AN APPROACH TO HOPF ALGEBRAS VIA FROBENIUS COORDINATES

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ABSTRACT. In Section 1 we introduce Frobenius coordinates in the general setting that includes Hopf subalgebras. In Sections 2 and 3 we review briefly the theories of Frobenius algebras and augmented Frobenius algebras with some new material in Section 3. In Section 4 we study the Frobenius structure of an FH-algebra H [25] and extend two recent theorems in [8]. We obtain two Radford formulas for the antipode in H and generalize in Section 7 the results on its order in [10]. We study the Frobenius structure on an FH-subalgebra pair in Sections 5 and 6. In Section 8 we show that the quantum double of H is symmetric and unimodular.

1. INTRODUCTION

Suppose A and S are noncommutative associative rings with S a unital subring in A, or stated equivalently, A/S is a ring extension. Given a ring automorphism $\beta : S \to S$, a left S-module M receives the β-twisted module structure ${}_\beta M$ by $s \cdot_\beta m := \beta(s)m$ for each $s \in S$ and $m \in M$. A/S is said to be a β-Frobenius extension if the natural module A_S is finite projective, and

$$ {}_S A_A \cong {}_\beta \mathrm{Hom_S(A_S, S_S)_A} $$

[10, 23]. A very useful characterization of β-Frobenius extensions is that they are the ring extensions having a Frobenius coordinate system. A *Frobenius coordinate system* for a ring extension A/S is data (E, x_i, y_i) where $E : {}_S A_S \to {}_\beta S_S$ is a bimodule homomorphism, called the *Frobenius homomorphism*, and elements $x_i, y_i \in A$ $(i = 1, \dots, n)$, called *dual bases*, such that for every $a \in A$:

$$ \sum_{i=1}^{n} \beta^{-1}(E(ax_i))y_i = a = \sum_i x_i E(y_i a). \tag{1} $$

One of the most important points about Frobenius coordinates for A/S is that any two of these, (E, x_i, y_i) and (F, z_j, w_j), differ by only an invertible $d \in C_A(S)$, the centralizer of S in A: viz. $F = Ed$ and $\sum_i x_i \otimes d^{-1}y_i = \sum_j z_j \otimes w_j$ [23]. The *Nakayama automorphism* η of $C_A(S)$ may be defined by

$$ E(\eta(c)a) = E(ac) $$

for every $a \in A, c \in C_A(S)$. Then from Equations 1, $\eta(c) = \sum_i \beta^{-1}(E(x_i c))y_i$, and

$$ \eta^{-1}(c) = \sum_i x_i E(cy_i). \tag{2} $$

The Nakayama automorphisms η and γ relative to two Frobenius homomorphisms E and $F = Ed$, respectively, are related by $\eta\gamma^{-1}(x) = dxd^{-1}$ for every $x \in C_A(S)$

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[23]. If A is a k-algebra and $S = k1_A$, then β is necessarily the identity by a short calculation [20] and $C_A(S) = A$.

For example, a Hopf subalgebra K in a finite dimensional Hopf algebra H over a field is a free β-Frobenius extension. The natural module H_K is free by the theorem of Nichols-Zoeller [21]. By a theorem of Larson-Sweedler in [18], the antipode is bijective, and H and K are Frobenius algebras with Frobenius homomorphisms which are left or right integrals in the dual algebra. From Oberst-Schneider [22, Satz 3.2] we have a formula (cf. Equation 40) that implies that the Nakayama automorphism of H, η_H, restricts to a mapping of $K \to K$. It follows from Pareigis [23, Satz 6] that H/K is a β-Frobenius extension, where the automorphism β of K is the following composition of the Nakayama automorphisms of H and K:

$$\beta = \eta_K \circ \eta_H^{-1} \tag{3}$$

(cf. Section 5).

This paper continues our investigations in [2, 3, 11, 12] on the interactions of Frobenius algebras/extensions with Hopf algebras. We apply Frobenius coordinates to a class of Hopf algebras over commutative rings called FH-algebras, which are Hopf algebras that are simultaneously Frobenius algebras (cf. Section 4). This class was introduced in [24, 25] and includes the finite dimensional Hopf algebras as well as the finite projective Hopf algebras over commutative rings with trivial Picard group (such as semi-local or polynomial rings). The added generality would apply for example to a Hopf algebra H over a Dedekind domain k satisfying the condition that the element represented by the k-module of left integrals $\int_{H^*}^{\ell}$ in the Picard group of k be trivial.

This paper is organized as follows. In Section 2, we review the basics of Frobenius algebras and Frobenius coordinates, as well as separability. In Section 3, we study norms, integrals and modular functions for augmented Frobenius algebras over a commutative ring, giving a lemma on the effect of automorphisms and anti-automorphism on s. In Section 4, we derive by means of different Frobenius coordinates Radford's Formula 32 for S^4 and Formula 27 relating S^2, t_1, t_2, where t is a right norm for H. This extends two formulas in [26, 28, k = field] to FH-algebras with different proofs. Then we generalize two recent results of Etingof and Gelaki [8], the main one stating that a finite dimensional semisimple and cosemisimple Hopf algebra is involutive. We show that with a small condition on $2 \in k$ a separable and coseparable Hopf k-algebra is involutive (Theorem 4.9). Furthermore, if H is separable and satisfies a certain bound on its local ranks, then H is coseparable and therefore involutive (Theorem 4.10).

In Section 5, we prove that a subalgebra pair of FH-algebras $H \supseteq K$ is a β-Frobenius extension, though not necessarily free. In Section 6, we derive by means of different Frobenius coordinates Equation 45 relating the different elements in a β-Frobenius coordinate system for a Hopf subalgebra pair $K \subset H$ given by Fischman-Montgomery-Schneider [10] . In Section 7, we prove that a group-like element in a finite projective Hopf algebra H over a Noetherian ring k has finite order dividing the least common multiple N of the P-ranks of H as a k-module. From the theorems in Section 4 it follows that S has order dividing $4N$, and, should H be an FH-algebra, that the Nakayama automorphism η has finite order dividing $2N$, as obtained for fields in [26] and [10], respectively. In Section 8, we extend the Drinfel'd notion of quantum double to FH-algebras, then prove that the quantum double of an FH-algebra H is a unimodular and symmetric FH-algebra.

2. A brief review of Frobenius algebras

All rings in this paper have 1, homomorphisms preserve 1, and unless otherwise specified k denotes a commutative ring. Given an associative, unital k-algebra A, A^* denotes the dual module $\mathrm{Hom_k(A,k)}$, which is an A-A bimodule as follows: given $f \in A^*$ and $a \in A$, af is defined by $(af)(b) = f(ba)$ for every $b \in A$, while fa is defined by $(fa)(b) = f(ab)$. We also consider the *tensor-square*, $A \otimes A$ as a natural A-bimodule given by $a(b \otimes c) = ab \otimes c$ and $(a \otimes b)c := a \otimes bc$ for every $a, b, c \in A$. An element $\sum_i z_i \otimes w_i$ in the tensor-square is called *symmetric* if it is left fixed by the transpose map given by $a \otimes b \mapsto b \otimes a$ for every $a, b \in A$.

We first consider some preliminaries on a Frobenius algebra A over a commutative ring k. A is a *Frobenius algebra* if the natural module A_k is *finite projective* (= finitely generated projective), and

$$A_A \cong A^*_A. \tag{4}$$

Suppose $f_i \in A^*$, $x_i \in A$ form a finite projective base, or dual bases, of A over k: i.e., for every $a \in A$, $\sum_i x_i f_i(a) = a$. Then there are $y_i \in A$ and a cyclic generator $\phi \in A^*$ such that the A-module isomorphism is given by $a \mapsto \phi a$, and

$$\sum_i x_i \phi(y_i a) = a = \sum \phi(a x_i) y_i, \tag{5}$$

for all $a \in A$. It follows that ϕ is nondegenerate (or faithful) in the following sense: a linear functional ϕ on an algebra A is *nondegenerate* if $a, b \in A$ such that $a\phi = b\phi$ or $\phi a = \phi b$ implies $a = b$.

We refer to ϕ as a *Frobenius homomorphism*, (x_i, y_i) as *dual bases*, and (ϕ, x_i, y_i) as a *Frobenius system* or *Frobenius coordinates*. It is useful to note from the start that $xy = 1$ implies $yx = 1$ in A, since an epimorphism of A onto itself is automatically bijective [24, 30].

It is equivalent to define a k-algebra A Frobenius if A_k is finite projective and ${}_AA \cong {}_AA^*$. In fact, with ϕ defined above, the mapping $a \mapsto a\phi$ is such an isomorphism, by an application of Equations 5.

Note that the bilinear form on A defined by $\langle a, b\rangle := \phi(ab)$ is a nondegenerate inner product which is associative: $\langle ab, c\rangle = \langle a, bc\rangle$ for every $a, b, c \in A$.

The Frobenius homomorphism is unique up to an invertible element in A. If ϕ and ψ are Frobenius homomorphisms for A, then $\psi = d\phi$ for some $d \in A$. Similarly, $\phi = d'\psi$ for some $d' \in A$, from which it follows that $dd' = 1$. The element d is referred to as the (left) *derivative* $\frac{d\psi}{d\phi}$ of ψ with respect to ϕ. Right derivatives in the group of units A° of A are similarly defined.

If (ϕ, x_i, y_i) is a Frobenius system for A, then $e := \sum_i x_i \otimes y_i$ is an element in the tensor-square $A \otimes_k A$ which is independent of the choice of dual bases for ϕ, called the *Frobenius element.* By a computation involving Equations 5, e is a Casimir element satisfying $ae = ea$ for every $a \in A$, whence $\sum_i x_i y_i$ is in the center of A.[1] It follows that A is k-separable if and only if there is a $a \in A$ such that

$$\sum_i x_i a y_i = 1. \tag{6}$$

For each $d \in A^\circ$, we easily check that $(\phi d, x_i, d^{-1}y_i)$ and $(d\phi, x_i d^{-1}, y_i)$ are the other Frobenius systems in a one-to-one correspondence. It follows that a Frobenius element is also unique, up to a unit in $A \otimes A$ (either $1 \otimes d^{\pm 1}$ or $d^{\pm 1} \otimes 1$).

