FROM FINITE DIFFERENCES TO FINITE ELEMENTS

A short history of numerical analysis of partial differential equations

VIDAR THOMÉE

ABSTRACT. This is an account of the history of numerical analysis of partial differential equations, starting with the 1928 paper of Courant, Friedrichs, and Lewy, and proceeding with the development of first finite difference and then finite element methods. The emphasis is on mathematical aspects such as stability and convergence analysis.

0. Introduction.

This article is an attempt to give a personal account of the development of numerical analysis of partial differential equations. We begin with the introduction in the 1930s and further development of the finite difference method and then describe the subsequent appearence around 1960 and increasing role of the finite element method. Even though clearly some ideas may be traced back further, our starting point will be the fundamental theoretical paper by Courant, Friedrichs, and Lewy (1928)[1] on the solution of problems of mathematical physics by means of finite differences. In this paper a discrete analogue of Dirichlet's principle was used to define an approximate solution by means of the five-point approximation of Laplace's equation, and convergence as the mesh width tends to zero was established by compactness. A finite difference approximation was also defined for the wave equation, and the CFL stability condition was shown to be necessary for convergence; again compactness was used to demonstrate convergence. Since the purpose was to prove existence of solutions, no error estimates or convergence rates were derived. With its use of a variational principle for discretization and its discovery of the importance of mesh-ratio conditions in approximation of time dependent problems this paper points forward and has had a great influence on numerical analysis of partial differential equations.

Error bounds for difference approximations of elliptic problems were first derived by Gerschgorin (1930) whose work was based on a discrete analogue of the maximum-principle for Laplace's equation. This approach was actively pursued through the 60s by, e.g., Collatz, Motzkin, Wasow, Bramble, and Hubbard, and various approximations of elliptic equations and associated boundary conditions were analyzed.

1991 *Mathematics Subject Classification.* 01A60, 65-03, 65M10, 65N10, 65M60, 65N30.

Key words and phrases. history, finite difference methods, finite element methods.

[1]We refer to original work with publication year; we sometimes also quote survey papers and textbooks which are numbered separately.

For time dependent problems considerable progress in finite difference methods was made during the period of, and immediately following, the Second World War, when large scale practical applications became possible with the aid of computers. A major role was played by the work of von Neumann, partly reported in O'Brien, Hyman and Kaplan (1951). For parabolic equations a highlight of the early theory was the important paper by John (1952). For mixed initial-boundary value problems the use of implicit methods was also established in this period by, e.g., Crank and Nicolson (1947). The finite difference theory for general initial value problems and parabolic problems then had an intense period of development during the 50s and 60s, when the concept of stability was explored in the Lax equivalence theorem and the Kreiss matrix lemmas, with further major contributions given by Douglas, Lees, Samarskii, Widlund and others. For hyperbolic equations, and particularly for nonlinear conservation laws, the finite difference method has continued to play a dominating role up until the present time, starting with work by, e.g., Friedrichs, Lax, and Wendroff.

Standard references on finite difference methods are the textbooks of Forsythe and Wasow [14] and Richtmyer and Morton [28].

The idea of using a variational formulation of a boundary value problem for its numerical solution goes back to Lord Rayleigh (1894, 1896) and Ritz (1908), see, e.g., Kantorovich and Krylov [21]. In Ritz's approach the approximate solution was sought as a finite linear combination of functions such as, for instance, polynomials or trigonometrical polynomials. The use in this context of continuous piecewise linear approximating functions based on triangulations adapted to the geometry of the domain was proposed by Courant (1943) in a paper based on an address delivered to the American Mathematical Society in 1941. Even though this idea had appeared earlier, also in work by Courant himself (see Babuška [5]), this is often thought of as the starting point of the finite element method, but the further development and analysis of the method would occur much later. The idea to use an orthogonality condition rather than the minimization of a quadratic functional is attributed to Galerkin (1915); its use for time-dependent problems is sometimes referred to as the Faedo-Galerkin method, cf. Faedo (1949), or, when the orthogonality is with respect to a different space, as the Petrov-Galerkin or Bubnov-Galerkin method.

As a computational method the finite element method originated in the engineering literature, where in the mid 50s structural engineers had connected the well established framework analysis with variational methods in continuum mechanics into a discretization method in which a structure is thought of as divided into elements with locally defined strains or stresses. Some of the pioneering work was done by Turner, Clough, Martin, and Topp (1956) and Argyris (1960) and the name of the finite element method appeared first in Clough (1960). The method was later applied to other classes of problems in continuum mechanics; a standard reference from the engineering literature is Zienkiewicz [43].

Independently of the engineering applications a number of papers appeared in the mathematical literature in the mid 60s which were concerned with the construction and analysis of finite difference schemes by the Rayleigh-Ritz procedure with piecewise linear approximating functions, by, e.g., Oganesjan (1962, 1966), Friedrichs (1962), Céa (1964), Demjanovič (1964), Feng (1965), and Friedrichs and Keller (1966) (who considered the Neumann problem). Although in fact special cases of the finite element method, the methods studied were conceived as finite difference methods; they were referred to in the Russian literature as variational

difference schemes.

In the period following this, the finite element method with piecewise polynomial approximating functions was analyzed mathematically in work such as Birkhoff, Schultz, and Varga (1968), in which the theory of splines was brought to bear on the development, and Zlámal (1968), with the first stringent *a priori* error analysis of more complicated finite elements. So called mixed finite element methods, which are based on variational formulations where, e.g., the solution of an elliptic equation and its gradient appear as separate variables and where the combined variable is a saddle-point of a Lagrangian functional, were introduced in Brezzi (1974); such methods have many applications in fluid dynamical problems and for higher order elliptic equations.

More recently, following Babuška (1976), Babuška and Rheinboldt (1978), much effort has been devoted to showing *a posteriori* error bounds which depend only on the data and the computed solution. Such error bounds can be applied to formulate adaptive algorithms which are of great importance in computational practice.

Comprehensive references for the analysis of the finite element method are Babuška and Aziz [4], Strang and Fix [34], Ciarlet [11], and Brenner and Scott [8].

Simultaneous with this development other classes of methods have arisen which are related to the above, and we will sketch four such classes: In a *collocation* method an approximation is sought in a finite element space by requiring the differential equation to be satisfied exactly at a finite number of collocation points, rather than by an orthogonality condition. In a *spectral* method one uses globally defined functions, such as eigenfunctions, rather than piecewise polynomials approximating functions, and the discrete solution may be determined by either orthogonality or collocation. A *finite volume* method applies to differential equations in divergence form. Integrating over an arbitrary volume and transforming the integral of the divergence into an integral of a flux over the boundary, the method is based on approximating such a boundary integral. In a *boundary integral* method a boundary value problem for a homogeneous elliptic equation in a d-dimensional domain is reduced to an integral equation on its $(d-1)$-dimensional boundary, which in turn can be solved by, e.g., the Galerkin finite element method or by collocation.

An important aspect of numerical analysis of partial differential equations is the numerical solution of the finite linear algebraic systems that are generated by the discrete equations. These are in general very large, but with sparse matrices, which makes iterative methods suitable. The development of convergence analysis for such methods has parallelled that of the error analysis sketched above. In the 50s and 60s particular attention was paid to systems associated with finite difference approximation of positive type of second order elliptic equations, particularly the five-point scheme, and starting with the Jacobi and Gauss-Seidel methods techniques were developed such as the Frankel and Young successive overrelaxation and the Peaceman-Rachford (1955) alternating direction methods, as described in the influential book of Varga [39]. In later years systems with positive definite matrices stemming from finite element methods have been solved first by the conjugate gradient method proposed by Hestenes and Stiefel (1952), and then making this more effective by preconditioning. The multigrid method was first introduced for finite difference methods in the 60s by Fedorenko and Bahvalov and further developed by Brandt in the 70s. For finite elements the multigrid method and the associated method of domain decomposition have been and are being intensely pursued by, e.g., Braess, Hackbusch, Bramble, and Widlund.

Many ideas and techniques are common to the finite difference and the finite element methods, and in some simple cases they coincide. Nevertheless, with its more systematic use of the variational approach, its greater geometric flexibility, and the way it more easily lends itself to error analysis, the finite element method has become the dominating approach both among numerical analysts and in applications. The growing need for understanding the partial differential equations modeling the physical problems has seen an increase in the use of mathematical theory and techniques, and has attracted the interest of many mathematicians. The computer revolution has made large scale real world problems accessible to simulation, and in later years the concept of computational mathematics has emerged with a somewhat broader scope than classical numerical analysis.

Our approach in this survey is to try to illustrate the ideas and concepts that have influenced the development, with as little technicalities as possible, by considering simple model situations. We emphasize the mathematical analysis of the discretization methods, involving stability and error estimates, rather than modeling and implementation issues. It is not our ambition to present the present state of the art but rather to describe the unfolding of the field. It is clear that it is not possible in the limited space available to do justice to all the many important contributions that have been made, and we apologize for omissions and inaccuracies; writing a survey such as this one is a humbling experience. In addition to references to papers which we have selected as important in the development we have quoted a number of books and survey papers where additional and more complete and detailed information can be found; for reason of space we have tried to limit the number of reference to any individual author.

Our presentation is divided into sections as follows: 1. The Courant-Friedrichs-Lewy paper; 2. Finite difference methods for elliptic problems; 3. Finite difference methods for initial value problems; 4. Finite difference methods for mixed initial-boundary value problems; 5. Finite element methods for elliptic problems; 6. Finite element methods for evolution equations; 7. Some other classes of approximation methods; 8. Numerical linear algebra for elliptic problems.

1. The Courant-Friedrichs-Lewy paper.

In this seminal paper from 1928 the authors considered difference approximations of both the Dirichlet problems for a second order elliptic equation and the biharmonic equation, and of the initial-boundary value problem for a second order hyperbolic equation, with a brief comment also about the model heat equation in one space variable. Their purpose was to derive existence results for the original problem by constructing finite dimensional approximations of the solutions, for which the existence was clear, and then showing convergence as the dimension grows. Although the aim was not numerical, the ideas presented have played a fundamental role in numerical analysis. The paper appears in English translation, together with commentaries by Lax, Widlund, and Parter concerning its influence on the subsequent development, see the quotation in References.

The first part of the paper treats the Dirichlet problem for Laplace's equation,

$$-\Delta u = 0 \quad \text{in } \Omega, \quad \text{with } u = g \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset R^2$ is a domain with smooth boundary $\partial\Omega$. Recall that by Dirichlet's principle the solution minimizes $\iint_\Omega |\nabla\varphi|^2\, dx$ over φ with $\varphi = g$ on $\partial\Omega$. For a

discrete analogue, consider mesh-points $x_j = jh$, $j \in Z^2$, and let Ω_h be the mesh-points in Ω for which the neighbors $x_{j\pm e_1}, x_{j\pm e_2}$ are in Ω ($e_1 = (1,0)$, $e_2 = (0,1)$), and let ω_h be those with at least one neighbor outside Ω. For $U_j = U(x_j)$ a mesh-function we introduce the forward and backward difference quotients

$$\partial_l U_j = (U_{j+e_l} - U_j)/h, \quad \bar{\partial}_l U_j = (U_j - U_{j-e_l})/h, \quad l = 1, 2. \tag{1.2}$$

By minimizing a sum of terms of the form $(\partial_1 U_j)^2 + (\partial_2 U_j)^2$ one finds a unique mesh-function U_j which satisfies

$$-\partial_1 \bar{\partial}_1 U_j - \partial_2 \bar{\partial}_2 U_j = 0 \quad \text{for } x_j \in \Omega_h, \quad \text{with} \quad U_j = g(x_j) \quad \text{on } \omega_h; \tag{1.3}$$

the first equation is the well-known five-point approximation

$$4U_j - (U_{j+e_1} + U_{j-e_1} + U_{j+e_2} + U_{j-e_2}) = 0. \tag{1.4}$$

It is shown by compactness that the solution of (1.3) converges to a solution u of (1.1) when $h \to 0$. By the same method it is shown that on compact subsets of Ω difference quotients of U converge to the corresponding derivatives of u as $h \to 0$. Also included are a brief discussion of discrete Green's function representation of the solution of the inhomogeneous equation, of discretization of the eigenvalue problem, and of approximation of the solution of the biharmonic equation.

The second part of the paper is devoted to initial-value problems for hyperbolic equations. In this case, in addition to the mesh-width h, a time step k is introduced and the discrete function values $U_j^n \approx u(x_j, t_n)$, with $t_n = nk, n \in Z_+$. The authors first consider the model wave equation

$$u_{tt} - u_{xx} = 0 \quad \text{for } x \in R^2,\ t \geq 0, \quad \text{with } u(\cdot, 0),\ u_t(\cdot, 0) \text{ given}, \tag{1.5}$$

and the approximate problem, with obvious modification of the notation (1.2),

$$\partial_t \bar{\partial}_t U_j^n - \partial_x \bar{\partial}_x U_j^n = 0 \quad \text{for } j \in Z,\ n \geq 1, \quad \text{with } U_j^n \text{ given for } n = 0, 1.$$

When $k = h$ the equation may also be expressed as $U_j^{n+1} + U_j^{n-1} - U_{j+1}^n - U_{j-1}^n = 0$, and it follows at once that in this case the discrete solution at $(x,t) = (x_j, t_n)$ depends only on the initial data in the interval $(x - t, x + t)$. For a general time-step k the interval of dependence becomes $(x - t/\lambda, x + t/\lambda)$, where $\lambda = k/h$ is the mesh-ratio. Since the exact solution depends on data in $(x - t, x + t)$ it follows that if $\lambda > 1$, not enough information is used by the scheme, and hence a necessary condition for convergence of the discrete solution to the exact solution is that $\lambda \leq 1$; this is referred to as the Courant-Friedrichs-Lewy or CFL condition. By an energy argument it is shown that the appropriate sums over the mesh-points of positive quadratic forms in the discrete solution are bounded, and compactness is used to show convergence as $h \to 0$ when $\lambda = k/h =$constant ≤ 1. The energy argument is a clever discrete analogue of an argument by Friedrichs and Lewy (1926): For the wave equation in (1.5) one may integrate the identity $0 = 2u_t(u_{tt} - u_{xx}) = (u_t^2 + u_x^2)_t - 2(u_x u_t)_x$ in x to show that $\int_R (u_t^2 + u_x^2)\, dx$ is independent of t, and thus bounded. The case of two spatial variables is also briefly discussed.

In an appendix brief discussions are included concerning a first order hyperbolic equation, the model heat equation in one space variable, and of wave equations with lower order terms.

2. Finite difference methods for elliptic problems.

The error analysis of finite difference methods for elliptic problems started with the work of Gerschgorin (1930). In contrast to the treatment in Courant, Friedrichs, and Lewy (1928) this work was based on a discrete version of the maximum principle. To describe this approach we begin with the model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where we first assume Ω to be the square $\Omega = (0,1) \times (0,1) \subset R^2$. For a finite difference approximation consider the mesh-points $x_j = jh$ with $h = 1/M$, $j \in Z^2$ and the mesh-function $U_j = U(x_j)$. With the notation (1.2) we replace (2.1) by

$$-\Delta_h U_j := -\partial_1 \bar{\partial}_1 U_j - \partial_2 \bar{\partial}_2 U_j = f_j \quad \text{for } x_j \in \Omega, \quad U_j = 0 \quad \text{for } x_j \in \partial\Omega. \tag{2.2}$$

This problem may be written in matrix form as $AU = F$, where A is a symmetric $(M-1)^2 \times (M-1)^2$ matrix whose elements are $4, -1$, or 0, with 0 the most common occurrence, cf. (1.4).

For the analysis one first shows a discrete maximum-principle: If U is such that $-\Delta_h U_j \le 0$ (≥ 0) in Ω, then U takes its maximum (minimum) on $\partial\Omega$; note that $-\Delta_h U_j \le 0$ is equivalent to $U_j \le (U_{j+e_1} + U_{j-e_1} + U_{j+e_2} + U_{j-e_2})/4$. Letting $W(x) = \frac{1}{2} - |x - x_0|^2$ where $x_0 = (\frac{1}{2}, \frac{1}{2})$ we have $W(x) > 0$ in Ω, and applying the discrete maximum principle to the function $V_j = \pm U_j - \frac{1}{4}|\Delta_h U|_\Omega W_j$ one concludes easily that, for any mesh-function U on $\bar{\Omega}$,

$$|U|_{\bar{\Omega}} \le |U|_{\partial\Omega} + C|\Delta_h U|_\Omega, \quad \text{where } |U|_S = \max_{x_j \in S} |U_j|.$$

Noting that the error $z_j = U_j - u(x_j)$ satisfies

$$-\Delta_h z_j = f_j + \Delta_h u(x_j) = (\Delta_h - \Delta)u(x_j) = \tau_j, \quad \text{with } |\tau_j| \le Ch^2 \|u\|_{C^4}, \tag{2.3}$$

one finds, since $z_j = 0$ on $\partial\Omega$, that

$$|U - u|_{\bar{\Omega}} = |z|_{\bar{\Omega}} \le C|\tau|_\Omega \le Ch^2 \|u\|_{C^4}.$$

The above analysis uses the fact that all neighbors $x_{j \pm e_l}$ of the interior mesh-points $x_j \in \Omega$ are either interior mesh-points or belong to $\partial\Omega$. When the boundary is curved, however, there will be mesh-points in Ω which have neighbors outside $\bar{\Omega}$. If for such a mesh-point $x = x_j$ we take a point $b_{h,x} \in \partial\Omega$ with $|b_{h,x} - x| \le h$ and set $U_j = u(b_{h,x}) = 0$, then it follows from Gerschgorin, loc. cit., that $|U - u|_{\bar{\Omega}} \le Ch\|u\|_{C^3}$. To retain second order accuracy Collatz (1933) proposed to use linear interpolation near the boundary: Assuming for simplicity that Ω is a convex plane domain with smooth boundary, we denote by Ω_h the mesh-points $x_j \in \Omega_h$ that are truly interior in the sense that all four neighbors of x_j are in $\bar{\Omega}$. (For the above case of a square, Ω_h simply consists of all mesh-points in Ω.) Let now $\tilde{\omega}_h$ be the mesh-points in Ω that are not in Ω_h. For $x_j \in \tilde{\omega}_h$ we may then find a neighbor $y = x_i \in \Omega_h \cup \tilde{\omega}_h$ such that the line through x and y cuts $\partial\Omega$ at a point $\bar{x}$ which is not a mesh-point. We denote by $\bar{\omega}_h$ the set of points $\bar{x} \in \partial\Omega$ thus associated with the points of $\tilde{\omega}_h$, and define for $x = x_j \in \tilde{\omega}_h$ the error in the linear interpolant

$$\ell_h u_j := u(x_j) - \alpha u(x_i) - (1 - \alpha)u(\bar{x}), \quad \text{where } \alpha = \gamma/(1+\gamma) \le \tfrac{1}{2} \text{ if } |x - \bar{x}| = \gamma h.$$

As $u = 0$ on $\partial\Omega$ we now pose the problem

$$-\Delta_h U_j = f_j \quad \text{in } \Omega_h, \quad \ell_h U_j = 0 \quad \text{in } \tilde{\omega}_h, \quad U(\bar{x}) = 0 \quad \text{on } \bar{\omega}_h,$$

and since $\alpha \leq 1/2$ it is not difficult to see that

$$|U|_{\Omega_h \cup \bar{\omega}_h} \leq C(|U|_{\bar{\omega}_h} + |\ell_h U|_{\bar{\omega}_h} + |\Delta_h U|_{\Omega}). \tag{2.4}$$

Using again (2.3) together with $|\ell_h z|_{\tilde{\omega}_h} \leq Ch^2 \|u\|_{C^3}$, $z = 0$ on $\bar{\omega}_h$, one finds that $|U - u|_{\Omega_h \cup \bar{\omega}_h} \leq Ch^2 \|u\|_{C^4}$. Another approximation near $\partial\Omega$ was proposed by Shortley and Weller (1938). For $x_j \in \tilde{\omega}_h$ it uses the points defined by the intersections of $\partial\Omega$ with the horizontal and vertical mesh-lines through x_j, which together with the neighbors that are in Ω form an irregular five-point star. This gives an approximation to $-\Delta$ which is only first order accurate, but, using it in a boundary operator ℓ_h similarly to the Collatz interpolation error it will yield a second order error bound.

Consider more generally the variable coefficient Dirichlet problem

$$Au := -\sum_{j,k=1}^{d} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{j=1}^{d} b_j \frac{\partial u}{\partial x_j} = f \quad \text{in } \Omega \subset R^d, \quad u = g \quad \text{on } \partial\Omega, \tag{2.5}$$

where the matrix (a_{jk}) is uniformly positive definite in $\bar{\Omega}$, and a corresponding finite difference operator with finitely many terms of the form

$$A_h u(x) = -h^{-2} \sum_j a_j u(x - jh), \quad a_j = a_j(x, h), \ j \in Z^2 \tag{2.6}$$

which is consistent with A so that $A_h u(x) \to Au(x)$ as $h \to 0$. Following Motzkin and Wasow (1953) such an operator is said to be of positive type if $a_j \geq 0$ for $j \neq 0$, with $a_0 < 0$.

For mesh-points x, let $N(x)$ be the convex hull of the set of neighbors of x defined by (2.6), i.e., the mesh-points $x - jh$ with $a_j(x, h) \neq 0$, and let Ω_h denote the set of mesh-points with $N(x) \subset \bar{\Omega}$. The remaining mesh-points in $\bar{\Omega}$ form the set $\tilde{\omega}_h$ of boundary mesh-points. We set $\bar{\Omega}_h = \Omega_h \cup \tilde{\omega}_h$. For $x \in \Omega_h$ we want to use the equation $A_h U(x) = M_h f(x)$ as an analogue of the differential equation in (2.5), where M_h is a linear operator approximating the identity operator I (in most cases $M_h = I$). For $x \in \tilde{\omega}_h$, $A_h U(x)$ is not defined by the values of U on $\bar{\Omega}_h$, and at such points we therefore want to choose an equation of the form

$$\ell_h u(x) := \sum_{x_j \in \bar{\Omega}} \tilde{a}_j u(x - jh) = m_h(g, f), \quad \tilde{a}_j = \tilde{a}_j(x, h),$$

where m_h is a suitable linear operator. The values of u at points in $\bar{\omega}_h$ will now be included in the right hand side by $u = g$ on $\partial\Omega$. Together these equations form our difference scheme, and we say (see Bramble, Hubbard, and Thomée (1969)) that this is of essentially positive type if A_h is of positive type and $\tilde{a}_0 = 1$, $\sum_{j \neq 0} |\tilde{a}_j| \leq \gamma < 1$ for $x \in \omega_h$. A discrete maximum-principle shows that the analogue of (2.4) remains valid in this case (with Δ_h replaced by A_h and without the term $|U|_{\bar{\omega}_h}$). The scheme is said to be accurate of order q if $A_h u - M_h A u = O(h^q)$ on

Ω_h, and $\ell_h u - m_h(u|_{\partial\Omega}, Au) = O(h^q)$ on $\tilde{\omega}_h$. Under somewhat more precise natural assumptions one may now conclude from (2.4) that $|U-u|_{\Omega_h} \le Ch^q\|u\|_{C^{q+2}}$. Error bounds may also be expressed in terms of data, and a $O(h^q)$ error bound holds if $f \in C^s(\bar{\Omega})$, $g \in C^s(\partial\Omega)$ with $s > q$. For homogeneous boundary conditions this follows easily using the Schauder estimate $\|u\|_{C^{q+2}} \le C\|f\|_{C^s}$. For inhomogeneous boundary conditions a more precise analysis may be based on a representation using a nonnegative discrete Green's function $G_{jl} = G(x_j, x_l)$,

$$U_j = h^d \sum_{x_l \in \Omega_h} G_{jl} A_h U_l + \sum_{x_l \in \omega_h} G_{jl} U_l, \quad \text{for } x_j \in \bar{\Omega}_h,$$

where $h^d \sum_{x_l \in \Omega_h} G_{jl} \le C$ and $\sum_{x_l \in \tilde{\omega}_h} G_{jl} \le 1$, and where also a discrete analogue of the estimate $\int_{\Gamma_\delta} G(x,y)\, ds \le C\delta$ for the continuous problem holds, where $\Gamma_\delta = \{y \in \Omega;\ \mathrm{dist}(y, \partial\Omega) = \delta\}$. The latter is related to the important observation by Bahvalov (1959) that the regularity demands on the solution u of the continuous problem can be relaxed in some cases by essentially two derivatives at the boundary without loosing the convergence rate. For less regular data one can obtain correspondingly weaker convergence estimates: When $f \in C^s(\bar{\Omega})$, $g \in C^s(\partial\Omega)$ with $0 \le s < q$, $O(h^s)$ order convergence may be shown by interpolation between C^s-spaces. The regularity demands of f may be further relaxed by choosing for M_h an averaging operator, see Tikhonov and Samarskii (1961); this paper also demonstrated how to construct finite difference approximations of elliptic equations with discontinuous coefficients by taking local harmonic averages. When the boundary itself is nonsmooth the exact solution may have singularities which make the above results not applicable. Laasonen (1957) showed that the presence of a corner with an accute inner angle does not affect the rate of convergence but if the angle is $\pi\alpha$ with $\alpha > 1$ he shows the weaker estimate $O(h^{1/\alpha-\epsilon})$ for any $\epsilon > 0$.

As an example of an operator of the form (2.6), consider the nine-point formula

$$-\Delta_h^{(9)} u(x) = \tfrac{1}{6} h^{-2}\big(20\, u(x) - 4 \sum_{|j|=1} u(x-jh) - \sum_{|j_1|=|j_2|=1} u(x-jh)\big).$$

With $M_h f = f + \frac{1}{12} h^2 \Delta_h f$ one finds $\Delta_h^{(9)} u + M_h \Delta u = O(h^4)$ and Bramble and Hubbard (1962) showed that the operator ℓ_h can be chosen so that the corresponding scheme is of essentially positive type and accurate of order $q = 4$. Further, Bramble and Hubbard (1963) constructed second order accurate schemes of essentially positive type in the case of a general A ($d = 2$), also with mixed derivative terms. Here the neighbors of x may be several mesh-widths away from x. Related results were also obtained in special cases by Wasow (1952), Laasonen (1958), and Volkov (1966), see also Bramble, Hubbard, and Thomée, loc. cit.

We shall now turn to some schemes that approximate elliptic equations containing mixed derivatives but are not generally of essentially positive type. Assume now that A is in divergence form, $A = -\sum_{i,k=1}^d (\partial/\partial x_i)(a_{ik}\partial u/\partial x_k)$. To our above notation ∂_i, $\bar{\partial}_i$ of (1.2) we add the symmetric difference quotient $\hat{\partial}_i = (\partial_i + \bar{\partial}_i)/2$ and set, with $a_{ik}^{(h)} = a_{ik}(x + \frac{1}{2} h e_i)$,

$$A_h^{(1)} u = -\sum_{i,k} \bar{\partial}_i (a_{ik}^{(h)} \partial_k u), \quad A_h^{(2)} u = -\sum_i \bar{\partial}_i (a_{ii}^{(h)} \partial_i u) - \sum_{i \ne k} \hat{\partial}_i (a_{ik} \hat{\partial}_k u).$$

These operators are obviously consistent with A and second order accurate. Except in special cases the $A_h^{(l)}$ are not of positive type and the above analysis does not apply. Instead one may use energy arguments to derive error estimates in discrete l_2-norms, see Thomée (1964). For $x \in \tilde{\omega}_h$ let as above $b_{h,x} \in \partial\Omega$, and consider the discrete Dirichlet problem

$$A_h U = f \quad \text{on } \Omega_h, \quad U = g(b_{h,x}) \quad \text{on } \tilde{\omega}_h, \quad \text{with } A_h = A_h^{(1)} \text{ or } A_h^{(2)}. \tag{2.7}$$

With $\mathcal{D}_h$ the mesh-functions which vanish outside Ω_h, we define for $U, V \in \mathcal{D}_h$

$$(U,V) = h^d \sum_j U_j V_j, \quad \|U\| = (U,U)^{1/2}, \quad \text{and} \quad \|U\|_1 = \|U\| + \sum_{j=1}^{d} \|\partial_i U\|.$$

By summation by parts one easily derives that $\|V\|_1^2 \le C(A_h V, V)$ for $V \in \mathcal{D}_h$, and this shows at once the uniqueness and hence the existence of a solution of (2.7). When $\tilde{\omega}_h \subset \partial\Omega$, application to $U - u$, using the second order accuracy of A_h shows that $\|U-u\|_1 \le C(u)h^2$. When $\tilde{\omega}_h \not\subset \partial\Omega$, h^2 has to be replaced by $\sqrt{h}$, but with $\|\cdot\|_1'$ a slightly weaker norm it was shown in Bramble, Kellogg, and Thomée (1968) that $\|U-u\| \le \|U-u\|_1' \le C(u)h$.

