A generalization of Littlewood's Tauberian theorem for the Laplace transform

Lennart Frennemo

0 Introduction

In this paper we will consider a Tauberian problem of the following kind:

Suppose that α and β are two functions of bounded variation on any finite interval $[0,T], T>0$, and suppose that $\alpha(0)=\beta(0)=0$. If furthermore

$$\int_0^\infty e^{-st}d\alpha(t) \sim \int_0^\infty e^{-st}d\beta(t), \quad s \to 0+, \tag{0.1}$$

where β satisfies some kind of regularity condition and α satisfies a Tauberian condition, then

$$\alpha(t) \sim \beta(t), \quad t \to \infty. \tag{0.2}$$

Here (0.2) means that

$$\alpha(t) = \beta(t) + o(\beta(t)), \quad t \to \infty$$

with a corresponding interpretation for (0.1).

The method used to prove this consists in making a certain substitution in (0.1) thereby making a convolution transform out of the Laplace transform. We then prove that the span of translates of this convolution kernel is dense in a certain weighted space, a result which might be of some independent interest. The solution of the problem depends on this. Our result is a generalisation of a classical Tauberian theorem due to Littlewood [4], but sometimes also referred to as Hardy and Littlewood's [1] or Karamata's [2] theorem.

As formulated above this kind of problems has been treated by Korenblum [3], Mohr [5] and Wagner [7, 8] but with stronger conditions than ours.

1 Preliminaries

Suppose that α is of local bounded variation on the positive real line with $\alpha(0) = 0$, and suppose that

(1.1) $$F(s) = \int_0^\infty e^{-st} d\alpha(t)$$

exists for all $s > 0$.

After a partial integration in (1.1) we see that

(1.2) $$F(s) = s\int_0^\infty e^{-st}\alpha(t)dt,$$

where the integral is absolutely convergent for all $s > 0$ (cf. e.g. [9, p. 41]). In (1.2) we make the substitutions

$$s = \exp(-x) \quad \text{and} \quad t = \exp u$$

and finally obtain

(1.3) $$F(s) = \int_{-\infty}^\infty K(x-u)\phi(u)du = K * \phi(x),$$

where

(1.4) $$K(x) = \exp(-\exp(-x) - x)$$

and

(1.5) $$\phi(u) = \alpha(\exp u).$$

We also note that the Laplace transform (1.1) exists for all $s > 0$ only if

$$\limsup_{t\to\infty} t^{-1}\log|\alpha(t)| = 0,$$

(cf. e.g. [9, p. 43]).

For the function ϕ of (1.5) this implies that

(1.6) $$|\phi(x)| = \exp(o(1)e^x),\ x \to \infty.$$

This fact leads us to introduce the following classes of functions:

DEFINITION. For any positive and non-increasing function p we let $L_e^1(p)$ consist of all measurable functions H such that

$$\|H\|_p^1 = \int_{-\infty}^\infty |H(-x)|\exp(p(x)e^x)dx < \infty$$

We also let $L_e^\infty(p)$ denote the space of all measurable functions ϕ such that

$$\|\phi\|_p^\infty = ess \sup_{-\infty<x<\infty} |\phi(x)|\exp(-p(x)e^x) < \infty.$$

We see that $L_e^1(p)$ is a Banachspace under this norm with $L_e^\infty(p)$ as its dual space, which means that any bounded linear functional on $L_e^1(p)$ is of the form

$$K \to \int_{-\infty}^{\infty} K(-x)\phi(x)dx$$

for some function $\phi \in L_e^\infty(p)$. (cf. e.g. [6 p. 136])

Every measurable function which satisfies (1.6) also belongs to $L_e^\infty(p)$ for some p such that $p(x) = o(1), x \to \infty$. We then perhaps have to change the function somewhat for negative values of its argument, which have no influence on our problem.

For convenience we let C stand for positive constants not necessarily the same in any two places.

2 Tauberian theorems

The following theorem is stronger than necessary for our applications to the Laplace transform. Perhaps it is of some interest in its own.