[1] e is the transpose of the element Q in [3].

A *symmetric algebra* is a Frobenius algebra A/k which satisfies the stronger condition:

$$ {}_A A_A \cong {}_A (A^*)_A. \tag{7} $$

Choosing an isomorphism Φ, the linear functional $\phi := \Phi(1)$ is a Frobenius homomorphism satisfying $\phi(ab) = \phi(ba)$ for every $a, b \in A$: i.e., ϕ is an trace on A. The dual bases x_i, y_i for this ϕ forms a symmetric element in the tensor-square, since for every $a \in A$,

$$ \begin{aligned} \sum_i a x_i \otimes y_i &= \sum_{i,j} y_j \otimes \phi(a x_i x_j) y_i \\ &= \sum_j y_j \otimes x_j a. \end{aligned} \tag{8} $$

A k-algebra A with $\phi \in A^*$ and $x_i, y_i \in A$ satisfying either $\sum_i x_i \phi(y_i a) = a$ for every $a \in A$ or $\sum_i \phi(a x_i) y_i = a$ for every $a \in A$ is automatically Frobenius. As a corollary, one of the dual bases equations implies the other. For if $\sum_{i=1}^n (x_i \phi) y_i = \mathrm{Id}_A$, then A is explicitly finite projective over k, and it follows that A^* is finite projective too. The homomorphism ${}_A A \to {}_A A^*$ defined by $a \mapsto a\phi$ for all $a \in A$ is surjective, since given $f \in A^*$, we note that $f = (\sum_i f(y_i) x_i)\phi$. Since A and A^* have the same P-rank for each prime ideal P in k, the epimorphism $a \mapsto a\phi$ is bijective [30], whence ${}_A A \cong {}_A A^*$. Starting with the other equation in the hypothesis, we similarly prove that $a \mapsto \phi a$ is an isomorphism $A_A \cong A^*_A$.

The Nakayama automorphism of a Frobenius algebra A is an algebra automorphism $\alpha : A \to A$ defined by

$$ \phi \alpha(a) = a \phi \tag{9} $$

for every $a \in A$. In terms of the associative inner product, $\langle x, a \rangle = \langle \alpha(a), x \rangle$ for every $a, x \in A$. α is an inner automorphism iff A is a symmetric algebra. The Nakayama automorphism η of another Frobenius homomorphism $\psi = \phi d$, where $d \in A^\circ$, is given by

$$ \eta(x) = \sum_i \phi(d x_i x) d^{-1} y_i = \sum_i d^{-1} \phi(\alpha(x) d x_i) y_i = d^{-1} \alpha(x) d, \tag{10} $$

so that $\alpha \eta^{-1}(x) = d x d^{-1}$. Thus the Nakayama automorphism is unique up to an inner automorphism. A Frobenius algebra A is a symmetric algebra if and only if its Nakayama automorphism is inner.

Another formula for α is obtained from Equations 9 and 5: for every $a \in A$,

$$ \alpha(a) = \sum_i \phi(x_i a) y_i. \tag{11} $$

If the Frobenius element $\sum_i x_i \otimes y_i$ is symmetric, it follows from this equation that $\alpha = \mathrm{Id}_A$. Together with Equation 8, this proves:

Proposition 2.1. *A Frobenius algebra A is a symmetric algebra if and only if it has a symmetric Frobenius element.*

Equation 7 generalizes to all Frobenius algebras as follows. A Frobenius isomorphism $\Psi : A_A \stackrel{\cong}{\to} A^*_A$ induces a bimodule isomorphism where one bimodule is twisted by the Nakayama automorphism α:

$$ {}_A A_A \cong {}_{\alpha^{-1}} A^*_A, \tag{12} $$

since with $\phi = \Psi(1)$ Equation 9 yields

$$\Psi(a_1 a a_2) = \phi a_1 a a_2 = \alpha^{-1}(a_1)\phi a a_2 = \alpha^{-1}(a_1)\Psi(a)a_2.$$

The left and right derivatives of a pair of Frobenius homomorphisms differ by an application of the Nakayama automorphism (cf. Equation 9). A computation applying Equations 5 and 9 proves that for every $a \in A$,

$$\sum_i x_i a \otimes y_i = \sum_i x_i \otimes \alpha(a) y_i. \tag{13}$$

In closing this section, we refer the reader to [6, 2, 12] for more on Frobenius algebras over commutative rings, and to [32] for a survey of the representation theory of Frobenius over fields and work on the Nakayama conjecture.

3. Augmented Frobenius algebras

A k-algebra A is said to be an *augmented algebra* if there is an algebra homomorphism $\epsilon : A \to k$, called an *augmentation*. An element $t \in A$ satisfying $ta = \epsilon(a)t$, $\forall$ a $\in$ A, is called a *right integral* of A. It is clear that the set of right integrals, denoted by $\int_A^r$, is a two-sided ideal of A, since for each $a \in A$, the element at is also a right integral. Similarly for the space of left integrals, denoted by $\int_A^\ell$. If $\int_A^r = \int_A^\ell$, A is said to be *unimodular*.

Now suppose that A is a Frobenius algebra with augmentation ϵ. We claim that a nontrivial right integral exists in A. Since $A^* \cong A$ as right A-modules, an element $n \in A$ exists such that $\phi n = \epsilon$ where ϕ is a Frobenius homomorphism. Call n the *right norm* in A with respect to ϕ. Given $a \in A$, we compute in A^*:

$$\phi n a = (\phi n)a = \epsilon a = \epsilon(a)\epsilon = \phi n \epsilon(a).$$

By nondegeneracy of ϕ, n satisfies $na = n\epsilon(a)$ for every $a \in A$.

Proposition 3.1. *If A is an augmented Frobenius algebra, then the set $\int_A^r$ of right integrals is a two-sided ideal which is free cyclic k-summand of A generated by a right norm.*

Proof. The proof is based on [24, Theorem 3], which assumes that A is also a Hopf algebra. Let $\phi \in A^*$ be a Frobenius homomorphism, and $n \in A$ satisfy $\phi n = \epsilon$, the augmentation. Given a right integral $t \neq 0$, we note that

$$\phi t = \phi(t)\epsilon = \phi(t)\phi n = \phi n \phi(t),$$

whence

$$t = \phi(t)n. \tag{14}$$

Then $\langle n \rangle := \{\rho n | \, \rho \in k\}$ coincides with the set of all right integrals.

Given $\lambda \in k$ such that $\lambda n = 0$, it follows that

$$\phi(n)\lambda = \epsilon(1)\lambda = \lambda = 0,$$

whence $\langle n \rangle$ is a free k-module. Moreover, $\langle n \rangle$ is a direct k-summand in A since $a \mapsto \phi(a)n$ defines a k-linear projection of A onto $\langle n \rangle$. □

The right norm in A is unique up to a unit in k, since norms are free generators of $\int_A^r$ by the proposition. The notions of norm and integral only coincide if k is a field.

Similarly the space $\int_A^\ell$ of left integrals is a rank one free summand in A, generated by any left norm. In general the spaces of right and left integrals do not coincide, and one defines an augmentation on A that measures the deviation from unimodularity. In the notation of the proposition and its proof, for every $a \in A$, the element an is a right integral since the right norm n is. From Equation 14 one concludes that $an = \phi(an)n = (n\phi)(a)n$. The function

$$m := n\phi : A \to k \tag{15}$$

is called the *right modular function*, which is an augmentation since $\forall\, \mathrm{a}, \mathrm{b} \in \mathrm{A}$ we have $(ab)n = m(ab)n = a(bn) = m(a)m(b)n$ and n is a free generator of $\int_A^r$.

The next proposition and corollary we believe has not been noted in the literature before.

Proposition 3.2. *If A is an augmented Frobenius algebra and α the Nakayama automorphism, then in the notation above,*

$$m \circ \alpha = \epsilon. \tag{16}$$

Proof. We note that $\phi \circ \alpha = \phi$ by evaluating each side of Equation 9 on 1. Then for each $x \in A$,

$$m(\alpha(x)) = (n\phi)(\alpha(x)) = (\phi\alpha(n))(\alpha(x)) = (\phi \circ \alpha)(x) = \phi(x). \quad \square$$

The next corollary follows from noting that if α is an inner automorphism, then $m = \epsilon$ from the proposition.

Corollary 3.3. *If A is an augmented symmetric algebra, then A is unimodular.*

We note two useful identities for the right norm,

$$n = \sum_i \phi(nx_i)y_i = \sum_i \epsilon(x_i)y_i \tag{17}$$

$$= \sum_i x_i(n\phi)(y_i) = \sum_i x_i m(y_i). \tag{18}$$

As an example, consider $A := k[X]/(X^n)$ where k is a commutative ring and $aX = X\alpha(a)$ for some automorphism α of k and every $a \in k$. Then A is an augmented Frobenius algebra with Frobenius homomorphism $\phi(a_0 + a_1X + \cdots + a_{n-1}X^{n-1}) := a_{n-1}$, dual bases $x_i = X^{i-1}$, $y_i = X^{n-i}$ $(i = 1, \cdots, n)$, and augmentation $\epsilon(a_0 + a_1X + \cdots + a_{n-1}X^{n-1}) := a_0$. It follows that a left and right norm is given by $n = X^{n-1}$, and A is symmetric and unimodular. A is not a Hopf algebra unless n is a prime p and the characteristic of k is p (cf. [10]).

The next proposition is well-known for finite dimensional Hopf algebras [19].