Consider now a constant coefficient finite difference operator of the form (2.6), which is consistent with A of (2.5). Introducing the symbol of A_h, the trigonometric polynomial $p(\xi) = \sum_j a_j e^{ij\cdot\xi}$, we say that A_h is elliptic if $p(\xi) \ne 0$ for $|\xi_l| \le \pi$, $l = 1, 2$, $\xi \ne 0$. For the five-point operator $-\Delta_h$ we have $p(\xi) = 4 - 2\cos\xi_1 - 2\cos\xi_2$ and $-\Delta_h$ is thus elliptic. For such operators A_h we have the following interior estimate by Thomée and Westergren (1968). Set $\|U\|_S = (h^d \sum_{x_j \in S} U_j^2)^{1/2}$ and $\|U\|_{k,S} = \left(\sum_{|\alpha|\le k} \|\partial_h^\alpha U\|_S^2\right)^{1/2}$ where $\partial_h^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and let $\bar{\Omega}_0 \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$. Then for any α and h small we have

$$|\partial_h^\alpha U|_{\Omega_0} \le C\big(\|A_h U\|_{l,\Omega_1} + \|U\|_{\Omega_1}\big), \quad \text{if } l \ge |\alpha| + [d/2] - 1.$$

Thus, the finite difference quotients of the solution of the equation $A_h U = f$ may be bounded in Ω_0 by the difference quotients of f in a slightly larger domain Ω_1 plus a discrete l_2-norm of U in Ω_1. Assuming u is a solution of (2.1) this may be used to show that if A_h is accurate of order q and Q_h is a finite difference operator which approximates the differential operator Q to order q, then

$$|Q_h U - Qu|_{\Omega_0} \le C(u)h^q + C\|U-u\|_{\Omega_1}. \tag{2.8}$$

Thus, if we already know that $U-u$ is of order $O(h^q)$ in the l_2-norm, then $Q_h u - Qu$ is of the same order in maximum-norm in the interior of the domain.

Finite difference approximations for elliptic equations of higher order were studied by, e.g., Saulev (1957); the work in Thomée (1964) concerns also such equations.

3. Finite difference methods for initial value problems.

In this section we sketch the development of the stability and convergence theory for finite difference methods applied to pure initial value problems. We first consider linear constant coefficient evolution equations and then specialize to parabolic and hyperbolic equations.

We begin with the initial-value problem for a general linear constant coefficient scalar equation

$$(3.1)\quad u_t = P(D)u \quad \text{for } x \in R^d,\ t > 0, \quad u(\cdot,0) = v, \quad \text{where } P(\xi) = \sum_{|\alpha| \le M} P_\alpha \xi^\alpha,$$

with $u = u(x,t)$, $v = v(x)$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$, and $D = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$. Such an initial-value problem is said to be well posed if it has a unique solution that depends continuously on the initial data, in some sense that has to be specified. For example, the one-dimensional wave equation $u_t = \rho u_x$ has the unique solution $u(x,t) = v(x+\rho t)$ and since $\|u(\cdot,t)\|_{L_p} = \|v\|_{L_p}$, this problem is well-posed in L_p for $1 \le p \le \infty$. Similarly, for the heat equation $u_t = u_{xx}$ we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} v(y)\, dy,$$

and $\|u(\cdot,t)\|_{L_p} \le \|v\|_{L_p}$, $1 \le p \le \infty$. More precisely, (3.1) is well posed in L_p if $P(D)$ generates a semigroup $E(t) = e^{tP(D)}$ in L_p which grows at most exponentially, so that the solution $u(t) = E(t)v$ satisfies $\|u(t)\|_{L_p} \le Ce^{\kappa t}\|v\|_{L_p}$ for $t \ge 0$, for some κ. For $p = 2$, which is the case we will concentrate on first, we see by Fourier transformation and Parseval's relation that this is equivalent to

$$(3.2)\qquad |e^{tP(i\xi)}| \le Ce^{\kappa t}, \quad \forall \xi \in R^d,\ t > 0,$$

or $\operatorname{Re} P(i\xi) \le \kappa$ for $\xi \in R^d$; if only the highest order terms are present in $P(D)$, this is equivalent to $\operatorname{Re} P(i\xi) \le 0$ for $\xi \in R^d$, in which case (3.2) holds with $\kappa = 0$, so that $E(t)$ is uniformly bounded in L_2 for $t \ge 0$. In the above examples $P(i\xi) = i\rho\xi$ and $P(i\xi) = -\xi^2$, respectively, and $\kappa = 0$.

Generalizing to systems of the form (3.1) with u and v N-vectors and the P_α $N \times N$ matrices, the condition for well-posedness in L_2 is again (3.2) where now $|\cdot|$ denotes any matrix norm. Here it is clear that a necessary condition for (3.2) is that $\operatorname{Re}\lambda_j(\xi) \le \kappa$ for $\xi \in R^d$, for any eigenvalue $\lambda_j(\xi)$ of $P(i\xi)$, and if $P(i\xi)$ is a normal matrix this is also sufficient. Necessary and sufficient conditions for (3.2) were given by Kreiss (1959b), and depend on the following lemma. Here, for any $N \times N$ matrix A with eigenvalues $\{\lambda_j\}_{j=1}^N$ we set $\Lambda(A) = \max_j \operatorname{Re}\lambda_j$ and $\operatorname{Re} A = (A + A^*)/2$, where A^* is the adjoint matrix. For Hermitian matrices $A \le B$ means $Au \cdot u \le Bu \cdot u$ for all $u \in R^N$. With this notation, (3.2) holds if and only if the set $\mathcal{F} = \{P(i\xi) - \kappa; \xi \in R^d\}$ satisfies the conditions of the following lemma:

Let $\mathcal{F}$ be a set of $N \times N$ matrices. Then the following conditions are equivalent:

(i) $|e^{tA}| \le C$ *for* $A \in \mathcal{F}$, $t \ge 0$, *for some* $C > 0$;

(ii) $\Lambda(A) \le 0$ *and* $Re\,(z\,|R(A;z)|) \le C$ *for* $A \in \mathcal{F}$, $Re\, z > 0$, *for some* $C > 0$;

(iii) $\Lambda(A) \le 0$ *for* $A \in \mathcal{F}$ *and there exist* C_1 *and* C_2 *and for each* $A \in \mathcal{F}$ *a matrix* $S = S(A)$ *such that* $\max(|S|, |S^{-1}|) \le C_1$ *and such that* $SAS^{-1} = B = (b_{jk})$ *is a triangular matrix with off-diagonal elements satisfying* $|b_{jk}| \le C_2 \min(|Re\,\lambda_j|, |Re\,\lambda_k|)$ *where* $\lambda_j = b_{jj}$;

(iv) Let $0 \le \gamma < 1$. *There exists* $C > 0$ *such that for each* $A \in \mathcal{F}$ *there is a Hermitian matrix* $H = H(A)$ *with* $C^{-1}I \le H \le CI$ *and* $Re\,(HA) \le \gamma\Lambda(A)H \le 0$.

Equations of higher order in the time variable such as the wave equation $u_{tt} = u_{xx}$ may be written in system form (3.1) by introducing the successive time derivatives of u as dependent variables, and is therefore covered by the above discussion.

For the approximate solution of the initial value problem (3.1), let h and k be (small) positive numbers. We want to approximate the solution at time level $t_n = nk$ by U^n for $n \geq 0$ where $U^0 = v$ and $U^{n+1} = E_k U^n$ for $n \geq 0$, and where E_k is an operator of the form

$$E_k v(x) = \sum_j a_j(h) v(x - jh), \quad \lambda = k/h^M = \text{constant}, \tag{3.3}$$

with summation over a finite set of multi-indices $j = (j_1, \ldots, j_d) \in Z^d$; such operators are called explicit. The purpose is to choose E_k so that $E_k^n v$ approximates $u(t_n) = E(t_n)v = E(k)^n v$. In numerical applications we would apply (3.3) only for $x = x_l = lh$, $l \in Z^d$, but for convenience in the analysis we shall think of $E_k v$ as defined for all x. As an example, for the heat equation $u_t = u_{xx}$ the simplest such operator is obtained by replacing derivatives by finite difference quotients, $\partial_t U_j^n = \partial_x \bar{\partial}_x U^n$. Solving for U^{n+1} we see that this defines $E_k v(x) = \lambda v(x+h) + (1 - 2\lambda)v(x) + \lambda v(x-h)$, $\lambda = k/h^2$. We shall consider further examples below.

We say that E_k is consistent with (3.1) if $u(x, t+k) = E_k u(x,t) + o(k)$ when $k \to 0$, for sufficiently smooth solutions $u(x,t)$ of (3.1), and accurate of order r if the term $o(k)$ may be replaced by $kO(h^r)$. By Taylor series expansion around (x,t) these conditions are seen to be equivalent to algebraic conditions between the coefficients $a_j(h)$. In our above example, $u(x,t+k) - E_k u(x,t) = ku_t + \frac{1}{2}k^2 u_{tt} - \lambda h^2 u_{xx} - \frac{1}{12}\lambda h^4 u_{xxxx} = kO(h^2)$ when $u_t = u_{xx}$, so that $r = 2$.

For the purpose of showing convergence of $E_k^n v$ to $E(t_n)v$ and to derive an error bound one needs some stability property of the operator E_k^n: This operator is said to be stable in L_2 if for any $T > 0$ there are constants C and κ such that $\|E_k^n v\| \leq Ce^{\kappa nk}\|v\|$ for $v \in L_2$, $n \geq 0$, where $\|\cdot\| = \|\cdot\|_{L_2}$. If this holds, and if (3.1) is well posed in L_2 and E_k is accurate of order r, then it follows from the identity $E_k^n - E(t_n) = \sum_{j=0}^{n-1} E_k^{n-1-j}(E_k - E(k))E(t_j)$ that, with $\|\cdot\|_s = \|\cdot\|_{H^s(R^d)}$,

$$\|(E_k^n - E(t_n))v\| \leq C \sum_{j=0}^{n-1} kh^r \|E(t_j)v\|_{M+r} \leq C_T\, h^r \|v\|_{M+r}, \quad \text{for } t_n \leq T, \tag{3.4}$$

where we have also used the fact that spatial derivatives commute with $E(t)$.

The sufficiency of stability for convergence of the solution of the discrete problem to the solution of the continuous initial value problem was shown in particular cases in many places, e.g., Courant, Friedrichs, and Lewy (1928), O'Brien, Hyman, and Kaplan (1951), Douglas (1956). It was observed by Lax and Richtmyer (1959) that stability is actually a necessary condition for convergence to hold for all $v \in L_2$; the general Banach space formulation of stability as a necessary and sufficient condition for convergence is known as the Lax equivalence theorem, see Richtmyer and Morton [28]. We note that for individual, sufficiently regular v convergence may hold without stability; for an early interesting example with analytic initial data and highly unstable difference operator, see Dahlquist (1954). Without stability, however, roundoff errors will then overshadow the theoretical solution in actual computation.

We shall see that the characterization of stability of finite difference operators (3.3) is parallel to that of well-posedness of (3.1). Introducing the trigonometric polynomial $E_k(\xi) = \sum_j a_j(h)e^{ij\cdot h\xi}$, the symbol of E_k, Fourier transformation shows that E_k is stable in L_2 if and only if, cf. (3.2),

$$|E_k(\xi)^n| \le Ce^{\kappa nk}, \quad \forall \xi \in R^d,\ n \ge 0.$$

In the scalar case this is equivalent to $|E_k(\xi)| \le 1 + Ck$ for $\xi \in R^d$ and small k, or $|E_k(\xi)| \le 1$ for $\xi \in R^d$ when the coefficients of E_k are independent of h, as is normally the case when no lower order terms occur in $P(D)$. For our above example $E_k(\xi) = 1 - 2\lambda + 2\lambda\cos h\xi$ and stability holds if and only if $\lambda = k/h^2 \le 1/2$. For the equation $u_t = \rho u_x$ and the scheme $\partial_t U_j^n = \hat{\partial}_x U_j^n$ we have $E_k(\xi) = 1 + i\rho\lambda \sin h\xi$ and this method is therefore seen to be unstable for any choice of $\lambda = k/h$.

Necessary for stability in the matrix case is the von Neumann condition

$$\rho(E_k(\xi)) \le 1 + Ck, \quad \forall \xi \in R^d, \tag{3.5}$$

where $\rho(A) = \max_j |\lambda_j|$ is the spectral radius of A, and for normal matrices $E_k(\xi)$ this is also sufficient. This covers the scalar case discussed above. Necessary and sufficient conditions are given in the following discrete version of the above Kreiss matrix lemma, see Kreiss (1962), where we denote $|u|_H = (Hu,u)^{1/2}$ and $|A|_H = \sup_{u\ne 0} |Au|_H/|u|_H$ for H positive definite:

Let $\mathcal{F}$ be a set of $N \times N$ matrices. Then the following conditions are equivalent:
(i) $|A^n| \le C$ *for* $A \in \mathcal{F},\ n \ge 0$, *for some* $C > 0$;

(ii) $\rho(A) \le 1$ *and* $(|z| - 1)|R(A; z)| \le C$ *for* $A \in \mathcal{F},\ |z| > 1$, *with* $C > 0$;

(iii) $\rho(A) \le 1$ *for* $A \in \mathcal{F}$ *and there are* C_1 *and* C_2 *and for each* $A \in \mathcal{F}$ *a matrix* $S = S(A)$ *such that* $\max(|S|, |S^{-1}|) \le C_1$ *and such that* $SAS^{-1} = (b_{jk})$ *is triangular with off-diagonal elements satisfying* $|b_{jk}| \le C_2 \min(1 - |\lambda_j|, 1 - |\lambda_k|)$ *where* $\lambda_j = b_{jj}$;

(iv) *Let* $0 \le \gamma < 1$. *There exists* $C > 0$ *and for each* $A \in \mathcal{F}$ *a Hermitian matrix* $H = H(A)$ *with* $C^{-1}I \le H \le CI$ *and* $|A|_H \le 1 - \gamma + \gamma\rho(A)$.

Application shows that if E_k is stable in L_2, then there is a κ such that $\mathcal{F} = \{e^{-\kappa k}E_k(\xi);\ k \le k_0,\ \xi \in R^d\}$ satisfies conditions $(i) - (iv)$. On the other hand, if one of these conditions holds for some κ, then E_k is stable in L_2.

Other related sufficient conditions were given in, e.g., Kato (1960), where it was shown that if the range of an $N \times N$ matrix is in the unit disc, i.e., if $|Av \cdot v| \le 1$ for $|v| \le 1$, then $|A^n v \cdot v| \le 1$, and hence, by taking real and imaginary parts, $|A^n| \le 2$ for $n \ge 1$.

Using the above characterizations one can show that a necessary and sufficient condition for the existence of an L_2-stable operator which is consistent with (3.1) is that (3.1) is well-posed in L_2, see Kreiss (1959a). It was also proved by Wendroff (1968) that for initial value problems which are well-posed in L_2 one may construct L_2-stable difference operators with arbitrarily high order of accuracy.

It may be shown that von Neumann's condition (3.5) is equivalent to growth of at most polynomial order of the solution operator in L_2, or $\|E_k^n v\|_{L_2} \le Cn^q \|v\|_{L_2}$ for $t_n \le T$, for some $q \ge 0$. This was used by Forsythe and Wasow [14] and Ryabenkii and Filippov [29] as a definition of stability.

For variable coefficients it was shown by Strang (1965) that if the initial-value problem for the equation $u_t = \sum_{|\alpha|\le M} P_\alpha(x) D^\alpha u$ is well posed in L_2, then the one for the equation without lower order terms and with coefficients fixed at $x \in R^d$ is also well-posed, and a similar statement holds for the stability of the finite difference scheme $E_k v(x) = \sum_j a_j(x,h) v(x - jh)$. However, Kreiss (1962) showed that well-posedness and stability with frozen coefficients is neither necessary nor sufficient for well-posedness and stability of a general variable coefficient problem. We shall return below to variable coefficients for parabolic and hyperbolic equations.

We now consider the special case when the system (3.1) is parabolic, and begin by quoting the fundamental paper of John (1952) in which maximum-norm stability was shown for finite difference schemes for second order parabolic equations in one space variable. For simplicity we restrict our presentation to the model problem

$$u_t = u_{xx} \quad \text{for } x \in R,\ t > 0, \quad \text{with } u(\cdot, 0) = v \quad \text{in } R, \tag{3.6}$$

and a corresponding finite difference approximation of the form (3.3) with $a_j(h) = a_j$ independent of h. Setting $a(\xi) = \sum_j a_j e^{-ij\xi} = E_k(h^{-1}\xi)$ one may write

$$U^n(x) = \sum_j a_{nj} v(x - jh), \quad \text{where } a_{nj} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\xi} a(\xi)^n \, d\xi. \tag{3.7}$$

Here the von Neumann condition reads $|a(\xi)| \le 1$, and is necessary and sufficient for stability in L_2. To show maximum-norm stability we need to estimate the a_{nj} in (3.7). It is easily seen that if the difference scheme is consistent with (3.6) then $a(\xi) = e^{-\lambda\xi^2} + o(\xi^2)$ as $\xi \to 0$, and if we assume that $|a(\xi)| < 1$ for $0 < |\xi| \le \pi$ it follows that

$$|a(\xi)| \le e^{-c\xi^2} \quad \text{for } |\xi| \le \pi, \quad \text{with } c > 0. \tag{3.8}$$

One then finds at once from (3.7) that $|a_{nj}| \le C n^{-1/2}$, and integration by parts twice, using $|a'(\xi)| \le C|\xi|$, shows $|a_{nj}| \le C n^{1/2} j^{-2}$. Thus

$$\sum_j |a_{nj}| \le C \sum_{j \le n^{1/2}} n^{-1/2} + \sum_{j > n^{1/2}} n^{1/2} j^{-2} \le C,$$

so that $\|U^n\|_\infty \le C \|v\|_\infty$ by (3.7) where $\|\cdot\|_\infty = \|\cdot\|_{L_\infty}$. We remark that for our simple example above we have $|E_k v(x)| \le (\lambda + |1 - 2\lambda| + \lambda) \|v\|_\infty = \|v\|_\infty$ for $\lambda \le 1/2$, which trivially yields maximum-norm stability.

In the general constant coefficient case the system (3.1) is said to be parabolic in Petrowski's sense if $\Lambda(P(\xi)) \le -\delta|\xi|^M + C$ for $\xi \in R^d$, and using (iii) of the Kreiss lemma one shows easily that the corresponding initial value problem is well posed in L_2. For well-posedness in L_∞ we may write

$$u(x,t) = (E(t)v)(x) = \int_{R^d} G(x - y, t) v(y) \, dy$$

where, cf., e.g., Friedman [15], with $D^\alpha = D_x^\alpha$,

$$|D^\alpha G(x,t)| \le C_T t^{-(|\alpha|+d)/M} e^{(-\delta(|x|^M/t)^{1/(M-1)})}, \quad \text{for } 0 < t \le T. \tag{3.9}$$

This implies that $\|D^\alpha u(\cdot,t)\|_\infty \le Ct^{-|\alpha|/M}\|v\|_\infty$, so that the solution, in addition to being bounded, is smooth for positive time even if the initial data are only bounded. Consider now a difference operator of the form (3.3) which is consistent with (3.1). Generalizing from (3.8) we define this operator to be parabolic in the sense of John if, for some positive δ and C,

$$\rho(E_k(h^{-1}\xi)) \le e^{-\delta|\xi|^M} + Ck, \quad \text{for } \xi \in Q = \{\xi;\, |\xi_j| \le \pi,\, j = 1,\dots,d\};$$

such schemes always exist when (3.1) is parabolic in Petrowski's sense. Extending the results of John (1952) to this situation Aronson (1963) and Widlund (1966) showed that if we write $U^n(x) = E_k^n v(x) = \sum_j a_{nj}(h)v(x - jh)$, then, denoting difference quotients corresponding to D^α by ∂_h^α, we have, cf. (3.9),

$$|\partial_h^\alpha a_{nj}(h)| \le Ch^d t_n^{-(|\alpha|+d)/M} e^{(-c(|j|^M/n)^{1/(M-1)})}, \quad \text{for } t_n \le T,$$

which implies that $\|\partial_h^\alpha E_k^n v\|_\infty \le Ct_n^{-|\alpha|/M}\|v\|_\infty$. In the work quoted also multistep methods and variable coefficients were treated.

From estimates of these types follow also convergence estimates such as, if D_h^α is a difference operator consistent with D^α and accurate of order r,

$$\|D_h^\alpha U^n - D^\alpha u(t_n)\|_\infty \le Ch^r \|v\|_{W_\infty^{r+|\alpha|}} \quad \text{for } t_n \le T.$$

Note in the convergence estimate for $\alpha = 0$ that, as a result of the smoothing property for parabolic equations, less regularity is required of initial data than for the general well-posed initial value problem, cf. (3.4). For even less regular initial data lower convergence rates have to be expected; by interpolation between the W_∞^s spaces one may show (see Peetre and Thomée (1967)), e.g.,

$$\|U^n - u(t_n)\|_\infty \le Ch^s \|v\|_{W_\infty^s} \quad \text{for } 0 \le s \le r,\ t_n \le T.$$

We remark that for nonsmooth initial data v it is possible to make a preliminary smoothing of v to recover full accuracy for t bounded away from zero: It was shown in Kreiss, Thomée, and Widlund (1970) that there exists a smoothing operator of the form $M_h v = \Phi_h * v$, where $\Phi_h(x) = h^{-d}\Phi(h^{-1}x)$, with Φ an appropriate function, such that if $M_h v$ is chosen as initial data for the difference scheme, then

$$\|U^n - u(t_n)\|_\infty = \|E_h^n M_h v - E(t_n)v\|_\infty \le Ch^r t_n^{-r/M}\|v\|_\infty.$$

Let us also note that from a known convergence rate for the difference scheme for fixed initial data v conclusions may be drawn about the regularity of v. For example, under the above assumptions, assume that for some p, s with $1 \le p \le \infty, 0 < s \le r$, we know that $\|U^n - u(t_n)\|_{L_p} \le Ch^s$ for $t_n \le T$. Then v belongs to the Besov space $B_p^{s,\infty}$ ($\approx W_p^s$), and, if $s > r$ then $v = 0$. Such inverse results were given in, e.g., Hedstrom (1968), Löfström (1970), and Thomée and Wahlbin (1974).

We now turn our attention to hyperbolic equations, and consider first systems with constant real coefficients

$$u_t = \sum_{j=1}^d A_j u_{x_j} \quad \text{for } t > 0, \quad \text{with } u(0) = v. \tag{3.10}$$

Such a system is said to be strongly hyperbolic if it is well-posed in L_2, cf. Strang (1967). With $P(\xi) = \sum_{j=1}^{d} A_j \xi_j$ this holds if and only if for each $\xi \in R^d$ there exists a nonsingular matrix $S(\xi)$, uniformly bounded together with its inverse, such that $S(\xi)P(\xi)S(\xi)^{-1}$ is a diagonal matrix with real elements. When the A_j are symmetric this holds with $S(\xi)$ orthogonal; the system is then said to be symmetric hyperbolic. The condition is also satisfied when the eigenvalues of $P(\xi)$ are real and distinct for $\xi \neq 0$; in this case the system is called strictly hyperbolic.

One important feature of hyperbolic systems is that the value $u(x,t)$ of the solution at (x,t) only depends on the initial values on a compact set $K(x,t)$, the smallest closed set such that $u(x,t) = 0$ when v vanishes in a neighborhood of $K(x,t)$. The convex hull of $K(x,t)$ may be described in terms of $P(\xi)$: if $K = K(0,1)$ we have for its support function $\phi(\xi) = \sup_{x \in K} x \cdot \xi = \lambda_{\max}(P(\xi))$.

Consider now the system (3.10) and a corresponding finite difference operator of the form (3.3) (with $M = 1$). Here we introduce the domain of dependence $\tilde{K}(x,t)$ of E_k as the smallest closed set such that $E_k^n v(x) = 0$ for all n, k with $nk = t$ when v vanishes in a neighborhood of $\tilde{K}(x,t)$. Corresponding to the above the support function of $\tilde{K}(x,t)$ satisfies $\tilde{\phi}(\xi) = \lambda^{-1} \max_{a_j \neq 0} j \cdot \xi$. Since clearly convergence, and hence stability, demands that $\tilde{K} = \tilde{K}(0,1)$ contains the continuous domain of dependence $K = K(0,1)$ it is necessary for stability that $\lambda_{\max}(P(\xi)) \leq \lambda^{-1} \max_{a_j \neq 0} j \cdot \xi$; this is the CFL-condition, cf. Section 1. In particular, $\min_{a_j \neq 0} j_l \leq \lambda \lambda_{\min}(A_l) \leq \lambda \lambda_{\max}(A_l) \leq \max_{a_j \neq 0} j_l$. For the equation $u_t = \rho u_x$ and a difference operator of the form (3.3) using only $j = -1, 0, 1$, this means $\lambda|\rho| \leq 1$: In this case $u(0,1) = v(\rho)$ so that $K = \{\rho\}$, and the condition is thus $\rho \in \tilde{K} = [-\lambda^{-1}, \lambda^{-1}]$.

We shall now give some sufficient conditions for stability. We first quote from Friedrichs (1954) that if a_j are symmetric and positive semidefinite, with $\sum_j a_j = I$, then $|E_k(h^{-1}\xi)| = |\sum_j a_j e^{ij \cdot \xi}| \leq 1$ so that the scheme is L_2-stable. As an example, the first order accurate Friedrichs operator

$$E_k v(x) = \frac{1}{2} \sum_{j=1}^{d} ((d^{-1} I + \lambda A_j) v(x + h e_j) + (d^{-1} I - \lambda A_j) v(x - h e_j)) \tag{3.11}$$

is L_2-stable if $0 < \lambda \leq (d\rho(A_j))^{-1}$. It was observed by Lax (1961) that this criterion is of limited value in applications because it cannot in general be combined with accuracy higher than first order.

The necessary and sufficient conditions for L_2-stability of the Kreiss stability lemma are of the nature that the necessary von Neumann condition (3.5) has to be supplemented by conditions assuring that the eigenvalues of $E_k(\xi)$ sufficiently well describe the growth behavior of $E_k(\xi)^n$. We now quote some such criteria which utilize relations between the behavior of the eigenvalues of $E_k(\xi)$ for small ξ and the accuracy of E_k. In Kreiss (1964) E_k is defined to be dissipative of order ν, ν even, if $\rho(\tilde{E}_k(h^{-1}\xi)) \leq 1 - \delta|\xi|^\nu$ for $\xi \in Q$ with $\delta > 0$, and it is shown that if E_k is consistent with the strongly hyperbolic system (3.10), accurate of order $\nu - 1$, and dissipative of order ν, then it is L_2-stable. Further, it was shown by Parlett (1966) that if the system is symmetric hyperbolic, and E_k is dissipative of order ν it suffices that it is accurate of order $\nu - 2$, and by Yamaguti (1967) that if the system is strictly hyperbolic, then dissipativity of some order ν is sufficient. For strictly hyperbolic systems Yamaguti also showed that if the eigenvalues of $E_k(\xi)$

are distinct for $Q \ni \xi \neq 0$, then von Neumann's condition is sufficient for stability in L_2.