THEOREM 1 *The span of translates of the Laplace kernel K is dense in $L_e^1(p)$ if $p(x) = o(1), x \to \infty$.*

PROOF. Take $\phi \in L_e^\infty(p)$ and suppose that

$$K * \phi(x) = 0, \quad -\infty < x < \infty. \tag{2.1}$$

Now let

$$\alpha(t) = \phi(\log t), \quad 0 < t < \infty.$$

By making appropriate substitutions in (2.1) we see that

$$\int_0^\infty e^{-st}\alpha(t)dt = 0, \quad 0 < s < \infty.$$

By the uniqueness of the Laplace transform (cf. [9 p. 62-63]) we obtain that

$$\alpha(t) = 0, \quad \text{a.e.}$$

Thus (2.1) implies that

$$\phi(x) = 0, \quad \text{a.e.} \tag{2.2}$$

It now follows from the Hahn-Banach theorem that the span of translates of K is dense in $L_e^1(p)$. The arguments for this is as follows:

Let M be the span of translates of K. Then M is a linear subspace in $L_e^1(p)$ such that all functionals

$$\phi \to \phi(H) = \int_{-\infty}^{\infty} H(-x)\phi(x)dx = 0$$

for all functions H in M.

Suppose now that M isn't dense in $L_e^1(p)$. Then there must exist a function $K_0 \in L_e^1(p)$ and a function $\phi \in L_e^\infty(p)$ such that

$$\int_{-\infty}^{\infty} K_0(-x)\phi(x)dx \neq 0. \quad \text{(cf. e.g. [6 p. 114])}$$

By (2.2) this is impossible.

THEOREM 2 *Let α and β be two real-valued functions of bounded variation on any finite interval $[0, T], T > 0$ with $\alpha(0) = \beta(0) = 0$. Let $\beta(t)$ be positive for $t > 0$ and let*

$$\int_0^{\infty} e^{-st} d\alpha(t) \sim \int_0^{\infty} e^{-st} d\beta(t), \quad s \to 0+. \tag{2.3}$$

Suppose that there exist positive numbers γ and m and that for any $\delta > 0$ we can find a number T such that

$$\gamma\beta(t) \leq \beta(rt) \leq (1+\delta)r^m\beta(t) \quad \textit{when } r \geq 1 \textit{ and } t \geq T. \tag{2.4}$$

Furthermore, suppose that

$$\lim_{\lambda \to 1+} \liminf_{x \to \infty} \inf_{x \leq t \leq \lambda x} \left(\frac{\alpha(t) - \alpha(x)}{\beta(x)}\right) = 0. \tag{2.5}$$

Then

$$\alpha(t) \sim \beta(t), \quad t \to \infty.$$

PROOF. By the method in Section 1 we transform (2.3) into

$$K * \phi(x) \sim K * \psi(x), \quad x \to \infty. \tag{2.6}$$

Here K and ϕ are defined by (1.4) and (1.5) respectively and

$$\psi(x) = \beta(\exp x). \tag{2.7}$$

By (2.4) we see that for any $\delta > 0$ there exist a constant X such that

$$\gamma\psi(x) \leq \psi(x+y) \leq (1+\delta)\exp(my)\psi(x) \quad \text{when } x \geq X, y \geq 0. \tag{2.8}$$

Condition (2.5) is equivalent to

$$\lim_{h\to 0+}\lim_{x\to\infty}\inf_{x\le y\le x+h}\left(\frac{\phi(y)-\phi(x)}{\psi(x)}\right)=0. \tag{2.9}$$

Now as a first step we prove that

$$\phi(x)=O(\psi(x)),\quad x\to\infty.$$

To do this, we combine (2.6), (2.8) and (2.9) with the fact that $\phi(x)$ and $\psi(x)$ are bounded on any interval $-\infty<x\le X$.

From (2.9) we easily see that

$$\phi(y)-\phi(x)\ge -C(1+\psi(x)),\quad x\le y\le x+2. \tag{2.10}$$

Now suppose that $x\le y\le x+1$ and write

$$K*\phi(y+1)-K*\phi(x)=\int_{-\infty}^{\infty}K(-u)(\phi(u+y+1)-\phi(u+x))du= \tag{2.11}$$
$$=\int_{-\infty}^{-1}+\int_{-1}^{0}+\int_{0}^{\infty}.$$

By (2.10)

$$\phi(u+y+1)-\phi(u+x)\ge -C(1+\psi(u+x)),$$

and hence

$$\begin{aligned}
&\int_{-\infty}^{-1}K(-u)(\phi(u+y+1)-\phi(u+x))du+\int_{0}^{\infty}K(-u)(\phi(u+y+1)-\phi(u+x))du\ge\\
&\ge -C\left(\int_{-\infty}^{-1}K(-u)(1+\psi(u+x))du+\int_{0}^{\infty}K(-u)(1+\psi(u+x))du\right)\ge\\
&\ge -C\left(\int_{-\infty}^{\infty}K(-u)(1+\psi(u+x))du\right)=\\
&=-C(1+K*\psi(x)).
\end{aligned}$$