Proposition 3.4. *Suppose A is a separable augmented Frobenius algebra. Then A is unimodular.*

Proof. The Endo-Watanabe theorem in [7] states that separable projective algebras are symmetric algebras. The result follows then from Corollary 3.3. $\square$

We will use repeatedly in Section 4 several general principles summarized in the next lemma. Items 1, 2 and 3 below are valid without the assumption of augmentation or ϵ-invariance.

Lemma 3.5. *Suppose (A, ϵ) is an augmented Frobenius algebra and α (respectively, β) is a k-algebra automorphism (resp. anti-automorphism) of A satisfying ϵ-invariance: viz. $\epsilon \circ \alpha = \epsilon$. Let (ϕ_A, x_i, y_i) be Frobenius coordinates of A. Then*

1. *The Frobenius system is transformed by* α *into a Frobenius system*
$$(\phi_A \circ \alpha^{-1}, \alpha(x_i), \alpha(y_i)).$$
2. *The Frobenius system is transformed by* β *into the Frobenius system*
$$(\phi_A \circ \beta^{-1}, \beta(y_i), \beta(x_i)).$$
3. *If* B *is another Frobenius* k*-algebra with Frobenius homomorphism* ϕ_B*, then* $A \otimes B$ *is a Frobenius algebra with Frobenius homomorphism* $\phi_A \otimes \phi_B : A \otimes B \to k$.
4. α *sends integrals to integrals and norms to norms, respecting chirality.*
5. β *sends integrals to integrals and norms to norms, reversing chirality.*

Proof. 1 is proven by applying α to $\sum_i \phi_A(ax_i)y_i = a$, obtaining
$$\sum_i \phi_A \alpha^{-1}(\alpha(a)\alpha(x_i))\alpha(y_i) = \alpha(a)$$
for every $a \in A$. 2 is proven similarly. 3 is easy. 4 is proven by applying first α to $ta = \epsilon(a)t$, obtaining that $\alpha(t) \in \int_A^r$ if t is too. Next, if $\phi_A n = \epsilon$, then $(\phi_A \circ \alpha^{-1})\alpha(n) = \epsilon$ as well, which together with 1 proves 4. 5 is proven similarly. □

4. FH-ALGEBRAS

We continue with k as a commutative ring. We review the basics of a Hopf algebra H which is finite projective over k [24]. A bialgebra H is an algebra and coalgebra where the comultiplication and the counit are algebra homomorphisms. We use a reduced Sweedler notation given by
$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} := \sum a_1 \otimes a_2$$
for the values of the comultiplication homomorphism $H \to H \otimes_k H$. The counit is the k-algebra homomorphism $\epsilon : H \to k$ and satisfies $\sum_i \epsilon(a_1)a_2 = \sum a_1 \epsilon(a_2) = a$ for every $a \in H$.

A Hopf algebra H is a bialgebra with antipode. The antipode $S : H \to H$ is an anti-homomorphism of algebras and coalgebras satisfying $\sum S(a_1)a_2 = \epsilon(a)1 = \sum a_1 S(a_2)$ for every $a \in H$.

A *group-like element* in H is defined to be a $g \in H$ such that $\Delta(g) = g \otimes g$ *and* $\epsilon(g) = 1$. It follows that $g \in H^\circ$ and $S(g) = g^{-1}$.

Finite projective Hopf algebras enjoy the duality properties of finite dimensional Hopf algebras. H^* is a Hopf algebra with convolution product $(fg)(x) := \sum f(x_1)g(x_2)$. The counit is given by $f \mapsto f(1)$. The unit of H^* is the counit of H. The comultiplication on H^* is given by $\sum f_1 \otimes f_2(a \otimes b) = f(ab)$ for every $f \in H^*$, $a, b \in H$. The antipode is the dual of S, a mapping of H^* into H^*, denoted again by S when the context is clear. Note that an augmentation f in H^* is a group-like element in H^*, and *vice versa*, with inverse given by $Sf = f \circ S$.

As Hopf algebras, $H \cong H^{**}$, the isomorphism being given by $x \mapsto \mathrm{ev}_x$, the evaluation map at x: we fix this isomorphism as an identification of H with H^{**}. The usual left and right action of an algebra on its dual specialize to the left action of H^* on $H^{**} \cong H$ given by $g \rightharpoonup a := \sum a_1 g(a_2)$, and the right action given by $a \leftharpoonup g := \sum g(a_1)a_2$.

We recall the definition of an equivalent version of Pareigis's FH-algebras [25].

Definition 4.1. A k-algebra H is an FH-algebra if H is a bialgebra and a Frobenius algebra with Frobenius homomorphism f a right integral in H^*. Call f the FH-homomorphism.[2]

The condition that $f \in \int_{H^*}^r$ is equivalent to

$$\sum f(a_1)a_2 = f(a)1 \tag{19}$$

for every $a \in H$. Note that H is an augmented Frobenius algebra with augmentation ϵ. Let $t \in H$ be a right norm such that $ft = \epsilon$. Note that $f(t) = 1$. Fix the notation f and t for an FH-algebra. We show below that an FH-homomorphism is unique up to an invertible scalar in k. If H is an FH-algebra and a symmetric algebra, we say that H is a *symmetric FH-algebra.*

It follows from [24, Theorem 2] that an FH-algebra H automatically has an antipode. With f its FH-homomorphism and t a right norm, define $S : H \to H$ by

$$S(a) = \sum f(t_1 a)t_2. \tag{20}$$

Then for every $a \in H$

$$\sum S(a_1)a_2 = \sum f(t_1 a_1)t_2 a_2 = f(ta)1 = \epsilon(a)1.$$

Now in the convolution algebra structure on $\mathrm{End_k(H)}$, this shows S has $\mathrm{Id_H}$ as right inverse. Since $\mathrm{End_k(H)}$ is finite projective over k, it follows that $\mathrm{Id_H}$ is also a left inverse of S; whence S is the unique antipode.

The Pareigis Theorem [24] generalizing the Larson-Sweedler Theorem [18] shows that a finite projective Hopf algebra H over a ground ring k with trivial Picard group is an FH-algebra. In detail, the theorem proves the following in the order given. The first two items are proven without the hypothesis on the Picard group of k. The last two items require only that $\int_{H^*}^{\ell}$ be free of rank 1.

1. There is a right Hopf H-module structure on H^*. Since all Hopf modules are trivial, $H^* \cong P(H^*) \otimes H$, for the coinvariants $P(H^*) = \int_{H^*}^{\ell}$.
2. The antipode S is bijective.
3. There exists a left integral f in H^* such that the mapping $\Theta : H \to H^*$ defined by
$$\Theta(x)(y) = f(yS(x)) \tag{21}$$
is a right Hopf module isomorphism.
4. H is a Frobenius algebra with Frobenius homomorphism f.

It follows from 2. above that an FH-algebra H possesses an ϵ-invariant anti-automorphism S. If $f \in H^*$ is an FH-homomorphism, then Sf is a Frobenius homomorphism and *left integral* in H^*. It is therefore equivalent to replace right with left in Definition 4.1.

Let $m : H \to k$ be the right modular function of H. Since m is an algebra homomorphism, it is group-like in H^*, whence m at times is called the *right distinguished group-like element* in H^*.

The next proposition is obtained in an equivalent form in [22], [10] and [2], though in somewhat different ways.

Proposition 4.2. *Let H be an FH-algebra with FH-homomorphism f and right norm t. Then $(f, S^{-1}t_2, t_1)$ is a Frobenius system for H.*

[2] The authors have called FH-algebras Hopf-Frobenius algebras in an earlier preprint.

Proof. Applying S^{-1} to both sides of Equation 20 yields

$$\sum S^{-1}(t_2)f(t_1 a) = a, \tag{22}$$

for every $a \in H$. It follows from the finite projectivity assumption on H that $(f, S^{-1}(t_2), t_1)$ is a Frobenius system. □

It follows from the proposition that $t \leftharpoonup f = 1$. Together with the corollary below this implies that f is a right norm in H^*, since 1 is the counit for H^*. It follows that g is another FH-homomorphism for H iff $g = f\lambda$ for some $\lambda \in k^\circ$. From Equation 18 and the proposition above it follows that

$$S(t) = t \leftharpoonup m. \tag{23}$$

Proposition 4.3. *H is an FH-algebra if and only if H^* is an FH-algebra.*

Proof. It suffices by duality to establish the forward implication. Suppose f is an FH-homomorphism for H and t a right norm. Now Equation 20 and the argument after it work for H^* and the right integrals t, f since $t \leftharpoonup f$ is the counit on H^*. It follows that

$$S(g) = \sum (f_1 g)(t) f_2 \tag{24}$$

is an equation for the antipode in H^*. By taking S^{-1} of both sides we see that

$$(t, S^{-1}f_2, f_1)$$

is a Frobenius system for H^* . Whence t is an FH-homomorphism for H^* with right norm f. □

It follows that H^* is also an augmented Frobenius algebra. Next, we simplify our criterion for FH-algebra.

Proposition 4.4. *If H is an FH-algebra if and only if H is a Frobenius algebra and a Hopf algebra.*

Proof. The forward direction is obvious. For the converse, we use the fact that the k-submodule of integrals of an augmented Frobenius algebra is free of rank 1 (cf. [24, Theorem 3] or Proposition 3.1. It follows that $\int_H^\ell \cong k$. From Pareigis's Theorem we obtain that the dual Hopf algebra H^* is a Frobenius algebra. Whence $\int_{H^*}^\ell \cong k$ and H is an FH-algebra. □

Let $b \in H$ be the right *distinguished group-like element* satisfying

$$gf = g(b)f \tag{25}$$

for every $g \in H^*$.