For $d = 1$ (with $A = A_1$) the well-known Lax-Wendroff (1960) scheme

$$E_k v(x) = \tfrac{1}{2}(\lambda^2 A^2 + \lambda A)v(x+h) + (1 - \lambda^2 A^2)v(x) + \tfrac{1}{2}(\lambda^2 A^2 - \lambda A)v(x-h)$$

is L_2-stable for $\lambda\rho(A) \leq 1$ if the system approximated is strongly hyperbolic. L_2-stable finite difference schemes of arbitrarily high order for such systems were constructed in, e.g., Strang (1962).

It is also possible to analyze multistep methods, e.g., by rewriting them as single step systems. A popular stable such method is the leapfrog method $\hat{\partial}_t U_j^n = \hat{\partial}_x U_j^n$ (or $U_j^{n+1} = U_j^{n-1} + \lambda\rho(U_{j+1}^n - U_{j-1}^n)$). The eigenvalues $\tau_1(\xi), \tau_2(\xi)$ appearing in the stability analysis then satisfy the characteristic equation $\tau - \tau^{-1} - 2\lambda\rho\sin\xi = 0$, and we find that the von Neumann condition, $|\tau_j(\xi)| \leq 1$ for $\xi \in R$, $j = 1, 2$, is satisfied if $\lambda\rho \leq 1$, and using the Kreiss lemma one can show that L_2-stability holds for $\lambda\rho < 1$.

As an example for $d = 2$ we quote the operator introduced in Lax and Wendroff (1964) defined by

$$\begin{aligned} E_k(h^{-1}\xi) &= I + i\lambda(A_1 \sin\xi_1 + A_2 \sin\xi_2) \\ &\quad - \lambda^2\big(A_1^2(1-\cos\xi_1) + \tfrac{1}{2}(A_1A_2 + A_2A_1)\sin\xi_1\sin\xi_2 + A_2^2(1-\cos\xi_2)\big). \end{aligned}$$

Using the above stability criterion of Kato they proved stability for $\lambda|A_j| \leq 1/\sqrt{8}$ in the symmetric hyperbolic case.

Consider now a variable coefficient symmetric hyperbolic system

$$u_t = \sum_{j=1}^{d} A_j(x) u_{x_j}, \quad \text{where } A_j(x)^* = A_j(x). \tag{3.12}$$

Using an energy argument Friedrichs (1954) showed that the corresponding initial value problem is well-posed; the boundedness follows at once by noting that after multiplication by u, integration over $x \in R^d$, and integration by parts,

$$\frac{d}{dt}\|u(t)\|^2 = -\int_{R^d} \big(\sum_j \partial A_j/\partial x_j u, u\big)\, dx \leq 2\kappa\|u(t)\|^2.$$

For finite difference approximations $E_k v(x) = \sum_j a_j(x) v(x - jh)$ which are consistent with (3.12) various sufficient conditions are available. Kreiss, Parlett, and Yamaguti, loc. cit., studied difference operators which are dissipative of order ν, i.e., such that $\rho(E(x,\xi)) \leq 1 - c|\xi|^\nu$ for $x \in R^d$, $\xi \in Q$, with $c > 0$, where $E(x,\xi) = \sum_j a_j(x) e^{-ij\xi}$. As in the constant coefficient case it was shown that E_k is stable in L_2 if the $a_j(x)$ are symmetric and if E_k is accurate of order $\nu - 1$ and dissipative of order ν; if (3.12) is strictly hyperbolic, accuracy of order $\nu - 2$ suffices. The proofs are based on the Kreiss stability lemma and a perturbation argument.

In an important paper it was proved by Lax and Nirenberg (1966) that if E_k is consistent with (3.12) and $|E(x,\xi)| \leq 1$ then E_k is strongly stable with respect to

the L_2-norm, i.e., $\|E_k\| \leq 1 + Ck$. In particular, if the $a_j(x)$ are symmetric positive definite stability follows by the result of Friedrichs quoted earlier for constant coefficients. Further, if E_k is consistent with (3.12) and $|E(x,\xi)v \cdot v| \leq 1$ for $|v| \leq 1$, then E_k is L_2-stable.

As mentioned above, higher order equations in time may be expressed as systems of the form (3.1), and finite difference methods may be based on such formulations. In Section 1 we described an example of a difference approximation of a second order wave equation in its original form.

We turn to estimates in L_p norms with $p \neq 2$. For the case when the equation in (3.10) is symmetric hyperbolic it was shown by Brenner (1966) that the initial value problem is well-posed in $L_p, p \neq 2$, if and only if the matrices A_j commute. This is equivalent to the simultaneous diagonalizability of these matrices, which in turn means that by introducing a new dependent variable it can be transformed to a system in which all the A_j are diagonal, so that the system consists of N uncoupled scalar equations.

Since stability of finite difference approximations can only occur for well-posed problems it is therefore natural to consider the scalar equations $u_t = \rho u_x$ with ρ real, and corresponding finite difference operators (3.3). With $a(\xi)$ as before we recall that such an operator is stable in L_2 if and only if $|a(\xi)| \leq 1$ for $\xi \in R$. Necessary and sufficient conditions for stability in $L_p, p \neq 2$, as well as rates of growth of $\|E_k^n\|_{L_p}$ in the nonstable case have been given, e.g., in Brenner and Thomée (1970), cf. also references in [9]. If $|a(\xi)| < 1$ for $0 < |\xi| \leq \pi$ and $a(0) = 1$ the condition is that $a(\xi) = e^{i\alpha\xi - \beta\xi^\nu(1+o(1))}$ as $\xi \to 0$, with α real, $\operatorname{Re}\beta > 0$, ν even. Thus E_k is stable in L_p, $p \neq 2$, if and only if there is an even number ν such that E_k is dissipative of order ν and accurate of order $\nu - 1$. As an example of an operator which is stable in L_2 but not in L_p, $p \neq 2$, we may take the Lax-Wendroff operator introduced above (with A replaced by ρ). For $0 < \lambda|\rho| < 1$, this operator is stable in L_2, is dissipative of order 4, but accurate only of order 2. It may be proved that for this operator $cn^{1/8} \leq \|E_k^n\|_\infty \leq Cn^{1/8}$ with $c > 0$, which shows a weak instability in the maximum-norm.

An area where finite difference techniques continue to flourish and to form an active research field is for nonlinear fluid flow problems. Consider the scalar nonlinear conservation law

$$u_t + F(u)_x = 0 \quad \text{for } x \in R,\ t > 0, \quad \text{with } u(\cdot,0) = v \quad \text{on } R, \tag{3.13}$$

where $F(u)$ is a nonlinear function of u, often strictly convex, so that $F''(u) > 0$. The classical example is Burger's equation $u_t + u\,u_x = 0$, where $F(u) = u^2/2$. Discontinuous solutions may arise even when v is smooth, and one therefore needs to consider weak solutions. Such solutions are not necessarily unique, and to select the unique physically relevant solution one has to require so called entropy conditions. This solution may also be obtained as the limit as $\epsilon \to 0$ of the diffusion equation with ϵu_{xx} replacing 0 on the right in (3.13).

One important class of methods for (3.13) are finite difference schemes in conservation form,

$$U_j^{n+1} = U_j^n - \lambda(H_{j+1/2}^n - H_{j-1/2}^n) \quad \text{for } n \geq 0, \quad \text{with } U_j^0 = v(jh), \tag{3.14}$$

where $H_{j+1/2}^n = H(U_j^n, U_{j+1}^n)$ is a numerical flux function which has to satisfy $H(V,V) = F(V)$ for consistency. Here stability for the linearized equation is neither

necessary nor sufficient for the nonlinear equation in the presence of discontinuities and much effort has been devoted to the design of numerical flux functions with good properties. When the right hand side of (3.14) is increasing in $U^n_{j+l}, l = -1, 0, 1$, the scheme is said to be monotone (corresponding to positive coefficients in the linear case) and such schemes converge for fixed λ to the entropy solution as $k \to 0$, but are at most first order accurate. Examples are the Lax-Friedrichs scheme which generalizes (3.11), the scheme of Courant, Isaacson, and Rees (1952), which is one-sided (upwinding), and the Engquist-Osher scheme which pays special attention to changes in sign of the characteristic direction. Godunov's method replaces U^n by a piecewise constant function, solves the corresponding problem exactly from t_n to t_{n+1} and defines U^{n+1} by an averaging process.

Higher order methods with good properties are also available, and are often constructed with an added artificial diffusion term which depends on the solution, or so called TVD (Total Variation Diminishing) schemes. Early work in the area was Lax and Wendroff (1960), and more recent contributions have been given by Engquist, Harten, Kuznetsov, MacCormack, Osher, Roe, Yee; for overviews and generalizations to systems and higher dimension, see Le Veque [23] and Godlewski and Raviart [16].

4. Finite differences for mixed initial-boundary value problems.

The pure initial value problem discussed in Section 3 is often not adequate to model a given physical situation and one needs to consider instead a problem whose solution is required to satisfy the differential equation in a bounded spatial domain $\Omega \subset R^d$, as well as boundary conditions on $\partial\Omega$ for positive time, and to take on given initial values. For such problems the theory of finite difference methods is less complete and satisfactory. In the same way as for the stationary problem treated in Section 2 one reason for this is that for $d \geq 2$ only very special domains may be well represented by mesh-domains, and even when $d = 1$ the transition between the finite difference approximation in the interior and the boundary conditions may be complex both to define and to analyze. Again there are three standard approaches to the analysis, namely methods based on maximum-principles, energy arguments, and spectral representation. We illustrate this first for parabolic and then for hyperbolic equations.

As a model problem for parabolic equations we shall consider the one-dimensional heat equation in $\Omega = (0, 1)$,

$$(4.1) \quad u_t = u_{xx} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega = \{0, 1\}, \quad \text{for } t \geq 0, \quad u(\cdot, 0) = v \quad \text{in } \Omega.$$

For the approximate solution we introduce mesh-points $x_j = jh$, where $h = 1/M$, and time levels $t_n = nk$, where k is the time step, and denote the approximate solution at (x_j, t_n) by U^n_j. As for the pure initial-value problem we may then approximate (4.1) by means of the explicit forward Euler difference scheme

$$\partial_t U^n_j = \partial_x \bar{\partial}_x U^n_j \quad \text{in } \Omega, \quad \text{with } U^{n+1}_j = 0 \quad \text{on } \partial\Omega, \quad \text{for } n \geq 0,$$

with $U^0_j = V_j = v(x_j)$ in Ω. For U^{n-1} given this defines U^n through

$$U^n_j = \lambda(U^{n-1}_{j-1} + U^{n-1}_{j+1}) + (1 - 2\lambda)U^{n-1}_j, \ 0 < j < M, \quad U^n_j = 0, \ j = 0, M.$$

For $\lambda = k/h^2 \le 1/2$ the coefficients are nonnegative and their sum is 1 so that we conclude that $|U^{n+1}| \le |U^n|$ where $|U| = \max_{x_j \in \Omega} |U_j|$. It follows that $|U^n| \le |V|$, and the scheme is thus stable in maximum-norm. Under these assumptions one shows as for the pure initial-value problem that $|U^n - u(t_n)| \le C(u)(h^2 + k) \le C(u)h^2$. It is easy to see that $k \le h^2/2$ is also a necessary condition for stability.

To avoid to have to impose the quite restrictive condition $\lambda \le 1/2$, Laasonen (1949) proposed the implicit backward Euler scheme

$$\bar\partial_t U_j^n = \partial_x \bar\partial_x U_j^n \quad \text{in } \Omega, \quad \text{with } U_j^n = 0 \quad \text{on } \partial\Omega, \quad \text{for } n \ge 1,$$

with $U_j^0 = v(x_j)$ as above. For U^{n-1} given one now needs to solve the linear system

$$(1+2\lambda)U_j^n - \lambda(U_{j-1}^n + U_{j+1}^n) = U_j^{n-1},\ 0 < j < M, \quad U_j^n = 0,\ j = 0, M.$$

This method is stable in maximum-norm without any restrictions on k and h. In fact, we find at once, for suitable k,

$$|U^n| = |U_k^n| \le \frac{2\lambda}{1+2\lambda}|U^n| + \frac{1}{1+2\lambda}|U^{n-1}|,$$

and hence $|U^n| \le |U^{n-1}| \le |V|$. Here, the error is of order $O(h^2 + k)$.

Although the backward Euler method is unconditionally stable, it is only first order accurate in time. A second order accurate method was proposed by Crank and Nicolson (1947) which uses the equation

$$\bar\partial_t U_j^n = \partial_x \bar\partial_x U_j^{n-1/2}, \quad 0 < j \le M, \quad \text{where } U_j^{n-1/2} = \tfrac{1}{2}(U_j^n + U_j^{n-1}). \tag{4.2}$$

The above approach will now show $|U^n| \le |U^{n-1}|$ only if $\lambda \le 1$.

For this problem, however, the energy method may be used to show unconditional stability in l_2-type norms: With $(V, W) = h\sum_{j=1}^{M-1} V_j W_j$ and $\|V\| = (V,V)^{1/2}$ one finds, upon multiplication of (4.2) by $2U_j^{n-1/2}$, summation, and summation by parts,

$$\bar\partial_t \|U^n\|^2 = 2(\bar\partial_t U^n, U^{n-1/2}) = -2h^{-2}\sum_{j=1}^{M}(U_j^{n-1/2} - U_{j-1}^{n-1/2})^2 \le 0,$$

which shows $\|U^n\| \le \|U^{n-1}\|$, i.e., stability in l_2 holds for any $\lambda > 0$. In the standard way this yields the error estimate $\|U^n - u(t_n)\| \le C(u)(h^2 + k^2)$; a corresponding estimate in a discrete H^1-norm may be obtained similarly, and this also yields a maximum-norm estimate of the same order by using a discrete Sobolev inequality. The energy approach was developed by, e.g., Kreiss (1959a), Lees (1960), and Samarskii (1961).

Stability in l_2 may also be deduced by spectral analysis, as observed in O'Brien, Hyman, and Kaplan (1951). We illustrate this for the Crank-Nicolson method: Representing the mesh-functions vanishing at $x = 0$ and 1 in terms of the eigenfunctions φ_p of $\partial_x\bar\partial_x$, the vectors with components $\varphi_{pj} = \sqrt{2}\sin \pi pjh,\ 0 \le j \le M$, and eigenvalues $\mu_p = -2h^{-2}(1 - \cos \pi ph)$, one finds

$$U^n = \sum_{p=1}^{M-1}(V, \varphi_p)E(\pi ph)^n \varphi_p \quad \text{where } E(\xi) = \frac{1 - \lambda(1-\cos\xi)}{1 + \lambda(1-\cos\xi)},$$

and $\|U^n\| \le \|V\|$ follows by Parseval's relation since $|E(\xi)| \le 1$ for $\xi \in R,\ \lambda > 0$. This is analogous to the Fourier analysis of Section 3 for pure initial value problems.

We remark that although the maximum-principle type argument for stability for the Crank-Nicolson method requires $\lambda \le 1$, it was shown by Serdjukova (1964) using Fourier analysis that the maximum-norm bound $|U^n| \le 23|V|$ holds. Precise convergence analyses in maximum-norm for initial data with low regularity were carried out in, e.g., Juncosa and Young (1957).

For the pure initial value problem, difference schemes or arbitrarily high order of accuracy can be constructed by including the appropriate number of terms in a difference operator of the form $U_j^{n+1} = \sum_{l=-q}^{s} a_l U_{j-l}^n$. For application to the mixed initial boundary value problem (4.1) such a formula would require additional equations for U_j^n near $x = 0$ or $x = 1$ when $s \ge 2$ or $q \ge 2$. For the semi-infinite interval $(0, \infty)$, Strang (1964) showed that with $s = 1$ any order of accuracy may be achieved together with stability by choosing an "unbalanced" operator with $q \ge 2$. The stability of schemes with additional boundary conditions has been analyzed in the parabolic case by Varah (1970) and Osher (1972) using the GKS-technique which we briefly describe for hyperbolic equations below.

We note that the above methods may be written as $U^{n+1} = E_k U^n$, where E_k acts in different spaces $\mathcal{N}_k$ of vectors $V = (V_0, \dots, V_M)^T$ with $V_0 = V_M = 0$, where M depends on k. In order to deal with the stability problem in such situations, Godunov and Ryabenkii (1963) introduced a concept of spectrum of a family of operators $\{E_k\}$, with E_k defined in a normed space $\mathcal{N}_k$ with norm $\|\cdot\|_k$, k small. The spectrum $\sigma(\{E_k\})$ is defined as the complex numbers z such that for any $\epsilon > 0$ and sufficiently small k there is a $U_k \in \mathcal{N}_k, U_k \ne 0$, such that $\|E_k U_k - z U_k\|_k \le \epsilon \|U_k\|_k$, and the following variant of von Neumann's criterion holds: If $\{E_k\}$ is stable in the sense that $\|E_k^n V\|_k \le C\|V\|_k$ for $t_n \le T$, with C independent of k, then $\sigma(\{E_k\}) \subset \{z; |z| \le 1\}$. It was demonstrated that the spectrum of a family such as one of the above is the union of three sets, one corresponding to the pure initial value problem and one to each of the one-sided boundary value problems for the differential equation in $\{x \ge 0, t \ge 0\}$ and in $\{x \le 1, t \ge 0\}$, with boundary conditions given at $x = 0$ and $x = 1$, respectively. For instance, in the example of the explicit method, $\sigma(\{E_k\})$ equals the set of eigenvalues of the operator $\tilde{E}_k$ associated with the pure initial-value problem, which is easily shown to be equal to the interval $[1 - 4\lambda, 1]$, and hence $\lambda \le 1/2$ is a necessary condition for stability. The proof of the equality between these sets is nontrivial as the eigenfunctions of $\tilde{E}_k$ do not satisfy the boundary conditions. Using instead a boundary condition of the form $u_0 - \gamma u_1 = 0$ at $x = 0$, will result in instability for certain choices of γ.

We now turn to the two-dimensional model problem in the square $\Omega = (0,1)^2$,

$$u_t = \Delta u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \quad \text{with } u(\cdot, 0) = v \quad \text{in } \Omega. \tag{4.3}$$

Again with $h = 1/M$ we use mesh-points $x_j = jh$, now with $j \in Z^2$. We consider the above methods collectively as the ϑ-method, with $0 \le \vartheta \le 1$,

$$\bar{\partial}_t U_j^n = \Delta_h(\vartheta U_j^n + (1 - \vartheta) U_j^{n-1}) \quad \text{in } \Omega, \quad U_j^n = 0 \quad \text{on } \partial\Omega,$$

where $\vartheta = 0, 1$, and $1/2$ for the forward and backward Euler methods and the Crank-Nicolson method, respectively. The above stability and error analysis carries over to this case; the ϑ-method is unconditionally stable in l_2 for $1/2 \le \vartheta \le 1$ whereas for $0 \le \vartheta < 1/2$ the mesh-ratio condition $\lambda(1 - 2\vartheta) \le 1/4$ has to be imposed.

For the model problem (4.3) we consider also the alternating direction implicit (ADI) scheme of Peaceman and Rachford (1955). Noting that the Crank-Nicolson scheme requires the solution at time t_n of the two-dimensional elliptic problem $(I - \frac{1}{2}k\Delta_h)U^n = (I + \frac{1}{2}k\Delta_h)U^{n-1}$, the purpose of the ADI method is to reduce the computational work by solving only one-dimensional problems. This is done by introducing the intermediate value $U^{n-1/2}$ for the solution at $t_{n-1/2} = t_n - k/2$ by the equations

$$\frac{U^{n-1/2} - U^{n-1}}{k/2} = \partial_1\bar{\partial}_1 U^{n-1/2} + \partial_2\bar{\partial}_2 U^{n-1}$$
$$\frac{U^{n} - U^{n-1/2}}{k/2} = \partial_1\bar{\partial}_1 U^{n-1/2} + \partial_2\bar{\partial}_2 U^{n}.$$

Elimination of $U^{n-1/2}$ gives, since the various operators commute,

$$U^n = E_k U^{n-1} = (I - \tfrac{k}{2}\partial_1\bar{\partial}_1)^{-1}(I + \tfrac{k}{2}\partial_1\bar{\partial}_1)(I - \tfrac{k}{2}\partial_2\bar{\partial}_2)^{-1}(I + \tfrac{k}{2}\partial_2\bar{\partial}_2)U^{n-1}.$$

By either energy arguments or by the spectral method one sees easily that this method is stable in the discrete l_2-norm and since it is second order accurate in both space and time one finds $\|U^n - u(t_n)\| \le C(u)(h^2 + k^2)$. We may also express the definition of E_k (using a different $U^{n-1/2}$) by

$$U^{n-j/2} = (I - \tfrac{k}{2}\partial_j\bar{\partial}_j)^{-1}(I + \tfrac{k}{2}\partial_j\bar{\partial}_j)U^{n-(j-1)/2}, \quad \text{for } j = 0, 1.$$

In this form, which generalizes in an obvious way to more dimensions, it is referred to as a fractional step method and depends on the splitting of the operator Δ_h into $\partial_1\bar{\partial}_1$ and $\partial_2\bar{\partial}_2$. This has been a very active area of research during the 60s, with contributions by, e.g., Douglas, Kellogg, Temam, Wachspress, Dyakonov, Samarskii, Marchuk, and Yanenko, see the survey article by Marchuk [25].

In the same way as for the elliptic problems studied in Section 2, complications arise when the boundary mesh-points do not fall exactly on $\partial\Omega$, which is the case in the presence of a curved boundary. Using, e.g., linear interpolation or Shortley-Weller type approximations of the Laplacian one may show $O(k+h^2)$ error estimates for the backward Euler method by means of a discrete maximum-principle, and one may also use energy arguments, with the crudest boundary approximation, to show $O(k + h^{1/2})$ error estimates, see Thomée [37].

We now turn to hyperbolic equations and consider first the spatially one-dimensional wave equation

$$u_{tt} = u_{xx} \quad \text{in } \Omega = (0,1), \quad \text{with } u(\cdot,0), u_t(\cdot,0) \text{ given.}$$

Here one may define the ϑ-method, $\vartheta \in [0,1]$, which is a special case of a family of schemes studied in Newmark (1959), by

$$\partial_t\bar{\partial}_t U_j^n = \partial_x\bar{\partial}_x\big((1-\vartheta)U_j^n + \tfrac{1}{2}\vartheta(U_j^{n+1} + U_j^{n-1})\big), \tag{4.4}$$

with $U_0^n = U_M^n = 0$ for $n \ge 0$ and U_j^0 and U_j^1 given. This scheme is unconditionally stable for $\frac{1}{2} \le \vartheta \le 1$, and for $0 \le \vartheta < \frac{1}{2}$ it is stable if $\lambda^2 = k^2/h^2 < 1/(1-2\vartheta)$; for

$\vartheta = 0$ we recognize the explicit scheme of Section 1, which can easily be shown to be unstable for $\lambda = 1$, see Raviart and Thomas [27].

As a simple first order hyperbolic model problem we consider, with $\rho > 0$,

$$u_t = \rho u_x \quad \text{in } \Omega = (0,1), \quad u(1,t) = 0, \quad \text{for } t > 0, \quad \text{with } u(\cdot,0) = v \quad \text{in } \Omega.$$

Note that since the solution is constant along the characteristics $x + \rho t =$ constant no boundary condition is needed at $x = 0$.

Consider first the "upwind" scheme, see Courant, Isaacson, and Rees (1952),

$$\partial_t U_j^n = \rho \partial_x U_j^n, \quad j = 0, \dots, M-1, \quad U_M^{n+1} = 0, \quad \text{for } n \geq 0, \quad \text{with } U_j^0 = v(x_j),$$

which may be written in explicit form as

$$U_j^{n+1} = (1 - \lambda\rho) U_j^n + \lambda\rho U_{j+1}^n, \quad 0 \leq j < M, \quad U_M^{n+1} = 0.$$

When $\lambda\rho \leq 1$ the method is stable in maximum-norm; this condition may be expressed by saying that the characteristic traced back from (x_j, t_{n+1}) cuts $t = t_n$ in $[x_j, x_{j+1}]$, which we recognize as the CFL condition. By the lack of symmetry it is only first order accurate. For $\rho\lambda > 1$ one can use instead the Carlson scheme (see Richtmyer and Morton [28])

$$\partial_t U_j^n = \rho \bar{\partial}_x U_j^{n+1}, \quad 1 \leq j \leq M, \quad U_M^{n+1} = 0, \quad \text{for } n \geq 0, \quad \text{with } U_j^0 = v(x_j),$$

which determines U_j^{n+1} for decreasing j by $U_{j-1}^{n+1} = (1 - (\rho\lambda)^{-1}) U_j^{n+1} + (\rho\lambda)^{-1} U_j^n$. Again this method is maximum-norm stable, but only first order accurate.

A second order method is the box scheme of Wendroff (1960),

$$\partial_t U_{j-1/2}^n = \rho \bar{\partial}_x U_j^{n+1/2}, \quad 1 \leq j \leq M, \quad U_M^{n+1} = 0, \quad U_{j-1/2} = \tfrac{1}{2}(U_j + U_{j-1}).$$

With U^n and U_M^{n+1} given this determines U_{j-1}^{n+1} for decreasing j as a combination of U_j^{n+1}, U_{j-1}^n, and U_j^n. Stability in l_2 may be shown by an energy argument.

We end this section with two examples where the finite difference operators used in the interior of Ω require modification near the boundary or additional artificial boundary conditions. In our first example we describe a special case of an energy argument proposed by Kreiss and Scherer (1972), see also [18]. We consider the initial-boundary value problem, with $\rho > 0$,

$$u_t - \rho u_x = f \quad \text{in } \Omega, \quad \text{with } u(1,t) = 0, \quad \text{for } t \geq 0, \quad u(\cdot,0) = v \quad \text{in } \Omega.$$

Assume that we want to apply the six-point Crank-Nicolson equation

$$\bar{\partial}_t U_j^n - \rho \hat{\partial}_x U_j^{n-1/2} = f_j^{n-1/2}, \quad \text{for } 1 \leq j \leq M-1. \tag{4.5}$$

At the right endpoint of Ω we set $U_M^n = 0$. For $x = 0$ the value of u is not given, and we therefore use the one-sided equation $\bar{\partial}_t U_0^n - \rho \partial_x U_0^{n-1/2} = f_0^{n-1/2}$. With the obvious definition of the composite difference operator D_x we may write $\bar{\partial}_t U_j^n - \rho D_x U_j^{n-1/2} = f_j^{n-1/2}$ for $0 \leq j < M$. Introducing temporarily the inner product $(U,V) = \frac{1}{2} h U_0 V_0 + h \sum_{j=1}^{M-1} U_j V_j$ we have $(D_x U, U) = -\frac{1}{2} U_0^2$ if $U_M = 0$,

which yields $(\bar\partial_t U^n, U^{n-1/2}) + \frac{1}{2}\rho(U_0^{n-1/2})^2 = (f^{n-1/2}, U^{n-1/2})$. Together with the inequality $hf_0U_0 \le \rho U_0^2 + Ch^2 f_0^2$ this easily shows the stability estimate

$$\|U^n\|^2 \le C\|U^0\|^2 + Ck\sum_{l=0}^{n-1}\|\tilde f^{l+1/2}\|^2, \quad \text{where } \tilde f = (hf_0, f_1, \cdots, f_{M-1}).$$

Note that the choice of the term with $j = 0$ in $(\cdot,\cdot)$ is essential for the argument.

Applying this to the error $U - u$, with the truncation error τ for f, and observing that $\tau_j = O(h^2)$ for $j \ge 1$, $\tau_0 = O(h)$, we find $\|U - u\| = O(h^2)$. This approach was also used in the reference quoted to construct higher order schemes. We note that the modification of (4.5) for $j = 0$ may also be interpreted as using (4.5) for $j = 0$, and adding the boundary condition $h^2\partial_x\bar\partial_x U_0^n = U_1^n - 2U_0^n + U_{-1}^n = 0$.

