq C(1 + \psi(x)), \quad x \leq y \leq x + 1. \]  
(2.13)

By aim of (2.13) and (2.8) we now see that for any \( u \geq 0 \)
\[ |\phi(u+x) - \phi(x)| \leq C(1 + u)\exp(mu)(1 + \psi(x)), \]
and for any \( u \leq 0 \)
\[ |\phi(u+x) - \phi(x)| \leq C(1 + |u|)(1 + \psi(x)). \]

Since
\[ \int_{-\infty}^{\infty} K(-u)du = 1 \]
we get the following estimate
\[ |K \ast \phi(x) - \phi(x)| = \int_{-\infty}^{0} K(-u)(\phi(u+x) - \phi(x))du + \int_{0}^{\infty} K(-u)(\phi(u+x) - \phi(x))du \leq C(1 + \psi(x))\left(\int_{-\infty}^{0} K(-u)(1 + |u|)du + \int_{0}^{\infty} K(-u)(1 + u)\exp(mu)du\right). \]

Thus
\[ |\phi(x)| \leq |K \ast \phi(x)| + C(1 + \psi(x)) \]
and by use of (2.6) and the right-hand side of (2.8) we have finally proved that
\[ \phi(x) = O(\psi(x)), \quad x \to \infty. \]  
(2.14)
Now take a non-increasing function \( p \) such that \( \psi \) belongs to \( L^\infty_c(p) \), perhaps after a change of \( \psi \) for negative values of its arguments. For any function \( H \) in \( L^1_c(p) \) and for any positive \( \epsilon \) we can, using Theorem 1, find a finite linear combination \( K_\epsilon \) of translates of \( K \),

\[
K_\epsilon(x) = \sum_{\nu=1}^n a_\nu K(x - \lambda_\nu),
\]

such that

\[
\|K_\epsilon - H\|_p^1 < \epsilon.
\]

Now write

\[
(2.15) \quad H \ast \phi = H \ast \psi + (H - K_\epsilon) \ast (\phi - \psi) + K_\epsilon \ast (\phi - \psi).
\]

By (2.6) and (2.8) we see that

\[
(2.16) \quad K_\epsilon \ast (\phi - \psi)(x) = o(1)K_\epsilon \ast \psi(x) = o(\psi(x)), \quad x \to \infty.
\]

By (2.14) we have that

\[
(H - K_\epsilon) \ast (\phi - \psi)(x) = O(1)(|H - K_\epsilon| \ast \psi(x)), \quad x \to \infty,
\]

and hence using (2.8) we have that

\[
(H - K_\epsilon) \ast (\phi - \psi)(x) = O(1)(\int_{-\infty}^0 |H(-u) - K_\epsilon(-u)|\psi(u + x)du + \\
+ \int_{0}^{\infty} |H(-u) - K_\epsilon(-u)|\psi(u + x)du) = O(1)\psi(x)(\int_{-\infty}^0 |H(-u) - K_\epsilon(-u)|du + \\
+ \int_{0}^{\infty} |H(-u) - K_\epsilon(-u)|(1 + u)\exp(mu)du) = O(1)\psi(x)\|H - K_\epsilon\|_p^1, \quad x \to \infty.
\]

Hence

\[
(2.17) \quad (H - K_\epsilon) \ast (\phi - \psi)(x) = O(1)\psi(x)\|H - K_\epsilon\|_p^1, \quad x \to \infty.
\]

Since \( \epsilon \) is arbitrary, it follows from (2.15), (2.16) and (2.17) that

\[
(2.18) \quad H \ast \phi(x) = H \ast \psi(x) + o(\psi(x)), \quad x \to \infty.
\]

For any positive number \( h \) we let \( H \) be the characteristic function on \((-h, 0)\) multiplied by \( h^{-1} \). Then by (2.18)

\[
h^{-1}\int_{-h}^0 \phi(x - u)du = h^{-1}\int_{-h}^0 \psi(x - u)du + o(\psi(x)), \quad x \to \infty.
\]

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We now write
\[
\frac{\phi(x)}{\psi(x)} = h^{-1} \int_{-h}^{0} \frac{\phi(x) - \phi(x - u)}{\psi(x)} du = \]
\[
h^{-1} \int_{-h}^{0} \frac{\phi(x) - \phi(x - u)}{\psi(x)} du + h^{-1} \int_{-h}^{0} \frac{\phi(x - u)}{\psi(x)} du = \]
\[
h^{-1} \int_{-h}^{0} \frac{\phi(x) - \phi(x - u)}{\psi(x)} du + h^{-1} \int_{-h}^{0} \frac{\psi(x - u)}{\psi(x)} du + o(1), x \to \infty.
\]
By use of (2.8) and (2.9) we see that if \(h\) is small enough then to any \(\epsilon > 0\) there exists a constant \(x_1\) such that
\[
\frac{\phi(x)}{\psi(x)} < 1 + \epsilon \quad \text{if } x \geq x_1.
\]
If on the other hand \(H\) is the characteristic function on \((0, h)\) multiplied by \(h^{-1}\), we can in an analogous way derive that there exists a constant \(x_2\) so that
\[
\frac{\phi(x)}{\psi(x)} > 1 - \epsilon \quad \text{if } x \geq x_2.
\]
By (2.19) and (2.20) it follows that
\[
\phi(x) \sim \psi(x), \quad x \to \infty,
\]
which implies that
\[
\alpha(t) \sim \beta(t), \quad t \to \infty.
\]
Hence we have proved Theorem 2.

**Remark 1** If \(\gamma\) is any non-negative number and if
\[
\beta(t) = L(t) \frac{A t^\gamma}{\Gamma(1 + \gamma)},
\]
where \(L\) is so-called slowly oscillating function, then
\[
F(s) \sim L(s) A s^{-\gamma}, \quad s \to 0 +.
\]
Hence Theorem 2 includes the almost classical Hardy-Littlewood-Karamata theorem in its most general version.

**Remark 2** Since our assumptions are weaker Theorem 2 sharpens the results of Korenblum [3], Mohr [5] and Wagner [7, 8] applied to the Laplace transform. Korenblum though treats a very general class of kernels. Mohr’s condition (2) is clearly unnecessary.
References