The convolution product inverse of m is $m^{-1} = m \circ S$. Given a left norm $v \in H$, we claim that

$$va = vm^{-1}(a).$$

Since t is a right norm, S an anti-automorphism and ϵ-invariant, it follows that St is a left norm. Then we may assume $v = St$. Then $S(at) = StSa = m(a)St$, whence $vx = vmS^{-1}(x)$ for every $a, x \in H$. The claim then follows from $m \circ S^2 = m$, since this implies that $m \circ S^{-1} = m^{-1}$. But $m \circ S^2 = m^{-1} \circ S = m$, since m^{-1} is group-like.

Lemma 4.5. *Given an FH-algebra H with right norm $f \in H^*$ and right norm $t \in H$ such that $f(t) = 1$, the Nakayama automorphism, relative to f, and its inverse are given by:*

$$\eta(a) = S^2(a \leftharpoonup m^{-1}) = (S^2 a) \leftharpoonup m^{-1}, \tag{26}$$

$$\eta^{-1}(a) = S^{-2}(a \leftharpoonup m) = (S^{-2} a) \leftharpoonup m.$$

Proof. Using the Frobenius coordinates $(f, S^{-1}t_2, t_1)$, we note that

$$\eta^{-1}(a) = \sum S^{-1}(t_2) f(t_1 \eta^{-1}(a)) = \sum S^{-1}(t_2) f(a t_1).$$

We make a computation as in [10, Lemma 1.5]:

$$\begin{aligned} S^2(\eta^{-1}(a)) &= \sum f(a t_1) S t_2 \\ &= \sum f(a_1 t_1) a_2 t_2 S t_3 \\ &= \sum f(a_1 t) a_2 \\ &= a \leftharpoonup m \end{aligned}$$

since $a \leftharpoonup f = f(a)1$, $at = m(a)t$ for every $a \in H$ and $f(t) = 1$. Whence $\eta^{-1}(a) = S^{-2}(a \leftharpoonup m)$. Since $mS^{-2} = m$, it follows that $\eta^{-1}(a) = (S^{-2}a) \leftharpoonup m$.

It follows that $a = (S^{-2}\eta a) \leftharpoonup m$, so let the convolution inverse m^{-1} act on both sides: $(a \leftharpoonup m^{-1}) = S^{-2}\eta(a)$. Whence $\eta(a) = S^2(a \leftharpoonup m^{-1}) = (S^2 a) \leftharpoonup m^{-1}$, since $m^{-1}S^2 = m^{-1}$. □

As a corollary, we obtain [22, Folg. 3.3] and [3, Proposition 3.8]: if H is a unimodular FH-algebra, then the Nakayama automorphism is the square of the antipode.

Now recall our definition of b after Proposition 4.3 as the right distinguished group-like in H. Equation 27 below was first established in [28] for finite dimensional Hopf algebras over fields by different means.

Theorem 4.6. *If H is an FH-algebra with FH-homomorphism f and right norm t, then*

$$\sum t_2 \otimes t_1 = \sum b^{-1} S^2 t_1 \otimes t_2. \tag{27}$$

Proof. On the one hand, we have seen that $(f, S^{-1}t_2, t_1)$ are Frobenius coordinates for H. On the other hand, the equation $f \rightharpoonup x = bf(x)$ for every $x \in H$ follows from Equation 25 and gives

$$\begin{aligned} \sum S(t_1) b f(t_2 a) &= \sum S(t_1) t_2 a_1 f(t_3 a_2) \\ &= \sum a_1 f(t a_2) \\ &= \sum a_1 \epsilon(a_2) f(t) = a. \end{aligned}$$

Then $(f, S(t_1)b, t_2)$ is another Frobenius system for H.

Since $(S^{-1}(t_2), t_1)$ and $(S(t_1)b, t_2)$ are both dual bases to f, it follows that $\sum S^{-1}t_2 \otimes t_1 = \sum S(t_1)b \otimes t_2$. Equation 27 follows from applying $S \otimes 1$ to both sides. □

Proposition 4.2 with $a = S^{-1}t$ gives

$$\sum S^{-1}t_2 f(t_1 S^{-1}t) = S^{-1}t f(S^{-1}t) = S^{-1}t.$$

Since $S^{-1}t$ is a left norm, it follows that

$$f(S^{-1}t) = 1. \tag{28}$$

The next proposition is not mentioned in the literature for Hopf algebras.

Proposition 4.7. *Given an FH-algebra H with FH-homomorphism f, the right distinguished group-like element b is equal to the derivative d of the left integral Frobenius homomorphism $g := S^{-1}f$ with respect to f:*

$$b = \frac{dg}{df} \tag{29}$$

Proof. By Lemma 3.5, another Frobenius system for H is given by (g, St_1, t_2), since S is an anti-automorphism. Then there exists a (derivative) $d \in H^\circ$ such that

$$df = g. \tag{30}$$

g is a left norm in H^* since S^{-1} is an ϵ-invariant anti-automorphism. Also bf is a left integral in H^* by the following argument. For any $g, g' \in H^*$, we have $b(gg') = (bg)(bg')$ as b is group-like. Then for every $h \in H^*$

$$\begin{aligned} h(bf) &= b[(b^{-1}h)f] \\ &= b[(b^{-1}h)(b)f] \\ &= h(1)(bf). \end{aligned}$$

Now both $g(t)$ and $bf(t)$ equal 1, since $f(S^{-1}t) = 1$, $f(tb) = \epsilon(b)f(t) = 1$ and b is group-like. Since bf is a scalar multiple of the norm g, it follows that

$$g = bf. \tag{31}$$

Finally, $d = b$ since $df = bf$ from Equations 30 and 31, and f is nondegenerate. □

We next give a different derivation for FH-algebras of a formula in [26] for the fourth power of the antipode of a finite dimensional Hopf algebra. The main point is that the Nakayama automorphisms associated with the two Frobenius homomorphisms $S^{-1}f$ and f differ by an inner automorphism determined by the derivative in Proposition 4.7.

Theorem 4.8. *Given an FH-algebra H with right distinguished group-like elements $m \in H^*$ and $b \in H$, the fourth power of the antipode is given by*

$$S^4(a) = b(m^{-1} \rightharpoonup a \leftharpoonup m)b^{-1} \tag{32}$$

for every $a \in H$.

Proof. Let $g := S^{-1}f$ and denote the left norm St by Λ. Note that $g(\Lambda) = 1 = g(S^{-1}\Lambda)$ since $f(t) = 1 = f(S^{-1}t)$. We note that $(g, \Lambda_2, S^{-1}\Lambda_1)$ are Frobenius coordinates for H, since S is an anti-automorphism of H

Then the Nakayama automorphism α associated with g has inverse satisfying

$$\alpha^{-1}(a) = \sum \Lambda_2 g(aS^{-1}\Lambda_1)$$

whence

$$\begin{aligned} S^{-1}\alpha^{-1}(a) &= \sum S^{-1}g(\Lambda_1 Sa)S^{-1}(\Lambda_2) \\ &= \sum S^{-1}(\Lambda_3)S^{-1}g(\Lambda_1 Sa_2)\Lambda_2 Sa_1 \\ &= \sum S^{-1}g(\Lambda Sa_2)Sa_1 \\ &= g(S^{-1}\Lambda)\sum m^{-1}(Sa_2)Sa_1 = S(m \rightharpoonup a), \end{aligned}$$

since $Sm^{-1} = m$. It follows that

$$\alpha^{-1}(a) = S^2(m \rightharpoonup a) = m \rightharpoonup S^2 a \tag{33}$$

$$\alpha(a) = m^{-1} \rightharpoonup S^{-2}a = S^{-2}(m^{-1} \rightharpoonup a). \tag{34}$$

From Proposition 4.7 we have $g = bf = f\eta(b)$, where η is the Nakayama automorphism of f. By Equation 10 and Lemma 4.5,

$$\begin{aligned} m^{-1} \rightharpoonup S^{-2}a &= \alpha(a) \\ &= \eta(b^{-1})\eta(a)\eta(b) \\ &= m^{-1}(b^{-1})b^{-1}(S^2(a) \leftharpoonup m^{-1})bm^{-1}(b) \\ &= b^{-1}S^2(a)b \leftharpoonup m^{-1}, \end{aligned}$$

since b and m are group-likes and S^2 leaves m and b fixed. It follows that

$$a = m \rightharpoonup b^{-1}S^4(a)b \leftharpoonup m^{-1},$$

for every $a \in H$. Equation 32 follows. □

The theorem implies [3, Corollary 3.9], which states that $S^4 = \mathrm{Id_H}$, if H and H^* are unimodular finite projective Hopf algebra over k. For localizing with respect to any maximal ideal $\mathcal{M}$, we obtain unimodular Hopf-Frobenius algebras $H_{\mathcal{M}} \cong H \otimes k_{\mathcal{M}}$ and its dual, since the local ring $k_{\mathcal{M}}$ has trivial Picard group. By Theorem 4.8, the localized antipode satisfies $(S_{\mathcal{M}})^4 = \mathrm{Id}$ for every maximal ideal $\mathcal{M}$ in k; whence $S^4 = \mathrm{Id_H}$ [30].