We finally give an example of the stability analysis based on the use of discrete Laplace transforms developed by Kreiss (1968), Gustafsson, Kreiss, and Sundström (1972), the so called GKS-theory. Consider the initial-boundary value problem, again with $\rho > 0$,

$$u_t = \rho u_x \quad \text{for } x > 0,\ t > 0, \quad \text{with } u(x,0) = v(x) \quad \text{for } x \ge 0.$$

Assume that we want to use the leapfrog scheme

$$\hat\partial_t U_j^n = \rho\hat\partial_x U_j^n \quad \text{for } j \ge 1, n \ge 1, \quad \text{with } U_j^0,\ U_j^1 \quad \text{given for } j \ge 0, \tag{4.6}$$

where we assume $\rho\lambda = \rho k/h < 1$ so that the corresponding scheme for the pure initial value problem is stable. Again an additional boundary condition is required for $j = 0$ in order to apply the equation at $j = 1$; following Strikwerda [35] we illustrate the theory by sketching the argument for stability in choosing the extrapolation $U_0^n = U_1^{n-1}$ for $n \ge 1$.

By subtracting a solution of the pure initial value problem one is reduced to assuming that $U_j^0 = U_j^1 = 0$ for $j \ge 0$, but then has to impose the inhomogeneous boundary condition $U_0^{n+1} = U_1^n + \beta^n$ for $n \ge 0$, and to show for this problem the stability estimate

$$\|U\|_\kappa \le C_\kappa|\beta|_\kappa, \quad \text{with } \|U\|_\kappa^2 = k\sum_{n=0}^{\infty} e^{-2\kappa t_n} h\sum_{j=0}^{\infty}|U_j^n|^2, \quad |\beta|_\kappa^2 = k\sum_{n=0}^{\infty} e^{-2\kappa t_n}|\beta^n|^2.$$

Note that κ is a parameter allowing for a certain exponential growth in time.

Applying discrete Laplace transforms in time to (4.6) one finds that the transformed solution $\tilde U_j(z)$ satisfies

$$(z - z^{-1})\tilde U_j = \lambda\rho(\tilde U_{j+1} - \tilde U_{j-1}), \quad \text{where } \tilde U(z) = k\sum_{n=0}^{\infty} z^{-n}U^n,$$

which is referred to as the resolvent equation. It is a second order difference equation in j, and provided the two roots $\tau_1(z), \tau_2(z)$ of its characteristic equation $\lambda\rho(\tau - \tau^{-1}) = z - z^{-1}$ are distinct, the general solution is $\tilde U_j(z) = c_1(z)\tau_1(z)^j + c_2(z)\tau_2(z)^j$. It follows from the stability of the scheme for the initial value problem that for $|z| > 1$, with the proper ordering, $|\tau_1(z)| < 1$ and $|\tau_2(z)| > 1$. In fact, if this

were not so and since $\tau_1(z)\tau_2(z) = 1$, we have $\tau_{1,2}(z) = e^{\pm i\xi}$ for some ξ, and z is therefore a solution of the characteristic equation $z - z^{-1} - 2i\lambda\rho\sin\xi = 0$ of the leapfrog scheme for the pure initial value problem. By von Neumann's condition we therefore have $|z| \leq 1$ which contradicts our assumption. Since we want $\tilde{U}_j$ to be in $l_2(Z_+)$ we must have $c_2(z) = 0$, and taking the Laplace transform also at $j = 0$, we find $c_1(z)(z - \tau_1(z)) = z\tilde{\beta}(z)$, and thus $\tilde{U}_j(z) = z\tilde{\beta}(z)\tau_1(z)^j/(z - \tau_1(z))$. With $z = e^{sk}$, $s = \kappa + i\eta$ we obtain using Parseval's relation

$$\|U\|_\kappa^2 = kh\sum_j \int_{-\pi/k}^{\pi/k} |\tilde{U}_j(z)|^2 d\eta = h\int_{-\pi/k}^{\pi/k} \frac{|z|^2\,|\tilde{\beta}(z)|^2}{|z - \tau_1(z)|^2(1 - |\tau_1(z)|^2)}\, d\eta.$$

By studying the behavior of $\tau_1(z)$ one may show that $|z - \tau_1(z)| \geq c$ and $1 - |\tau_1(z)|^2 \geq 1 - |\tau_1(z)| \geq c(|z| - 1) = c(e^{\kappa k} - 1) \geq c\kappa k$, with $c > 0$. Hence

$$\|U\|_\kappa^2 \leq Ch(k\kappa)^{-1}\int_{-\pi/k}^{\pi/k} |\tilde{\beta}(z)|^2\, d\eta = C(\lambda\kappa)^{-1}|\beta|_\kappa^2,$$

which shows that the method is stable. Using similar arguments it is possible to show that the alternative extrapolation defined by $U_0^n = U_1^n$ is unstable.

5. Finite element methods for elliptic problems.

In this section we summarize the basic definitions, properties, and successive development of the finite element method for elliptic problems. As a model problem we consider Dirichlet's problem for Poisson's equation in a domain $\Omega \subset R^d$,

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

The standard finite element method uses a variational formulation to define an approximate solution u_h of (5.1) in a finite-dimensional linear space S_h, normally consisting of continuous, piecewise polynomial functions on some partition of Ω: By Dirichlet's principle the solution u of (5.1) may be characterized as the function which minimizes $J(v) = \|\nabla v\|^2 - 2(f, v)$ over $H_0^1 = H_0^1(\Omega)$, where $(\cdot,\cdot)$ and $\|\cdot\|$ are the standard inner product and norm in $L_2 = L_2(\Omega)$. The Euler equation for this minimization problem is

$$(\nabla u, \nabla\varphi) = (f, \varphi), \quad \forall\varphi \in H_0^1; \tag{5.2}$$

this weak or variational form of (5.1) may also be derived by multiplying the elliptic equation in (5.1) by $\varphi \in H_0^1$, integrating over Ω, and applying Green's formula in the left-hand side. The standard finite element method assumes $S_h \subset H_0^1$ and defines the approximate solution u_h as the minimizer of $J(v)$ over S_h, or, equivalently,

$$(\nabla u_h, \nabla\chi) = (f, \chi), \quad \forall\chi \in S_h. \tag{5.3}$$

In terms of a basis $\{\Phi_j\}_{j=1}^{N_h}$ for S_h, our discrete problem (5.3) may be stated in matrix form as $A\alpha = \tilde{f}$, where A is the matrix with elements $a_{jk} = (\nabla\Phi_j, \nabla\Phi_k)$ (the stiffness matrix), $\tilde{f}$ the vector with entries $f_j = (f, \Phi_j)$, and α the vector of unknown coefficients α_j in $u_h = \sum_{j=1}^{N_h} \alpha_j\Phi_j$. Here A is a Gram matrix and thus,

in particular, positive definite and invertible, so that (5.3) has a unique solution. From (5.2) and (5.3) follows that $(\nabla(u_h - u), \nabla\chi) = 0$ for $\chi \in S_h$, that is, u_h is the orthogonal projection of u onto S_h with respect to the Dirichlet inner product $(\nabla v, \nabla w)$.

We recall that defining u_h as the minimizer of $J(\chi)$ is referred to as the Ritz method, and using instead (5.3), which is suitable also for nonsymmetric differential equations, as Galerkin's method. Some further historical remarks are collected in the introduction to this paper.

For the purpose of error analysis we briefly consider the approximation in S_h of smooth functions in Ω which vanish on $\partial\Omega$. We first exemplify by the Courant elements in a convex plane domain Ω. For such a domain, let $\mathcal{T}_h$ denote a partition into disjoint triangles τ such that no vertex of any triangle lies on the interior of a side of another triangle and such that the union of the triangles determine a polygonal domain $\Omega_h \subset \Omega$ with boundary vertices on $\partial\Omega$. Let h denote the maximal length of the sides of the triangles of $\mathcal{T}_h$, and assume that the angles of the $\mathcal{T}_h$ are bounded below by a positive constant, independently of h. Let now S_h denote the continuous functions on the closure $\bar{\Omega}$ of Ω which are linear in each triangle of $\mathcal{T}_h$ and which vanish outside Ω_h. With $\{P_j\}_{j=1}^{N_h}$ the interior vertices of $\mathcal{T}_h$, a function in S_h is then uniquely determined by its values at the points P_j and thus $\dim(S_h) = N_h$. Let Φ_j be the "pyramid" function in S_h which takes the value 1 at P_j but vanishes at the other vertices; these functions form a basis for S_h. A given smooth function v on Ω which vanishes on $\partial\Omega$ may now be approximated by, e.g., its interpolant $I_h v = \sum_{j=1}^{N_h} v(P_j)\Phi_j \in S_h$, which agrees with v at the interior vertices, and one may show

$$(5.4) \quad \|I_h v - v\| \le Ch^2\|v\|_2 \quad \text{and} \quad \|\nabla(I_h v - v)\| \le Ch\|v\|_2, \quad \text{for } v \in H^2 \cap H_0^1,$$

where $\|\cdot\|_r$ denotes the norm in the Sobolev space $H^r = H^r(\Omega)$.

More generally we consider the case when $\Omega \subset R^d$ and $\{S_h\}$ is a family of finite-dimensional subspaces of H_0^1 such that, for some integer $r \ge 2$,

$$(5.5) \quad \inf_{\chi \in S_h} \{\|v - \chi\| + h\|\nabla(v - \chi)\|\} \le Ch^s\|v\|_s, \quad \text{for } 1 \le s \le r, \quad v \in H^s \cap H_0^1.$$

The spaces S_h are thought of as consisting of piecewise polynomials of degree at most $r - 1$ on a partition $\mathcal{T}_h$ of Ω, and the bound (5.5) shown by exhibiting a $\chi = I_h u$ where $I_h : H^r \cap H_0^1 \to S_h$ is an interpolation type operator, see Zlámal (1968). The proof often involves the lemma of Bramble and Hilbert (1970):

Let $D \subset R^d$ and assume that F is a bounded linear functional on $H^r(D)$ which vanishes for all polynomials of degree $< r$. Then $|F(u)| \le C \sum_{|\alpha|=r} \|D^\alpha u\|_{L_2(D)}$.

To use this to show (5.4), e.g., one considers the difference $I_h u - u$ on an individual $\tau \in \mathcal{T}_h$, transforms this to a unit size reference triangle $\hat{\tau}$, invokes the Bramble-Hilbert lemma with $D = \hat{\tau}$, noting that $I_h u - u$ vanishes for linear functions, and transforms back to τ, using the fact that the bound for $|F(u)|$ in the lemma only contains the highest order derivatives. In this example $\Omega_h \ne \Omega$ but the width of $\Omega \setminus \Omega_h$ is of order $O(h^2)$ and the contribution from this set is bounded appropriately. When $\partial\Omega$ is curved and $r > 2$, however, there are difficulties in the construction and analysis of such operators I_h near the boundary; we shall return to this problem below. When Ω is polygonal and $\Omega_h = \Omega$, the Bramble-Hilbert argument for (5.5)

may be used also for $r > 2$, but in this case the solution of (5.1) will not normally have the regularity required. For comprehensive accounts of various choices of partitions $\mathcal{T}_h$ and finite element spaces S_h we refer to, e.g., Ciarlet [11] and Brenner and Scott [8].

We return to the finite element equation (5.3) using Courant elements. One way of triangulating $\Omega \subset R^2$ is to start with the three families of straight lines $x_1 = lh,\ x_2 = lh,\ x_1 + x_2 = lh, l \in Z$. The triangles thus formed may be used in the interior of Ω and then supplemented by other triangles near $\partial\Omega$ to form a triangulation $\mathcal{T}_h$ with the desired properties. With the notation (1.2) the equation corresponding to an interior vertex $x_j = jh$, $j \in Z^2$ then takes the form

$$-\partial_1\bar{\partial}_1 U_j - \partial_2\bar{\partial}_2 U_j = h^{-2}(f, \Phi_j), \quad \text{where } U_j = u_h(x_j). \tag{5.6}$$

We recognize this as essentially the five-point finite difference equation (2.2), but with the right hand side $f_j = f(x_j)$ replaced by an average of f over a neighborhood of x_j. Taking $f(x_j)$ may be considered as a quadrature rule for the right hand side of (5.6). Recall that such averages were proposed also for finite difference methods.

Whereas a finite difference method may be obtained by replacing derivatives by finite differences, with some *ad hoc* modification near the boundary, the basic finite element method thus uses a variational formulation in a way that automatically accomodates the boundary conditions. We recall that the error analysis for the finite difference method uses a local estimate for the truncation error, together with some stability property, such as a discrete maximum-principle. The finite element error analysis, as we shall now see, is based directly on the variational formulation and is global in nature. The difficulties in the construction of finite difference equations near the boundary are even greater for Neumann type boundary conditions, whereas in the variational approach these are natural boundary conditions which do not have to be imposed on the approximating functions.

Under assumption (5.5) we now demonstrate the optimal order error estimate

$$\|u_h - u\| + h\|\nabla(u_h - u)\| \le Ch^s\|u\|_s, \quad \text{for } 1 \le s \le r. \tag{5.7}$$

Starting with the error in the gradient we note that since u_h is the orthogonal projection of u onto S_h with respect to $(\nabla v, \nabla w)$, we have, by (5.5),

$$\|\nabla(u_h - u)\| = \inf_{\chi \in S_h} \|\nabla(\chi - u)\| \le Ch^{s-1}\|u\|_s, \quad \text{for } 1 \le s \le r; \tag{5.8}$$

for linear finite elements this was observed in Oganesjan (1963). For the L_2-error we apply a duality argument by Aubin (1967) and Nitsche (1968): Let φ be arbitrary in L_2, take $\psi \in H^2 \cap H_0^1$ as the solution of

$$-\Delta\psi = \varphi \quad \text{in } \Omega, \quad \text{with } \psi = 0 \quad \text{on } \partial\Omega, \tag{5.9}$$

and recall the elliptic regularity inequality $\|\psi\|_2 \le C\|\Delta\psi\| = C\|\varphi\|$. We then have for the error $e = u_h - u$, for any $\chi \in S_h$,

$$(e, \varphi) = -(e, \Delta\psi) = (\nabla e, \nabla\psi) = (\nabla e, \nabla(\psi - \chi)) \le \|\nabla e\|\,\|\nabla(\psi - \chi)\|, \tag{5.10}$$

and hence, using (5.8) and (5.5) with $s = 2$, the desired result follows from

$$(e, \varphi) \le (Ch^{s-1}\|u\|_s)\,(Ch\|\psi\|_2) \le Ch^s\|u\|_s\|\varphi\|.$$

In the case of a more general, not necessarily symmetric, elliptic equation, and an approximation by Galerkins method, the estimate for the gradient may be obtained by application with $V = H_0^1$ of the lemma of Céa (1964):

Let V be a Hilbert space with norm $|\cdot|$ and let $A(u,v)$ be a continuous bilinear form on V such that $|A(u,v)| \le M\,|u|\,|v|$ and $A(u,u) \ge \mu|u|^2$, $\mu > 0$. For F a continuous linear functional on V, consider the equation

$$A(u,\varphi) = F(\varphi), \quad \forall \varphi \in V. \tag{5.11}$$

Let $S_h \subset V$ and let $u_h \in S_h$ be the solution of $A(u_h,\chi) = F(\chi)$ for $\chi \in S_h$. Then $|u_h - u| \le M\mu^{-1} \inf_{\chi\in S_h} |\chi - u|$.

Since $A(u_h - u, \chi) = 0$ for $\chi \in S_h$ this follows at once from

$$\mu|u_h - u|^2 \le A(u_h - u, u_h - u) = A(u_h - u, \chi - u) \le M|u_h - u|\,|\chi - u|.$$

Note that the problem (5.11) has a unique solution in V by the Lax-Milgram lemma.

We remark that the finite element error estimate for, e.g., the Courant elements, will require the solution to have two derivatives, whereas four derivatives were needed in the five-point finite difference method. This advantage of finite elements stems from the use of averages and disappears when a quadrature rule is used.

The error analysis given above assumed the approximation property (5.5) for some $r \ge 2$. The most natural example of such a family in a plane domain Ω would be to take for S_h the continuous piecewise polynomials of degree at most $r-1$ on a triangulation $\mathcal{T}_h$ of Ω of the type described above, which vanish on $\partial\Omega$. However, for $r > 2$ and in the case of a domain with curved boundary, it is then not possible, in general, to satisfy the homogeneous boundary conditions exactly, and the above analysis therefore does not apply. One method to deal with this difficulty is to consider elements near $\partial\Omega$ that are polynomial maps of a reference triangle $\hat{\tau}$, so called isoparametric elements, such that these elements define a domain Ω_h which well approximates Ω, and to use the corresponding maps of polynomials on $\hat{\tau}$ as approximating functions. Such finite element spaces were proposed by Argyris and by Fried, Ergatoudis, Irons, and Zienkiewicz, and Felipa and Clough, and analyzed in, e.g., Ciarlet and Raviart (1972), and other types of curved finite elements were considered by, e.g., Zlámal and Scott, see Ciarlet [11].

Another example of how to deal with the boundary condition is provided by the following method proposed by Nitsche (1971), again in a plane domain Ω. It uses a family $\mathcal{T}_h$ of triangulations which is quasi-uniform in the sense that $\text{area}(\tau) \ge ch^2$ for $\tau \in \mathcal{T}_h$, with $c > 0$ independent of h. In this case certain inverse inequalities hold, such as $\|\nabla\chi\| \le Ch^{-1}\|\chi\|$ for $\chi \in S_h$; this follows at once from the corresponding result for each $\tau \in \mathcal{T}_h$, for which it is shown by transformation to a fixed reference triangle and using the fact that all norms on a finite dimensional space are equivalent, see, e.g., Ciarlet, loc. cit. With $\langle\cdot,\cdot\rangle$ the L_2-inner product on $\partial\Omega$, the solution of (5.1) satisfies, for $\chi \in S_h$,

$$N_\gamma(u,\chi) := (\nabla u, \nabla\chi) - \langle\frac{\partial u}{\partial n}, \chi\rangle - \langle u, \frac{\partial\chi}{\partial n}\rangle + \gamma h^{-1}\langle u,\chi\rangle = -(\Delta u, \chi) = (f,\chi).$$

Using inverse and trace inequalities, the bilinear form $N_\gamma(\cdot,\cdot)$ is seen to be positive definite on S_h for γ fixed and sufficiently large, and we may therefore pose the discrete problem $N_\gamma(u_h,\chi) = (f,\chi)$ for $\chi \in S_h$. Nitsche showed

$$\|u_h - u\| + h\|\nabla(u_h - u)\| + h^{1/2}\|u_h\|_{L_2(\partial\Omega)} \le Ch^r\|u\|_r.$$

The bound for the third term expresses that u_h almost vanishes on $\partial\Omega$.

Other examples of methods used to deal with curved boundaries for which $S_h \not\subset H_0^1$ include a method of Babuška (1973) with Lagrangian multipliers, the method of interpolated boundary conditions by Berger, Scott, and Strang (1972), Scott (1975), and an approach by Bramble, Dupont, and Thomée (1972) and Dupont (1974) where the finite element method is based on an approximating polygonal domain with a correction built into the boundary conditions.

In some situations one may want to use finite element spaces S_h defined by piecewise polynomial approximating functions on a partition $\mathcal{T}_h$ of Ω which are not continuous across interelement boundaries, so called nonconforming elements. Assuming Ω polygonal so that it is exactly a union of elements τ, one may introduce a discrete bilinear form by $D_h(\psi,\chi) = \sum_{\tau\in\mathcal{T}_h}(\nabla\psi,\nabla\chi)_\tau$. Provided S_h is such that $\|\chi\|_{1,h} = D_h(\chi,\chi)^{1/2}$ is a norm on S_h, a unique nonconforming finite element solution u_h of (5.1) is now defined by $D_h(u_h,\chi) = (f,\chi)$ for $\chi \in S_h$, and it was shown in Strang (1972) that

$$\|u_h - u\|_{1,h} \le C \inf_{\chi\in S_h} \|u-\chi\|_{1,h} + C \sup_{\chi\in S_h} \frac{|D_h(u,\chi)-(f,\chi)|}{\|\chi\|_{1,h}}. \tag{5.12}$$

As an example, consider an axes parallel rectangular domain, partitioned into smaller such rectangles with longest edge $\le h$, and let S_h be piecewise quadratics which are continuous at the corners of the partition. Then $\|\cdot\|_{1.h}$ is a norm on S_h. In Wilson's rectangle, the six parameters involved on each small rectangle are determined by the values at the corners plus the (constant) values of $\partial^2\chi/\partial x_l^2,\ l = 1,2$. The functions in S_h are not in $C(\bar\Omega)$ but using (5.12) one may still show $\|u_h - u\|_{1,h} \le C(u)h$.

The analysis above assumes that all inner products are calculated exactly. An analysis where quadrature errors are permitted was also worked out by Strang (1972). For instance, if (f,χ) is replaced by a quadrature formula $(f,\chi)_h$, a term of the form $C\sup_{\chi\in S_h}|(f,\chi)-(f,\chi)_h|/\|\nabla\chi\|$ has to be added to the bound for $\|\nabla(u_h-u)\|$. For example, if the quadrature formula is exact on each element for constants and if $f \in W_q^1(\Omega)$ with $q > 2$, then the $O(h)$ error for $\|\nabla(u_h-u)\|$ is maintained. The situations when curved boundaries, nonconforming elements, or quadrature errors occur, so that the basic assumptions of the variational formulation are not satisfied, are referred to in Strang, loc. cit., as variational crimes.

Because of the variational formulation of Galerkin's method, the natural error estimates are expressed in L_2-based norms. In the maximum-norm it was shown by Natterer (1975), Nitsche (1975), and Scott (1976), see Schatz and Wahlbin (1982), that, for piecewise linear approximating functions on a quasi-uniform family $\mathcal{T}_h$ in a plane domain Ω, we have

$$\|u_h - u\|_{L_\infty} \le Ch^2\log(1/h)\|u\|_{W_\infty^2}, \qquad \|\nabla(u_h-u)\|_{L_\infty} \le Ch\|u\|_{W_\infty^2}.$$

For polygonal domains and with piecewise polynomials of degree $r-1 > 1$,

$$\|u_h - u\|_{L_\infty} + h\|\nabla(u_h-u)\|_{L_\infty} \le Ch^r\|u\|_{W_\infty^r},$$

but Haverkamp (1984) has proved that the above factor $\log(1/h)$ for piecewise linears may not be removed, even though it is not needed when estimating $I_hu - u$.

We shall now consider a finite element method for our model problem (5.1) which is based on a so called mixed formulation of this problem. Here the gradient of the solution u is introduced as a separate dependent variable whose approximation is sought in a different finite element space than the solution itself. This may be done in such a way that ∇u may be approximated to the same order of accuracy as u. With ∇u as a separate variable, (5.1) may thus be formulated

$$(5.13) \qquad -\operatorname{div}\sigma = f \quad \text{in } \Omega, \quad \sigma = \nabla u \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega.$$

With $H = \{\omega = (\omega_1, \omega_2) \in L_2 \times L_2;\ \operatorname{div}\omega \in L_2\}$ we note that the solution $(u, \sigma) \in L_2 \times H$ also solves the variational problem

$$(5.14) \quad (\operatorname{div}\sigma, \varphi) + (f, \varphi) = 0, \quad \forall \varphi \in L_2, \qquad (\sigma, \omega) + (u, \operatorname{div}\omega) = 0, \quad \forall \omega \in H,$$

where the $(\cdot, \cdot)$ denote the appropriate L_2 inner products, and a smooth solution of (5.14) satisfies (5.13). Setting $L(v, \mu) = \frac{1}{2}\|\mu\|^2 + (\operatorname{div}\mu + f, v)$ the solution (u, σ) of (5.13) may also be characterized as the saddle-point satisfying

$$(5.15) \qquad L(v, \sigma) \le L(u, \sigma) \le L(u, \mu), \quad \forall v \in L_2,\ \mu \in H,$$

and the key to the existence of a solution is the inequality

$$(5.16) \qquad \inf_{v \in L_2} \sup_{\mu \in H} \frac{(v, \operatorname{div}\mu)}{\|v\|\,\|\mu\|_H} \ge c > 0, \quad \text{where } \|\mu\|_H^2 = \|\mu\|^2 + \|\operatorname{div}\mu\|^2.$$

With S_h and H_h certain finite-dimensional subspaces of L_2 and H we shall consider the discrete analogue of (5.14) to find $(u_h, \sigma_h) \in S_h \times H_h$ such that

$$(5.17) \quad (\operatorname{div}\sigma_h, \chi) + (f, \chi) = 0,\ \forall \chi \in S_h, \quad (\sigma_h, \psi) + (u_h, \operatorname{div}\psi) = 0, \quad \forall \psi \in H_h.$$

As in the continuous case this problem is equivalent to the discrete analogue of the saddle point problem (5.15), and in order for this discrete problem to have a solution with the desired properties the choice of combinations $S_h \times H_h$ has to be such that the analogue of (5.16) holds, in this context referred to as the Babuška-Brezzi *inf-sup* condition (Babuška (1971), Brezzi (1974).

One family of pairs of spaces which satisfy the *inf-sup* condition was introduced in Raviart and Thomas (1977); the first order accurate pair of this family is as follows: With $\mathcal{T}_h$ a quasi-uniform family of triangulation of Ω, which we assume here to be polygonal, we set $S_h = \{\chi \in L_2; \chi|_\tau \text{ linear }, \forall \tau \in \mathcal{T}_h\}$, with no continuity required across inter-element boundaries. We then define $H_h = \{\psi = (\psi_1, \psi_2) \in H; \psi|_\tau \in H(\tau), \quad \forall \tau \in \mathcal{T}_h\}$, where $H(\tau)$ denotes affine maps of quadratics on a reference triangle $\hat{\tau}$ of the form $(l_1(\xi) + \alpha\xi_1(\xi_1 + \xi_2), l_2(\xi) + \beta\xi_2(\xi_1 + \xi_2))$, with $l_1(\xi), l_2(\xi)$ linear, $\alpha, \beta \in R$. This space thus consists of piecewise quadratics on the triangulation $\mathcal{T}_h$ which are of the specific form implied by the definition of $H(\tau)$, and $\dim H(\tau) = 8$. As degrees of freedom for H_h one may use the values of $\psi \cdot n$ at two points on each side of τ (6 conditions) and in addition the mean-values of ψ_1 and ψ_2 over τ (2 conditions). We note that the condition $\psi \in H$ in the definition of H_h requires that $\operatorname{div}\psi \in L_2$, which is equivalent to the continuity of $\chi \cdot n$ across inter-element boundaries. For the solutions of (5.17) and (5.13) holds

$$\|u_h - u\| \le Ch^2\|u\|_2 \quad \text{and} \quad \|\sigma_h - \sigma\| \le Ch^s\|u\|_{s+1}, \quad s = 1, 2,$$

and correspondingly higher order estimates were derived for higher order Raviart-Thomas elements.

We now turn to negative norm estimates and superconvergence. Recalling the error estimate (5.7) which holds for the model problem under the approximation assumption (5.5), we shall now see that for $r > 2$, the duality argument used to show the L_2-norm estimate yields an error estimate in a negative order norm. Introducing such negative norms by $\|v\|_{-s} = \sup_{\varphi \in H^s} (v,\varphi)/\|\varphi\|_s$ for $s \geq 0$, the error in u_h satisfies

$$\|u_h - u\|_{-s} \leq Ch^{q+s}\|u\|_q, \quad \text{for } 0 \leq s \leq r-2,\ 1 \leq q \leq r. \tag{5.18}$$

In particular, $\|u_h - u\|_{-(r-2)} \leq Ch^{2r-2}\|u\|_r$. Since $2r-2 > r$ for $r > 2$ the power of h in this estimate is higher than in the standard $O(h^r)$ error estimate in the L_2-norm. To show (5.18), we use the solution ψ of (5.9) and recall that $\|\psi\|_{s+2} \leq C\|\varphi\|_s$. This time (5.10) yields, for $0 \leq s \leq r-2$,

$$|(e,\varphi)| \leq \|\nabla e\| \inf_{\chi \in S_h} \|\nabla(\psi - \chi)\| \leq \|\nabla e\|\,(Ch^{s+1}\|\psi\|_{s+2}) \leq Ch^{s+1}\|\nabla e\|\,\|\varphi\|_s.$$

By (5.8) this gives $|(e,\varphi)| \leq Ch^{q+s}\|u\|_q\|\varphi\|_s$ for $\varphi \in H^s$, which shows (5.18). As an application of (5.18), assume we want to evaluate the integral $F(u) = \int_\Omega u\,\psi\,dx = (u,\psi)$, where u is the solution of (5.1) and $\psi \in H^{r-2}$. Then for the obvious approximation $F(u_h) = (u_h,\psi)$ we find the superconvergent order error estimate

$$|F(u_h) - F(u)| = |(u_h - u,\psi)| \leq \|u_h - u\|_{-(r-2)}\|\psi\|_{r-2} \leq Ch^{2r-2}\|u\|_r\,\|\psi\|_{r-2}.$$

One more example of these ideas is provided by Douglas and Dupont (1974), which concerns superconvergent nodal approximation in the two-point boundary value problem

$$Au = -\frac{d}{dx}(a\frac{du}{dx}) + a_0 u = f \quad \text{in } I = (0,1), \quad \text{with } u(0) = u(1) = 0. \tag{5.19}$$

Defining the partition $0 = x_0 < x_1 < \cdots < x_M = 1$, with $x_{i+1} - x_i \leq h$, we set

$$S_h = \{\chi \in C(\bar{I});\ \chi|_{I_i} \in \Pi_{r-1},\ 1 \leq i \leq M;\ \chi(0) = \chi(1) = 0\},$$

where $I_i = (x_{i-1}, x_i)$. Clearly this family satisfies our assumption (5.5). The finite element solution is now defined by $A(u_h,\chi) = (f,\chi)$ for $\chi \in S_h$, where $A(v,w) = (av_x, w_x) + (a_0 v, w)$, and the error estimate (5.18) holds.