This inequality we combine with (2.11), (2.6) and the right-hand side of (2.8) and see that

$$\begin{aligned}
&\int_{-1}^{0}K(-u)(\phi(u+y+1)-\phi(u+x))du\le\\
&\le K*\phi(y+1)-K*\phi(x)+C(1+K*\psi(x))\le\\
&\le C(1+\psi(x)).
\end{aligned} \tag{2.12}$$

For $-1\le u\le 0$ we write

$$\begin{aligned}
&\phi(u+y+1)-\phi(u+x)=\phi(u+y+1)-\phi(y)+\phi(x)-\phi(u+x)+\phi(y)-\phi(x)\ge\\
&\ge -C(1+\psi(y))-C(1+\psi(u+x))+\phi(y)-\phi(x)\ge -C(1+\psi(x))+\phi(y)-\phi(x).
\end{aligned}$$

Here the last inequality follows from the left-hand side of (2.8).

From this we obtain that

$$\int_{-1}^{0} K(-u)(\phi(u+y+1)-\phi(u+x))du \geq (-C(1+\psi(x))+\phi(y)-\phi(x)) \int_{-1}^{0} K(-u)du,$$

which implies that

$$\phi(y)-\phi(x) \leq C(1+\psi(x)+\int_{-1}^{0} K(-u)(\phi(u+y+1)-\phi(u+x))du).$$

Combining this inequality with (2.12) we see that

$$\phi(y)-\phi(x) \leq C(1+\psi(x)), \quad x \leq y \leq x+1,$$

which together with (2.10) gives that

$$|\phi(y)-\phi(x)| \leq C(1+\psi(x)), \quad x \leq y \leq x+1. \tag{2.13}$$

By aim of (2.13) and (2.8) we now see that for any $u \geq 0$

$$|\phi(u+x)-\phi(x)| \leq C(1+u)\exp(mu)(1+\psi(x)),$$

and for any $u \leq 0$

$$|\phi(u+x)-\phi(x)| \leq C(1+|u|)(1+\psi(x)).$$

Since

$$\int_{-\infty}^{\infty} K(-u)du = 1$$

we get the following estimate

$$\begin{aligned} |K*\phi(x)-\phi(x)| &= \int_{-\infty}^{0} K(-u)(\phi(u+x)-\phi(x))du + \\ &+ \int_{0}^{\infty} K(-u)(\phi(u+x)-\phi(x))du| \leq \\ &\leq C(1+\psi(x))(\int_{-\infty}^{0} K(-u)(1+|u|)du + \int_{0}^{\infty} K(-u)(1+u)\exp(mu)du). \end{aligned}$$

Thus

$$|\phi(x)| \leq |K*\phi(x)| + C(1+\psi(x))$$

and by use of (2.6) and the right-hand side of (2.8) we have finally proved that

$$\phi(x) = O(\psi(x)), \quad x \to \infty. \tag{2.14}$$

Now take a non-increasing function p such that ψ belongs to $L_e^\infty(p)$, perhaps after a change of ψ for negative values of its arguments. For any function H in $L_e^1(p)$ and for any positive ϵ we can, using Theorem 1, find a finite linear combination K_ϵ of translates of K,

$$K_\epsilon(x) = \sum_{\nu=1}^{n} a_\nu K(x - \lambda_\nu),$$

such that

$$\|K_\epsilon - H\|_p^1 < \epsilon.$$

Now write

$$H * \phi = H * \psi + (H - K_\epsilon) * (\phi - \psi) + K_\epsilon * (\phi - \psi). \tag{2.15}$$

By (2.6) and (2.8) we see that

$$K_\epsilon * (\phi - \psi)(x) = o(1) K_\epsilon * \psi(x) = o(\psi(x)), \quad x \to \infty. \tag{2.16}$$

By (2.14) we have that

$$(H - K_\epsilon) * (\phi - \psi)(x) = O(1)(|H - K_\epsilon| * \psi(x)),\ x \to \infty,$$

and hence using (2.8) we have that

$$(H - K_\epsilon) * (\phi - \psi)(x) = O(1)(\int_{-\infty}^{0} |H(-u) - K_\epsilon(-u)| \psi(u + x) du +$$
$$+ \int_0^\infty |H(-u) - K_\epsilon(-u)| \psi(u + x) du) = O(1)\psi(x)(\int_{-\infty}^{0} |H(-u) - K_\epsilon(-u)| du +$$
$$+ \int_0^\infty |H(-u) - K_\epsilon(-u)|(1 + u)\exp(mu) du) = O(1)\psi(x)\|H - K_\epsilon\|_p^1,\ x \to \infty.$$