Theorem 4.9. *Let k be a commutative ring in which* 2 *is not a zero divisor and H a finite projective Hopf algebra. If H is separable and coseparable, then $S^2 = \mathrm{Id}$.*

Proof. First we note that H is unimodular and counimodular. Then it follows from the theorem above that $S^4 = \mathrm{Id}$. Localizing with respect to the set $T = \{2^n, n = 0, 1, ..\}$ we may assume that 2 is invertible in k. Then $H = H_+ \oplus H_-$ where $H_\pm = \{h \in H : S^2(h) = \pm h\}$, respectively. We have to prove that $H_- = 0$. It suffices to prove that $(H_-)_{\mathcal{M}} = 0$ for any maximal ideal $\mathcal{M}$ in k. Since $H_{\mathcal{M}}/\mathcal{M}H_{\mathcal{M}}$ is separable and coseparable over the field $k/\mathcal{M}$, we deduce from the main theorem in [8] that $(H_-)_{\mathcal{M}} \subset \mathcal{M}H_{\mathcal{M}}$ and therefore $(H_-)_{\mathcal{M}} \subset \mathcal{M}(H_-)_{\mathcal{M}}$. The desired result follows from the Nakayama Lemma because H_- is a direct summand in H. □

In [3] it was established that if H is separable over a ring k with no torsion elements, then $S^2 = \mathrm{Id}$. We may improve on this and similar results by an application of the last theorem. If k is a commutative ring and M is a finitely generated projective k-module, we let $rank_M : Spec\, k \to \mathcal{Z}$ be the *rank function*, which is defined on a prime ideal $\mathcal{P}$ in k by

$$rank_M(\mathcal{P}) := dim_{\,\overline{k/\mathcal{P}}}\, (M \otimes_k k/\mathcal{P}) \otimes_{k/\mathcal{P}} \overline{k/\mathcal{P}}$$

where $\overline{k/\mathcal{P}}$ is the field of fractions of $k/\mathcal{P}$. The range of $rank_M$ is finite and consists of a set of positive integers $n_1, n_2, ..., n_k$.

Now for any prime $p \in \mathcal{Z}$, let $Spec^{(p)}k \subseteq Spec\, k$ be the subset of prime ideals P for which the characteristic $char\,\overline{(k/P)} = p$. Suppose that $Spec^{(p)}k$ is non-empty and

$$rank_M(Spec^{(p)}k) = \{n_{i_1}, \dots, n_{i_s}\}.$$

For such p and ϕ the Euler function, we define

$$N(M,p) := \max_{m=1,\dots,s} \{n_{i_m}^{\frac{\phi(n_{i_m})}{2}}\}.$$

Theorem 4.10. *Let k be a commutative ring in which 2 is not a zero divisor and H be a f.g. projective Hopf algebra. If H is k-separable such that $N(H,p) < p$ for every odd prime p, then H is coseparable and $S^2 = \mathrm{Id}$.*

Proof. First we note that 2 may be assumed invertible in k without loss of generality by localization with respect to powers of 2. Let $\mathcal{M}$ in k be a maximal ideal. The characteristic of $k/\mathcal{M}$ is not 2 by our assumption.

It is known that an algebra A is separable iff $A/\mathcal{M}A$ is separable over $k/\mathcal{M}$ for every maximal ideal $\mathcal{M} \subset k$ [4]: whence $H/\mathcal{M}H$ is $k/\mathcal{M}$-separable. Furthermore note that if $d(\mathcal{M}) := \dim_{k/\mathcal{M}} H/\mathcal{M}H$ is greater than 2, then

$$d(\mathcal{M})^{\frac{\phi(d(\mathcal{M}))}{2}} < char\,(k/\mathcal{M}).$$

It then follows from [8] that $H^*/\mathcal{M}H^* \cong (H/\mathcal{M}H)^*$ is $k/\mathcal{M}$-separable for such $\mathcal{M}$.

If $d(\mathcal{M}) = 2$ and $\overline{k/\mathcal{M}}$ denotes the algebraic closure of $k/\mathcal{M}$, then $H^*/mH^* \otimes_k \overline{k/\mathcal{M}}$ is either semisimple or isomorphic to the ring of dual numbers. But the latter is impossible since it is not a Hopf algebra in characteristic different from 2. Hence $H^*/\mathcal{M}H^*$ is $k/\mathcal{M}$-separable for all maximal ideal $\mathcal{M}$. Hence H^* is k-separable by [4]. By Theorem 4.9 then, $S^2 = \mathrm{Id}$. □

In closing this section, we note that Schneider [29] has established Equation 32 by different methods for k a field. Equation 32 is generalized in a different direction by Koppinen [16]. Waterhouse sketches a different method of how to extend the Radford formula to a finite projective Hopf algebra [31].

5. FH-Subalgebras

In this section we prove that a Hopf subalgebra pair of FH-algebras $B \subseteq A$ form a β-Frobenius extension. The first results of this kind were obtained by Oberst and Schneider in [22] under the assumption that H is cocommutative.

The proposition below *sans* Equation 36 is more general than [9, Theorem 1.3] and a special case of [23, Satz 7] : the proof simplifies somewhat and is needed for establishing Equation 36.

Proposition 5.1. *Suppose A and B are Frobenius algebras over the same commutative ring k with Frobenius coordinates (ϕ, x_i, y_i) and (ψ, z_j, w_j), respectively. If B is a subalgebra of A such that A_B is projective and the Nakayama automorphism η_A of A satisfies $\eta_A(B) = B$, then A/B is a β-Frobenius extension with β the relative Nakayama automorphism,*

$$\beta = \eta_B \circ \eta_A^{-1}, \tag{35}$$

and β-Frobenius homomorphism

$$F(a) = \sum_j \phi(az_j)w_j, \tag{36}$$

for every $a \in A$.

Proof. Since B is finite projective over k, it follows that A_B is a *finite* projective module.

It remains to check that ${}_BA_A \cong {}_\beta(A_B)^*_A$, which we do below by using the Hom-Tensor Relation and Equation 12 twice (for A and for B). Let η_A^{-1} denote the restriction of η_A^{-1} to B below.

$$\begin{aligned} {}_BA_A &\cong {}_{\eta_A^{-1}}\mathrm{Hom_k(A,k)_A} \\ &\cong \mathrm{Hom_k(A \otimes_B B_{\eta_A^{-1}}, k)_A} \\ &\cong \mathrm{Hom_B(A_B, {}_{\eta_A^{-1}}Hom_k(B,k)_B)_A} \\ &\cong {}_{\eta_A^{-1}}\mathrm{Hom_B(A_B, {}_{\eta_B}B_B)} \\ &\cong {}_{\eta_B \circ \eta_A^{-1}}\mathrm{Hom_B(A_B, B_B)_A}. \end{aligned}$$

By sending 1_A along the isomorphisms in the last set of equations, we compute that the Frobenius homomorphism $F : {}_BA_B \to {}_\beta B_B$ is given by Equation 36. One may double check that $F(bab') = \beta(b)F(a)b'$ for every $b, b' \in B, a \in A$ by applying Equation 13. □

Given a commutative ground ring k, we assume H and K are Hopf algebras with H a finite projective k-module. K is a *Hopf subalgebra* of H if it is a *pure* k-submodule of H [17] and a subalgebra of H for which $\Delta(K) \subseteq K \otimes_k K$ and $S(K) \subseteq K$. It follows that K is finite projective as a k-module [17]. The next lemma is a corollary of the Nicholls-Zoeller freeness theorem.

Lemma 5.2. *If H is a finitely generated free Hopf algebra over a local ring k with K a Hopf subalgebra, then the natural modules H_K and ${}_KH$ are free.*

Proof. It will suffice to prove that H_K is free, the rest of the proof being entirely similar. First note that H_K is finitely generated since H_k is. If $\mathcal{M}$ is the maximal ideal of k, then the finite dimensional Hopf algebra $\overline{H} := H/\mathcal{M}H$ is free over the Hopf subalgebra $\overline{K} := K/\mathcal{M}K$ by purity and the freeness theorem in [21]. Suppose $\theta : \overline{K}^n \stackrel{\cong}{\to} \overline{H}$ is a $\overline{K}$-linear isomorphism. Since K is finitely generated over k, $\mathcal{M}K$ is contained in the radical of K. Now θ lifts to a right K-homomorphism $K^n \to H$ with respect to the natural projections $H \to \overline{H}$ and $K^n \to \overline{K}^n$. By Nakayama's lemma, the homomorphism $K^n \to H$ is epi (cf. [30]). Since H_k is finite projective, τ is a k-split epi, which is bijective by Nakayama's lemma applied to the underlying k-modules. Hence, H_K is free of finite rank. □

Over a non-connected ring $k = k_1 \times k_2$, it is easy to construct examples of Hopf subalgebra pairs

$$K := k[H_1 \times H_2] \subseteq H := k[G_1 \times G_2]$$

where $G_1 > H_1$, $G_2 > H_2$ are subgroup pairs of finite groups and H_K is not free (by counting dimensions on either side of $H \cong K^n$). The next proposition follows from the lemma.

Proposition 5.3. *If H is a finite projective Hopf algebra and K is a finite projective Hopf subalgebra of H, then the natural modules H_K and ${}_K H$ are finite projective.*

Proof. We prove only that H_K is finite projective since the proof that ${}_K H$ is entirely similar. First note that H_K is finitely generated.

If k is a commutative ground ring, $Q \to P$ is an epimorphism of K-modules, then it will suffice to show that the induced map $\Psi : \mathrm{Hom_K(H_K, Q_K)} \to \mathrm{Hom_K(H_K, P_K)}$ is epi too. Localizing at a maximal ideal $\mathcal{M}$ in k, we obtain a homomorphism denoted by $\Psi_{\mathcal{M}}$. By adapting a standard argument such as in [30], we note that for every module M_K

$$\mathrm{Hom_K(H_K, M_K)}_{\mathcal{M}} \cong \mathrm{Hom^r_{K_{\mathcal{M}}}(H_{\mathcal{M}}, M_{\mathcal{M}})} \tag{37}$$

since H_k is finite projective. Then $\Psi_{\mathcal{M}}$ maps

$$\mathrm{Hom^r_{K_{\mathcal{M}}}(H_{\mathcal{M}}, Q_{\mathcal{M}})} \to \mathrm{Hom^r_{K_{\mathcal{M}}}(H_{\mathcal{M}}, P_{\mathcal{M}})}.$$

By Lemma 5.2, $H_{\mathcal{M}}$ is free over $K_{\mathcal{M}}$. It follows that $\Psi_{\mathcal{M}}$ is epi for each maximal ideal $\mathcal{M}$, whence Ψ is epi. □

Suppose $K \subseteq H$ is a pair of FH-algebras where K is a Hopf subalgebra of H: call $K \subseteq H$ a *FH-subalgebra pair.* We now easily prove that H/K is a β-Frobenius extension.