Let $g = g^{\bar{x}}$ denote the Green's function of the two-point boundary value problem (5.19) with singularity at the partition point $\bar{x}$, which we now consider fixed, so that $w(\bar{x}) = A(w,g)$ for $w \in H_0^1 = H_0^1(I)$. Applied to the error $e = u_h - u$, and using the orthogonality of e to S_h with respect to $A(\cdot,\cdot)$, we find $e(\bar{x}) = A(e,g) = A(e, g-\chi)$ for $\chi \in S_h$, and hence that

$$|e(\bar{x})| \leq C\|e\|_1 \inf_{\chi \in S_h} \|g - \chi\|_1 \leq Ch^{r-1}\|u\|_r \inf_{\chi \in S_h} \|g - \chi\|_1.$$

Although $g^{\bar{x}}$ is not a smooth function at $\bar{x}$ it may still be approximated well by a function in S_h since it is smooth except at $\bar{x}$ and the discontinuity of the derivative at $\bar{x}$ can be accommodated in S_h. In particular, we have

$$\inf_{\chi \in S_h} \|g - \chi\|_1 \leq Ch^{r-1}(\|g\|_{H^r((0,\bar{x}))} + \|g\|_{H^r((\bar{x},1))}) \leq Ch^{r-1},$$

so that $|e(\bar{x})| \leq Ch^{2r-2}\|u\|_r$. Note that for $A = -d^2/dx^2$ the Green's function $g^{\bar{x}}$ is linear outside $\bar{x}$ and so $g^{\bar{x}} \in S_h$. We may then conclude that $e(\bar{x}) = 0$, which is a degenerate case.

We now touch on some superconvergent order estimates for the gradient in the two-dimensional model problem (5.1) using piecewise linear approximations for S_h in (5.3). It was shown in Oganesjan and Ruhovec (1969) that if the triangulations $\mathcal{T}_h$ are uniform then $\|\nabla(u_h - I_h u)\|_{L_2(\Omega_h)} \leq Ch^2\|u\|_{H^3}$, where as above I_h denotes the interpolant into S_h. This implies that at the midpoints of the edges of $\mathcal{T}_h$ the average of ∇u_h from the two adjacent triangles is a $O(h^2)$ approximation to ∇u in a discrete l_2 sense. Such results have been improved to maximum-norm estimates and to triangulations that are perturbations in various ways of uniform triangulations by C.M. Chen, Q. Lin, J.C. Xu, Zhou, Zhu, and others, and the approximation at other points than midpoints of edges has also been studied, see, e.g., references in Križek and Neittaanmäki [22] or Wahlbin [41]. We remark that for uniform, axes parallel triangulations it follows from (5.6) that finite differences may be used as in (2.8) to approximate both the gradient and higher order derivatives to order $O(h^2)$ in the interior of Ω.

All error estimates quoted above are *a priori* error estimates in that they depend on certain norms of the exact solution of the problem. In principle, these norms could be bounded in terms of norms of the data of the problem, but generally such bounds would be rather crude. During the last decades so called *a posteriori* error estimates have been developed which depend directly on the computed solution, and on the data. Such estimates may be applied to include an adaptive aspect in the solution method, by detecting areas in a computational domain where the error is larger than elsewhere, and using this information to refine the mesh locally to reduce the error by an additional computation. Pioneering work is Babuška (1976) and Babuška and Rheinboldt (1978); for a recent survey, see Verfürth [40].

We illustrate this approach for the two-dimensional problem (5.1) in a polygonal domain Ω, using piecewise linear finite element approximations. With $\{\Phi_j\}$ the basis of pyramid functions we define Ω_j by $\bar{\Omega}_j = \operatorname{supp}\Phi_j$. Given the finite element solution $u_h \in S_h$ of (5.3), we now consider the local error equation

$$-\Delta w_j = f \quad \text{in } \Omega_j, \quad \text{with } w_j = u_h \quad \text{on } \partial\Omega_j.$$

It is then proved in Babuška and Rheinboldt, loc cit., that, with c and C positive constants which depend on geometrical properties of the triangulations $\mathcal{T}_h$,

$$c\sum_j \eta_j^2 \leq \|\nabla(u_h - u)\|^2 \leq C\sum_j \eta_j^2, \quad \text{where } \eta_j = \|\nabla(w_j - u_h)\|.$$

The error in ∇u_h is thus bounded both above and below in terms of the local quantities η_j, which can be approximately determined. It is argued that a triangulation for which the quantities η_j are of essentially the same size gives a small error in ∇u_h, and this therefore suggests an adaptive strategy for the solution of (5.1).

Another approach was taken in Eriksson and Johnson (1991), showing an *a posteriori* error estimate of the form

$$\|\nabla(u_h - u)\| \leq C\big((\sum_\tau h_\tau^2\|f\|^2_{L_2(\tau)})^{1/2} + (\sum_\gamma h_\gamma^2|[\frac{\partial u_h}{\partial n}]_\gamma|)^{1/2}\big), \tag{5.20}$$

where the γ with length h_γ are the edges of $\mathcal{T}_h$ and $[\cdot]_\gamma$ denotes the jump across γ. Under a certain assumption on the local variation of h_τ, which is weaker than quasi-uniformity the *a priori* estimate

$$\|\nabla(u_h - u)\| \le C\Big(\sum_\tau h_\tau^2 \|u\|^2_{H^2(\tau)}\Big)^{1/2} \tag{5.21}$$

is also derived. Together with (5.20) this may be used to justify an adaptive scheme with a given tolerance and an essentially minimal number of triangles. Analogous *a posteriori* and *a priori* bounds are demonstrated for $\|u_h - u\|$ and, in Eriksson (1994), also in maximum-norm, where the analogue of (5.21) reads $\|\nabla(u_h - u)\|_{L_\infty} \le C \max_\tau (h_\tau \|u\|_{W^2_\infty(\tau)})$.

Superconvergence of the error in the gradient has been used in, e.g., Zienkiewicz and Zhu (1992) to derive *a posteriori* error bounds for adaptive purposes.

We finally mention the p- and h-p-versions of the finite element method: So far it has been assumed that the approximating subspaces S_h are piecewise polynomial spaces of a fixed degree based on partitions $\mathcal{T}_h$ with $\max_{\tau\in\mathcal{T}_h} \operatorname{diam}(\tau) \le h$, and higher accuracy is achieved by refining the partition. An alternative approach proposed in Babuška, Szabó, and Katz (1981) is to fix the mesh and then let the degree of the polynomials grow. The two approaches are referred to as the h-version and the p-version of the finite element method, respectively. A combination of the two methods, the h-p-version has been studied in Babuška and Dorr (1981). For more material about the p- and h-p-method, see Babuška and Suri (1990).

6. Finite element methods for evolution equations.

This section is concerned with the application of the finite element method to time dependent problems. We begin with the model heat equation and discuss then the wave equation and finally some simple first order hyperbolic model problems.

We consider thus first the approximate solution of the parabolic problem

$$u_t - \Delta u = f(t) \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega,\ t > 0, \quad u(\cdot, 0) = v \quad \text{in } \Omega, \tag{6.1}$$

in a finite dimensional space S_h belonging to a family satisfying (5.5). As a first step we discretize this problem in the spatial variable by writing it in variational form and defining $u_h = u_h(\cdot, t) \in S_h$ for $t \ge 0$ by

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla\chi) = (f(t), \chi), \quad \forall \chi \in S_h,\ t > 0, \quad u_h(0) = v_h \approx v. \tag{6.2}$$

With respect to a basis $\{\Phi_j\}_{j=1}^{N_h}$ of S_h this may be written as a system of ordinary differential equations $B\alpha' + A\alpha = \tilde{f}$ where A is the stiffness matrix introduced in Section 5 and where $B = (b_{jk})$, $b_{jk} = (\Phi_j, \Phi_k)$, is referred to as the mass matrix. A fully discrete time stepping scheme may then be obtained by discretization of this system in time, using, e.g., the single step ϑ-method: With k the time step, $t_n = nk$, $\bar\partial_t U^n = (U^n - U^{n-1})/k$, and with $\vartheta \in [0, 1]$, the approximation $U^n \in S_h$ of $u(t_n)$ for $n \ge 1$ is then defined by

$$(\bar\partial_t U^n, \chi) + (\nabla(\vartheta U^n + (1-\vartheta)U^{n-1}), \nabla\chi) = (f^{n-1+\vartheta}, \chi), \quad \forall \chi \in S_h,\ n \ge 1, \tag{6.3}$$

with $f^s = f(sk)$ and U^0 given. For $\vartheta = 0$ and 1 these are the forward and backward Euler methods and for $\vartheta = 1/2$ the Crank-Nicolson method. Note that the forward Euler method is not explicit because the matrix B is nondiagonal.

For the semidiscrete problem (6.2) Douglas and Dupont (1970) showed that

$$\|u_h(t) - u(t)\| + \Big(\int_0^t \|\nabla(u_h - u)\|^2 ds\Big)^{1/2} \le \|v_h - v\| + C(u)h^{r-1}. \tag{6.4}$$

For a proof we note that the error $e = u_h - u$ satisfies $(e_t, \chi) + (\nabla e, \nabla\chi) = 0$ for $\chi \in S_h, t > 0$, and hence $(e_t, e) + (\nabla e, \nabla e) = (e_t, \chi - u) + (\nabla e, \nabla(\chi - u))$, from which the result follows after integration and with the appropriate choice of χ. Because of the contribution from $\|\nabla(\chi - u)\|$ on the right, (6.4) is of suboptimal order in L_2-norm. In this regard the estimate was improved by Wheeler (1973) to

$$\|u_h(t) - u(t)\| \le \|v_h - v\| + Ch^r\Big(\|v\|_r + \int_0^t \|u_t\|_r ds\Big), \quad \text{for } t > 0. \tag{6.5}$$

This was done by introducing the elliptic or Ritz projection, the orthogonal projection $R_h : H_0^1 \to S_h$ with respect to the Dirichlet inner product, thus defined by $(\nabla(R_h u - u), \nabla\chi) = 0$ for $\chi \in S_h$, and writing $e = (u_h - R_h u) + (R_h u - u) = \theta + \rho$. Here, by the error estimate (5.7) for the elliptic problem, $\|\rho(t)\| \le Ch^r \|u(t)\|_r$, which is bounded as desired, and one also finds $(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = -(\rho_t, \chi)$ for $\chi \in S_h,\ t > 0$. Choosing $\chi = \theta$ and integrating this yields $\|\theta(t)\| \le \|\theta(0)\| + \int_0^t \|\rho_t\|\, ds$ which is easily bounded as desired. In particular, for $v_h \in S_h$ suitably chosen, this shows an optimal order error estimate in L_2.

Defining the discrete Laplacian $\Delta_h : S_h \to S_h$ by $-(\Delta_h\psi, \chi) = (\nabla\psi, \nabla\chi)\ \forall \psi, \chi \in S_h$ and using the L_2-projection P_h onto S_h the above equation for θ may be written as $\theta_t - \Delta_h\theta = -P_h\rho_t$, and, with $E_h(t) = e^{\Delta_h t}$ the solution operator of (6.2) with $f = 0$, we find by Duhamel's principle that $\theta(t) = E_h(t)\theta(0) + \int_0^t E_h(t-s)P_h\rho_t(s)\, ds$. An obvious energy argument shows the stability property $\|E_h(t)v_h\| \le \|v_h\|$, which again gives the above bound for θ. The error estimate for the semidiscrete problem thus follows from the stability of $E_h(t)$ together with error estimates for the elliptic problem; for finite difference methods stability was similarly combined with a bound for the truncation error.

The use of the elliptic projection also yields an estimate of superconvergent order for $\nabla\theta$. In fact, by choosing this time $\chi = \theta_t$ in the variational equation for θ, we find after integration and simple estimates that $\|\nabla\theta(t)\| \le C(u)h^r$ if $v_h = R_h v$. For piecewise linears ($r = 2$) this may be combined with the superconvergent second order estimate for $\nabla(R_h u - I_h u)$ quoted in Section 5 to bound $\nabla(u_h - I_h u)$, with similar consequences as in the elliptic case, see Thomée, Xu, and Zhang (1989).

Estimates for the fully discrete ϑ-method (6.3) were also shown in Douglas and Dupont (1970) and Wheeler (1973). The contribution from the time discretization that has to be added to (6.5) at $t = t_n$ is then $Ck\int_0^{t_n} \|u_{tt}\| ds$, with a stability condition $k \le \gamma_\vartheta h^2$ for $0 \le \vartheta < 1/2$, and $Ck^2 \int_0^{t_n} (\|u_{ttt}\| + \|\Delta u_{tt}\|) ds$ for $\vartheta = 1/2$.

The ϑ-method for the homogeneous equation may be defined by $U^n = E_{kh}^n v_h$ where $E_{kh} = r(-k\Delta_h)$, $r(\lambda) = (1 + (1-\vartheta)\lambda)/(1 + \vartheta\lambda)$. Two-level schemes using more general rational functions $r(\lambda)$ of arbitrary order of accuracy were constructed in Baker, Bramble, and Thomée (1977), under the stability assumption $|r(\lambda)| \le 1$

for $\lambda \in \sigma(-k\Delta_h)$. Stable two-level time-stepping methods for the inhomogeneous equation of arbitrary order of accuracy may be constructed in the form $U^{n+1} = r(-k\Delta_h)U^n + k\sum_{j=1}^m q_j(-k\Delta_h)f(t_n + \tau_j k)$ where $r(\lambda)$ and the $q_j(\lambda)$ are rational functions, with, e.g., the backward Euler method included for $m = 1, \tau_1 = 0, r(\lambda) = q_1(\lambda) = 1/(1+\lambda)$, see Brenner, Crouzeix, and Thomée (1982). Stable multistep time discretization schemes of accuracy of order $q \leq 6$ have also been derived by LeRoux (1979) and others, see Thomée [38].

The regularity requirements needed for optimal order convergence in some of the above error estimates make it natural to enquire about error estimates under weaker regularity assumptions on data or on the solution. To illustrate this we now consider the solution of the homogeneous equation, i.e. (6.1) with $f = 0$, and recall that the solution of this problem is smooth for $t > 0$ even if v is only in L_2, say, and satisfies $\|u(t)\|_s \leq Ct^{-s/2}\|v\|$. Similarly, for the semidiscrete solution, $\|\Delta_h^{s/2}u_h(t)\| \leq Ct^{-s/2}\|v_h\|$, and using this one may show the nonsmooth data error estimate

$$\|u_h(t) - u(t)\| \leq Ch^r t^{-r/2}\|v\|, \quad \text{for } t > 0, \quad \text{if } v_h = P_h v,$$

so that optimal order $O(h^r)$ convergence holds for $t > 0$, without any regularity assumptions on v. The corresponding result for the backward Euler method reads

$$\|U^n - u(t_n)\| \leq C(h^r t_n^{-r/2} + kt_n^{-1})\|v\|, \quad \text{for } n \geq 1, \quad \text{if } v_h = P_h v.$$

Results of this type were shown by spectral representation in, e.g., Blair (1970), Helfrich (1974), and later by energy methods, permitting also time-dependent coefficients, in Huang and Thomée (1981), Luskin and Rannacher (1982), Sammon (1983a,b), see Thomée [38]. For stable fully discrete approximations of the form $U^n = E_{kh}^n v_h$ with $E_{kh} = r(-k\Delta_h)$ one then has to require $|r(\infty)| < 1$, see Baker, Bramble, and Thomée, loc. cit. The Crank-Nicolson method lacks this smoothing property, but Rannacher (1984) showed that using this method with two initial steps of the backward Euler method, one has $\|U^n - u(t_n)\| \leq C(h^r t_n^{-r/2} + k^2 t_n^{-2})\|v\|$ for $n \geq 1$.

The methods quoted in Section 5 for handling the difficulty of incorporating homogeneous Dirichlet boundary conditions in the approximating spaces S_h have been carried over from the elliptic to the semidiscrete parabolic case in Bramble, Schatz, Thomée, and Wahlbin (1977). This may be accomplished by replacing the gradient term in (6.2) and (6.3) by more general bilinear forms such as $N_\gamma(\cdot,\cdot)$ described there, or by using other approximations of the Laplacian than the above Δ_h. Within this framework negative norm estimates and superconvergence results were derived in Thomée (1980). We also quote Johnson and Thomée (1981) where the mixed method discussed in Section 5 is applied to (6.1).

In the fully discrete schemes discussed above, Galerkin's method was applied in space but a finite difference type method was used in time. We shall now describe an approach which uses a Galerkin type method also in time, the discontinuous Galerkin time-stepping method. This method was introduced and analyzed in Lesaint and Raviart (1974) and Jamet (1978), and generalized in the case of ordinary differential equations in Delfour, Hager, and Trochu (1981). In the present context it was studied in Eriksson, Johnson, and Thomée (1985). With a not

necessarily uniform partition of $[0,\infty)$ into intervals $J_n = [t_{n-1}, t_n)$, $n \geq 1$, let $\mathcal{S}_h = \{X = X(x,t); X|_{J_n} = \sum_{j=0}^{q-1} \chi_j t^j, \chi_j \in S_h\}$, where S_h are finite element spaces satisfying (5.5). Since the elements $X \in \mathcal{S}_h$ are not required to be continuous at the t_j we set $U_\pm^n = U(t_n \pm 0)$ and $[U^{n-1}] = U_+^{n-1} - U_-^{n-1}$. The discontinuous Galerkin method may then be stated: With $U_-^0 = v_h$ given, find $U \in \mathcal{S}_h$ such that, for $n \geq 1$,

$$\int_{J_n} \big[(U_t, X) + (\nabla U, \nabla X)\big] ds + ([U^{n-1}], X_+^{n-1}) = \int_{J_n} (f, X) ds, \ \forall X \in \mathcal{S}_h. \tag{6.6}$$

In the piecewise constant case, $q = 0$, this may be written, with $k_n = t_n - t_{n-1}$,

$$(\bar\partial_{t_n} U^n, \chi) + (\nabla U^n, \nabla \chi) = (k_n^{-1} \int_{J_n} f ds, \chi), \quad \bar\partial_{t_n} U^n = (U^n - U^{n-1})/k_n; \tag{6.7}$$

this reduces to the standard backward Euler method when the average of f is replaced by $f(t_n)$. It was shown in Eriksson and Johnson (1991) that for the error in the time-discretization in (6.7)

$$\|U^n - u_h(t_n)\| \leq C \ell_n \max_{j \leq n} (k_j \|u_{h,t}\|_{J_j}), \quad \text{where } \ell_n = (1 + \log(t_n/k_n))^{1/2},$$

with u_h the solution of the semidiscrete problem (6.2) and $\|\varphi\|_{J_j} = \sup_{t \in J_j} \|\varphi(t)\|$.

For $q = 1$ the method requires the determination on J_n of $U(t)$ of the form $U(t) = U_+^{n-1} + (t - t_{n-1})/k_n V_n$ with $U_+^{n-1}, V_n \in S_h$, and such that (6.6) holds, which gives a 2×2 system for these elements in S_h. In this case we have

$$\|U - u_h\|_{J_n} \leq C \ell_n \max_{j \leq n} (k_j^2 \|u_{h,tt}\|_{J_j}), \quad \|U_-^n - u_h(t_n)\| \leq C \ell_n \max_{j \leq n} (k_j^3 \|\Delta_h u_{h,tt}\|_{J_j}),$$

thus with third order superconvergence at the nodal points. For the total error in the fully discrete scheme, with, e.g., piecewise linear elements in space, one has

$$\|U_-^n - u(t_n)\| \leq C \ell_n \max_{j \leq n} (k_j^3 \|\Delta u_{tt}\|_{J_j} + h^2 \|u\|_{2,J_j}), \quad \|u\|_{2,J_j} = \sup_{t \in J_j} \|u(t)\|_2.$$

All our error estimates so far have been *a priori* error estimates, expressed in terms of the unknown exact solution of our parabolic problem. We close by mentioning briefly some *a posteriori* estimates by Eriksson and Johnson, loc. cit., based on an idea of Lippold (1991), where the error bounds are expressed in terms of the data and the computed solution. For $q = 0$, $r = 2$ such an estimate is

$$\|U_-^n - u(t_n)\| \leq C \ell_n \max_{j \leq n} \big((h^2 + k_j)\|f\|_{J_j} + k_j \|\bar\partial_t U^j\| + h^2 \|U^j\|_{2,h}\big), \tag{6.8}$$

where $\|\cdot\|_{2,h}$ is a discrete H^2-norm defined by $\|U\|_{2,h} = \big(\sum_\gamma \big|\big[\partial U/\partial n\big]_\gamma\big|^2\big)^{1/2}$, with γ denoting the edges of $\mathcal{T}_h$ and $[\cdot]_\gamma$ the jumps across γ. Error bounds are also available for $q = 1$, and the estimates generalize to variable h.

Estimates such as (6.8) may be used to design adaptive schemes in which the time-step is successively chosen so that the error is bounded by a given tolerance. The earlier *a priori* estimates are then needed to show that such a procedure will end

in a finite number of steps, cf. Eriksson and Johnson, loc. cit. This approach was further developed in a sequence of paper by Eriksson and Johnson, see the survey paper Eriksson, Estep, Hansbo, and Johnson [13]. A Petrov-Galerkin method with continuous in time approximations was studied by Aziz and Monk (1989).

We now briefly consider the question of maximum-norm stability for the finite element scheme. For the solution operator $E(t)$ of the homogeneous equation in (6.1) ($f = 0$) the maximum-principle shows at once that $\|E(t)v\|_{L_\infty} \le \|v\|_{L_\infty}$, and the smoothing estimate $\|\Delta^{s/2}E(t)v\|_{L_\infty} \le Ct^{-s/2}\|v\|_{L_\infty}$ also holds for $s > 0$. However, considering the case $r = d = 2$ one can easily see (cf. Thomée [38]) that the maximum-principle does not apply for the semidiscrete finite element analogue. This is in contrast to the corresponding finite difference method and is related to the fact that the mass matrix B is nondiagonal. In this regard, it was shown by Fujii (1973) that if B is replaced by a diagonal matrix whose diagonal elements are the row sums of B, and if all angles of the triangulation are nonobtuse, then the maximum-principle holds and hence $\|u_h(t)\|_{L_\infty} \le \|v_h\|_{L_\infty}$ for $t > 0$. This method is called the lumped mass method and can also be defined by

$$(u_{h,t}, \chi)_h + (\nabla u_h, \nabla \chi) = 0, \quad \forall \chi \in S_h,\ t > 0,$$

where the first term has been obtained by replacing the first term in (6.2) by using the simple quadrature expression $Q_{h\tau}(u_{h,t}\chi)$ on each τ, where $Q_{h\tau}(\varphi) = \frac{1}{3}\text{area}(\tau)\sum_{j=1}^{3}\varphi(P_j)$ with P_j the vertices of τ.

Even though the maximum-principle does not hold for (6.2), it was shown in Schatz, Thomée, and Wahlbin (1980) that, for $d = r = 2$ and quasi-uniform $\mathcal{T}_h$,

$$\|E_h(t)v_h\|_{L_\infty} + t\|E_h'(t)v_h\|_{L_\infty} \le C\ell_h\|v_h\|_{L_\infty}, \quad \text{for } t > 0, \quad \ell_h = \log(1/h).$$

The proof uses a weighted norm technique to estimate a discrete fundamental solution. For $d = 1, 2, 3$ and $r \ge 4$, Nitsche and Wheeler (1981-82) subsequently proved stability without the factor ℓ_h, and this and the corresponding smoothing result were shown for $d = 1$ and $r \ge 2$ in Crouzeix, Larsson, and Thomée (1994). Recently logarithm free stability and smoothness bounds have been shown for general d and r, first for Neumann boundary conditions in Schatz, Thomée, and Wahlbin (1998), and then for Dirichlet boundary conditions in Thomée and Wahlbin (1998). We note that the combination of stability and smoothing shows that the semigroup $E_h(t)$ is analytic, and via a resolvent estimate for its infinitesimal generator Δ_h this may be used to derive stability estimates also for fully discrete approximations of the form $U^n = r(-k\Delta_h)^n v_h$ where $r(z)$ is a rational function with the appropriate stability and consistency properties, see Palencia (1992) and Crouzeix, Larsson, Piskarev, and Thomée (1993). Other maximum-norm error bounds have been given in the literature by, e.g., Dobrowolski (1978), Nitsche (1979), and Rannacher (1991).

We now turn to hyperbolic equations and begin with a brief discussion of semidiscrete and fully discrete finite element schemes for the initial boundary value problem for the wave equation,

$$\begin{aligned} u_{tt} - \Delta u &= f \quad \text{in } \Omega \subset R^d, \quad \text{with } u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0 \\ u(\cdot, 0) &= v, \quad u_t(\cdot, 0) = w \quad \text{in } \Omega. \end{aligned}$$

Assuming as usual that $S_h \subset H_0^1$ satisfies (5.5), the semidiscrete analogue of our problem is to find $u_h(t) \in S_h$ for $t \geq 0$ from

$$(u_{h,tt}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi) \quad \forall \chi \in S_h,\ t > 0, \quad \text{with } u_h(0) = v_h,\ u_{h,t}(0) = w_h.$$

Similarly to the parabolic case this problem may be written in matrix form, this time as $B\alpha'' + A\alpha = \tilde{f}$ for $t > 0$, with $\alpha(0)$ and $\alpha'(0)$ given, where B and A are the mass and stiffness matrices.

Analogously to the analysis in the parabolic case it was shown in Dupont (1973a), with a certain improvement in Baker (1976), that under natural regularity assumptions and with appropriate choices of v_h and w_h,

$$\|u_h(t) - u(t)\| + h\|\nabla(u_h(t) - u(t))\| \leq C(u)h^r.$$

One possible fully discrete method for the wave equation is, cf. the case $\vartheta = 1/2$ of the Newmark type method (4.4),

$$(\partial_t \bar{\partial}_t U^n, \chi) + (\nabla(\tfrac{1}{4}U^{n+1} + \tfrac{1}{2}U^n + \tfrac{1}{4}U^{n-1}), \nabla\chi) = (f(t_n), \chi), \quad \forall \chi \in S_h, \quad n \geq 1,$$

where U^0 and U^1 are given approximations of $u(0) = v$ and $u(k)$, respectively. Setting $U^{n+1/2} = (U^n + U^{n+1})/2$ one shows for the homogeneous equation ($f = 0$) that the energy $\|\partial_t U^n\|^2 + \|\nabla U^{n+1/2}\|^2$ is conserved for $n \geq 0$, and, also in the general case, that $\|U^{n+1/2} - u(t_n + \frac{1}{2}k)\| = O(h^r + k^2)$ for appropriate initial values U^0 and U^1, u sufficiently regular, and t_n bounded. Although the error is then estimated at the points $t_n + \frac{1}{2}k$ it is easy to derive approximations also at the points t_n. For the homogeneous equaton more general time stepping schemes based on rational functions of the discrete Laplacian Δ_h were studied in Baker and Bramble (1979), where the second order wave equation was written as a first order system.