Hence

$$(H - K_\epsilon) * (\phi - \psi)(x) = O(1)\psi(x)\|H - K_\epsilon\|_p^1,\ x \to \infty. \tag{2.17}$$

Since ϵ is arbitrary, itfollows from (2.15), (2.16) and (2.17) that

$$H * \phi(x) = H * \psi(x) + o(\psi(x)),\ x \to \infty. \tag{2.18}$$

For any positive number h we let H be the characteristic function on $(-h, 0)$ multiplied by h^{-1}. Then by (2.18)

$$h^{-1}\int_{-h}^{0} \phi(x - u) du = h^{-1}\int_{-h}^{0} \psi(x - u) du + o(\psi(x)),\ x \to \infty.$$

We now write

$$\begin{aligned}
\frac{\phi(x)}{\psi(x)} &= h^{-1}\int_{-h}^{0}\frac{\phi(x)}{\psi(x)}du = \\
&= h^{-1}\int_{-h}^{0}\frac{\phi(x)-\phi(x-u)}{\psi(x)}du + h^{-1}\int_{-h}^{0}\frac{\phi(x-u)}{\psi(x)}du = \\
&= h^{-1}\int_{-h}^{0}\frac{\phi(x)-\phi(x-u)}{\psi(x)}du + h^{-1}\int_{-h}^{0}\frac{\psi(x-u)}{\psi(x)}du + o(1),\ x\to\infty.
\end{aligned}$$

By use of (2.8) and (2.9) we see that if h is small enough then to any $\epsilon > 0$ there exists a constant x_1 such that

$$\frac{\phi(x)}{\psi(x)} < 1+\epsilon \quad \text{if } x \geq x_1. \tag{2.19}$$

If on the other hand H is the characteristic function on $(0,h)$ multiplied by h^{-1}, we can in an analogous way derive that there exists a constant x_2 so that

$$\frac{\phi(x)}{\psi(x)} > 1-\epsilon \quad \text{if } x \geq x_2. \tag{2.20}$$

By (2.19) and (2.20) it follows that

$$\phi(x) \sim \psi(x), \quad x\to\infty,$$

which implies that

$$\alpha(t) \sim \beta(t), \quad t\to\infty.$$

Hence we have proved Theorem 2.

Remark 1 If γ is any non-negative number and if

$$\beta(t) = L(t)\frac{At^{\gamma}}{\Gamma(1+\gamma)},$$

where L is so-called slowly oscillating function, then

$$F(s) \sim L(s)As^{-\gamma}, \quad s\to 0+.$$

Hence Theorem 2 includes the almost classical Hardy-Littlewood-Karamata theorem in its most general version.

Remark 2 Since our assumptions are weaker Theorem 2 sharpens the results of Korenblum [3], Mohr [5] and Wagner [7, 8] applied to the Laplace transform. Korenblum though treats a very general class of kernels. Mohr's condition (2) is clearly unnecessary.

References

1. G.H. Hardy and J.E. Littlewood, *Notes on the theory of series XI. On Tauberian theorems*, Proc. L. M. S. (2) 30 (1929), 23-37.

2. J. Karamata, *Neuer Beweis und verallgemenerungen einiger Tauberian-Sätze*, Math. Z. 33 (1931), 294-299.

3. B. Korenblum, *A generalisation of Wiener's Tauberian theorem and harmonic analysis of rapidly increasing functions*, Trudy Moskov, Mat. Obsc. 7 (1958), 121-148. (Russian).

4. J. E. Littlewood, *The converse of Abel's theorem*, Proc. L. M. S. (2) (1910), 434-448.

5. E. Mohr, *Ein Umkehrsaats Tauberscher Art für Laplace-Stieltjes-Integrale*, Math. Annalen 166 (1966), 257-263.

6. W. Rudin, *Real and Complex Analysis*, McGraw-Hill 1974.

7. E. Wagner, *Taubersche Sätze reeller Art für die Laplace-Transformation*, Math. Nachr. 31 (1966), 153-168.

8. E. Wagner, *Taubersche Sätze reeller Art für Integraltransformen mit Kernen der form K(st)*, Math. Nachr. 47 (1970), 303-317.

9. D. V. Widder, *The Laplace transform*, Princeton University Press 1946.