Theorem 5.4. *If H/K is a FH-subalgebra pair, then H/K is a β-Frobenius extension where $\beta = \eta_K \circ \eta_H^{-1}$.*

Proof. The Nakayama automorphism η_H sends K into K by Equation 26, since K is a Hopf subalgebra of H. H_K is projective by Proposition 5.3. The conclusion follows then from Proposition 5.1 □

From the theorem and Lemma 4.5 we readily compute β in terms of the relative modular function $\chi := m_H * m_K^{-1}$, obtaining the formula [10, 1.6]: for every $x \in K$,

$$\beta(x) = x \leftharpoonup \chi. \tag{38}$$

Applying m_K to both sides of this equation, we obtain

$$m_H(x) = m_K(\beta(x)), \tag{39}$$

a formula which extends that in [10, Corollary 1.8] from the case $\beta = \mathrm{Id_K}$.

6. Some Formulas for a Hopf Subalgebra Pair

It follows from Theorem 5.4 and Lemma 5.2 that a Hopf subalgebra pair $K \subseteq H$ *over a local ring k* is a free β-Frobenius extension. Since H_K is free and therefore faithfully flat, the proof in [10] that $(E, S^{-1}(\Lambda_2), \Lambda_1)$, defined below, is a Frobenius system carries through word for word as described next.

From Proposition 4.2 it follows that $(f, S^{-1}(t_{H(2)}), t_{H(1)})$ is a Frobenius system for H where $f \in H^*$ and t_H in H are right integrals such that $f(t_H) = 1$. Given right and left modular functions m_H and m_H^{-1}, a computation using Equation 2 determines that

$$\eta_H^{-1}(a) = S^{-2}(a \leftharpoonup m_H), \tag{40}$$

for every $a \in H$. Let t_K be a right integral for K. Now by a theorem in [21], H_K and ${}_K H$ are free. Then there exists $\hat{\Lambda} \in H$ such that $t_H = \hat{\Lambda} t_K$. Let $\Lambda := \eta_H(S^{-1}(\hat{\Lambda}))$. Then a β-Frobenius system for H/K is given by $(E, S^{-1}\Lambda_{(2)}, \Lambda_{(1)})$ where

$$E(a) = \sum_{(a)} f(a_{(1)} S^{-1}(t_K)) a_{(2)}, \tag{41}$$

for every $a \in H$ [10]. For example, if K is the unit subalgebra, $E = f$ and $\Lambda = t$.

The rest of this section is devoted to comparing the different Frobenius systems for a Hopf subalgebra pair $K \subseteq H$ over a local ring k implied by our work in Sections 4 and 5. Suppose that $f \in \int^r_{H^*}$ and $t \in \int^r_H$ such that $ft = \epsilon$, and that $g \in \int^r_{K^*}$ and $n \in \int^r_K$ satisfy $gn = \epsilon|_K$. Then by Section 4 $(f, S^{-1}(t_2), t_1)$ is a Frobenius system for H, and $(g, S^{-1}(n_2), n_1)$ is a Frobenius system for K, both as Frobenius algebras.

By Equation 36, we note that a Frobenius homomorphism $F : H \to K$ of the β-Frobenius extension H/K is given by

$$F(a) = \sum f(aS^{-1}(n_2)) n_1. \tag{42}$$

Comparing E in Equation 41 and F above, we compute the (right) derivative d such that $F = Ed$:

$$\begin{aligned} d &= \sum F(S^{-1}(\Lambda_2)\Lambda_1 \\ &= \sum f(S^{-1}(\Lambda_2)S^{-1}(n_2))n_1\Lambda_1 \\ &= \sum (S^{-1}f)(n_2\Lambda_2)n_1\Lambda_1 = (S^{-1}f)(n\Lambda)1_H, \end{aligned}$$

since $S^{-1}f \in \int^\ell_{H^*}$. Hence, $(S^{-1}f)(n\Lambda) \in k^\circ$.

We next make note of a transitivity lemma for Frobenius systems, which adds Frobenius systems to the transitivity theorem, [23, Satz 6].

Lemma 6.1. *Suppose A/S is a β-Frobenius extension with system (E_S, x_i, y_i) and S/T is a γ-Frobenius extension with system (E_T, z_j, w_j). If $\beta(T) = T$, then A/T is a $\gamma \circ \beta$-Frobenius extension with system,*

$$(E_T \circ E_, \ x_i z_j, \ \beta^{-1}(w_j) y_i).$$

Proof. The mapping $E_T E_S$ is clearly a bimodule homomorphism ${}_T A_T \to {}_{\gamma\circ\beta} T_T$. We compute for every $a \in A$:

$$\begin{aligned} \sum_{i,j} x_i z_j E_T E_S(\beta^{-1}(w_j) y_i a) &= \sum_i x_i \sum_j z_j E_T(w_j E_S(y_i a)) \\ &= \sum_i x_i E_S(y_i a) = a, \\ \sum_{i,j} (\gamma\beta)^{-1}(E_T E_S(a x_i z_j))\beta^{-1}(w_j) y_i &= \sum_i \beta^{-1}(\sum_j \gamma^{-1}(E_T(E_S(a x_i) z_j)) w_j) y_i \\ &= \sum_i \beta^{-1}(E_S(a x_i)) y_i = a. \quad \square \end{aligned}$$

Applying the lemma to the Frobenius system $(E, S^{-1}(\Lambda_2), \Lambda_1)$ for H/K and Frobenius system $(g, S^{-1}(n_2), n_1)$ for K yields the Frobenius system for the algebra H,

$$(g \circ E,\ S^{-1}(\Lambda_2)S^{-1}(n_2),\ \beta^{-1}(n_1)\Lambda_1).$$

Comparing this with the Frobenius system $(f, S^{-1}(t_2), t_1)$, we compute the derivative $d' \in H^\circ$ such that $(gE)d' = f$:

$$d' = \sum f(S^{-1}(n_2\Lambda_2))\beta^{-1}(n_1)\Lambda_1 \tag{43}$$

We note that $f = gF$, since for every $a \in H$,

$$g(\sum f(aS^{-1}(n_2))n_1) = f(a\sum S^{-1}(n_2)g(n_1)) = f(a).$$

Now appy g from the left to $F = Ed$ and conclude that $d = d'$. It follows that gE is a right norm in H^* like f, since $d \in k^\circ$.

Since $ft = \epsilon$ and $m_H(d') = d = (S^{-1}f)(n\Lambda)1_H$, we see that dt is a right norm for gE. Using Equation 17, we compute that

$$\begin{aligned} dt &= \sum \epsilon(S^{-1}(n_2\Lambda_2))\beta^{-1}(n_1)\Lambda_1 \\ &= \beta^{-1}(n)\Lambda. \end{aligned} \tag{44}$$

Recalling from Section 1 that $t = \hat{\Lambda}n$, we note that

$$\beta^{-1}(n)\Lambda = \hat{\Lambda}nd. \tag{45}$$

Multiplying both sides of the equation $\beta^{-1}(n)\Lambda = td$ from the left by $\beta^{-1}(x)$, where $x \in K$, derives Equation 39 by other means for local ground rings.

7. Finite order elements

Let M be a finite projective module over a commutative ring k. Let $rank_M$: $Spec(k) \to \mathcal{Z}$ be the rank function as in Section 4. We introduce the *rank number* $\hat{D}(M,k)$ of M as the least common multiple of the integers in the range of the rank function on M:

$$\hat{D}(M,k) = l.c.m.\{n_1, n_2, \ldots, n_k\}.$$

Let H be a finite projective Hopf algebra over a Noetherian ring k. Let $d \in H$ be a group-like element. In this section we provide a proof that $d^N = 1$ where N divides $\hat{D}(H,k)$ (Theorem 7.7). In particular if H has constant rank n, such as when $Spec(k)$ is connected, then N divides n. Then we establish in Corollaries 7.8 and 7.9 that the antipode S and Nakayama automorphism η satisfy $S^{4N} = \eta^{2N} = \mathrm{Id_H}$ as corollaries of Theorem 4.8.

Let $k[d, d^{-1}]$ denote the subalgebra of H generated over k by 1 and the negative and positive powers of d. Let $k[d]$ denote only the k-span of 1 and the positive powers of d. Clearly $k[d, d^{-1}]$ is Hopf subalgebra of H. d has a *minimal polynomial* $p(x) \in k[x]$ if $p(x)$ is a polynomial of least degree such that $p(d) = 0$ and the gcd of all the coefficients is 1. We first consider the case where k is a domain.

Lemma 7.1. *If k is a domain, each group-like $d \in H$ has a minimal polynomial of the form $p(x) = x^s - 1$ for some integer s. Moreover, s divides $dim_{\overline{k}}(H \otimes_k \overline{k})$ and $\overline{f}(d^s) \neq 0$, where $\overline{k}$ denote the field of fractions of k $\overline{f}$ is FH-homomorphism for $\overline{k}[d, d^{-1}]$.*

Proof. We work at first in the Hopf algebra $H \otimes_k \overline{k}$ in which H is embedded. Since $\overline{k}[d, d^{-1}]$ is a finite dimensional Hopf algebra, there is a unique minimal polynomial of d, given by $\overline{p}(x) = x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_0 1$. Since d is invertible, $\lambda_0 \neq 0$ and $\overline{k}[d, d^{-1}] = \overline{k}[d]$.

$\overline{k}[d]$ is a Hopf-Frobenius algebra with FH-homomorphism $f : \overline{k}[d] \to \overline{k}$. Then $f(d^k)d^k = f(d^k)1$ for every integer k, since each d^k is group-like. If $f(d^k) \neq 0$, then $k \geq s$, since otherwise d is root of $x^k - 1$, a polynomial of degree less than s.