We proceed with some results for first order hyperbolic equations, and begin with the periodic model problem

$$u_t + u_x = f \quad \text{for } x \in R,\ t > 0, \quad \text{with } u(\cdot, 0) = v \quad \text{on } R, \tag{6.9}$$

where f, v, and the solution sought are 1-periodic in x; the L_2 inner products and norms used below are based on intervals of length 1.

To define an approximate solution, let $S_h \subset C^k(R)$, with $k \geq 0$, denote 1-periodic splines of order r (i.e., piecewise polynomials of degree $r-1$) based on a partition with maximal interval length h. The standard Galerkin method for (6.9) is then to find $u_h(t) \in S_h$ for $t \geq 0$ such that

$$(u_{h,t}, \chi) + (u_{h,x}, \chi) = (f, \chi), \quad \forall \chi \in S_h,\ t > 0, \quad \text{with } u_h(0) = v_h.$$

The equation may again be written in the form $B\alpha' + A\alpha = \tilde{f}$ where as usual B is the mass matrix but where the matrix A now has elements $a_{jk} = (\Phi_j', \Phi_k)$ and is skew-symmetric.

We first establish the simple error estimate, cf. Swartz and Wendroff (1969),

$$\|u_h(t) - u(t)\| \leq \|v_h - v\| + C(u)h^{r-1}, \quad \text{for } t \geq 0. \tag{6.10}$$

For this, we use an interpolation operator Q_h into S_h which commutes with time differentiation and is such that $\|Q_h v - v\| + h\|(Q_h v - v)_x\| \le Ch^r \|v\|_r$. It remains to bound $\theta = u_h - Q_h u$, which satisfies $(\theta_t, \chi) + (\theta_x, \chi) = -(\rho_t + \rho_x, \chi)$ for $\chi \in S_h$. Setting $\chi = \theta$ and observing that $(\theta_x, \theta) = 0$ by periodicity, we conclude

$$\tfrac{1}{2}\frac{d}{dt}\|\theta\|^2 = \|\theta\|\frac{d}{dt}\|\theta\| \le \big(\|\rho_t\| + \|\rho_x\|\big)\|\theta\| \le C(u)(h^r + h^{r-1})\|\theta\|, \tag{6.11}$$

which shows $\|\theta(t)\| \le \|\theta(0)\| + C(u)h^{r-1} \le \|v_h - v\| + C(u)h^{r-1}$, and yields (6.10).

We observe that the estimate (6.10) is of non-optimal order $O(h^{r-1})$, because the first derivative of the error in $Q_h u$ occurs on the right side of (6.11). For special cases more accurate results are known. For example, for the homogeneous equation, with S_h consisting of smooth splines ($k = r - 2$) on a uniform partition, the last term in (6.10) may be replaced by the optimal order term $C(u)h^r$. In this case superconvergence takes place at the nodes in the sense that $\|u_h(t) - I_h u(t)\| \le C(u)h^{2r}$ if $v_h = I_h v$, where $I_h v$ denotes the interpolant of v in S_h. This follows from Fourier arguments, see Thomée (1973), after observing that the Galerkin method may be interpreted as a finite difference method for the coefficients with respect to a basis for S_h. This was generalized to variable coefficients in Thomée and Wendroff (1974). It was shown by Dupont (1973b), however, that the improvement to optimal order is not always possible. In fact, if S_h is defined by a uniform partition with $r = 4$, $k = 1$ (Hermite cubics), and if v is a non-constant analytic 1-periodic function and $v_h \in S_h$ is arbitrary, then $\sup_{t\in(0,t^*)} \|u_h(t) - u(t)\| \ge ch^3$, $c > 0$, for any $t^* > 0$.

Leaving the standard Galerkin method, it was shown in Wahlbin (1974) that with S_h defined on a uniform partition and $0 \le k \le r - 2$, optimal order convergence holds for the Petrov-Galerkin method

$$(u_{h,t} + u_{h,x}, \chi + h\chi_x) = (f, \chi + h\chi_x), \quad \forall \chi \in S_h,\ t > 0, \quad \text{with } u_h(0) = v_h.$$

In Dendy (1974) and Baker (1975) nonstandard variational schemes with optimal order convergence without requiring uniform meshes were exhibited for the initial boundary value problem

$$u_t + u_x = f \quad \text{for } x \in I = (0,1),\ u(0,t) = 0, \quad \text{for } t \ge 0, \quad \text{with } u(\cdot,0) = v.$$

We now quote some space-time methods for this initial boundary value problem for $(x,t) \in \Omega = I \times J\,(J = (0,T))$, and introduce the characteristic directional derivative $Du = u_t + u_x$. Let $\mathcal{T}_h = \{\tau\}$ be a quasi-uniform triangulation of Ω with $\max \operatorname{diam}(\tau) = h$ and let $\mathcal{S}_h = \{\chi \in C(\bar{\Omega}); \chi|_\tau \in \Pi_{r-1} : \chi = 0 \quad \text{on } \partial\Omega^-\}$, where $\partial\Omega^-$ is the inflow boundary $(\{0\} \times J) \cup (I \times \{0\})$. With $((v,w)) = \int_0^T \int_0^1 v\, w\, dx\, dt$, the standard Galerkin method in this context is then to find $u_h \in \mathcal{S}_h$ such that

$$((Du_h, \chi)) = ((f, \chi)), \quad \forall \chi \in \mathcal{S}_h. \tag{6.12}$$

Standard arguments show as above the non-optimal order error estimate

$$\|u_h - u\| + \|u_h - u\|_{L_2(\partial\Omega^+)} \le Ch^{r-1}\|u\|_r, \tag{6.13}$$

where $\partial\Omega^+ = \partial\Omega \setminus \partial\Omega^-$ is the outflow boundary. This method does not work well in the case of discontinuous solutions. To stabilize the scheme one could consider

an artificial dissipation in the form of an additional term $h((\nabla u_h, \nabla\chi))$ on the left in (6.12), but such a method would be at most first order accurate.

The so called streamline diffusion method was introduced by Hughes and Brooks (1979) and analyzed in Johnson, Nävert, and Pitkäranta (1986). It consists in substituting $\chi + hD\chi$ for the test function χ in (6.12), and (6.13) then holds with h^{r-1} replaced by $h^{r-1/2}$. In the discontinuous Galerkin method studied by Lesaint and Raviart (1974) and Johnson and Pitkäranta (1986), cf. the corresponding method for parabolic equations introduced above, one determines $u_h \in \mathcal{S}_h = \{\chi \in L_2(\Omega); \chi|_\tau \in \Pi_{r-1}; \chi = 0 \text{ on } \partial\Omega^-\}$, thus without requiring $u_h \in C(\bar\Omega)$, from

$$((Du_h, \chi))_\tau - \int_{\partial\tau^-} [u_h]_{\partial\tau}(n_t + n_x)\, ds = ((f,\chi))_\tau, \quad \forall\chi \in \Pi_{r-1},\ \tau \in \mathcal{T}_h,$$

where $((\cdot,\cdot))_\tau$ denotes restriction to τ. This method also satisfies (6.12), with $h^{r-1/2}$ instead of h^{r-1}, and here the error in Du_h is of optimal order $O(h^{r-1})$.

Winther (1981) investigated a Petrov-Galerkin method defining u_h in continuous, piecewise Π_{r-1} spaces $\mathcal{S}_h$ based on rectangles $\tau = I_j \times J_l$, where I_j and J_l are partitions of I and J, by the equation $((Du_h,\chi))_\tau = ((f,\chi))_\tau$ for all $\chi \in \Pi_{r-2}$ and all τ, and proved optimal order $O(h^r)$ convergence. This method coincides with the cell vertex finite volume method, and is associated with the Wendroff box scheme, see Morton [26].

These types of approaches have been developed also for advection dominated diffusion problems, see, e.g., Johnson, Nävert, and Pitkäranta (1984), Hughes, Franca and Mallet (1987), and to nonlinear conservation laws, Johnson and Szepessy (1987), Szepessy (1989). For such time dependent problems Pirroneau (1982), Douglas and Russel (1982), and others have also analyzed the so called characteristic Galerkin method in which one adopts a Lagrangian point of view in the time-stepping, following an approximate characteristic defined by the advection term, in combination with a finite element approximation in the diffusive term.

7. Some other classes of approximation methods.

Methods other than finite difference and finite element methods, but often closely related to these, have also been developed, and in this section we sketch briefly four such classes of methods, namely collocation methods, spectral methods, finite volume methods, and boundary element methods.

In a *collocation method* one seeks an approximate solution of a differential equation in a finite dimensional space of sufficiently regular functions by requiring that the equation is satisfied exactly at a finite number of points. Such a procedure for parabolic equations in one space variable was analyzed by Douglas (1972), Douglas and Dupont (1974b); we describe it for the model problem

$$u_t = u_{xx} \text{ in } I = (0,1), \quad u(0,t) = u(1,t) = 0 \quad \text{for } t > 0, \quad \text{with } u(\cdot,0) = v \text{ in } I.$$

Setting $h = 1/M$, $x_j = jh$, $j = 0,\dots,M$, and $I_j = (x_{j-1}, x_j)$, we introduce the piecewise polynomial space $S_h = \{\chi \in C^1(\bar I); v\big|_{I_j} \in \Pi_{r-1},\ v(0) = v(1) = 0\}$, with $r \geq 4$. Letting ξ_i, $i = 1,\dots,r-2$, be the Gaussian points in $(0,1)$, the zeros of the Lagrange polynomial P_{r-2}, we define the collocation points $\xi_{ji} = x_{j-1} + h\xi_i$ in I_j, and pose the spatially semidiscrete problem to find $u_h \in S_h$ such that

$$u_{h,t}(\xi_{ji},t) = u_{h,xx}(\xi_{ji},t), \quad \text{for } j = 1,\dots,M,\ i = 1,\dots,r-2, \quad t > 0,$$

with $u_h(\cdot,0) = v_h$ an approximation of v. This method may be considered as a Galerkin method using a discrete inner product based on a Gauss quadrature rule. For v_h appropriately chosen one may then show the global error estimate

$$\|u_h(t) - u(t)\|_{L_\infty} \le Ch^r \Big(\max_{s\le t} \|u(s)\|_{r+2} + \big(\int_0^t \|u_t\|_{r+2}^2 ds \big)^{1/2} \Big).$$

Further, for $r > 4$, and with a more refined choice of initial approximation v_h, superconvergence takes place at the nodes,

$$|u_h(x_j,t) - u(x_j,t)| \le C_T h^{2r-4} \sup_{s\le t} \sum_{p+2q\le 2r-1} \|u^{(q)}(s)\|_p, \quad \text{for } t \le T.$$

We note the more stringent regularity requirements than for the Galerkin methods discussed in Section 6. These results carry over to fully discrete methods using both finite difference approximations and collocation in time.

For a two-point boundary value problem, results of a similar natur were derived by de Boor and Swartz (1973).

Spectral methods are in many ways similar to Galerkin/collocation methods. The main difference is in the choice of finite dimensional approximating spaces. We begin by considering an evolution equation in a Hilbert space framework.

Let thus H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and assume L is a nonnegative operator defined in $D(L) \subset H$, so that $(Lu,u) \ge 0$. Consider the initial value problem

$$u_t + Lu = f \quad \text{for } t > 0, \quad \text{with } u(0) = v. \tag{7.1}$$

Let now $\{\varphi_j\}_{j=1}^\infty \subset H$ be a sequence of linearly independent functions in $D(L)$ which span H and set $S_N = \operatorname{span}\{\varphi_j\}_{j=1}^N$. We define a "spatially" semidiscrete approximation $u_N = u_N(t) \in S_N$ of (7.1) by

$$(u_{N,t},\chi) + (Lu_N,\chi) = (f,\chi) \quad \forall \chi \in S_N,\ t \ge 0, \quad \text{with } u_N(0) = v_N. \tag{7.2}$$

Introducing the orthogonal projection $P_N : H \to S_N$ we may write (7.2) as

$$u_{N,t} + L_N u_N = f_N := P_N f, \quad \text{for } t \ge 0, \quad \text{where } L_N = P_N L P_N.$$

Clearly $(L_N\chi,\chi) = (LP_N\chi, P_N\chi) \ge 0$. With $u_N(t) = \sum_{j=1}^N \alpha_j(t)\varphi_j$, this equation may be written $B\alpha' + A\alpha = \tilde{f}$ for $t \ge 0$, where the elements of the matrices A and B are $(L\varphi_i,\varphi_j)$ and (φ_i,φ_j), respectively. Clearly B is a Gram matrix and so positive definite.

As a simple example, let $L = -(d/dx)^2$ on $I = (0,1)$ and $H = L_2(I)$, $D(L) = H^2 \cap H_0^1$, and let the $\varphi_j(x) = c_j \sin \pi j x$ be the normalized eigenfunctions of L. Then $B = I$, A is positive definite and P_N is simply the truncation of the Fourier series, $P_N v = \sum_{j=1}^N (v,\varphi_j)\varphi_j$, with $L_N = \sum_{j=1}^N (\pi j)^2 (v,\varphi_j)\varphi_j = P_N L v$.

We note that the error $e_N = u_N - u$ satisfies

$$e_{N,t} + L_N e_N = f_N - f + (L_N - L)u \quad \text{for } t > 0, \quad e_N(0) = v_N - v,$$

and hence, since $E_N(t) = e^{-L_N t}$ is bounded,

$$\|e_N\| \le \|v_N - v\| + \int_0^t (\|(P_N - I)f\| + \|(L_N - L)u\|)\, ds. \tag{7.3}$$

It follows that the error is small with $v_N - v$, $(P_N - I)f$, and $(L_N - L)u$.

In our above example we see that if $v_N = P_N v$, and if the Fourier series for v, f, and Lu converge, then the error is small. In particular, the convergence is of order $O(N^{-r})$ for any r provided the solution is sufficiently regular.

Another way to define a semidiscrete numerical method employing the space S_N of our example is to make S_N a Hilbert space with the inner product $(v, w)_N = h\sum_{j=0}^{N-1} v(x_j)w(x_j)$ where $x_j = j/(N-1)$. This gives rise to a projection P_N defined by $P_N u(x_j) = u(x_j), j = 0, \ldots, N-1$, and the semidiscrete equation (7.2) now becomes the collocation equation

$$u_{N,t}(x_j, t) + Lu_N(x_j, t) = f(x_j, t) \quad \text{for } j = 0, \ldots, N-1,\ t \ge 0.$$

This is also referred to as a pseudospectral method and the error estimate (7.3) will be valid in the discrete norm corresponding to $(\cdot, \cdot)_N$.

Spectral and pseudospectral methods using the above sinusoidal basis functions are particularly useful for periodic problems. For initial-boundary value problems for hyperbolic equations basis functions related to Chebyshev and Lagrange polynomials are sometimes useful. Such methods are successfully applied in fluid dynamics calculations. Spectral methods have been studied since the 70s, see Gottlieb and Orszag [17], Canuto, Hussaini, Quarteroni, and Zang [10], Boyd [6], and references therein.

We now turn to the *finite volume method* which we exemplify for the model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega, \tag{7.4}$$

where Ω is a convex polygonal domain in R^2. The basis for this approach is the observation that for any $V \subset \Omega$ we have by Green's formula that

$$\int_{\partial V} \frac{\partial u}{\partial n}\, ds = \int_V f\, dx. \tag{7.5}$$

Let now $\mathcal{T}_h = \{\tau_j\}_{j=1}^{N_h}$ be a triangulation of Ω and consider (7.5) with $V = \tau_j, j = 1, \ldots, N_h$. Let Q_j be the center of the circumscribed circle of τ_j. If τ_i has an edge γ_{ji} in common with τ_j, then $Q_i - Q_j$ is orthogonal to γ_{ji}, and $\partial u/\partial n$ in (7.5) may be approximated by the difference quotient $(u(Q_i) - u(Q_j))/|Q_i - Q_j|$. This produces a finite difference scheme on the nonuniform mesh $\{Q_j\}$; for the boundary triangles one may use the boundary values in (7.4). Writing the discrete problem as $AU = F$ the matrix A is symmetric positive definite, and the solution satisifies a discrete maximum-principle. When the $\mathcal{T}_h$ is quasi-uniform (and such that the Q_j are on $\partial\Omega$) one has $\|U - u\|_{1,h} \le Ch^{s-1}\|u\|_s$ for $s = 2$ in a certain discrete H^1-norm, and, under an additional symmetry assumption on $\mathcal{T}_h$, also for $s = 3$. This method may be described as cell centered and goes back to Tikhonov and Samarskii (1961) in the case of rectangular meshes; for further developments, see Samarskii, Lazarov,

and Makarov [30]. For such meshes it was used in Varga [39] to construct finite difference schemes.

An associated method is the following vertex centered method, also referred to as the finite volume element method: Let $S_h \subset H_0^1$ be the piecewise linear finite element space defined by $\mathcal{T}_h$. The straight lines connecting a vertex of $\tau \in \mathcal{T}_h$ with the midpoint of the opposite edge intersect at the barycenter of τ and divide τ into six triangles. Let $B_{j,\tau}$ be the union of the two of these which have P_j as a vertex. For each interior vertex P_j we let B_j be the union of the corresponding $B_{j,\tau}$, and let $\bar{S}_h$ denote the associated piecewise constant functions. Motivated by (7.5) we then pose the Petrov-Galerkin method to find $u_h \in S_h$ such that

$$\bar{A}(u_h, \psi) := \sum_j \psi_j \int_{\partial B_j} \frac{\partial u_h}{\partial n}\, ds = (f, \psi) \quad \forall \psi \in \bar{S}_h; \tag{7.6}$$

this may also be thought of as a finite difference scheme on the irregular mesh $\{P_j\}$. The B_j are referred to as control volumes; they were called mesh regions in Mac Neal (1953). Associating with $\chi \in S_h$ the function $\bar{\chi} \in \bar{S}_h$ which agrees with χ at the vertices of $\mathcal{T}_h$ one finds that $\bar{A}(\psi, \bar{\chi}) = A(\psi, \chi)$ so that (7.6) may be written $A(u_h, \chi) = (f, \bar{\chi})$ for $\chi \in S_h$. icular, (7.6) has a unique solution, and using the Babuška-Brezzi *inf-sup* condition it was shown in Bank and Rose (1987) that the standard error estimate $\|u_h - u\|_1 \le Ch\|u\|_2$ holds for this method.

Finite volume methods are useful for operators in divergence form and have also been applied to time dependent conservation laws, see Heinrich [20], Morton [26]. For the model heat equation the vertex centered method is similar to the lumped mass finite element method.

In a *boundary integral method* a boundary value problem for a homogeneous partial differential equation in a domain Ω is reformulated as an integral equation over the boundary $\partial\Omega$. This equation may then be used as a basis for numerical approximation. We shall illustrate this approach for the model problem

$$\Delta u = 0 \quad \text{in } \Omega \subset R^d, \quad \text{with } u = g \quad \text{on } \partial\Omega, \tag{7.7}$$

$\partial\Omega$ smooth. To pose the boundary integral equation, let $\Gamma(x) = -(2\pi)^{-1} \log|x|$ for $d = 2$ and $\Gamma(x) = c_d|x|^{-d+2}$ for $d > 2$ be the fundamental solution of the Laplacian in R^d. For any u with $\Delta u = 0$ on $\partial\Omega$ we have by Green's formula

$$u(x) = \int_{\partial\Omega} \Gamma(x-y) \frac{\partial u}{\partial n_y}\, ds_y - \int_{\partial\Omega} \frac{\partial \Gamma}{\partial n_y}(x-y) u(y)\, ds_y, \quad x \in \Omega. \tag{7.8}$$

With x on $\partial\Omega$ the integrals on the right define the single and double layer potentials $V\partial u/\partial n$ and Wu (note that $K(x,y) = (\partial\Gamma/\partial n_y)(x-y) = O(|x-y|^{-(d-2)})$ for $x, y \in \partial\Omega$). For $x \in \Omega$ approaching $\partial\Omega$ the two integrals tend to $V\partial u/\partial n$ and $\frac{1}{2}u + Wu$, respectively, so that (7.8) yields $\frac{1}{2}u = V\partial u/\partial n + Wu$. With $u = g$ on $\partial\Omega$ this is a Fredholm integral equation of the first kind to determine $\partial u/\partial n$ on $\partial\Omega$, which inserted into (7.8) together with $u = g$ on $\partial\Omega$ gives the solution of (7.7).

Instead of this direct method one may use the indirect method of assuming that the solution of (7.8) may be represented as a potential of a function on $\partial\Omega$, so that

$$u(x) = \int_{\partial\Omega} \Gamma(x-y) v(y)\, ds_y \quad \text{or} \quad u(x) = \int_{\partial\Omega} \frac{\partial \Gamma}{\partial n_y}(x-y) w(y)\, ds_y, \quad x \in \Omega.$$

With V and W as above, if such functions v and w exist, they satisfy the first and second kind Fredholm integral equations

$$Vv = g \quad \text{and} \quad \tfrac{1}{2}w + Ww = g. \tag{7.9}$$

Writing $H^s = H^s(\partial\Omega)$, V and W are pseudodifferential operators of order -1, bounded operators $H^s \to H^{s+1}$, in particular compact on H^s; for $d = 2$ the kernel of W is actually smooth. The first kind equation is uniquely solvable provided a certain measure, the transfinite diameter $\delta_{\partial\Omega}$ of $\partial\Omega$, is such that $\delta_{\partial\Omega} \neq 1$, and the second kind equation in (7.8) always has a unique solution. Similar reformulations may be used also for Neumann boundary conditions, for a large number of other problems involving elliptic type equations, and for exterior problems; in fact, this approach to the numerical solution is particularly useful in the latter cases.

The use of boundary integral equations, particularly of the second kind, to study boundary value problems for elliptic equations has a long history, and includes work of Neumann and Fredholm. We shall not dwell on this here but refer to, e.g., Atkinson [1].

In the Boundary Element Method (BEM) one determines the approximate solution in a piecewise polynomial finite element type space of a boundary integral formulation such as the above using the Galerkin or the collocation method.

The numerical solution of second kind equations by projection methods, which include both Galerkin and collocation methods, were studied in an abstract Banach space setting in the important paper by Kantorovich (1948), and their convergence was established under the appropriate assumptions on the projection operator involved. Consider, e.g., the second kind equation in (7.9) (with $d = 2$) in $C(\partial\Omega)$ with the maximum-norm $|\cdot|_\infty$, and let $S_h \subset C(\partial\Omega)$ be finite-dimensional. With $P_h : C(\partial\Omega) \to S_h$ a projection operator, the corresponding discrete problem is $\frac{1}{2}w_h + P_hWw_h = P_hg$, and if $|P_hW - W|_\infty \to 0$ in operator norm one may show that $|w_h - w|_\infty \leq C|P_hw - w|_\infty$, so that the discrete solution converges as fast as the projection as $h \to 0$.

The collocation method may also be combined with quadrature, as suggested in Nyström (1930): as an example we may use, e.g., the composite trapezoidal rule on a uniform partition, and then apply collocation at the nodal points so that, with $\partial\Omega = \{x(s);\ 0 \leq s \leq l\}$ and $K(s,t)$ the kernel of W, the discrete solution is

$$w_h(x(s)) = 2g(x(s)) - 2h\sum_{j=1}^{N_h} K(s, s_j)w_j, \quad \text{for } 0 \leq s \leq l,$$

where the w_j are determined by setting $w_h(x(s_i)) = w_i$ for $i = 1, \ldots, N_h$. It is not difficult to see that since the trapezoidal rule is infinitely accurate for smooth periodic functions we have $|w_h - w|_\infty = O(h^r)$ for any $r > 0$.

For the second kind equation in (7.9), using Galerkin's method and a finite dimensional subspace S_h of $L_2(\partial\Omega)$, the discrete approximation $w_h \in S_h$ to w is determined from

$$\tfrac{1}{2}\langle w_h, \chi\rangle + \langle Ww_h, \chi\rangle = \langle g, \chi\rangle, \quad \forall\chi \in S_h, \quad \text{where } \langle\cdot,\cdot\rangle = (\cdot,\cdot)_{L_2(\partial\Omega)}.$$

Writing $|\cdot|_s$ for the norm in H^s, one has $|w_h - w|_0 \leq C_r(u)h^r$ if S_h is accurate of order $O(h^r)$, and by a duality argument one may show the superconvergent order

negative norm estimate $|w_h - w|_{-r} \le C_r(u)h^{2r}$, see Sloan and Thomée (1985); using an iteration argument by Sloan (1976) this may be used, in principle, to define an approximate solution $\tilde{w}_h$ with $|\tilde{w}_h - w|_0 = O(h^{2r})$.

After early work of Wendland (1968) and Nedelec and Planchard (1973) the study of the Galerkin finite element approach using first kind equations was pursued from the mid 70s by Hsiao, Le Roux, Nedelec, Stephan, Wendland, and others, see the surveys in Atkinson [1], Sloan [31], and Wendland [42]. Within this framework we consider the numerical solution of the first kind equation in (7.9) with $d = 2$ in the finite dimensional space S_h of periodic smoothest splines of order r, i.e., $S_h \subset C^{r-2}$ consists of piecewise polynomials in Π_{r-1}. Our discrete problem is then to find $v_h \in S_h$ such that

$$\langle Vv_h, \chi \rangle = \langle g, \chi \rangle, \quad \forall \chi \in S_h.$$

It was shown in the work quoted above that the bilinear form $\langle Vv, w \rangle$ associated with $V : H^{-1/2} \to H^{1/2}$ is symmetric, bounded, and coercive in $H^{-1/2}$, i.e.

$$\langle Vv, w \rangle = \langle v, Vw \rangle \le C|v|_{-1/2}|w|_{-1/2} \quad \text{and } \langle Vv, v \rangle \ge c|v|^2_{-1/2}, \quad \text{with } c > 0.$$

An application of Céa's lemma and approximation properties of S_h then show

$$|v_h - v|_{-1/2} \le C \inf_{\chi \in S_h} |\chi - v|_{-1/2} \le Ch^{r+1/2}|v|_r,$$

and an Aubin-Nitsche type duality argument first used by Hsiao and Wendland (1981) implies $|v_h - v|_{-r-1} \le Ch^{2r+1}|v|_r$. For x an interior point of Ω we therefore find for $u_h = Vv_h$ that $|u_h(x) - u(x)| \le C_x|v_h - v|_{-r-1} \le Ch^{2r+1}$, since $\Gamma(x - y)$ is smooth when $y \ne x$.

Expressed in terms of a basis $\{\Phi_j\}$ of S_h this problem may be written in matrix form as $A\alpha = \tilde{g}$ where A is symmetric positive definite. However, although the dimension of A has been reduced by the reduction of the original two-dimensional problem to a one-dimensional one, in contrast to the finite element method for a differential equation problem, the matrix A is now not sparse. We also note that the elements $\langle V\Phi_i, \Phi_j \rangle$ require two integrations, one in forming $V\Phi_i$ and one in forming the inner product.

In order to reduce this work the collocation method has again been considered by Arnold and Wendland (1983); here v_h is determined from $Vv_h(x(s_j)) = g(x(s_j))$ at N_h quadrature points s_j in $[0, l]$, where $N_h = \dim S_h$. Applied to our above model problem this method, using smoothest splines of even order r, has a lower order of maximal convergence rate, $O(h^{r+1})$ rather than $O(h^{2r+1})$; if r is odd and the mesh uniform Saranen (1988) has shown $O(h^{r+1})$ in $\|\cdot\|_{-2}$. A further step in the development is the qualocation method proposed by Sloan (1988), which is a Petrov-Galerkin method, thus with different trial and test spaces. For S_h the smoothest splines of order r on a uniform mesh (so that Fourier analysis may be applied) and with the quadrature rule suitably chosen, negative norm estimates of order $O(h^{r+3})$ for even r and $O(h^{r+4})$ for odd r may be shown.