Thus, $f(d) = \cdots = f(d^{s-1}) = 0$, but $f(1) \neq 0$ since $f \neq 0$ on $\overline{k}[d]$. Then $f(\overline{p}(d)) = f(d^s) + \lambda_0 f(1) = 0$, so that $f(d^s) = -\lambda_0 f(1) \neq 0$. Since $f(d^s)d^s = f(d^s)1$, it follows that $d^s - 1 = 0$. Clearly $\overline{k}[d]$ is a Hopf subalgebra of $H \otimes_k \overline{k}$ of dimension s over $\overline{k}$ and it follows from the Nichols-Zoeller theorem that s divides $dim_{\overline{k}}(H \otimes_k \overline{k})$.

For H over an integral domain we arrive instead at $r(d^s - 1) = 0$ for some $0 \neq r \in k$. Since H is finite projective over an integral domain, it follows that $d^s - 1 = 0$. □

It follows easily from the proof that if $g(x) \in k[x]$ such that $g(d) = 0$, then $d^s = 1$ for some integer $s \leq \deg g$.

Theorem 7.2. *Let H be a finite projective Hopf algebra over a commutative ring k, which contains no additive torsion elements. If $d \in H$ is a group-like element, then $d^N = 1$ for some N that divides $\hat{D}(H, k)$.*

Proof. Let $\mathcal{P}$ be a prime ideal in k and $rank_H(\mathcal{P}) = n_i$. Let $D = \hat{D}(H, k)$.

Note that $H/\mathcal{P}H \cong H \otimes_k (k/\mathcal{P})$ is a finite projective Hopf algebra over the domain $k/\mathcal{P}$. By Lemma 7.1, there is an integer $s_{\mathcal{P}}$ such that $d^{s_{\mathcal{P}}} - 1 \in \mathcal{P}H$ and $s_{\mathcal{P}}$ divides n_i. It follows that

$$d^D - 1 \in \mathcal{P}H$$

for each prime ideal $\mathcal{P}$ of k. Since H is a finite projective over k, a standard argument gives $\mathrm{Nil}(k)H = \cap(\mathcal{P}H)$ over all prime ideals, where the nilradical $\mathrm{Nil}(k) = \cap\mathcal{P}$ is equal to the intersection of all prime ideals in k. Thus, $d^D - 1 = \sum r_i a_i$ where $r_i \in \mathrm{Nil}(k)$. Let k_i be integers such that $r_i^{k_i} = 0$. Then

$$(d^D - 1)^{(\sum_{i=1}^{n} k_i)+1} = 0. \tag{46}$$

It is clear that $P(x) := (x^D - 1)^{(\sum_{i=1}^{n} k_i)+1}$ is a monic polynomial with integer coefficients.

In general, let $m \in \mathcal{Z}$ be the least number such that $m \cdot 1 = 0$: we will call m the characteristic of k. Clearly in this case $m = 0$ and $\mathcal{Z} \subseteq k$. Again by Equation 46, $\mathcal{Z}[d, d^{-1}] = \mathcal{Z}[d]$ is a Hopf algebra over $\mathcal{Z}$. Moreover, since k has no additive torsion elements, we conclude that $\mathcal{Z}[d]$ is a free module over $\mathcal{Z}$.

Now by Lemma 7.1 $q(x) = x^s - 1$ is the minimal polynomial for d over $\mathcal{Z}$ and therefore $x^s - 1$ divides $P(x)$ in $\mathcal{Z}[x]$. Since $P(x)$ has the same roots in $\mathcal{C}$ as $x^D - 1$ it follows that a primitive s-root of unity is a D-root of unity and whence s divides D. □

In preparation for the next theorem, observe that if $(k, \mathcal{M})$ is a local ring of positive characteristic, then $char(k) = p^t$ for some positive power of a prime number p and $char(k/\mathcal{M}) = p$.

Theorem 7.3. *Let k be a local ring of positive characteristic and H be a finite projective Hopf algebra over k of rank n. If $d \in H$ is group-like, then $d^s = 1$ where s is the order of the image of d in $H/\mathcal{M}H$ (and therefore s divides n).*

Proof. $\mathcal{Z}_{p^t}$ is clearly a subring of k. Since we can choose a spanning set of H over k of the form $1, d, \dots, d^{s-1}, t_s, \cdots t_n$ where the elements of the set are linearly independent *modulo* $\mathcal{M}$, it follows that $1, d, \dots, d^{s-1}$ generate a free module over $\mathcal{Z}_{p^t}$. We claim that $\mathcal{Z}_{p^t}[d]$ coincides with this module and therefore is free over $\mathcal{Z}_{p^t}$.

First we need to prove that $\mathcal{M}H \cap \mathcal{Z}_{p^t}[d] = p\mathcal{Z}_{p^t}[d]$. To do this, we observe that $\mathcal{M} \cap \mathcal{Z}_{p^t} = p\mathcal{Z}_{p^t}$ and thus $p\mathcal{Z}_{p^t}[d] \subset \mathcal{M}H \cap \mathcal{Z}_{p^t}[d]$. Then there is a canonical epimomorphism of algebras over $\mathcal{Z}_p : \frac{\mathcal{Z}_{p^t}[d]}{p\mathcal{Z}_{p^t}} \to \frac{\mathcal{Z}_{p^t}[d]}{\mathcal{M}H \cap \mathcal{Z}_{p^t}[d]}$. Let $\bar{d}$ denote the image of d in $H/\mathcal{M}H$. Since $\bar{d}^s = 1$ over $k/\mathcal{M}$, we deduce that $\bar{d}^s = 1$ over $\mathcal{Z}_p$ from the fact that $\mathcal{Z}_p$ is the prime subfield of $k/\mathcal{M}$. Thus there is a canonical algebra epimorphism $\mathcal{Z}_p[\pi_s] \to \frac{\mathcal{Z}_{p^t}[d]}{p\mathcal{Z}_{p^t}}$, where π_s is a cyclic group of order s. Since $1, d, \dots, d^{s-1}$ are linearly independent *modulo* $\mathcal{M}$, it follows that $dim_{\mathcal{Z}_p}(\frac{\mathcal{Z}_{p^t}[d]}{\mathcal{M}H \cap \mathcal{Z}_{p^t}[d]}) \geq s$ while $dim_{\mathcal{Z}_p}(\mathcal{Z}_p[\pi_s]) = s$. Therefore all three $\mathcal{Z}_p$-algebras above have dimension s and $\mathcal{M}H \cap \mathcal{Z}_{p^t}[d] = p\mathcal{Z}_{p^t}[d]$.

The next step we need is to prove that d satisfies a monic polynomial equation of degree s over k. Clearly $d^s - 1 \in \mathcal{M}H \cap \mathcal{Z}_{p^t}[d]$ and hence $d^s - 1 = \Sigma a_i d^i$ with $a_i \in p\mathcal{Z}_{p^t}$. If all $i < s$ then this is exactly what we need. Otherwise we can replace $a_{s+k}d^{s+k}$ by $a_{s+k}d^k + a_{s+k}\Sigma a_i d^{i+k}$. However new coefficients $a_{s+k}a_i$ are divisible by p^2. Continuing this process we will arrive at a monic polynomial in d of degree s because $p^t = 0$ and all the monomials of degree greater than s will be eliminated.

Now it follows that $\mathcal{Z}_{p^t}[d, d^{-1}] = \mathcal{Z}_{p^t}[d]$ is a Hopf algebra over $\mathcal{Z}_{p^t}$ and a free module of rank s. Then $\mathcal{Z}_{p^t}[d]$ is a Hopf-Frobenius algebra because $\mathcal{Z}_{p^t}$ is a local ring. Let f be a Hopf-Frobenius homomorphism for $\mathcal{Z}_{p^t}[d]$. If $\bar{f} = f \bmod p$, it is clear that this is a Hopf-Frobenius homomorphism for $\mathcal{Z}_p[\bar{d}]$. Since $\bar{f}(\bar{d}^s) \neq 0$, it follows that $f(d^s)$ is an invertible element of $\mathcal{Z}_{p^t}$. Hence, the relation $f(d^s)d^s = f(d^s) \cdot 1$ implies that $d^s = 1$, which proves the theorem. ☐

Definition 7.4. We say that a commutative ring k is a GH-ring if any group-like element d of any finite projective Hopf algebra H satisfies $d^{\hat{D}(H,k)} = 1$.

We have already proved that fields, rings without additive torsion, and local rings of positive characteristic are GH-rings. Next we make the following easy remarks:

Remark 7.5. If $k_1, \dots, k_n$ are GH-rings then $\oplus_{i=1}^n k_i$ is a GH-ring.

Let $g : k \to K$ be a ring homomorphism ($g(1) = 1$) and let $g^* : Spec(K) \to Spec(k)$ be the induced continuous mapping. Then it is well-known (see for instance [1]) that $rank_{M \otimes_k K} = g^* \circ rank_M$ for any projective k-module M and it follows that $\hat{D}(M \otimes_k K, K)$ divides $\hat{D}(M, k)$. Then we can make the following

Remark 7.6. If $g : k \to K$ is an embedding and K is a GH-ring, then k is a GH-ring.

Theorem 7.7. *Noetherian rings are GH-rings.*

Proof. Let k be a Noetherian ring and $T(k) \subset k$ be the set of all torsion elements, i.e. for any $a \in T(k)$ there exists a positive integer m such that $ma = 0$. $T(k)$ is clearly an ideal of k. Since $T(k)$ is finitely generated over k, there exists a positive integer $t(k)$ such that $t(k)T(k) = 0$. Let $\pi_1 : k \to \frac{k}{T(k)}$ and $\pi_2 : k \to \frac{k}{t(k)\cdot k}$ be canonical surjections. We claim that $\pi_1 \oplus \pi_2 : k \to \frac{k}{T(k)} \oplus \frac{k}{t(k)\cdot k}$ is an embedding. Indeed, if $(\pi_1 \oplus \pi_2)(x) = 0$ then $x \in T(k)$ and $x = t(k)a$ for some $a \in k$. Obviously then $a \in T(k)$ and $x = t(k)a = 0$.