In the vast literature on the numerical boundary integral methods much attention has been paid to the complications arising when our above regularity assumptions fail to be satisfied, such as for domains with corners in which case V and W are not compact.

8. Numerical linear algebra for elliptic problems.

Both finite difference and finite element methods for elliptic problems such as (2.1) lead to linear algebraic systems

$$AU = F, \tag{8.1}$$

where A is a nonsingular matrix. When Ω is a d-dimensional domain, using either finite differences or finite elements based on quasi-uniform triangulations, the dimension N of the corresponding finite dimensional problem is of order $O(h^{-d})$, where h is the mesh-width, and for $d > 1$ direct solution by Gauss elimination is normally not feasible as this method requires $O(N^3) = O(h^{-3d})$ algebraic operations. Except in special cases one therefore turns to iterative methods. In this section we summarize the historical development of such methods.

As a basic iterative method we consider the Picard method

$$U^{n+1} = U^n - \tau(AU^n - F) \quad \text{for } n \geq 0, \quad \text{with } U^0 \text{ given}, \tag{8.2}$$

where τ is a positive parameter. With U the exact solution of (8.1) we have

$$U^n - U = R(U^{n-1} - U) = \cdots = R^n(U^0 - U), \quad \text{where } R = I - \tau A,$$

and hence the rate of convergence of the method depends on $\|R^n\|$ where $\|\cdot\|$ is the matrix norm subordinate to the Euclidean norm in R^N. When A is symmetric positive definite (SPD) we have $\|R^n\| = \rho^n$ where $\rho = \rho(R) = \max_i |1 - \tau\lambda_i|$ denotes the spectral radius of R, and (8.2) converges if $\rho < 1$. The optimal choice is $\tau = 2/(\lambda_1 + \lambda_N)$, which gives $\rho = (\kappa - 1)/(\kappa + 1)$, where $\kappa = \kappa(A) = \lambda_N/\lambda_1$ is the condition number or A; note, however, that this choice of τ requires knowledge of λ_1 and λ_N which is not normally at hand. In applications to second order elliptic problems one often has $\kappa = O(h^{-2})$ so that $\rho \leq 1 - ch^2$ with $c > 0$. Hence with the optimal choice of τ the number of iterations required to reduce the error to a small $\epsilon > 0$ is of order $O(h^{-2}|\log\epsilon|)$. Since each iteration uses $O(h^{-d})$ operations in the application of $I - \tau A$ this shows that the total number of operations needed to reduce the error to a given tolerance is of order $O(h^{-d-2})$, which is smaller than for the direct solution when $d \geq 2$.

The early more refined methods were designed for finite difference methods of positive type for second order elliptic equations, particularly the five-point operator (2.2). The corresponding matrix may then be written $A = D - E - F$ where D is diagonal and E and F are (elementwise) nonnegative and strictly lower and upper triangular. The analysis was often based on the Perron-Frobenius theory of positive matrices. A commonly used property is diagonal dominance: $A = (a_{ij})$ is diagonally dominant if $\sum_{j\neq i} |a_{ij}| \leq |a_{ii}|$, $i = 1\ldots, N$, irreducibly diagonally dominant if it is also irreducible, so that (8.1) cannot be written as two lower order systems, and strictly diagonally dominant if there is strict inequality for at least one i. Examples are the Jacobi (after Jacobi (1845)) and Gauss-Seidel (Gauss (1823), Seidel (1874)) or Liebmann (1918) methods which are defined by

$$U^{n+1} = U^n - B(AU^n - F) = RU^n + BF, \quad \text{with } R = I - BA, \tag{8.3}$$

in which $B = B_J = D^{-1}$ or $B = B_{GS} = (D - E)^{-1}$ with $R_J = D^{-1}(E + F)$ and $R_{GS} = (D - E)^{-1}F$, respectively. In the application to the model problem (2.1)

in the unit square, using the five-point operator, the equations may be normalized so that $D = 4I$ and the application of R_J simply means that the new value at any interior mesh-point x_j is obtained by replacing it by the average of the old values at the four neighboring points $x_{j\pm e_l}$. The Gauss-Seidel method also takes averages, but with the mesh-points taken in a given order, and successively uses the values already obtained in forming the averages. The methods were referred to in Geiringer (1949) as the methods of simultaneous and successive displacements, respectively. For the model problem one may easily determine the eigenvalues and eigenvectors of A and show that with $h = 1/M$ one has $\rho(R_J) = \cos \pi h = 1 - \frac{1}{2}\pi^2 h^2 + O(h^4)$ and $\rho(R_{GS}) = \rho(R_J)^2 = 1 - \pi^2 h^2 + O(h^4)$ so that the number of iterates needed to reduce the error to ϵ is of the orders $2h^{-2}\pi^2 |\log \epsilon|$ and $h^{-2}\pi^2 |\log \epsilon|$. The Gauss-Seidel method thus requires about half as many iterations as the Jacobi method.

If A is irreducibly diagonally dominant then $\rho(R_J) < 1$ and $\rho(R_{GS}) < 1$ so that both methods converge, see Geiringer, loc.cit.; for A strictly diagonally dominant this was shown in Collatz (1942). Further, Stein and Rosenberg (1948) showed that if D is positive and E and F nonnegative and $\rho(B_J) < 1$ then $\rho(B_{GS}) < \rho(B_J)$, i.e., the Gauss-Seidel method converges faster than the Jacobi method.

Forming the averages in the Jacobi and Gauss-Seidel methods may be thought as relaxation; in the early work by Gauss and Seidel this was not done in a cyclic order as described above, and which is convenient on computers, but according to the size of the residual or other criteria, see Southwell (1946, 1956) and Fox (1948). It turns out that one may obtain better results than those described above by overrelaxation, i.e., choosing $B_\omega = (D - \omega E)^{-1}$ and $R_\omega = (D - \omega E)^{-1}((1-\omega)E + F)$ with $\omega > 1$. These methods were first studied by Frankel (1950) in the case of the model problem, and Young (1950, 1954) in more general cases of matrices satisfying his property $\mathcal{A}$, which holds for a large class of difference approximations of elliptic problems in general domains. Frankel proved that for the model problem the optimal choice of the parameter is $\omega_{opt} = 2/(1 + \sqrt{1 - \rho^2})$ where $\rho = \rho(B_J) = \cos \pi h$, i.e., $\omega_{opt} = 2/(1 + \sin \pi h) = 2 - 2\pi h + O(h^2)$, and that correspondingly $\rho(R_{\omega_{opt}}) = \omega_{opt} - 1 = 1 - 2\pi h + O(h^2)$. The number of iterations required is thus then of order $O(h^{-1})$, which is significally smaller than for the above methods. It was shown by Kahan (1958), also for nonsymmetric A, that $\rho(R_\omega) \geq |\omega - 1|$ so that convergence can only occur for $0 < \omega < 2$. On the other hand, Ostrowski (1954) showed that if A is SPD, then $\rho(R_\omega) < 1$ if and only if $0 < \omega < 2$.

We consider again an iterative method of the form (8.3) with $\rho(R) < 1$, and introduce now the new sequence $V^n = \sum_{j=0}^n \beta_{nj} U^j$ where the β_{nj} are real. Setting $p_n(\lambda) = \sum_{j=0}^n \beta_{nj} \lambda^j$, and assuming $p_n(1) = \sum_{j=0}^n \beta_{nj} = 1$ for $n \geq 0$, we obtain easily $V^n - U = p_n(R)(U^0 - U)$. For V^n to converge fast to U one wants to choose the β_{nj} in such a way that $\rho(p_n(R))$ becomes small with n. By the Cayley-Hamilton theorem $p_n(R) = 0$ if p_n is the characteristic polynomial of R, and hence $V^n = U$ if $n \geq N$, but this is a prohibitively large number of iterations. For $n < N$ we have by the spectral mapping theorem that $\rho(p_n(R)) = \max_{\lambda \in \sigma(R)} |p_n(\lambda)|$. In particular, if R is symmetric and $\rho = \rho(R)$, a simple calculation shows that, taking the maximum instead over $[-\rho, \rho] \supset \sigma(R)$, the optimal polynomial is $p_n(\lambda) = T_n(\lambda/\rho)/T_n(1/\rho)$ where T_n is the nth Chebyshev polynomial, and the corresponding value of $\rho(p_n(R))$ is bounded by

$$T_n(1/\rho)^{-1} = 2\Big(\big(\frac{1 + \sqrt{1 - \rho^2}}{\rho}\big)^n + \big(\frac{1 + \sqrt{1 - \rho^2}}{\rho}\big)^{-n}\Big)^{-1} \leq 2\big(\frac{\rho}{1 + \sqrt{1 - \rho^2}}\big)^n.$$

For the model problem using the Gauss-Seidel basic iteration we have as above $\rho = 1 - \pi^2 h^2 + O(h^4)$ and we find that the average error reduction factor per iteration step in our present method is bounded by $1 - \sqrt{2}\pi h + O(h^2)$, which is of the same order of magnitude as for SOR. The use of the sequence V^n instead of the U^n was called linear acceleration by Forsythe (1953) and is sometimes attributed to Richardson (1910); in Varga [39] it is referred to as a semi-iterative method.

We now describe the Peaceman-Rachford alternating direction implicit iterative method for the model problem (2.1) on the unit square, using the five-point discrete elliptic equation with $h = 1/M$. In this case we may write $A = H + V$ where H and V correspond to the horizontal and vertical difference operators $-h^2\partial_1\bar{\partial}_1$ and $-h^2\partial_2\bar{\partial}_2$. Note that H and V are positive definite and commute. Introducing an acceleration parameter τ and an intermediate value $U^{n+1/2}$ we may consider the scheme defining U^{n+1} from U^n by

$$(\tau + H)U^{n+1/2} = (\tau - V)U^n + F, \quad (\tau + V)U^{n+1} = (\tau - H)U^{n+1/2} + F,$$

or after elimination, with G_τ appropriate and using that H and V commute,

$$U^{n+1} = R_\tau U^n + G_\tau, \quad \text{where } R_\tau = (\tau I - H)(\tau I + H)^{-1}(\tau I - V)(\tau I + V)^{-1}.$$

The error satisfies $U^n - U = R_\tau^n(U^0 - U)$, and with μ_i the (common) eigenvalues of H and V, $\|R_\tau\| \le \max_i |(\tau - \mu_i)/(\tau + \mu_i)|^2 < 1$, and it is easy to see that the maximum occurs for $i = 1$ or M. With $\mu_1 = 4\sin^2(\frac{1}{2}\pi h)$, $\mu_M = 4\cos^2(\frac{1}{2}\pi h)$ the optimal τ is $\tau_{opt} = (\mu_1\mu_M)^{1/2}$ with the maximum for $i = 1$, so that, with $\kappa = \kappa(H) = \kappa(V) = \mu_M/\mu_1$,

$$\|R\|_{\tau_{\rm opt}} \le \Big(\frac{(\mu_1\mu_M)^{1/2} - \mu_1}{(\mu_1\mu_M)^{1/2} + \mu_1}\Big)^{1/2} = \frac{\kappa^{1/2} - 1}{\kappa^{1/2} + 1} = 1 - \pi h + O(h^2).$$

This again shows the same order of convergence rate as for SOR.

A more efficient procedure is obtained by using varying acceleration parameters $\tau_j, j = 1, 2, \ldots$, corresponding to the n step error reduction matrix $\tilde{R}_n = \prod_{j=1}^n R_{\tau_j}$. It can be shown that the τ_j can be chosen cyclically with period m in such a way that $m \approx c\log\kappa \approx c\log(1/h)$ and

$$\|\tilde{R}_m\|^{1/m} = \max_{1\le i\le M}\Big(\prod_{j=0}^{m-1} \Big|\frac{\tau_j - \mu_i}{\tau_j + \mu_i}\Big|\Big)^{2/m} \le 1 - c(\log(1/h))^{-1}, \quad c > 0.$$

The analysis indicated depends strongly on the fact that H and V commute, which only happens for rectangles and constant coefficients, but the method may be defined and shown convergent for more general cases, see Birkhoff and Varga (1959). We remark that these iterative schemes may often be associated with time-stepping methods for parabolic problems and that our discussion in Section 4 of fractional step and splitting methods are relevant also in the present context. For a comprehensive account of the above methods for solving systems associated with finite difference methods, including historical remarks, see Varga [39].

We now turn to the development of iterative methods for systems mainly associated with the emergence of the finite element method. We begin by describing

the conjugate gradient method by Hestenes and Stiefel (1952), and assume that A is SPD. Considering the iterative method

$$U^{n+1} = (I - \tau_n A)U^n + \tau_n F \quad \text{for } n \geq 0, \quad \text{with } U^0 = 0,$$

we find at once that, for any choice of the parameters τ_j, U^n belongs to the Krylov space $K_n(A;F) = \text{span}\{F, AF, \ldots, A^{n-1}F\}$. The conjugate gradient method defines these parameters so that U^n is the best approximation of U in $K_n(A;F)$ with respect to the norm defined by $|U| = (AU,U)^{1/2}$, i.e., as the orthogonal projection of U onto $K_n(A;F)$ with respect to the inner product (AV,W). By our above discussion it follows that, with $\kappa = \kappa(A)$ the condition number of A,

$$|U^n - U| \leq (T_n(1/\rho))^{-1}|U| \leq 2\Big(\frac{\kappa^{1/2}-1}{\kappa^{1/2}+1}\Big)^n |U|. \tag{8.4}$$

The computation of U^n can be done by a two term recurrence relation, for instance in the following form using the residuals $r^n = F - AU^n$ and the auxiliary vectors $q^n \in K_{n+1}(A;F)$, orthogonal to $K_n(A;F)$,

$$U^{n+1} = U^n + \frac{(r^n,q^n)}{(Aq^n,q^n)}q^n, \quad q^{n+1} = r^{n+1} - \frac{(Ar^{n+1},q^n)}{(Aq^n,q^n)}q^n, \quad U^0 = 0,\ q^0 = F.$$

In the preconditioned conjugate gradient (PCG) method the conjugate gradient method is applied to equation (8.1) after multiplication by some easy to determine SPD approximation B of A^{-1} and using the inner product $(B^{-1}V,W)$; we note that BA is SPD with respect to this inner product. The error estimate (8.4) is now valid with $\kappa = \kappa(BA)$; B would be chosen so that this condition number is smaller than $\kappa(A)$. For the recursion formulas the only difference is that now $r^n = B(F - AU^n)$ and $q^0 = BF$. An early application of PCG to partial differential equations is Wachspress (1963) and it is systematically presented in Marchuk (1975) and Axelsson and Barker (1982), where reference to other work can be found.

One way of defining a preconditioner is by means of the multigrid method. This method is based on the observation that large components of the errors are associated with low frequencies in a spectral representation. The basic idea is then to work in a systematic way with a sequence of triangulations and reduce the low frequency errors on coarse triangulations, which corresponds to small size problems, and higher frequency residual errors on finer triangulations by a smoothing operator, such as a step of the Jacobi method, which is relatively inexpensive.

One common situation is as follows: Assuming Ω is a plane polygonal domain we first perform a coarse triangulation of Ω. Each of the triangles is then divided into four similar triangles, and this process is repeated, which after a finite number M of steps leads to a fine triangulation with each of the original triangles devided into 4^M small triangles. Going from one level of fineness to the next the procedure may be described in three steps: 1. Presmoothing on the finer triangulation, 2. Correction on the coarser triangulation by solving a residual equation, 3. Postsmooting on the finer triangulation. This procedure is then used recursively between the levels of the refinement leading to, e.g. the V-cycle or W-cycle algorithms. It turns out that under some assumptions the error reduction matrix R corresponding to one sweep of the algorithm satisfies $\|R\| \leq \rho < 1$, with ρ independent of M, i.e., of h, and that

the number of operations is of order $O(N)$ where $N = O(h^{-2})$ is the dimension of the matrix associated with the finest triangulation.

The multigrid method was first introduced for finite difference methods in the 60s by Fedorenko (1964) and Bahvalov (1966) and further developed and advocated by Brandt in the 70s, see, e.g., Brandt (1977). For finite elements it has been intensely pursued by, e.g., Braess and Hackbusch, Bramble and Pasciak, Mandel, McCormick and Bank; for overviews with further references, see Hackbusch [18] and Bramble [7].

A class of iterative methods that have attracted a lot of attention recently is the so called domain decomposition methods. These assume that the domain Ω in which we want to solve our elliptic problem may be decomposed into subdomains $\Omega_j,\ j = 1, \ldots, M$, which could overlap. The idea is to reduce the boundary value problem on Ω into problems on each of the Ω_j, which are then coupled by their values on the intersections. The problems on the Ω_j could be solved independently on parallel processors. This is particularly efficient when the individual problems may be solved very fast, e.g., by fast transform methods. Such a case is provided by the model problem (2.1) on the unit square which may be solved directly by using the discrete Fourier transform, defined by $\hat{F}_m = \sum_j F^j e^{-2\pi i\, m\cdot j\, h}$. In fact, we then have $(-\Delta_h U)\hat{}_m = 2\pi^2|m|^2\hat{U}_m$ and hence $\hat{U}_m = (2\pi^2|m|^2)^{-1}\hat{F}_m$ so that by the inverse discrete Fourier transform $U^j = \sum_m (2\pi^2|m|^2)^{-1}\hat{F}_m e^{2\pi i\, m\cdot j\, h}$. Using the Fast Fourier Transform both $\hat{F}_m$ and U^j may be calculated in $O(N\log N)$ operations.

The domain decomposition methods go back to the Schwarz alternating procedure (1869), in which $\Omega = \Omega_1 \cup \Omega_2$. Considering the Dirichlet problem (2.1) on Ω one defines a sequence $\{u^k\}$ starting with a given u^0 vanishing on $\partial\Omega$, by

$$-\Delta u^{2k+1} = f \text{ in } \Omega_1, \quad \text{with } u^{2k+1} = u^{2k} \text{ on } \partial\Omega_1 \cap \Omega_2,\ u^{2k+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega,$$
$$-\Delta u^{2k+2} = f \text{ in } \Omega_2, \text{ with } u^{2k+2} = u^{2k+1} \text{ on } \partial\Omega_2 \cap \Omega_1,\ u^{2k+2} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega,$$

and this procedure can be combined with numerical solution by, e.g., finite elements. A major step in the analysis of this so called multiplicative form of the Schwarz alternating procedure was taken by Lions (1988). A modification referred to as the additive form was first studied by Matsokin and Nepomnyashchikh (1985) and Dryja and Widlund (1987).

The following alternative approach may be pursued when Ω_1 and Ω_2 are disjoint but with a common interface $\partial\Omega_1 \cap \partial\Omega_2$: If u_j denotes the solution in $\Omega_j, j = 1, 2$, transmission conditions $u_1 = u_2,\ \partial u_1/\partial n = \partial u_2/\partial n$ have to be satisfied on the interface. One method is then to reduce the problem to an integral type equation on the interface and use this as a basis of an iterative method. For a survey of domain decomposition techniques, see Smith, Bjørstad, and Gropp [32].

References

D.N. Arnold and W.L. Wendland (1983), *On the asymptotic convergence of collocation methods*, Math. Comp. **41**, 349-381.

D.G. Aronson (1963), *The stability of finite difference approximations to second order linear parabolic differential equations*, Duke Math. J. **30**, 117-128.

J. P. Aubin (1967), *Behavior of the error of the approximate solution of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **21**, 599-637.

A.K. Aziz and P. Monk (1989), *Continuous finite elements in space and time for the heat equation*, Math. Comp. **52**, 255-274.

I. Babuška (1971), *Error bounds for finite element method*, Numer. Math. **16**, 322-333.

I. Babuška (1973), *The finite element method with Lagrangian multipliers*, Numer. Math. **20**, 179-192.

I. Babuška (1976), *The selfadaptive approach in the finite element method*, The Mathematics of Finite Elements and Applications, II (J.R. Whiteman, ed.), Academic Press, New York, pp. 125-142.

I. Babuška and M.R. Dorr (1981), *Error estimates for the combined h and p-version of the finite element method*, Numer. Math. **37**, 257-277.

I. Babuška and W.C. Rheinboldt (1978), *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal. **15**, 736-754.

I. Babuška and M. Suri (1990), *The p- and h-p versions of the finite element method, an overview*, Spectral and High Order Methods for Partial Differential Equations (C. Canuto and A. Quarteroni, eds.), North-Holland, Amsterdam, pp. 5-26.

I. Babuška, B.A. Szabó, and I.N. Katz (1981), *The p-version of the finite element method*, SIAM J. Numer. Anal. **18**, 512-545.

N.S. Bahvalov (1959), *Numerical solution of the Dirichlet problem for Laplace's equation*, Vestnik Moskov. Univ. Ser. Math. Meh. Astronom. Fiz. Him. **3**, 171-195.

N.S. Bahvalov (1966), *On the convergence of a relaxation method with natural constraints on the elliptic operator*, USSR Comput. Math. and Math. Phys. **6**, 101-135.

G.A. Baker (1975), *A finite element method for first order hyperbolic equations*, Math. Comp. **20**, 995-1006.

G.A. Baker (1976), *Error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal. **13**, 564-576.

G.A. Baker and J.H. Bramble (1979), *Semidiscrete and single step fully discrete approximations for second order hyperbolic equations*, RAIRO Anal. Numér. **13**, 75-100.

G.A. Baker, J.H. Bramble, and V. Thomée (1977), *Single step Galerkin approximations for parabolic problems*, Math. Comp. **31**, 818-847.

R.E. Bank and D.J. Rose, *Some error estimates for the box method*, SIAM J. Numer. Anal. **24**, 777-787.

A.E. Berger, L.R. Scott, and G. Strang (1972), *Approximate boundary conditions in the finite element method*, Symposia Mathematica **10**, Academic Press, New York, 295-313.

G. Birkhoff, M.H. Schultz, and R.S. Varga (1968), *Piecewise Hermite interpolation in one and two variables with applications to partial differential equations*, Numer. Math., 232-256.

J. Blair (1970), *Approximate solution of elliptic and parabolic boundary value problems*, PhD thesis, University of California, Berkeley.

J.H. Bramble and S.R. Hilbert (1970), *Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation*, SIAM J. Numer. Anal. **7**, 113-124.

J.H. Bramble and B.E. Hubbard (1962), *On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation*, Numer. Math. **4**, 313-327.

J.H. Bramble and B.E. Hubbard (1963), *A theorem on error estimation for finite difference analogues of the Dirichlet problem for elliptic equations*, Contributions to Differential Equations **2**, 319-340.

J.H. Bramble, T. Dupont, and V. Thomée (1972), *Projection methods for Dirichlet's problem in approximating polygonal domains with boundary value corrections*, Math. Comp. **26**, 869-879.

J.H. Bramble, B.E. Hubbard, and V. Thomée (1969), *Convergence estimates for essentially positive type discrete Dirichlet problems*, Math. Comp. **23**, 695-710.

J.H. Bramble, R.B. Kellogg, and V. Thomée (1968), *On the rate of convergence of some difference schemes for second order elliptic equations*, Nordisk Tidskr. Informations-Behandling (BIT) **8**, 154-173.

J.H. Bramble, A.H. Schatz, V. Thomée, and L.B. Wahlbin (1977), *Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations*, SIAM J. Numer. Anal. **14**, 218-241.

A. Brandt (1977), *Multi-level adaptive solution to boundary-value problems*, Math. Comp. **31**, 333-390.

P. Brenner (1966), *The Cauchy problem for symmetric hyperbolic systems in L_p*, Math. Scand. **19**, 27-37.

P. Brenner and V. Thomée (1970), *Stability and convergence rates in L_p for certain difference schemes*, Math. Scand. **27**, 5-23.

P. Brenner, M. Crouzeix, and V. Thomée (1982), *Single step methods for inhomogeneous linear differential equations in Banach space*, RAIRO Anal. Numér. **16**, 5-26.

F. Brezzi (1974), *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian mulitpliers*, RAIRO Anal. Numér. **8**, 129-151.

J. Céa (1964), *Approximation variationelle des problèmes aux limites*, Ann. Inst. Fourier. (Grenoble) **14**, 345-444.

R.W. Clough (1960), *The finite element method in plane stress analysis*, Proc. Second ASCE Conf. on Electronic Computation (Pittsburg, Pennsylvania), vol. 8, pp. 345-378.

L. Collatz (1933), *Bemerkungen zur Fehlerabschätzung für das Differenzenverfahren bei partiellen Differentialgleichungen*, Z. Angew. Math. Mech. **13**, 56-57.

L. Collatz (1942), *Fehlerabschätzung für das Iterationsverfahren zur Auflösung linearer Gleichungssysteme*, Z. Angew. Math. Mech. **22**, 357-361.

R. Courant (1943), *Variational methods for the solution of problems of equilibrium and vibrations*, Bull. Amer. Math. Soc. **49**, 1-23.

R. Courant, K.O. Friedrichs, and H. Lewy (1928), *Über die partiellen Differenzengleichungen der Mathematischen Physik*, Math. Ann. **100**, 32-74; English translation, with commentaries by P.D. Lax, O.B. Widlund, and S.V. Parter, in IBM J. Research and Development **11** (1967).

R. Courant, E. Isaacson, and M. Rees (1952), *On the solution of nonlinear hyperbolic differential equations by finite differences*, Comm. Pure Appl. Math. **5**, 243-255.

J. Crank and P. Nicolson (1947), *A practical method for numerical integration of solution of partial differential equations of heat-conduction type*, Proc. Cambridge Philos. Soc. **43**, 50-67.

M. Crouzeix, S. Larsson, and V. Thomée (1994), *Resolvent estimates for elliptic finite element operators in one dimension*, Math. Comp. **63**, 121-140.

M. Crouzeix, S. Larsson, S. Piskarev, and V. Thomée (1993), *The stability of rational approximations of analytic semigroups*, BIT **33**, 74-84.

G. Dahlquist (1954), *Convergence and stability for a hyperbolic difference equation with analytic initial-values*, Math. Scand. **2**, 91-102.

D. de Boor and B. Swartz (1973), *Collocation at Gaussian points*, SIAM J. Numer. Anal. **10**, 582-606.

M. Delfour, W. Hager, and F. Trochu (1981), *Discontinuous Galerkin method for ordinary differential equations*, Math. Comp. **36**, 453-473.

Ju.K. Demjanovič (1964), *The net method for some problems in mathematical physics*, Dokl. Akad. Nauk SSSR **159**, 250-253.

J.E. Dendy (1974), *Two methods of Galerkin type achieving optimum L^2 rates of convergence for first order hyperbolics*, SIAM J. Numer. Anal. **11**, 637-653.

M. Dobrowolski (1978), *L^∞-convergence of linear finite element approximations to quasi-linear initial boundary value problems*, RAIRO Anal. Numér. **12**, 247-266.

J. Douglas, Jr. (1956), *On the relation between stability and convergence in the numerical solution of linear parabolic and hyperbolic differential equations*, J. Soc. Indust. Appl. Math. **4**, 20-37.

J. Douglas, Jr. (1972), *A superconvergence result for the approximate solution of the heat equation by a collocation method*, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A.K. Aziz, ed.), Academic Press, New York, pp. 475-490.

J. Douglas, Jr. and T. Dupont (1970), *Galerkin methods for parabolic equations*, SIAM J. Numer. Anal. **7**, 575-626.

J. Douglas, Jr. and T. Dupont (1974a), *Galerkin approximations for the two point boundary problem using continuous, piecewise polynomial spaces*, Numer. Math. **22**, 99-109.

J. Douglas, Jr. and T. Dupont (1974b), *Collocation Methods for Parabolic Equations in a Single Space Variable*, Lecture Notes in Mathematics, vol. 385, Springer, Berlin, Heidelberg, New York.

J. Douglas, Jr. and T.F. Russel (1982), *Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures*, SIAM J. Numer. Anal. **19**, 871-885.

M. Dryja and O.B. Widlund (1987), *An additive variant of the Schwarz alternating method for the case of many subregions*, Technical Report, Department of Computer Science, Courant Institute of Mathematical Sciences New York University.

T. Dupont (1973a), *L^2-estimates for Galerkin methods for second order hyperbolic equations*, SIAM J. Numer. Anal. **10**, 880-889.

T. Dupont (1973b), *Galerkin methods for first order hyperbolics: an example*, SIAM J. Numer. Anal. **10**, 890-899.

T. Dupont (1974), *L_2 error estimates for projection methods for parabolic equations in approximating domains*, Mathematical Aspects of Finite Elements in Partial Differential Equations (C. de Boor, ed.), Academic Press, New York, San Fransisco, London, pp. 313-352.

K. Eriksson (1994), *An adaptive finite element method with efficient maximum norm error control for elliptic problems*, Math. Models Methods Appl. Sciences **4**, 313-329.

K. Eriksson, and C. Johnson (1991), *Adaptive finite element methods for parabolic problems. I: A linear model problem*, SIAM J. Numer. Anal. **28**, 43-77.

K. Eriksson, C. Johnson, and V. Thomée (1985), *A discontinuous in time Galerkin method for parabolic type problems*, RAIRO Math.Anal. **19**, 611-643.

S. Faedo (1949), *Un nuovo metodo de l'analisi esistenziale e quantitative dei problemi di propagazione*, Ann. Scuola Norm. Sup Pisa Cl. Sci. **1 (3)**, 1-40.

R. Fedorenko (1964), *The speed of convergence of one interative process*, USSR Comput. Math. and Math. Phys. **4**, 227-235.

K. Feng (1965), *Finite difference schemes based on variational principles*, Appl. Math. Comput. Math. **2**, 238-262.

L. Fox (1948), *A short account of relaxation methods*, Quart. J. Mech. Appl. Math. **1**, 253-280.

S.P. Frankel (1950), *Convergence rates of iterative treatments of partial differential equations*, Math. Tables Aids Comput. **4**, 65-75.

K.O. Friedrichs (1954), *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math **7**, 345-392.

K.O. Friedrichs (1962), *A finite difference scheme for the Neumann and Dirichlet problem*, Report No NYO-9760, AEC Comp. and Appl. Math. Center, Courant Inst., Math. Sci., New York Univ.

K.O. Friedrichs and H.B. Keller (1966), *A finite difference scheme for generalized Neumann problems*, Numerical Solution of Partial Differential Equations, Academic Press, New York, London, pp. 1-19.

K.O. Friedrichs and H. Lewy (1926), *Über die Eindeutigkeit und das Abhängigkeitsgebiet der Lösungen beim Anfangswertproblem linearer hyperbolischer Differentialgleichungen*, Math. Ann. **98**, 192-204.

H. Fujii (1973), *Some remarks on finite element analysis of time-dependent field problems*, Theory and Practice in Finite Element Structural Analysis, University of Tokyo, pp. 91-106.

B.G. Galerkin (1915), *Rods and plates. Series occurring in various questions concerning the elastic equilibrium of rods and plates*, Vestnik Insz. **19**, 897-908.

C.F. Gauss (1823), *Brief an Gerling*, Werke, vol. 9, pp. 278-281.

H. Geiringer (1949), *On the solution of systems of linear equations by certain iterative methods*, Reissner Anniversary Volume, J.W. Edwards, Ann Arbor, Mich., pp. 365-393.

S. Gerschgorin (1930), *Fehlerabschätzung für das Differenzenverfahren zur Lösung partieller Differentialgleichungen*, Z. Angew. Math. Mech. **10**, 373-382.

S.K. Godunov and V.S. Ryabenkii, *Special criteria of stability of boundary-value problems for non-selfadjoint difference equations*, Uspekhi Mat. Nauk **18**, 3-14.

B. Gustafsson, H.O. Kreiss, and A. Sundström (1972), *Stability theory for difference approximations of mixed initial boundary value problems*, Math. Comp. **26**, 649-686.

R. Haverkamp (1984), *Eine Aussage zur L_∞-Stabilität und zur genauen Konvergenzordnung der H_0^1-Projektionen*, Numer. Math. **44**, 393-405.

G.W. Hedstrom (1968), *The rate of convergence of some difference schemes*, SIAM J. Numer. Anal. **5**, 363-406.

H.-P. Helfrich (1974), *Fehlerabschätzungen für das Galerkinverfahren zur Lösung von Evolutionsgleichungen*, Manuscr. Math. **13**, 219-235.

M. Hestenes and E. Stiefel (1952), *Methods of conjugate gradients for solving linear systems*, J. Res. Nat. Bur. Standards **49**, 409-436.

G. Hsiao and W. Wendland (1981), *The Aubin-Nitsche lemma for integral equations*, J. Integral Equations **3**, 299-315.

T.J.R. Hughes and A. Brooks (1979), *A multidimensional upwind scheme with no crosswind diffusion*, Finite Element Methods for Convection Dominated Flows (T.J.R. Hughes, ed.), vol. 34, American Society of Mechanical Engineers, pp. 19-35.

T.J.R. Hughes, L.P. Franca, and M. Mallet (1987), *A new finite element formulation for computational fluid dynamics: VI. Convergence analysis of the generalized SUPG formulation for linear time dependent multi-dimensional advective-diffusive systems*, Comput. Meth. Appl.Mech. Engrg. **63**, 97-112.

P. Jamet (1978), *Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain*, SIAM J. Numer. Anal. **15**, 912-928.

F. John (1952), *On integration of parabolic equations by difference methods*, Comm. Pure Appl. Math. **5**, 155-211.

C. Johnson and J. Pitkäranta (1986), *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp. **46**, 1-26.

C. Johnson and A. Szepessy (1987), *On the convergence of a finite element method for a nonlinear hyperbolic conservation law*, Math. Comp. **49**, 427-444.

C. Johnson and V. Thomée (1981), *Error estimates for some mixed finite element methods for parabolic type problems*, RAIRO Modél. Math. Anal. Numér. **15**, 41-78.

C. Johnson, U. Nävert, and J. Pitkäranta (1984), *Finite element methods for linear hyperbolic problems*, Comput. Methods Appl. Mech. Engrg. **45**, 285-312.

M.L. Juncosa and D. M. Young (1957), *On the Crank-Nicolson procedure for solving parabolic partial differential equations*, Proc. Cambridge Philos. Soc. **53**, 448-461.

L. Kantorovich (1948), *Functional analysis and applied mathematics*, Uspehi Mat. Nauk **3**, 89-185.

T. Kato (1960), *Estimation of iterated matrices, with application to the von Neumann condition*, Numer. Math. **2**, 22-29.

H.O. Kreiss (1959a), *Über die Lösung des Cauchyproblems für lineare partielle Differenzialgleichungen mit Hilfe von Differenzengleichungen*, Acta Math. **101**, 179-199.

H.O. Kreiss (1959b), *Über Matrizen die beschränkte Halbgruppen erzeugen*, Math. Scand. **7**, 72-80.

H.O. Kreiss (1962), *Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differentialgleichungen approximieren*, Nordisk Tidskr. Informations-Behandling **2**, 153-181.

H.O. Kreiss (1964), *On difference approximations of dissipative type for hyperbolic differential equations*, Comm. Pure Appl. Math. **17**, 335-353.

H.O. Kreiss (1968), *Stability theory for difference approximations of mixed initial boundary value problems*, Math. Comp. **22**, 703-714.

H.O. Kreiss and G. Scherer (1974), *Finite element and finite difference methods for hyperbolic partial differential equations*, Academic Press, New York, San Fransisco, London, 195-212.

H.O. Kreiss, V. Thomée, and O.B. Widlund (1970), *Smooting of initial data and rates of convergence for parabolic difference equations*, Comm. Pure Appl. Math. **23**, 241-259.

P. Laasonen (1949), *Über eine Methode zur Lösung der Wärmeleitungsgleichung*, Acta Math. **81**, 309-317.

P. Laasonen (1958), *On the solution of Poisson's difference equation*, J. Assoc. Comput. Mach. **5**, 370-382.

P.D. Lax (1961), *On the stability of difference approximations to solutions of hyperbolic differential equations*, Comm. Pure Appl. Math. **14**, 497-520.

P.D. Lax and L. Nirenberg (1966), *On stability for difference schemes: a sharp form of Gårding's inequality*, Comm. Pure Appl. Math. **19**, 473-492.

P.D. Lax and R.D. Richtmyer (1959), *Survey of the stability of linear finite difference equations*, Comm. Pure Appl. Math. **9**, 473-492.

P.D. Lax and B. Wendroff (1960), *Systems of conservation laws*, Comm. Pure Appl. Math. **13**, 217-237.

P.D. Lax and B. Wendroff (1964), *Difference schemes for hyperbolic equations with high order of accuracy*, Comm. Pure Appl. Math. **17**, 381-398.

M. Lees (1960), *A priori estimates for the solution of difference approximations to parabolic differential equations*, Duke Math. J. **27**, 297-311.

M. Lees (1961), *The solution of positive-symmetric hyperbolic systems by difference methods*, Proc. Amer. Math. Soc. **12**, 195-204.

M.N. Le Roux (1979), *Semidiscretization in time for parabolic problems*, Math. Comp. **33**, 919-931.

P. Lesaint and P.-A. Raviart (1974), *On the finite element method for solving the neutron transport equation*, Mathematical Aspects of Finite Elements in Partial Differential Equations (C. de Boor, ed.), Academic Press, New York, pp. 89 - 123.

H. Liebmann (1918), *Die angenäherte Ermittlung harmonischer Funktionen und konformer Abbildung (nach Ideen von Boltzmann und Jacobi)*, Sitzungsberichte der Beyer. Akad. Wiss., Math.-Phys. Kl. **47**, 385-416.

P.-L. Lions (1988), *On the Schwarz alternating method I*, First International Symposium on Domain Decomosition Methods (R. Glowinski, G.H. Golub, G.A. Meurant, and J. Périaux, eds.), SIAM, Philadelphia, pp. 1-42.

G. Lippold (1991), *Error estimates and step-size control for the approximate solution of a first order evolution equation*, RAIRO Modél. Math. Anal. Numér. **25**, 111-128.

J. Löfström (1970), *Besov spaces in the theory of approximation*, Ann. Mat. Pura Appl. **85**, 93-184.

M. Luskin and R. Rannacher (1982), *On the smoothing property of the Crank-Nicolson scheme*, Appl. Anal. **14**, 117-135.

R.H. Mac Neal (1953), *An asymmetric finite difference network*, Quart. Appl. Math. **1**, 295-310.

A.M. Matsokin and S.V. Nepomnyashchikh (1985), *A Schwarz alternating method in a subspace*, Izv. VUZ Mat. **29 (10)**, 61-66.

T.S. Motzkin and W. Wasow (1953), *On the approximation of linear elliptic differential equations by difference equations with positive coefficients*, J. Math. Phys. **31**, 253-259.

F. Natterer (1975), *Über die punktweise Konvergenz finiter Elemente*, Numer. Math. **25**, 67-77.

J.C. Nedelec and J. Planchard (1973), *Une méthode variationelle d'élements finis pour la résolution numérique d'un problème extérieur dans* R^3, RAIRO, Anal. Numér. **7**, 105-129.

N.M. Newmark (1959), *A method of computation for structural dynamics*, Proceedings of ASCE, Journal of Engineering Mechanics (EM3) **85**, 67-94.

J. Nitsche (1968), *Ein Kriterium für die Quasioptimalität des Ritzschen Verfahrens*, Numer. Math. **11**, 346-348.

J. Nitsche (1971), *Über ein Variationsprinzip zur Lösung von Dirichlet-problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Semin. Univ. Hamb. **36**, 9-15.

J.A. Nitsche (1975), L_∞*-convergence of finite element approximation*, Second conference on finite elements, Rennes, France.

J.A. Nitsche (1979), L_∞*-convergence of finite element Galerkin approximations for parabolic problems*, RAIRO Anal. Numér. **13**, 31-54.

J.A. Nitsche and M.F. Wheeler (1981-82), L_∞*-boundedness of the finite element Galerkin operator for parabolic problems*, Numer. Funct. Anal. Optimization **4**, 325-353.

E. Nyström (1930), *Über die praktische Auflösung von Integralgleichungen mit Anwendungen auf Randwertaufgaben*, Acta Math. **54**, 185-204.

G.G. O'Brien, M.A. Hyman, and S. Kaplan (1951), *A study of the numerical solution of partial differential equations*, J. Math. and Phys. **29**, 223-251.

L.A. Oganesjan (1963), *Numerical solution of plates*, Lensovnarchoz CBTI, Sbornik.

L.A. Oganesjan (1966), *Convergence of difference schemes in case of improved approximation of the boundary*, Ž. Vyčisl. Mat. i Mat. Fiz. **6**, 1029-1042.

L.A. Oganesjan and L.A. Ruhovec (1969), *An investigation of the rate of convergence of variational-difference schemes for second order elliptic equations in a two-dimensional region with smooth boundary*, Zh. Vychisl. Mat. i Mat. Fiz. **9**, 1102-1120.

S. Osher (1972), *Stability of parabolic difference approximations to certain mixed initial boundary value problems*, Math. Comp. **26**, 13-39.

A.M. Ostrowski (1954), *On the linear iteration procedures for symmetric matrices*, Rend. Mat. e Appli. **14**, 140-163.

C. Palencia (1992), *A stability result for sectorial operators in Banach spaces*, SIAM J. Numer. Anal. **30**, 1373-1384.

B. Parlett (1966), *Accuracy and dissipation in difference schemes*, Comm. Pure Appl. Math. **19**, 111-123.

D.W. Peaceman and H.H. Rachford, Jr. (1955), *The numerical solution of parabolic and elliptic differential equations*, J. Soc. Indust. Appl. Math. **3**, 28-41.

J. Peetre and V. Thomée (1967), *On the the rate of convergence for discrete initial-value problems*, Math. Scand. **21**, 159-176.

O. Pirroneau (1982), *On the transport-diffusion algorithm and its applications to the Navier-Stokes equations*, Numer. Math. **38**, 309-332.

R. Rannacher (1984), *Finite element solution of diffusion problems with irregular data*, Numer. Math. **43**, 309-327.

R. Rannacher (1991), *L^∞-stability and asymptotic error expansion for parabolic finite element equations*, Bonner Math. Schriften **228**.

P. Raviart and J. Thomas (1977), *A mixed finite element method for 2nd order elliptic problems*, Proc. of the Symposium on the Mathematical Aspects of the Finite Element Method, Rome, December, 1975, vol. 606, Springer Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, pp. 292-315.

Lord Rayleigh, *Theory of sound (1894, 1896)*, vol. I, II, Macmillan, London.

W. Ritz (1908), *Über eine neue Methode zur Lösing gewisser Variationsprobleme der mathematischen Physik*, J. Reine Angew. Math. **135**, 1-61.

J. Saranen (1988), *The convergence of even degree spline collocation solution for potential problems in smooth domains of the plane*, Numer. Math. **53**, 299-314.

A.A. Samarskii (1961), *A priori estimates for the solution of the difference analogue of a parabolic differential equation*, Zh. Vychisl. Mat. i Mat. Fiz **1**, 441-460 (Russian); English translation in U.S.S.R. Comput. Math. and Math. Phys. **1**, 487-512.

P. Sammon (1983a), *Convergence estimates for semidiscrete parabolic equation approximations*, SIAM J. Numer. Anal. **19**, 68-92.

P. Sammon (1983b), *Fully discrete approximation methods for parabolic problems with nonsmooth initial data*, SIAM J. Numer. Anal. **20**, 437-470.

V.K. Saulev (1957), *On a class of elliptic equations solvable by the method of finite-differences*, Vycisl. Mat. **1**, 81-86.

A.H. Schatz and L.B. Wahlbin (1982), *On the quasi-optimality in L_∞ of the $\overset{\circ}{H}{}^1$-projection into finite element spaces*, Math. Comp. **38,**, 1-22.

A. H. Schatz, V. Thomée, and L.B. Wahlbin (1980), *Maximum-norm stability and error estimates in parabolic finite element equations*, Comm. Pure Appl. Math. **33**, 265-304.

A. H. Schatz, V. Thomée, and L. B. Wahlbin (1998), *Stability, analyticity, and almost best approximation in maximum-norm for parabolic finite element equations*, Comm. Pure Appl. Math. **51**, 1349-1385.

H.A. Schwarz (1869), *Über einige Abbildungsaufgaben*, J. Reine Angew. Math **70**, 105-120.

L.R. Scott (1975), *Interpolated boundary conditions in the finite element method*, SIAM J. Numer. Anal. **12**, 404-427.

L.R. Scott (1976), *Optimal L^∞ estimates for the finite element method*, Math. Comp. **30**, 681-697.

L. Seidel (1874), *Über ein Verfahren, die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, sowie lineäre Gleichungen überhaupt, durch successive Annäherung aufzulösen*, Abh. Math.-Phys. Kl., Bayerische Akad. Wiss. München **11**, 81-108.

S. J. Serdjukova (1964), *The uniform stability with respect to the initial data of a sixpoint symmetrical scheme for the heat conduction equation*, Numerical Methods for the Solution of Differential Equations and Quadrature Formulae, Nauka, Moscow, pp. 212-216.

G.H. Shortley and R. Weller (1938), *The numerical solution of Laplace's equation*, J. Appl. Phys. **9**, 334-344.

I.H. Sloan (1976), *Improvement by iteration for compact operator equations*, Math. Comp. **30**, 758-764.

I.H. Sloan (1988), *A quadrature-based approach to improving the collocation method*, Numer. Math. **54**, 41-56.

I.H. Sloan and V. Thomée (1985), *Superconvergence of the Galerkin iterates for integral equations of the second kind*, J. Integral Equations **9**, 1-23.

P. Stein and R.L. Rosenberg (1948), *On the solution of linear simultaneous equations by iteration*, J. London Math. Soc. **23**, 111-118.

G. Strang (1962), *Trigonometric polynomials and difference methods of maximum accuracy*, J. Math. and Phys. **41**, 147-154.

G. Strang (1964), *Unbalanced polynomials and difference methods for mixed problems*, SIAM J. Numer. Anal. **2**, 46-51.

G. Strang (1965), *Necessary and insufficient conditions for well-posed Cauchy problems*, J. Differential Equations **2**, 107-114.

G. Strang (1967), *On strong hyperbolicity*, J. Math. Kyoto Univ. **6**, 397-417.

G. Strang (1972), *Variational crimes in the finite element method*, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A.K. Aziz, ed.), Academic Press, New York, pp. 689-710.

B. Swartz and B. Wendroff (1969), *Generalized finite-difference schemes*, Math. Comp. **23**, 37-49.

A. Szepessy (1989), *Convergence of a shock-capturing streamline diffusion finite element method for a scalar conservation law in two space dimensions*, Math. Comp. **53**, 527-545.

A.N. Tikhonov and A.A. Samarskii (1961), *Homogeneous difference schemes*, Zh. Vychisl. Mat. i Mat. Fiz. **1**, 5-63; U.S.S.R. Comput. Math. and Math. Phys. **1**, 5-67.

V. Thomée (1964), *Elliptic difference equations and Dirichlet's problem*, Contributions to Differential Equations **3**, 301-324.

V. Thomée (1973), *Convergence estimates for semi-discrete Galerkin methods for initial-value problems*, Numerische, insbesondere approximationstheoretische Behandlung von Funktionalgleichungen, vol. 333, Springer Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, pp. 243-262.

V. Thomée (1974), *Some convergence results for Galerkin methods for parabolic boundary value problems*, Mathematical Aspects of Finite Elements (C. de Boor, ed.), Academic Press, New York, pp. 55-88.

V. Thomée (1980), *Negative norm estimates and superconvergence in Galerkin methods for parabolic problems*, Math. Comp. **34**, 93-113.

V. Thomée and L.B. Wahlbin (1974), *Convergence rates of parabolic difference schemes for non-smooth data*, Math. Comp. **28**, 1-13.

V. Thomée and L.B. Wahlbin (1998), *Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions*, Preprint no. 1998-44, Department of Mathematics, Göteborg.

V. Thomée and B. Wendroff (1974), *Convergence estimates for Galerkin methods for variable coefficient initial value problems*, SIAM J. Numer. Anal. **1**, 1059-1068.

V. Thomée and B. Westergren (1968), *Elliptic difference equations and interior regularity*, Numer. Math. **11**, 196-210.

V. Thomée, J.C. Xu, and N.Y. Zhang (1989), *Superconvergence of the gradient in piecewise linear finite-element approximation to a parabolic problem*, SIAM J. Numer. Anal. **26**, 553-573.

M.J. Turner, R.W. Clough, H.C. Martin, and L. Topp (1956), *Stiffness and deflection analysis of complex structures*, J. Aero. Sci. **23**, 805-823.

J.M. Varah (1970), *Maximum norm stability of difference approximations to the mixed initial boundary value problem for the heat equation*, Math. Comp. **24**, 31-44.

E.A. Volkov (1966), *Obtaining an error estimate for a numerical solution of the Dirichlet problem in terms of known quantities*, Ž. Vyčisl. Mat. i Mat. Fiz. **6, no. 4, suppl**, 5-17.

L. B. Wahlbin (1974), *A dissipative Galerkin method applied to some quasilinear hyperbolic equations*, RAIRO Anal. Numér. **8**, 109-117.

W. Wasow (1952), *On the truncation error in the solution of Laplace's equation by finite differences*, J. Res. Nat. Bur Standards **48**, 345-348.

E. Wachspress (1963), *Extended application of alternating direction implicit iteration model problem theory*, SIAM J. Appl. Math. **11**, 994–1016.

W. Wendland (1968), *Die Behandlung von Randwertaufgaben im R_3 mit Hilfe von Einfach- und Doppelschichtpotentialen*, Numer. Math. **11**, 380-404.

B. Wendroff (1960), *On central difference equations for hyperbolic systems*, J. Soc. Indust. Appl. Math. **8**, 549-555.

B. Wendroff (1968), *Well-posed problems and stable difference operators*, SIAM J. Numer. Anal. **5**, 71-82.

M.F. Wheeler (1973), *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal. **10**, 723-759.

O.B. Widlund (1966), *Stability of parabolic difference schemes in the maximum-norm*, Numer. Math. **8**, 186-202.

R. Winther (1981), *A stable finite element method for initial-boundary value problems for first order hyperbolic equations*, Math. Comp. **36**, 65-86.
M. Yamaguti (1967), *Some remarks on the Lax-Wendroff finite-difference scheme for nonsymmetric hyperbolic systems*, Math. Comp. **21**, 611-619.
D. M. Young, Jr., *Iterative Methods for Solving Partial Differential Equations of Elliptic Type*, thesis, Harvard University, 1950, reprinted in `http://www-sccm.stanford.edu/pub/sccm/`.
D. M. Young (1954), *Iterative methods for soliving partial difference equations of ellipic type*, Trans. Amer. Math. Soc. **76**, 92-111.
O.C. Zienkiewicz and J.Z. Zhu (1992), *The superconvergent patch recovery and a posteriori error estimates*, Internat. J. Numer. Methods Engrg. **33**, 1331-1382.
M. Zlámal (1968), *On the finite element method*, Numer. Math. **12**, 394-409.

Survey articles and books.

1. K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, UK, 1997.
2. J.H. Argyris, *Energy theorems and structural analysis. A generalised discourse with applications on energy principles of structural analysis including the effects of temperature and non-linear stress-strain relations*, Butterworths, London, Toronto, Sydney, 1960.
3. O. Axelsson and V.A. Barker, *Finite Element Solution of Boundary Value Problems*, Academic Press, 1984.
5. I. Babuška, *Courant element: before and after*, Finite Element Methods: Fifty Years of the Courant Element (M. Křížek, P. Neittaanmäki, R. Stenberg, eds.), Decker, New York, pp. 37-51.
4. A.K. Aziz (ed.), *Survey lectures on the mathematical foundation of the finite element method*, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, Academic Press, New York, pp. 5-359.
6. J.P. Boyd, *Chebyshev & Fourier Spectral Methods*, Lecture Notes in Engineering, vol. 49, Springer, Berlin, 1989.
7. J.H. Bramble, *Multigrid Methods*, Pitman Research Notes in Mathematics, Longman, London, 1993.
8. S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, New York, 1994.
9. P. Brenner, V. Thomée, and L.B. Wahlbin, *Besov Spaces and Applications to Difference Methods for Initial Value Problems*, Springer Lecture Notes in Mathematics, Vol. 434, Springer, Berlin, Heidelberg, New York, 1975.
10. C. Canuto, Y.M. Hussaini, A. Quarteroni, and T.A. Zang, *Spectral Methods in Fluid Dynamics*, Springer, Berlin, 1988.
11. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, Oxford, 1978.
13. K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, *Introduction to adaptive methods for differential equations*, Acta Numerica, 105-158.
14. G.E. Forsythe and W.R. Wasow, *Finite difference methods for partial differential equations*, Wiley, New York, 1960.
15. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
16. E. Godlewski and P.-A. Raviart, *Hyperbolic Systems of Conservation Laws*, Ellipse, Paris, 1991.
17. D. Gottlieb and S.A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM, Philadelphia, 1977.
18. B. Gustafsson, H.-O. Kreiss, and J. Oliger, *Time Dependent Problems and Difference Methods*, Wiley, 1995.
18. W. Hackbusch, *Multi-Grid Methods and Applications*, Springer series in computational mathematics, vol. 4, Springer-Verlag, New York, 1985.
20. B. Heinrich, *Finite Difference Methods on Irregular Networks*, Akademie-Verlag, Berlin, 1987.
21. L.V. Kantorovich and V.I. Krylov, *Approximate methods in higher analysis*, Interscience, 1958.

22. M. Křižek and P. Neittaanmäki, *On superconvergence techniques*, Acta Appl. Math. **9** (1987), 175-198.
23. R.J. Le Veque, *Numerical Methods for Conservation Laws*, Birkhäuser Verlag, Basel, 1990.
24. G.I. Marchuk, *Methods of Numerical Mathematics*, Springer, New York, 1975.
25. G.I. Marchuk, *Splitting and Alternating Direction Methods*, Handbook of Numerical Analysis, Vol. I. Finite Difference Methods (Part 1) (P. G. Ciarlet and J. L. Lions, eds.), North-Holland, 1990, pp. 197-462.
26. K.W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Chapman & Hall, London, 1996.
27. P.A. Raviart and J.M. Thomas, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson, Paris, 1983.
28. R.D. Richtmyer and K.W. Morton, *Difference Methods for Initial-Value Problems*, Interscience, New York, 1967.
29. V.S. Ryabenkii and A. F. Filippov, *Über die Stabilität von Differenzengleichungen*, Deutscher Verlag der Wissenschaften, Berlin, 1960.
30. A.A. Samarskii, R.D. Lazarov, and V.L. Makarov, *Difference Schemes for Differential Equations with Generalized Solutions*, Vysshaya Shkola, Moscow, 1987. (Russian)
31. I.H. Sloan, *Error analysis of boundary integral methods*, Acta Numerica (1992), 287-339.
32. B.F. Smith, P.E. Bjørstad, and W.D. Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, 1996.
33. R.V. Southwell, *Relaxation Methods in Theoretical Physics, I, II*, Clarendon Press, Oxford, 1946, 1956.
34. G. Strang and G.J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, NJ, 1973.
35. J.C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadsworth & Brooks/Cole, 1989.
36. V. Thomée, *Stability theory for partial difference operators*, SIAM Rev. **11** (1969), 152-195.
37. V. Thomée, *Finite Difference Methods for Linear Parabolic Equations*, Handbook of Numerical Analysis, Vol. I. Finite Difference Methods (Part 1) (P. G. Ciarlet and J. L. Lions, eds.), North-Holland, 1990, pp. 5-196.
38. V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer Series in Computational Mathematics, Vol. 25, Springer Verlag, 1997.
39. R.S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
40. R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley Teubner, 1996.
41. L.B. Wahlbin, *Superconvergence in Galerkin Finite Element Methods*, Lecture Notes in Mathematics, vol. 1605, Springer, 1995.
42. W.L. Wendland, *Boundary Element Methods for Elliptic Problems*, Mathematical Theory of Finite and Boundary Element Methods (A.H. Schatz, V. Thomée, and W.L. Wendland, ed.), Birkhäuser, Basel, Boston, Berlin, 1990.
43. O.C. Zienkiewicz, *The Finite Element Method in Engineering Science*, McGraw-Hill, London, New York, 1977.

Department of mathematics, Chalmers University of Technology,
S-412 96 Göteborg, Sweden
E-mail address: thomee@math.chalmers.se