Since $\frac{k}{T(k)}$ has no additive torsion and therefore is a GH-ring, it remains to prove that a Noetherian ring of a positive characteristic is a GH-ring. For that let us consider a multiplicatively closed set $\mathcal{S}$ consisting of all the non-divisors of zero of k. It is well-known that $k \to \mathcal{S}^{-1}k$ is an embedding and $\mathcal{S}^{-1}k$ is a semi-local ring if k is Noetherian (see [1]). So, it is sufficient to prove that a semi-local ring A of positive characterestic is a GH-ring. Let $\mathcal{M}_1, .., \mathcal{M}_n$ be the set of all maximal ideals of A and $A_{\mathcal{M}_i}$ be the corresponding localizations. Now let us consider a homomorphism $f : A \to \oplus A_{\mathcal{M}_i}$ induced by canonical homomorphisms $f_i : A \to A_{\mathcal{M}_i}$. We claim that f is an embedding. Let in contrary $f(x) = 0$. Then $f_i(x) = 0$ for any i and there exists $a_i \in A \backslash \mathcal{M}_i$ such that $a_i x = 0$. Let us consider the ideal I generated by all a_i. Clearly $Ix = 0$. On the other hand I cannot belong to $\mathcal{M}_i$ because a_i is not in $\mathcal{M}_i$. Therefore we get that $I = A$ and consequently $x = 0$. Since any $A_{\mathcal{M}_i}$ is a local ring of positive characteristic, Theorem 3.2 implies the required result. □

As a consequence of Proposition 4.8, Theorem 7.2 and Equation 26, we obtain the following corollaries.

Corollary 7.8. *Let H be an FH-algebra over a Noetherian ring k. Then* $S^{4\hat{D}(H,k)} = \eta^{2\hat{D}(H,k)} = \mathrm{Id_H}$

Proof. Note that $\hat{D}(H,k) = \hat{D}(H^*,k)$. □

Corollary 7.9. *Let H be a finite projective Hopf algebra over a Noetherian ring k. Then* $S^{4\hat{D}(H,k)} = \mathrm{Id_H}$.

Proof. Localizing with respect to $\mathcal{S} = k - \{\text{zero divisors}\}$ we reduce the statement to a semi-local ring A. Then it is well-known (see [1]) that $Pic(A) = 0$ and hence the statement follows from the Hopf-Frobenius case. □

8. The quantum double of an FH-algebra

Let k be a commutative ring. We note that the *quantum double* $D(H)$, due to Drinfel'd [5], is definable for a finite projective Hopf algebra H over k: at the level of coalgebras it is given by

$$D(H) := H^{*\,\mathrm{cop}} \otimes_k H,$$

where $H^{*\,\mathrm{cop}}$ is the co-opposite of H^*, the coproduct being Δ^{op}.

The multiplication on $D(H)$ is described in two equivalent ways as follows [19, Lemma 10.3.11]. In terms of the notation gx replacing $g \otimes x$ for every $g \in H^*, x \in H$, both H and $H^{*\,\mathrm{cop}}$ are subalgebras of $D(H)$, and for each $g \in H^*$ and $x \in H$,

$$xg := \sum (x_1 g S^{-1} x_3) x_2 = \sum g_2 (S^{-1} g_1 \rightharpoonup x \leftharpoonup g_3). \tag{47}$$

The algebra $D(H)$ is a Hopf algebra with antipode $S'(gx) := SxS^{-1}g$, the proof proceeding as in [15]. A Hopf algebra H' is *almost cocommutative*, if there exists $R \in H' \otimes H'$, called the *universal R-matrix*, such that

$$R\Delta(a)R^{-1} = \Delta^{\mathrm{op}}(a) \tag{48}$$

for every $a \in H'$. A *quasi-triangular Hopf algebra* H' is almost cocommutative with universal R-matrix satisfying the two equations,

$$(\Delta \otimes \mathrm{Id})\mathrm{R} = R_{13}R_{23} \tag{49}$$

$$(\mathrm{Id} \otimes \Delta)\mathrm{R} = R_{13}R_{12}. \tag{50}$$

By a proof like that in [15, Theorem IX.4.4], $D(H)$ is a quasi-triangular Hopf algebra with universal R-matrix

$$R = \sum_i e_i \otimes e^i \in D(H) \otimes D(H), \tag{51}$$

where (e_i, e^i) is a finite projective base of H [5].

The next theorem is now a straightforward generalization of [27, Theorem 4.4].

Theorem 8.1. *If H is an FH-algebra, then the quantum double $D(H)$ is a unimodular FH-algebra.*

Proof. Let f be an FH-homomorphism with t a right norm. Then $T := S^{-1}f$ is a left norm in H^*. Let b^{-1} be the left distinguished group-like element in H satisfying $Tg = g(b^{-1})T$ for every $g \in H^*$. Moreover, note that $\ell := S^{-1}(t)$ is a left norm in H.

In this proof we denote elements of $D(H)$ as tensors in $H^* \otimes H$. We claim that $T \otimes t$ is a left and right integral in $D(H)$. We first show that it is a right integral.

The transpose of Formula 27 in Theorem 4.6 is $\sum t_1 \otimes t_2 = \sum t_2 \otimes b^{-1}S^2 t_1$. Applying $\Delta \otimes S^{-1}$ to both sides yields $\sum t_1 \otimes t_2 \otimes S^{-1}t_3 = \sum t_2 \otimes t_3 \otimes (St_1)b$. It follows easily that

$$\sum S^{-1}t_3 b^{-1} t_1 \otimes t_2 = 1 \otimes t. \tag{52}$$

We next make a computation like that in [19, 10.3.12]. Given a simple tensor $g \otimes x \in D(H)$, note that in the second line below we use $Tg = g(b^{-1})T$ for each $g \in H^*$, and in the third line we use Equation 52:

$$\begin{aligned}(T \otimes t)(g \otimes x) &= \sum Tg(S^{-1}t_3(-)t_1) \otimes t_2 x \\ &= Tg(S^{-1}t_3 b^{-1} t_1) \otimes t_2 x \\ &= g(1)T \otimes tx \\ &= g(1)\epsilon(x) T \otimes t\end{aligned}$$

In order to show that $T \otimes t$ is also a left integral, we note that Formula 27 applied to the right norm $T' = S^{-1}T$ in H^* is $\sum T'_1 \otimes T'_2 = \sum T'_2 \otimes m^{-1}S^2 T'_1$. Apply $S \otimes S$ to obtain

$$\sum T_2 \otimes T_1 = \sum T_1 \otimes S^2 T_2 m. \tag{53}$$

Applying $\Delta \otimes S^{-1}$ to both sides yields $\sum T_2 \otimes T_3 \otimes mS^{-1}T_1 = \sum T_1 \otimes T_2 \otimes ST_3$. Whence

$$\begin{aligned}\sum T_2 \otimes T_3 m S^{-1} T_1 &= \sum T_1 \otimes T_2 S T_3 \\ &= T \otimes 1. \end{aligned} \tag{54}$$

Then

$$\begin{aligned}(g\otimes x)(T\otimes t) &= \sum gT_2\otimes (S^{-1}T_1 \rightharpoonup x \leftharpoonup T_3)t \\ &= \sum gT_2 \otimes S^{-1}T_1(x_3)T_3(x_1)x_2t \\ &= \sum gT_2\otimes [T_3 m S^{-1}T_1](x)t \\ &= gT\otimes \epsilon(x)t = g(1)\epsilon(x)T\otimes t\end{aligned}$$

Thus $T\otimes t$ is also a left integral.

Next we note that $T\otimes t$ is an FH-homomorphism for $D(H)^*$, since $D(H)^*\cong H^{\mathrm{op}}\otimes H^*$, the ordinary tensor product of algebras (recall that $D(H)$ is the ordinary tensor product of coalgebras $(H^{\mathrm{op}})^*\otimes H$). This follows from $T\otimes t$ being a right integral in $D(H)$ on the one hand, while, on the other hand, H^{op} and H^* are FH-algebras with FH-homomorphisms $T = S^{-1}f$ and t.

Since $T\otimes t$ is an FH-homomorphism for $D(H)^*$, it follows that $T\otimes t$ is a right norm in $D(H)$. Since $T\otimes t$ is a left integral in $D(H)$, it follows that it is a left norm too. Hence, $D(H)$ is unimodular. □

Corollary 8.2. *$S(t)\otimes f$ is an FH-homomorphism for $D(H)$.*

Proof. Note that $S(t)\otimes f$ is a right integral in $D(H)^*\cong H^{\mathrm{op}}\otimes H^*$, since $S(t)$ and f are right integrals in H^{op} and H^*, respectively. Then

$$(T\otimes t)(S(t)\otimes f) = \epsilon_{D(H)^*}T(S(t))f(t) = \epsilon_{D(H)^*}. \tag{55}$$

so that $S(t)\otimes f$ is a right norm in $D(H)^*$. By Proposition 4.3, $D(H)$ is an FH-algebra with FH-homomorphism $S(t)\otimes f$. □

The next theorem implies that the quantum double $D(H)$ of an FH-algebra is a symmetric algebra, of which [3, Corollary 3.12] is a special case.

Theorem 8.3. *A unimodular almost commutative FH-algebra H' is a symmetric algebra.*

Proof. Since H' is unimodular, Lemma 4.2 shows that H' has Nakayama automorphism S^2. Since H' is almost commutative, a computation like Drinfeld's (cf. [19, Proposition 10.1.4]) shows that S^2 of an almost commutative Hopf algebra H is an inner automorphism: if $R=\sum_i z_i\otimes w_i$ is the universal R-matrix satisfying Equation 48, then $S^2(a) = uau^{-1}$ where $u=\sum_i (Sw_i)z_i$. Thus, the Nakayama automorphism is inner, and H' is a symmetric algebra. □

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